Eksamen 2010

Oppg. 1

$$P(z) = z^3 - 3z^2 - 9z - 5$$

a) $P(\sqrt{5}) = \sqrt{5}^3 - 3\cdot\sqrt{5}^2 - 9\sqrt{5} - 5 =$

$5\cdot\sqrt{5} - 3\cdot 5 - 9\cdot\sqrt{5} - 5 = \underline{\underline{-20 - 4\cdot\sqrt{5}}}$

b) Ser $P(-1) = (-1)^3 - 3\cdot(-1)^2 - 9\cdot(-1) - 5 =$

$-1 - 3 + 9 - 5 = 0$

$\Rightarrow$ $z - (-1) = z + 1$ er faktor i $P(z)$

Polynomdivisjon:

$$
\begin{array}{l}
(z^3 - 3z^2 - 9z - 5) : (z+1) = z^2 - 4z - 5 \\
\underline{-(z^3 + z^2)} \\
\quad -4z^2 - 9z \\
\quad \underline{-(-4z^2 - 4z)} \\
\qquad -5z - 5 \\
\qquad \underline{-(-5z - 5)} \\
\qquad\qquad 0
\end{array}
$$

$z^2 - 4z - 5 = 0$

$z = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4\cdot 1\cdot(-5)}}{2\cdot 1} = 2 \pm 3$

$z = 2 - 3 = -1 \quad \vee \quad z = 2 + 3 = 5$

$\Rightarrow P(z) = (z+1)^2 (z-5)$

$P(z) = 0 \Leftrightarrow$

$(z+1)^2 (z-5) = 0 \Leftrightarrow$

$z+1 = 0 \vee z-5 = 0 \Leftrightarrow$

$\underline{z = -1 \vee z = 5}$

c) $P(z) \leq 0$

Fortegnsskjema:

		-1		5	
$(z+1)^2$	+	0	+		+
$z-5$	−		−	0	+
$P(z)$	−	0	−	0	+

Ser:

$P(z) < 0 \Leftrightarrow \underline{\underline{z \leq 5}}$

Oppg. 2

$\vec{u} = [0, 1, 2], \quad \vec{v} = [1, -3, -2]$

a) $\vec{u} \cdot \vec{v} = 0 \cdot 1 + 1 \cdot (-3) + 2 \cdot (-2) = -3 - 4 = \underline{\underline{-7}}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 0 & 1 & 2 \\ 1 & -3 & -2 \end{vmatrix} = [-2 - 2 \cdot (-3), 2, -1] = \underline{\underline{[4, 2, -1]}}$$

$\vec{v} \times \vec{u} = -\vec{u} \times \vec{v} = \underline{\underline{[-4, -2, 1]}}$

b) Dersom $\vec{u} \perp \vec{v}$: $\vec{u} \cdot \vec{v} = 0$

Sidan $\vec{u} \cdot \vec{v} \neq 0$, er $\vec{u}$ og $\vec{v}$ ikkje ortogonale

Dersom $\vec{u} \parallel \vec{v}$: $\vec{u} \times \vec{v} = \vec{0}$

Sidan $\vec{u} \times \vec{v} \neq \vec{0}$, er $\vec{u}$ og $\vec{v}$ ikkje parallelle.

c) Dersom $\vec{w}$ ligg i planet spent ut av $\vec{u}$ og $\vec{v}$, må det finnes en x og en y slik at

