Obligatorisk innlevering nr. 7

- Løysingsforslag

Oppg. 1

a) $\int_0^\infty x e^{-x^2} dx$

Variabelbyte: $u = x^2$, $\frac{du}{dx} = 2x$, $dx = \frac{1}{2x} du$

$u(0) = 0$

$\lim_{x\to\infty} u(x) = \infty$

$$\int_0^\infty x \cdot e^{-x^2} dx = \int_0^\infty x e^{-u} \frac{1}{2x} du = \frac{1}{2} \int_0^\infty e^{-u} du =$$

$$-\frac{1}{2} \lim_{b\to\infty} \left[e^{-u} \right]_0^b = -\frac{1}{2} \left(\lim_{b\to\infty} e^{-b} - 1 \right) =$$

$$-\frac{1}{2} (0-1) = \underline{\underline{\frac{1}{2}}}$$

b) $\int_0^3 \frac{x^3+1}{x^2+x-2} dx$

Polynomdivisjon: $(x^3+1):(x^2+x-2) = x - 1 + \frac{3x-1}{x^2+x-2}$

$$\begin{array}{l} -(x^3+x^2-2x) \\ \hline \quad -x^2+2x+1 \\ \quad -(-x^2-x+2) \\ \hline \quad\quad 3x-1 \end{array}$$

Faktoriserar nemnaren: $x^2+x-2 = 0 \Leftrightarrow x = \frac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot(-2)}}{2}$

$= \frac{-1\pm 3}{2}$

$x = \frac{-1-3}{2} = -2 \lor x = \frac{-1+3}{2} = 1$

$\Rightarrow \quad x^2+x-2 = -1(x-(-2))(x-1) = (x-1)(x+2)$

Delbrøksopspalting:

$$\frac{3x-1}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$3x-1 = A(x+2) + B(x-1)$$

$x=1$: $\quad 2 = A\cdot 3 + B\cdot 0, \quad A = \frac{2}{3}$

$x=-2$: $\quad -7 = A\cdot 0 + B\cdot(-3), \quad B = \frac{7}{3}$

$$\int_0^3 \frac{x^3+1}{x^2+x-2}dx = \int_0^3 \left(x-1+\frac{7/3}{x+2}\right)dx + \frac{2}{3}\int_0^3 \frac{dx}{x-1}$$

$$\int_0^3 \frac{dx}{x-1} = \lim_{b\to 1^-}\int_0^b \frac{dx}{x-1} + \lim_{a\to 1^+}\int_a^3 \frac{dx}{x-1}$$

$$\lim_{b\to 1^-}\int_0^b \frac{dx}{x-1} = \lim_{b\to 1^-}\left[\ln|x-1|\right]_0^b = \lim_{b\to 1^-}\ln(1-b) - 0 = -\infty$$

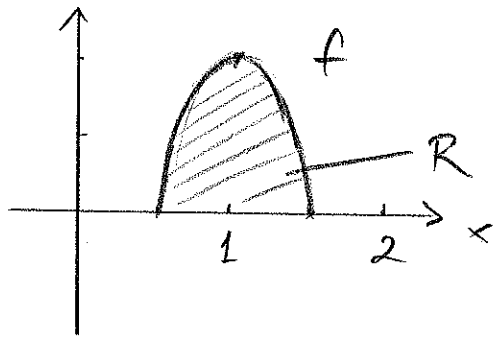
$$\lim_{a\to 1^+}\int_a^3 \frac{dx}{x-1} = \lim_{a\to 1^+}\left[\ln|x-1|\right]_a^3 = \ln 2 - \lim_{a\to 1^+}\ln(a-1) = +\infty$$

Integrala $\int_0^1 \frac{dx}{x-1}$ og $\int_1^3 \frac{dx}{x-1}$ er udefinerte.
Difor er også integralet $\int_0^3 \frac{x^3+1}{x^2+x-2}dx$ udefinert.

c) $\int_{-1}^0 \frac{dx}{\sqrt{1-x^2}} = \lim_{a\to -1^+}\left[\arcsin x\right]_a^0 = \arcsin 0 - \lim_{a\to -1^+}\arcsin a =$

$0 - \arcsin(-1) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$

Oppg. 2



a) Volumet er: $\pi \int_{1/2}^{3/2} (f(x))^2 \, dx =$

$\pi \int_{1/2}^{3/2} (-\cos(\pi x))^2 \, dx = \pi \int_{1/2}^{3/2} \cos^2(\pi x)\, dx =$

