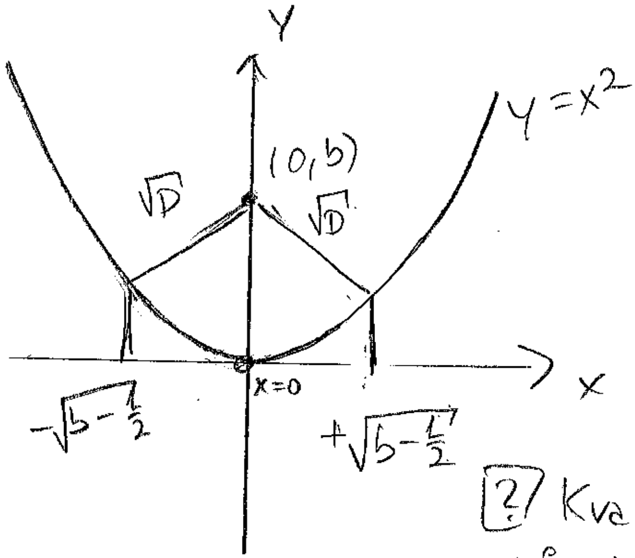
# Førelesing 1/2

① Presisering: Det siste eksemplet i går var vanskeleg.

Oppsummering:

Avstand frå $(0,b)$ til parabel: $\sqrt{D}$, der




$$D(x) = x^2 + (x^2-b)^2$$
$$= x^4 - (2b-1)x^2 + b^2$$

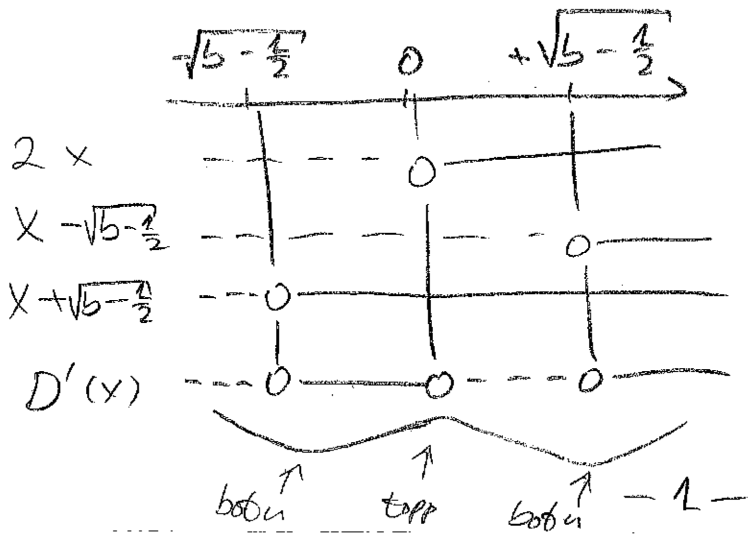
$\boxed{?}$ Kva når $b<0$?

$$D'(x) = 4x^3 - 2(2b-1)x = 2x(2x^2-2b+1)$$

$D'(x) = 0$ når $x=0$ og når $x = \pm\sqrt{b-\frac{1}{2}}$

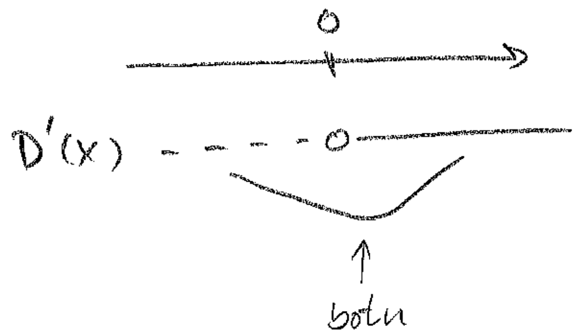
når $b \geq \frac{1}{2}$

$\boxed{?}$ Men er desse punkta botnpunkt?