$$\vec{w} = x \cdot \vec{u} + y \cdot \vec{v}$$

$$[1, 3, 3] = x \cdot [0, 1, 2] + y \cdot [1, -3, -2]$$

$$[1, 3, 3] = [y, x - 3y, 2x - 2y]$$

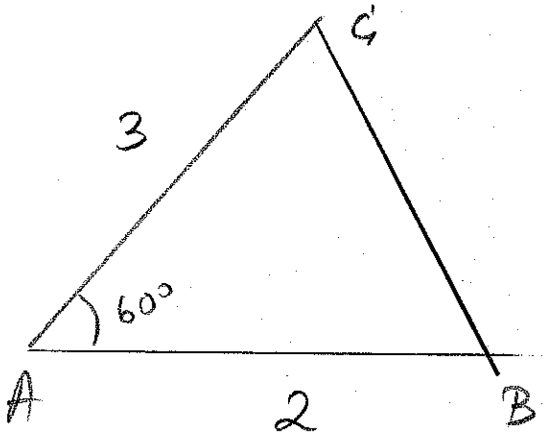
$$\begin{cases} y = 1 & (i) \\ x - 3y = 3 & (ii) \\ 2x - 2y = 3 & (iii) \end{cases}$$

(i) i (ii): $x - 3 \cdot 1 = 3 \Leftrightarrow x = 6$

(i) i (iii): $2x - 2 \cdot 1 = 3 \Leftrightarrow x = \frac{5}{2}$

x kan ikkje vere både 6 og $\frac{5}{2}$. Likninga $\vec{w} = x\vec{u} + y \cdot \vec{v}$ har inga løysing. Difor ligg $\vec{w}$ ikkje i planet spent ut av $\vec{u}$ og $\vec{v}$. (q.e.d.)

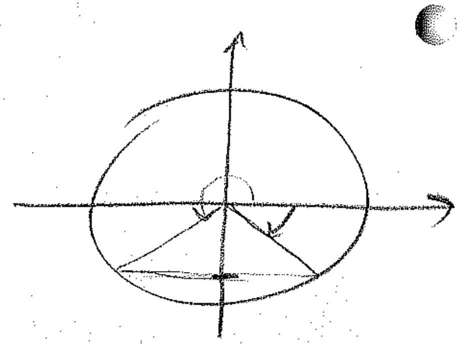
Oppg. 3



a) Area: $\frac{1}{2} AB \cdot AC \cdot \sin = \frac{1}{2} \cdot 2 \cdot 3 \cdot \sin 60° = 3 \cdot \frac{\sqrt{3}}{2} =$

$3 \cdot \frac{\sqrt{3}}{2} = \underline{\underline{\frac{3^{3/2}}{2}}}$

b) $5 \sin\alpha + 3 = 0, \quad \alpha \in [0, 2\pi\rangle$



$\sin\alpha = -\frac{3}{5}$

$\alpha = \sin^{-1}(-\frac{3}{5}) + n \cdot 2\pi \quad \vee \quad \alpha = \pi - \sin^{-1}(-\frac{3}{5}) + n \cdot 2\pi$

$n \in \mathbb{Z}$

$\sin^{-1}(-\frac{3}{5}) \approx -0{,}644$

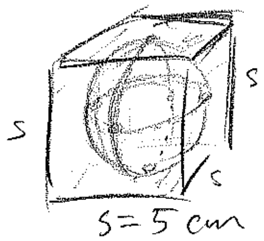
$\alpha \in [0, 2\pi\rangle$

$\alpha \approx -0{,}644 + 2\pi = 5{,}64 \quad \vee$

$\alpha \approx \pi - (-0{,}644) = 3{,}79$

$\underline{\alpha \approx 5{,}64 \quad \vee \quad \alpha \approx 3{,}79}$

c)



Radien til den største kula vi kan fjerne er $r = \frac{s}{2} = \frac{5}{2}$ cm

Volum av kube: $V_{\square} = s^3 = 5^3\,cm^3 = 125\,cm^3$

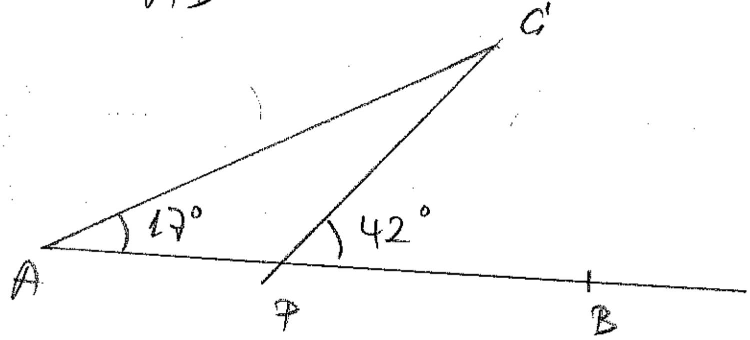
— " — kule: $V_{O} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \left(\frac{5}{2}\right)^3 cm^3 =$