$\frac{\pi}{2} \int_{1/2}^{3/2} (1 + \cos(2\pi x))\, dx = \frac{\pi}{2} \left[x + \frac{1}{2\pi} \sin(2\pi x) \right]_{1/2}^{3/2} =$

$\frac{\pi}{2} (3/2 - 1/2) + \frac{1}{4} (\sin(3\pi) - \sin \pi) = \underline{\underline{\frac{\pi}{2}}}$

b) Volumet er: $V = 2\pi \int_{1/2}^{3/2} x \, f(x) \, dx =$

$-2\pi \int_{1/2}^{3/2} x \cos(\pi x)\, dx$

Delvis integrasjon: $u = x \Rightarrow u' = 1$

$v' = \cos(\pi x) \Leftarrow v = \frac{1}{\pi} \sin(\pi x)$

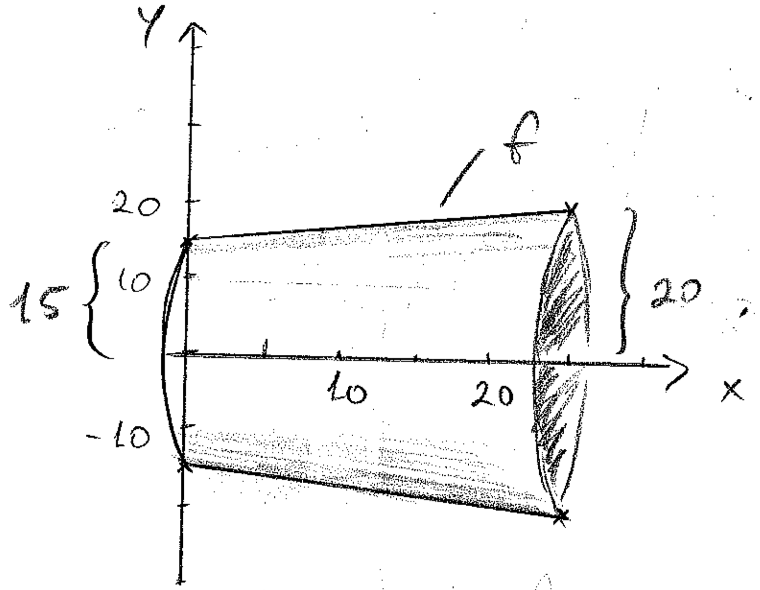
$V = -2\pi \left(\left[x \cdot \frac{1}{\pi} \sin(\pi x) \right]_{1/2}^{3/2} - \int_{1/2}^{3/2} 1 \cdot \frac{1}{\pi} \sin(\pi x)\, dx \right) =$

$-2 \left[x \sin(\pi x) \right]_{1/2}^{3/2} + 2 \int_{1/2}^{3/2} \sin(\pi x)\, dx =$

$-2 \left(\frac{3}{2} \sin \frac{3\pi}{2} - \frac{1}{2} \sin \frac{\pi}{2} \right) - \frac{2}{\pi} \left[\cos(\pi x) \right]_{1/2}^{3/2} =$

$-2 \left(-\frac{3}{2} - \frac{1}{2} \right) - \frac{2}{\pi} \left(\cos \frac{3\pi}{2} - \cos \frac{\pi}{2} \right) = \underline{\underline{4}}$

c)



Ser:

Grafen til f er ei rett linje;

$f(x) = ax + b$

$f(0) = 15 \Rightarrow b = 15$

$f(25) = 20 \Rightarrow$

$a = \frac{20-15}{25} = \frac{1}{5}$

Volum, i cm^3:

$$\pi \int_0^{25} (f(x))^2 dx = \pi \int_0^{25} \left(\frac{1}{5}x + 15\right)^2 dx =$$

$$\pi \int_0^{25} \left(\frac{1}{25}x^2 + 6x + 225\right) dx =$$

$$\pi \left[\frac{1}{3 \cdot 25}x^3 + 3x^2 + 225x\right]_0^{25} =$$

$$\pi \left(\frac{25^3}{3 \cdot 25} + 3 \cdot 25^2 + 225 \cdot 25 - 0\right) =$$

$$25^2 \pi \left(\frac{1}{3} + 3 + 9\right) = 25^2 \pi \cdot \frac{37}{3} \approx 24.2 \cdot 10^3$$

Volumet er ca 24.2 l.