$b > \frac{1}{2}$

$b < \frac{1}{2}$: $2x^2 - 2b + 1$ alltid positivt




Alternativt:

Sjå på $D''(x)$

$D''(x) = 12x^2 - 4b + 2$

$D''(0) = -4b + 2$

$D''\left(\pm\sqrt{b - \frac{1}{2}}\right) = 12\left(b - \frac{1}{2}\right) - 4b + 2 = 8b - 4$

$b < \frac{1}{2}$

$D''(0) = -4b + 2 > 0$ $\cup$ botn

$b > \frac{1}{2}$

$D''(0) = -4b + 2 < 0$ $\cap$ topp

$D''\left(\pm\sqrt{b - \frac{1}{2}}\right) = 8b - 4 > 0$ $\cup$ botn

## ② Deriverte av produkt

Veit: $(u(x) + v(x))' = u'(x) + v'(x)$

Stemmer dette:

$$(u(x) \cdot v(x))' = u'(x) \cdot v'(x) \quad ?$$

- Sjekkar med $x^3$:

$$(x^3)' = 3x^2$$

Også: $(x^3)' = (x^2 \cdot x)' \overset{?}{=} (x^2)' \cdot (x)' = 2x \cdot 1 = 2x$

- Ser ikkje ut til å stemme.

Dette stemmer:

$$(u(x) \cdot v(x))' = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

Sjekk:

$$(x^3)' = (x^2 \cdot x)' = (x^2)' \cdot x + x^2 \cdot (x)' =$$

$$2x \cdot x + x^2 \cdot 1 = 2x^2 + x^2 = 3x^2 \qquad \text{OK}$$

## ③ Bevis

Namn: $f(x) = u(x) \cdot v(x)$

$$(u(x) \cdot v(x))' = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$\lim_{h\to 0} \frac{1}{h}\left(u(x+h)\cdot v(x+h) - u(x)\cdot v(x)\right) =$$

$$\lim_{h\to 0} \frac{1}{h}\left(u(x+h)\cdot v(x+h) \underbrace{- u(x)v(x+h) + u(x)v(x+h)}_{0} - u(x)\cdot v(x)\right) =$$

$$\lim_{h\to 0}\left(\frac{u(x+h)-u(x)}{h}\cdot v(x+h) + u(x)\,\frac{v(x+h)-v(x)}{h}\right) =$$

$$\underbrace{\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}}_{u'(x)}\cdot\underbrace{\lim_{h\to 0} v(x+h)}_{v(x)} + u(x)\cdot\underbrace{\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}}_{v'(x)}$$

$$= u'(x)\cdot v(x) + u(x)\cdot v'(x)$$ (q.e.d.)

## (3) Eksempel

Derivēr desse funksjonane ved å bruke produktregelen:

a) $a(x) = x^2$

b) $b(x) = \sqrt[3]{x}\cdot\frac{x^5}{5}$

c) $c(x) = \frac{x^2-1}{x}$

L:

a) $a'(x) = 2x$ — det veit vi jo.

$a'(x) = (x\cdot x)' = 1\cdot x + x\cdot 1 = \underline{2x}$ OK

b) $b'(x) = \left(x^{1/3}\cdot\frac{x^5}{5}\right)' = \frac{1}{3}x^{\frac{1}{3}-1}\cdot\frac{x^5}{5} + x^{1/3}\,\frac{5x^4}{5} =$

$\frac{1}{15}x^{-\frac{2}{3}}\cdot x^5 + x^{1/3}\cdot x^4 = \frac{1}{15}x^{13/3} + x^{13/3} = \underline{\underline{\frac{16}{15}x^{13/3}}}$

c) $c'(x) = \left((x^2-1)\cdot x^{-1}\right)' = 2x\cdot x^{-1} + (x^2-1)\cdot(-1)x^{-2} =$

$$\underline{\underline{2 - \frac{x^2-1}{x^2}}}$$

## (4) Kjerneregelen

Dersom ein funksjon $f(x)$ kan skrivast som ein funksjon (g) av ein annan funksjon (u):

$f(x) = g(u(x))$:

$$f'(x) = g'(u)\cdot u'(x)$$

Eksempel: $f(x) = (x^2+2)^3$

Kan velge $u(x) = x^2+2$

$f(x) = (u(x))^3$

$g(u) = u^3$

Kjerneregelen påstår:

$f'(x) = g'(u)\cdot u'(x)$

$g'(u) = 3u^2$

$u'(x) = (x^2+2)' = 2x$

Altså: $f'(x) = 3u^2\cdot 2x = 3(x^2+2)^2\cdot 2x = 6x(x^2+2)^2$

Stemmer dette?