$\frac{4 \cdot 125\pi}{3 \cdot 8} cm^3 = \frac{125\pi}{6} cm^3$

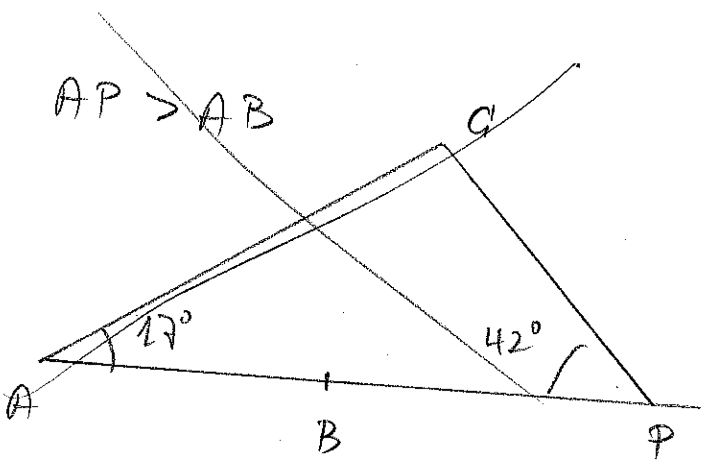
Volumet av det som vert att:

$V_{\square} - V_{O} = \left(125 - \frac{\pi}{6} \cdot 125\right) cm^3 = \underline{\underline{\left(1 - \frac{\pi}{6}\right) 125\ cm^3}}$

d) 1) $AP < AB$



2) $AP > AB$



1) Sinus-setningen:

$$\frac{AC}{\sin \angle APC} = \frac{AP}{\sin \angle ACP}$$

$\angle APC = 180° - 42° = 138°$

$\angle ACP = 180° - \angle A - \angle APC =$

$180° - 17° - (180 - 42°) = 42° - 17° = 25°$

$$AC = AP \cdot \frac{\sin \angle APC}{\sin \angle ACP} = 2 \cdot \frac{\sin 138°}{\sin 25°} \approx \underline{\underline{3,17}}$$

2) Sinus-setningen:

$$\frac{AC}{\sin \angle APC} = \frac{AP}{\sin \angle ACP}$$

$\angle APC = 42°$

$\angle ACP = 180° - \angle A - \angle APC = 180° - 17° - 42° = 121°$

$$AC = AP \frac{\sin \angle APC}{\sin \angle ACP} = 2 \cdot \frac{\sin 42°}{\sin 121°} \approx \underline{\underline{1,56}}$$

Oppg. 4

a) i) $f(x) = 2 + \sqrt{x} + \frac{1}{x}$

$$f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} + \frac{1}{-1} x^{-1-1} = \underline{\underline{\frac{1}{2\sqrt{x}} - \frac{1}{x^2}}}$$

ii) $g(x) = (x^2+1)\cos(\pi x)$

$$g'(x) = (2x+0)\cos(\pi x) + (x^2+1)\cdot(-\sin(\pi x)\cdot \pi) =$$

$$\underline{\underline{2x \cos(\pi x) - \pi (x^2+1)\sin(\pi x)}}$$

iii) $h(x) = \ln(e^{2x}\sin x)$

$u(x) = e^{2x}\sin x \,, \quad h(x) = \ln(u(x))$

$\frac{du}{dx} = e^{2x}\cdot 2\sin x + e^{2x}\cos x = e^{2x}(2\sin x + \cos x)$

$$\frac{dh}{dx} = \frac{dh}{du}\cdot\frac{du}{dx} = \frac{1}{u(x)}\cdot\frac{du}{dx} = \frac{e^{2x}(2\sin x+\cos x)}{e^{2x}\sin x} =$$

$$\frac{2\sin x + \cos x}{\sin x} = \underline{\underline{2 + \frac{\cos x}{\sin x}}}$$