Oppg 3

a) $\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = 1 \cdot 2 - 3 \cdot (-1) = \underline{5}$

b) $\begin{vmatrix} 2 & 4 & -2 \\ -1 & -2 & 2 \\ -2 & -4 & 0 \end{vmatrix} \begin{matrix} 1 \\ \downarrow \\ \end{matrix} = \begin{vmatrix} 2 & 4 & -2 \\ 1 & 2 & 0 \\ -2 & -4 & 0 \end{vmatrix} = (-2) \cdot (-1)^{1+3} \begin{vmatrix} 1 & 2 \\ -2 & -4 \end{vmatrix} =$

$$-2\,(1\cdot(-4)-2\cdot(-2)) = \underline{0}$$

c)

$$\begin{vmatrix} -1 & 3 & 0 & 4 \\ 7 & -1 & 2 & 3 \\ -2 & 3 & 0 & 6 \\ 4 & -1 & -4 & 1 \end{vmatrix} \; 2 \;=\; \begin{vmatrix} -1 & 3 & 0 & 4 \\ 7 & -1 & 2 & 3 \\ -2 & 3 & 0 & 6 \\ 18 & -3 & 0 & 7 \end{vmatrix} =$$

$$2\cdot(-1)^{2+3} \begin{vmatrix} -1 & 3 & 4 \\ -2 & 3 & 6 \\ 18 & -3 & 7 \end{vmatrix} \; \begin{matrix} -2 \\ 9 \end{matrix} \;=\; -2 \begin{vmatrix} -1 & 3 & 4 \\ 0 & -3 & -2 \\ 0 & 24 & 61 \end{vmatrix} =$$

$$-2\cdot(-1)\cdot(-1)^{1+1} \begin{vmatrix} -3 & -2 \\ 24 & 61 \end{vmatrix} = 2\,(-3\cdot 61 + 2\cdot 24) = \underline{\underline{-270}}$$

Oppg. 4

$$\det(AB) = \det A \cdot \det B$$

a) $\det A = 5$ (jmf. 3a)

$$\det B = \begin{vmatrix} 0 & 1 \\ 2 & -2 \end{vmatrix} = 0 - 2 = -2$$

$$AB = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 3\cdot 2 & 1-3\cdot 2 \\ 2\cdot 2 & -1-2\cdot 2 \end{pmatrix} = \begin{pmatrix} 6 & -5 \\ 4 & -5 \end{pmatrix}$$

$$\det(AB) = \begin{vmatrix} 6 & -5 \\ 4 & -5 \end{vmatrix} = 6\cdot(-5) + 5\cdot 4 = -10$$

$$\underline{\det A \cdot \det B =}\; 5\cdot(-2) = -10 \;\underline{= \det(AB)}$$

b) A er invertibel: $A^{-1}A = I$

$\det I = 1$

$1 = \det(I) = \det(A^{-1}A) = \det A^{-1} \cdot \det A \Leftrightarrow$

$\det A = \dfrac{1}{\det A^{-1}}$ (q.e.d.).

c) A og B similære:

Det finst ei invertibel matrise P slik at

$A = PBP^{-1}$

$\det A = \det(PBP^{-1}) = \det P \cdot \det(BP^{-1}) =$

$\det P \cdot \det B \cdot \det P^{-1} = \det P \cdot \det B \cdot \dfrac{1}{\det P} =$

$\dfrac{\det P}{\det P} \cdot \det B = \det B$

Altså: $\underline{\det A = \det B}$ (q.e.d.).

d) Eigenverdiane til ei matrise A er gitt som løysingane til den karakteristiske likninga

$\det(A - \lambda I) = 0.$

$0 = \det(A - \lambda I) = \det(PBP^{-1} - \lambda I) =$

$\det(PBP^{-1} - \lambda PIP^{-1}) = \det(P(B - \lambda I)P^{-1}) =$

$\det P \cdot \det(B - \lambda I) \cdot \det P^{-1} =$

$\dfrac{\det P}{\det P} \cdot \det(B - \lambda I) = \det(B - \lambda I)$

Altså:

A og B har den same karakteristiske likninga $\Leftrightarrow$ <u>A og B har dei same eigenverdiane</u> (q.e.d.)

