Førelesing 4/4

① Ny oblig ute

② Kjapp rep. av det siste vi har gått gjennom:

1) Integrasjon ved variabelskifte

$\int f(x)\,dx$

Dersom $f(x) = g(u(x)) \cdot u'(x)$

$\int f(x)\,dx = \int g(u)\,du$

2) Delvis integrasjon

$\int u \cdot v'\,dx = u \cdot v - \int u' \cdot v\,dx$

Eksempel:

Bestem integralet $\int (2x+3)\sin x\,dx$

$u = 2x+3, \quad v' = \sin x$

$u' = 2, \quad v = -\cos x$

$\int (2x+3)\sin x\,dx = (2x+3)(-\cos x) - \int 2 \cdot (-\cos x)\,dx =$

$-(2x+3)\cos x + 2\int \cos x\,dx = \underline{-(2x+3)\cos x + 2\sin x + C}$

③ Eksempel (pent triks)

Finn dette ubestemte integralet ved delvis integrasjon:

$$\int \sin(3x+1)\, e^x \, dx.$$

$u = \sin(3x+1)$

$v' = e^x$

$u' = \cos(3x+1) \cdot 3$

$v = e^x$

$$I = \int \sin(3x+1)\, e^x = \sin(3x+1)\, e^x - 3\int \cos(3x+1)\, e^x dx$$

$$\int \cos(3x+1)\, e^x \, dx$$

$u = \cos(3x+1)$

$v = e^x$

$u' = -\sin(3x+1) \cdot 3$

$v' = e^x$

$$\int \cos(3x+1)\, e^x \, dx = \cos(3x+1)\, e^x - 3\int (-\sin(3x+1))\, e^x \, dx =$$

$$\cos(3x+1)\, e^x + 3 \underbrace{\int \sin(3x+1)\, e^x \, dx}_{I + C_1}$$

Altså:

$$I = \sin(3x+1)\, e^x - 3 \cdot \left(\cos(3x+1)\, e^x + 3I + 3C_1 \right) =$$

$$\sin(3x+1)\, e^x - 3\cos(3x+1)\, e^x - 9I - 9C_1 \iff$$

$$I + 9I = \sin(3x+1)\, e^x - 3\cos(3x+1)\, e^x - 9C_1$$

$$10I = e^x(\sin(3x+1) - 3\cos(3x+1)) - 9C_1'$$

$$I = \frac{e^x}{10}(\sin(3x+1) - 3\cos(3x+1)) \underbrace{- \frac{9C_1'}{10}}_{C'}$$

$$\int \sin(3x+1)\, e^x = \underline{\frac{e^x}{10}(\sin(3x+1) - 3\cos(3x+1)) + C'}$$

Sjekker svaret ved derivasjon

$$\left[\frac{e^x}{10}(\sin(3x+1) - 3\cos(3x+1)) + C\right]' =$$

$$\frac{e^x}{10}(\sin(3x+1) - 3\cos(3x+1)) + \frac{e^x}{10}(3\cos(3x+1) + 9\sin(3x+1))$$

$$= \frac{e^x}{10}(\sin(3x+1) - \underline{3\cos(3x+1) + 3\cos(3x+1)} + 9\sin(3x+1)) =$$

$$\frac{e^x}{\cancel{10}} \cdot \cancel{10}\sin(3x+1) = \underline{\sin(3x+1)\, e^x}$$

④ Eksempel

Finn desse bestemte integrala

a) $\int_{-2}^{2} x^2 e^{x^3+3} dx$

b) $\int_{-2}^{-1} (x^2+2) \ln|x| \, dx$

c) $\int_0^1 \sin(3x+1) e^x dx$ (jmf. tidlegare eksempel)

a) $\int_{-2}^{2} x^2 e^{x^3+3} dx$

$u = x^3+3$

$\frac{du}{dx} = 3x^2$

$dx = \frac{1}{3x^2} du$

$u(-2) = (-2)^3+3 = -5$

$u(2) = 2^3+3 = 11$

$\int_{-2}^{2} x^2 e^{x^3+3} dx = \int_{-5}^{11} e^u \cdot x^2 \frac{1}{3x^2} du =$