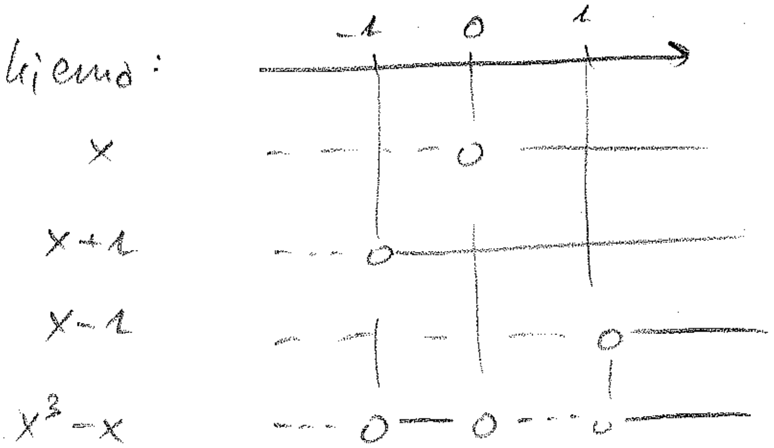
b) $f(x) = \ln(x^3 - x)$

i) krav: $x^3 - x > 0$

$x(x^2-1) > 0$

$x(x+1)(x-1) > 0$

Forteiknskjema:



$x^3 - x > 0 \iff x \in \langle -1, 0\rangle \cup \langle 1, \rightarrow\rangle$

Størst mogeleg definisjonsmengd: $\underline{\underline{D_f = \langle -1, 0\rangle \cup \langle 1, \rightarrow\rangle}}$

ii) Stasjonære punkt: $f'(x) = 0$

$$f'(x) = \frac{1}{x^3-x}\cdot(x^3-x)' = \frac{3x^2-1}{x^3-x} = \frac{3\left(x+\frac{1}{\sqrt{3}}\right)\left(x-\frac{1}{\sqrt{3}}\right)}{x(x+1)(x-1)}$$

$f'(x) = 0$

$3x^2 - 1 = 0$

$x = \pm \frac{1}{\sqrt{3}}$

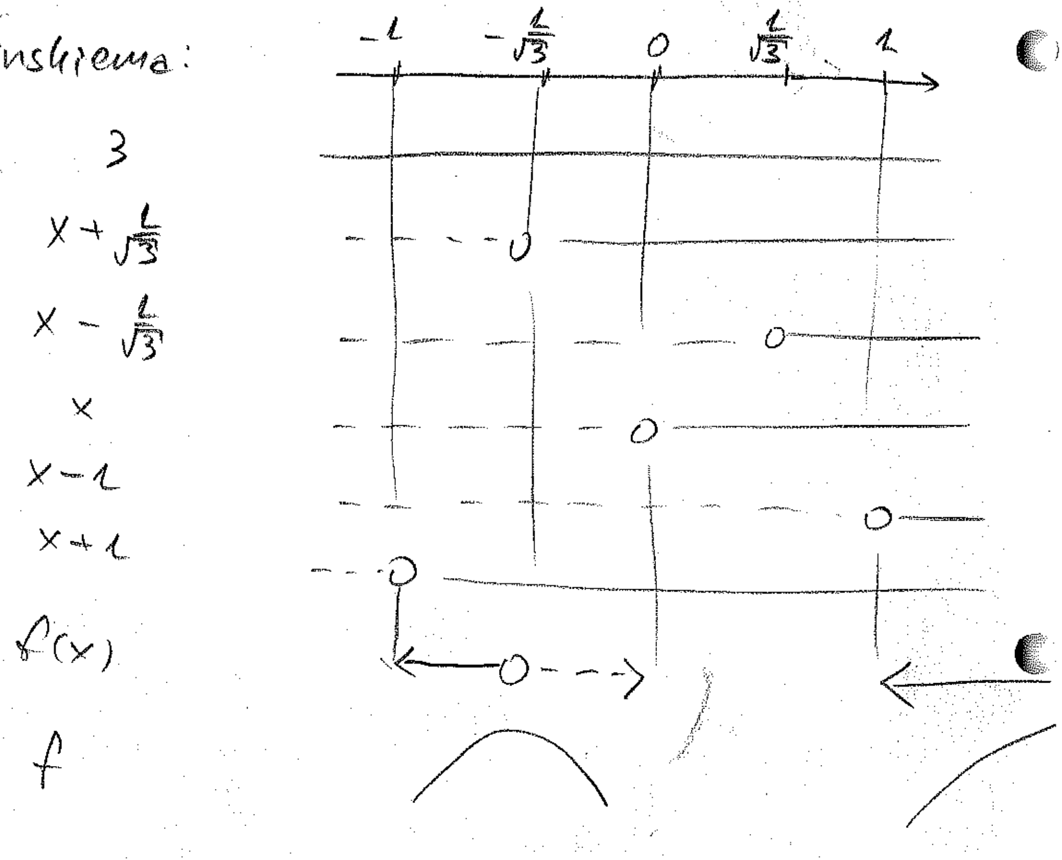
(ser at nemnaren i $f'(x)$ er ulik 0 for $x = \pm \frac{1}{\sqrt{3}}$).