$f(x) = (x^2+2)^3 = (x^2+2)(x^2+2)^2 = (x^2+2)(x^4+4x^2+4) =$

$x^6+4x^4+4x^2+2x^4+8x^2+8=$

$x^6+6x^4+12x^2+8$

$f'(x)=6x^5+6\cdot 4x^3+12\cdot 2x+0=6x^5+24x^3+24x=$

$6x(x^4+4x^2+4)=6x(x^2+2)^2$

– Ser: Det gjelde kjappare med kjerneregelen.

## (6) Bevis for kjerneregelen

$f(x)=g(u(x))$

Antar at $f$, $g$ og $u$ alle er deriverbare

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{g(u(x+h))-g(u(x))}{h}=$$

$$\lim_{h\to 0}\frac{[g(u(x+h))-g(u(x))]\cdot(u(x+h)-u(x))}{h\cdot(u(x+h)-u(x))}=$$

$$\lim_{h\to 0}\frac{g(u(x+h))-g(u(x))}{u(x+h)-u(x)}\cdot\frac{u(x+h)-u(x)}{h}=\lim_{h\to 0}\frac{g(u(x+h))-g(u(x))}{u(x+h)-u(x)}\cdot\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

Namn: $u(x+h)-u(x)=k$

$\Leftrightarrow u(x+h)=u(x)+k$

Ser også: $h\to 0 \Rightarrow k\to 0$

Vi får:

$$f'(x)=\underbrace{\lim_{k\to 0}\frac{g(u+k)-g(u)}{k}}_{g'(u)}\cdot\underbrace{\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}}_{u'(x)}=g'(u)\cdot u'(x)$$

(q.e.d.)

## 5 Eksempel med kjerneregelen

Deriver desse funksjonane:

$a(x) = (x+1)^2$

$b(x) = \sqrt{x^2-1}$

$c(x) = x^3 \cdot (x-3)^{1/3}$

$d(x) = \sqrt{x^2 + (x^2-b)^2}$

a)

$u = x+1$

$g(u) = u^2$

$a'(x) = g'(u) \cdot u'(x) = 2u \cdot 1 = 2u = \underline{\underline{2(x+1)}}$

b)

$u = x^2-1$

$g(u) = \sqrt{u} = u^{1/2}$

$b'(x) = \frac{1}{2} u^{1/2-1} \cdot u'(x) = \frac{1}{2} u^{-1/2} \cdot 2x = \frac{x}{\sqrt{u}} = \underline{\underline{\frac{x}{x^2-1}}}$

c)

$c'(x) = (x^3)' \cdot (x-3)^{1/3} + x^3 \cdot ((x-3)^{1/3})'$ (produktregelen)

$u = x-3$

$g(u) = u^{1/3}$

$((x-3)^{1/3})' = \frac{1}{3} u^{\frac{1}{3}-1} \cdot u'(x) = \frac{1}{3} u^{-2/3} \cdot 1 = \frac{1}{3} (x-3)^{-2/3}$

$c'(x) = 3x^2 \cdot (x-3)^{1/3} + x^3 \cdot \frac{1}{3} (x-3)^{-2/3} =$

$x^2 \left[ 3(x-3)^{1/3} + \frac{1}{3} x (x-3)^{-2/3} \right]$

d)

$\cdot\, d(x) = \sqrt{x^2 + (x^2-5)^2}$

$u = x^2 + (x^2-5)^2$

$g(u) = \sqrt{u}$

$d'(x) = g'(u) \cdot u'(x) = \frac{1}{2} u^{1/2-1} \cdot u'(x) = \frac{u'(x)}{2\sqrt{u}}$

$u'(x) = 2x + 2(x^2-5)^{2-1} \cdot 2x =$

$2x + (2x^2 - 25) \cdot 2x = 2x + 4x^3 - 45x$

$d'(x) = \frac{2x + 4x^3 - 45x}{2\sqrt{x^2 + (x^2-5)^2}} = \underline{\underline{\frac{2x^3 - (25-1)x}{\sqrt{x^2+(x^2-5)^2}}}}$