$\frac{1}{3}\int_{-5}^{11} e^u \, du = \frac{1}{3}[e^u]_{-5}^{11} = \underline{\frac{1}{3}(e^{11}-e^{-5})}$

b) $\int_{-2}^{-1} (x^2+2) \ln|x| \, dx$

$u = \ln|x|$, $v' = x^2+2$

$u' = \frac{1}{x}$, $v = \frac{1}{3}x^3+2x$

$\int_{-2}^{-1} (x^2+2) \ln|x| dx = [\ln|x| (\frac{1}{3}x^3+2x)]_{-2}^{-1} - \ldots$

$\int_{-2}^{-1} \frac{1}{x} \cdot (\frac{1}{3}x^3+2x) \, dx =$

$\ln|-1| \cdot \left(\frac{1}{3}\cdot(-1)^3 + 2\cdot(-1)\right) - \ln|-2|\left(\frac{1}{3}\cdot(-2)^3 + 2\cdot(-2)\right) -$

$\int_{-2}^{-1}\left(\frac{1}{3}x^2+2\right)dx = -\ln 2\left(-\frac{8}{3}-4\right) -$

$\left[\frac{1}{9}x^3 + 2x\right]_{-2}^{-1} = \frac{20}{3}\ln 2 - \left(\frac{1}{9}\cdot(-1)^3 + 2\cdot(-1) - \left(\frac{1}{9}\cdot(-2)^3 + 2\cdot(-2)\right)\right)$

$= \frac{20}{3}\ln 2 - \left(-\frac{1}{9} - 2 + \frac{8}{9} + 4\right) = \underline{\frac{20}{3}\ln 2 - \frac{25}{9}}$

$(\approx 1,843)$

c) $\int_0^1 \sin(3x+1)\, e^x\, dx$

$u = \sin(3x+1), \quad v' = e^x$

$u' = 3\cos(3x+1), \quad v = e^x$

$\int_0^1 \sin(3x+1)\, e^x\, dx = \left[\sin(3x+1)\, e^x\right]_0^1 -$

$3\int \cos(3x+1)\, e^x\, dx$

Ny u og v':

$u = \cos(3x+1), \quad v' = e^x$

$u' = -3\sin(3x+1), \quad v = e^x$

$\int_0^1 \sin(3x+1)\, e^x\, dx = \sin 4\cdot e - \sin 1\cdot e^0 -$

$3\left(\left[\cos(3x+1)\, e^x\right]_0^1 - \int_0^1 (-3\sin(3x+1))\cdot e^x\, dx\right) =$

$e\cdot\sin 4 - \sin 1 - 3\left(\cos 4\cdot e - \cos 1\cdot e^0 +\right.$

$\left.3\int_0^1 \sin(3x+1)\cdot e^x\, dx\right) =$

$e\cdot\sin 4 - \sin 1 - 3e\cos 4 + 3\cos 1 - 9\int_0^1 \sin(3x+1)\, e^x dx$

Altså:

$\int_0^1 \sin(3x+1)e^x dx = e(\sin 4 - 3\cos 4) + 3\cos 1 - \sin 1 - 9\int_0^1 \sin(3x+1)\, e^x\, dx$

$10\int_0^1 \sin(3x+1)e^x dx = e(\sin 4 - 3\cos 4) + 3\cos 1 - \sin 1$

$\underline{\int_0^1 \sin(3x+1)\, e^x\, dx = \frac{1}{10}\left(e(\sin 4 - 3\cos 4) + 3\cos 1 - \sin 1\right)} \; (\approx 0,4053)$

5 Integrasjon ved delbrøkoppspalting

Vi veit korleis vi finn integral av typen $\int \frac{1}{ax+b}\,dx \overset{?}{=} \frac{1}{a}\ln|ax+b|+C$

Vi veit også korleis vi kan finne integral av typen $\int \frac{p(x)}{ax+b}\,dx$ der $p(x)$ er eit polynom.