$+\frac{1}{\sqrt{3}} \notin D_f$

$f\left(-\frac{1}{\sqrt{3}}\right) = \ln\left(\left(-3^{-1/2}\right)^3 - \left(-3^{-1/2}\right)\right) = \ln\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}^3}\right)$

Kritisk punkt: $\underline{\left(-\frac{1}{\sqrt{3}}, \ln\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}^3}\right)\right)}$

iii) Forteiknskjema:



$x \to 0^-$: $x^3 - x \to 0^+ \Rightarrow f(x) = \ln(x^3 - x) \to -\infty$

$x \to \infty$: $x^3 - x \to \infty \Rightarrow f(x) \to +\infty$

$f(x)$ har ingen veldefinert største eller minste verdi.

c) $\int \sqrt{x}\,dx = \int x^{1/2}dx = \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + C = \underline{\frac{2}{3} x^{3/2} + C}$

$\int \frac{3}{7-x}\,dx = -3\int \frac{1}{x-7}\,dx = \underline{-3\ln|x-7| + C}$

$\int \frac{3x}{x^2-2}\,dx$

Variabelskifte:

$u(x) = x^2-2$

$\frac{du}{dx} = 2x$

$dx = \frac{1}{2x}du$

$\int \frac{3x}{x^2-2}\,dx = \int \frac{3x}{u}\cdot\frac{1}{2x}du = \frac{3}{2}\int\frac{du}{u} = \frac{3}{2}\ln|u| + C =$

$\underline{\frac{3}{2}\ln|x^2-2| + C}$

$\int (x+1)\sin(2x)\,dx$

Delvis integrasjon:

$u = x+1, \quad v' = \sin(2x)$

$u' = 1, \quad v = -\frac{1}{2}\cos(2x)$

$\int (x+1)\sin(2x)dx = (x+1)\cdot\left(-\frac{1}{2}\right)\cos(2x) - \int 1\cdot\left(-\frac{1}{2}\cos(2x)\right)dx =$

$-\frac{1}{2}(x+1)\cos(2x) + \frac{1}{2}\int\cos(2x)dx = \underline{-\frac{1}{2}(x+1)\cos(2x) + \frac{1}{4}\sin(2x) + C}$

d) $\int_{-1}^{1}(x^2 - x(x+1))dx = \int_{-1}^{1}(x^2-x^2+x)dx = -\int_{-1}^{1}x\,dx =$