[?] Polynomdivisjon

Men korleis finn vi slike integral:

$$\int \frac{x+2}{x^2-5x+6}\,dx \;?$$

Vi faktoriserer nemnaren og prøver å skrive brøken som ein sum av brøkar der alle nemnarane er 1.-grads-polynom.

1) Faktoriserar nemnaren:

$$x^2-5x+6=0$$

$$x=\frac{5\pm\sqrt{25-4\cdot 6}}{2}=\frac{5\pm 1}{2}$$

$$x=2 \lor x=3$$

$$x^2-5x+6=1\cdot(x-2)(x-3)$$

2) Vis kan skrive integranden som

$$\frac{x+2}{(x-2)(x-3)}=\frac{A}{x-2}+\frac{B}{x-3}$$

$$\frac{A}{x-2}+\frac{B}{x-3}=\frac{A(x-3)+B(x-2)}{(x-2)(x-3)}=\frac{(A+B)x-3A-2B}{(x-2)(x-3)}=\frac{x+2}{(x-2)(x-3)}$$

Altså:

$$\frac{(A+B)x-3A-2B}{(x-2)(x-3)}=\frac{x+2}{(x-2)(x-3)}$$

Merk: Dette skal vere oppfyld uansett kva x er.
Difor må dette vere oppfyld:

$$A+B=1 \qquad (I)$$

$$-3A-2B=2 \qquad (II)$$

I: $B=1-A$

I ; II:

$$-3A-2(1-A)=2$$

$$-A-2=2$$

$$A=-4$$

I: $B=1-A=1-(-4)=5$

Altså: $\frac{x+2}{x^2-5x+6}=\frac{-4}{x-2}+\frac{5}{x-3}$

$$\int\frac{x+2}{x^2-5x+6}\,dx=\int\frac{-4}{x-2}\,dx+\int\frac{5}{x-3}\,dx=$$

$$-4\ln|x-2|+5\ln|x-3|+C$$

⑥ Eksempel

Finn integralet $\int \frac{x^3+2x^2+2x+1}{x^2-4}\,dx$

Polynomdivisjon:

$$\begin{array}{l} (x^3+2x^2+2x+1) : (x^2-4) = x+2+\frac{6x+9}{x^2-4} = x+2+3\,\frac{2x+3}{x^2-4} \\ \underline{-(x^3-4x)} \\ \quad 2x^2+6x \\ \quad \underline{-(2x^2-8)} \\ \qquad 6x+9 \end{array}$$

Delbrøksoppspalting:

$$\frac{2x+3}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$\frac{A(x+2)+B(x-2)}{(x-2)(x+2)} = \frac{(A+B)x+2A-2B}{x^2-4} = \frac{2x+3}{x^2-4}$$

$A+B=2 \quad \wedge \quad 2A-2B=3$

$B=2-A$

$2A-2(2-A)=3$

$4A-4=3$

$A=\frac{7}{4}$

$B=2-A=2-\frac{7}{4}=\frac{1}{4}$

$$\int \frac{x^3+2x^2+2x+1}{x^2-4}\, dx =$$

$$\int \left(x+2+3\left(\frac{\frac{7}{4}}{x-2}+\frac{\frac{1}{4}}{x+2}\right)\right) dx =$$

$$\frac{1}{2}x^2+2x+\frac{21}{4}\int \frac{1}{x-2}\,dx+\frac{3}{4}\int \frac{1}{x+2}\,dx =$$

$$\frac{1}{2}x^2+2x+\frac{21}{4}\ln|x-2|+\frac{3}{4}\ln|x+2|+C$$