$-\left[\frac{1}{2}x^2\right]_{-1}^{1} = -\left(\frac{1}{2}\cdot 1^2 - \frac{1}{2}\cdot(-1)^2\right) = \underline{0}$

$$\int_{-\pi}^{\pi} x^{17} \cos x \, dx$$

Integranden, $x^{17}\cos x$, er ein odde funksjon, og integrasjons intervallet er symmetrisk kring $x=0$

$\Rightarrow$ Integralet får like store bidrag med motsatt forteikn på kvar side av y-aksen

$$\int_{-\pi}^{\pi} x^{17} \cos x \, dx = \underline{\underline{0}}$$

$$\int_0^3 x e^{1-x} dx$$

Delvis int.:

$u = x, \quad v' = e^{1-x}$

$u' = 1, \quad v = -e^{1-x}$

$$\int_0^3 x e^{1-x} dx = [x \cdot (-e^{1-x})]_0^3 - \int_0^3 1 \cdot (-e^{1-x}) dx =$$

$$[-x e^{1-x}]_0^3 + \int_0^3 e^{1-x} dx =$$

$$(-3 \cdot e^{1-3} - (-0 \cdot e^{1-0})) + [-e^{1-x}]_0^3 =$$

$$-3e^{-2} + (-e^{1-3} - (-e^{1-0})) =$$

$$-3e^{-2} - e^{-2} + e = \underline{\underline{e - \frac{4}{e^2}}}$$

$$\int_{-\pi}^{\pi} \sin^6\left(\frac{x}{2}\right)\cos\frac{x}{2}\,dx$$

Delvis int.:

$u = \sin^6\frac{x}{2}$, $v' = \cos\frac{x}{2}$

$u' = 6\sin^5\frac{x}{2}\cos\frac{x}{2}\cdot\frac{1}{2} = 3\sin^5\frac{x}{2}\cos\frac{x}{2}$, $v = 2\sin\frac{x}{2}$

$$\int_{-\pi}^{\pi}\sin^6\frac{x}{2}\cos\frac{x}{2} = \left[\sin^6\frac{x}{2}\cdot 2\sin\frac{x}{2}\right]_{-\pi}^{\pi} -$$

$$\int_{-\pi}^{\pi} 3\sin^5\frac{x}{2}\cdot\cos\frac{x}{2}\cdot 2\sin\frac{x}{2}\,dx =$$

$$2\left[\sin^7\frac{x}{2}\right]_{-\pi}^{\pi} - 6\int_{-\pi}^{\pi}\sin^6\frac{x}{2}\cos\frac{x}{2}\,dx =$$

$$2\left(\sin^7\frac{\pi}{2} - \sin^7\left(-\frac{\pi}{2}\right)\right) - 6\int_{-\pi}^{\pi}\sin^6\frac{x}{2}\cos\frac{x}{2}\,dx$$

$$7\int_{-\pi}^{\pi}\sin^6\frac{x}{2}\cos\frac{x}{2}\,dx = 2\cdot(1-(-1)) = 4$$

$$\int_{-\pi}^{\pi}\sin^6\frac{x}{2}\cos\frac{x}{2}\,dx = \underline{\underline{\frac{4}{7}}}$$

Enklare: Variabelbyte

$u(x) = \sin\frac{x}{2}$, $\frac{du}{dx} = \frac{1}{2}\cos\frac{x}{2}$

$dx = \frac{1}{\frac{1}{2}\cos\frac{x}{2}}\,du$

$u(-\pi) = -1$, $u(\pi) = 1$

$\int_{-\pi}^{\pi}\sin^6\frac{x}{2}\cos\frac{x}{2}\,dx =$

$\int_{-1}^{1} u^6\cos\frac{x}{2}\,\frac{1}{\frac{1}{2}\cos\frac{x}{2}}\,du =$

$2\int_{-1}^{1} u^6\,du = 2\left[\frac{1}{7}u^7\right]_{-1}^{1} =$

$\frac{2}{7}(1^7 - (-1)^7) = \underline{\underline{\frac{4}{7}}}$

Oppg. 5 $p(x,t) = \frac{1}{\sqrt{2\pi t}}\,e^{-\frac{x^2}{2t}}$, $t > 0$

a) - Ser på t som ein konstant.

$x \to \infty \Rightarrow e^{-\frac{x^2}{2t}} \to 0$

Altså: $p(x,t) \to 0$ når $x \to \infty$

b) - Ser på x som ein konstant og lar

$t \to \infty$: $e^{-\frac{x^2}{2t}} \to e^{-0} = 1$, $\frac{1}{\sqrt{2\pi t}} \to 0$

Altså: $\rho(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \to 0 \cdot 1 = 0$ når $t \to \infty$

c) Resultatet i a) seier at langt borte frå boksen er der lite gass. Dette verkar rimeleg. Resultatet i b) seier at gassen vert veldig tynn etter lang tid. Dette verkar også rimeleg; etterkvart som gassen spreier seg over „heile verda", vert han uendeleg tynn.

d) $\rho(2,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{2^2}{2t}} = \frac{1}{\sqrt{2\pi t}} e^{-\frac{2}{t}}$

- Kallar $\rho(2,t)$ for $f(t)$.

$f'(t) = \left((2\pi t)^{-1/2} e^{-\frac{2}{t}}\right)' = -\frac{1}{2}(2\pi t)^{-3/2} \cdot 2\pi e^{-2/t} +$

$(2\pi t)^{-1/2} \cdot e^{-\frac{2}{t}} \cdot \left(-\frac{2}{t}\right)' =$

$-\frac{\pi}{\sqrt{2\pi t}^3} e^{-2/t} + \frac{1}{\sqrt{2\pi t}} e^{-2/t} \cdot (-2 \cdot (-1) t^{-2}) =$

$\frac{1}{\sqrt{2\pi t}} e^{-2/t} \left(\frac{2}{t^2} - \frac{\pi}{2\pi t}\right) = \frac{1}{\sqrt{2\pi t}} e^{-2/t} \left(\frac{2}{t^2} - \frac{1}{2t}\right)$

$f'(t) = 0$

$\frac{1}{\sqrt{2\pi t}} e^{-2/t} \left(\frac{2}{t^2} - \frac{1}{2t}\right) = 0$

$t^{-2}\left(2 - \frac{1}{2}t\right) = 0$

$2 - \frac{1}{2}t = 0$

$t = 4$

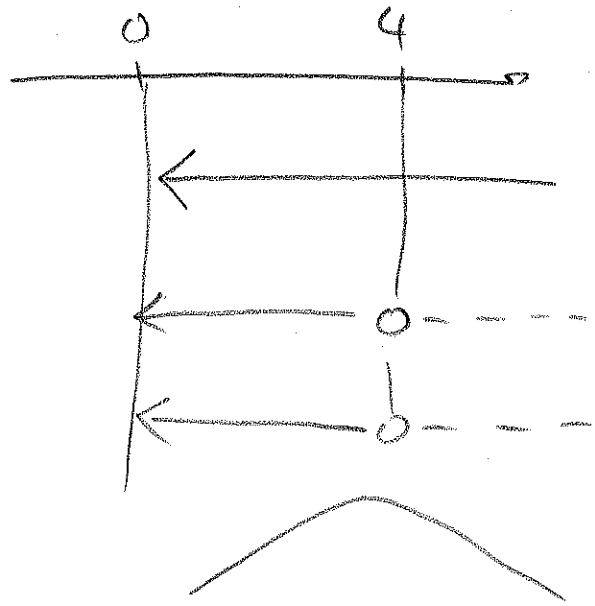
Forteiknskjema

$\frac{1}{\sqrt{2\pi t}} e^{-2/4} t^{-2}$

$2 - \frac{4}{2} t$

$f'(t)$

f



Av forteiknskjemaet ser vi at $f(t) = p(2,t)$ er maksimal for $\underline{\underline{t=4}}$.