Oppg. 5

a) $A = \begin{pmatrix} -3 & 2 \\ 3 & 2 \end{pmatrix}$

Karakteristisk likning: $\det(A - \lambda I_2) = 0$

$$0 = \begin{vmatrix} -3-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (-3-\lambda)(2-\lambda) - 2\cdot 3 = \lambda^2 + \lambda - 12 \Leftrightarrow$$

$$\lambda = \frac{-1 \pm \sqrt{(-1)^2 - 4\cdot 1\cdot(-12)}}{2\cdot 1} = \frac{-1\pm 7}{2} \Leftrightarrow$$

$$\lambda = \frac{-1-7}{2} = -4 \quad \vee \quad \lambda = \frac{-1+7}{2} = 3$$

Eigenverdiar: $\underline{\underline{\lambda_1 = -4 \text{ og } \lambda_2 = 3}}$

Eigenvektor for $\underline{\lambda_1 = -4}$:

$(A - \lambda_1 I_2)\vec{x} = \vec{0}$, $\quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$A - \lambda_1 I_2 = \begin{pmatrix} -3-(-4) & 2 \\ 3 & 2-(-4) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$x_1 + 2x_2 = 0$, $\quad x_1 = -2x_2$

$\vec{x} = \begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, Eigenvektor: $\underline{\underline{\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}}$

Eigenvektor for $\underline{\lambda_2 = 3}$:

$$A - \lambda_2 I_2 = \begin{pmatrix} -3-3 & 2 \\ 3 & 2-3 \end{pmatrix} = \begin{pmatrix} -6 & 2 \\ 3 & -1 \end{pmatrix} \sim \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix}$$

$3x_1 - x_2 = 0, \quad x_2 = 3x_1$

$$\vec{x} = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \text{eigenvektor: } \underline{\underline{\vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}}$$

b) $$B = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

Karakteristisk likning: $\det(B - \lambda I_3) = 0$

$$0 = \begin{vmatrix} 2-\lambda & 1 & 0 \\ -1 & -\lambda & 1 \\ 1 & 3 & 1-\lambda \end{vmatrix} = (2-\lambda)\cdot(-1)^{1+1} \begin{vmatrix} -\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} +$$

$$1\cdot(-1)^{1+2} \begin{vmatrix} -1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 0 =$$

$$(2-\lambda)(-\lambda(1-\lambda)-3) - (-1(1-\lambda)-1) =$$

$$(2-\lambda)(\lambda^2-\lambda-3) - (\lambda-2) =$$

$$(2-\lambda)(\lambda^2-\lambda-3) + (2-\lambda) = (2-\lambda)(\lambda^2-\lambda-3+1) =$$

$$(2-\lambda)(\lambda^2-\lambda-2) \Leftrightarrow$$

$$2-\lambda = 0 \lor \lambda^2-\lambda-2 = 0 \Leftrightarrow$$

$$\lambda = 2 \lor \lambda = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\cdot 1\cdot(-2)}}{2\cdot 1} = \frac{1 \pm 3}{2}$$

$$\lambda = 2 \lor \lambda = \frac{1-3}{2} = -1 \lor \lambda = \frac{1+3}{2} = 2$$

Eigenverdiar: $\underline{\underline{\lambda_1 = -1 \text{ og } \lambda_2 = 2}}$

λ_2 har multiplisitet 2 og kan ha opp til 2 linerert uavhengige eigenvektorar.

Eigenvektor for $\underline{\lambda_1 = -1}$:

$$(B - \lambda_1 I_3)\vec{x} = \vec{0}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$B - \lambda_1 I_3 = \begin{pmatrix} 2-(-1) & 1 & 0 \\ -1 & -(-1) & 1 \\ 1 & 3 & 1-(-1) \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{matrix} \\ \\ 1 \; -3 \end{matrix} \sim$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & 3 \\ 0 & -8 & -6 \end{pmatrix} \begin{matrix} \\ 2 \leftarrow \frac{1}{4} \\ \\ \end{matrix} \sim \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ -3 \\ \\ \end{matrix} \sim \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - \frac{1}{4}x_3 = 0, \quad x_1 = \frac{1}{4}x_3$

$x_2 + \frac{3}{4}x_3 = 0, \quad x_2 = -\frac{3}{4}x_3$

$$\vec{x} = \begin{pmatrix} \frac{1}{4}x_3 \\ -\frac{3}{4}x_3 \\ x_3 \end{pmatrix} = \frac{1}{4}x_3 \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}, \quad \text{eigenvektor: } \underline{\underline{\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}}}$$

Eigenvektorer for $\underline{\lambda_2 = 2}$:

$$B - 2I_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ 1 & 3 & -1 \end{pmatrix} \begin{matrix} \\ \\ 1 \end{matrix} \sim \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{matrix} \\ -3 \; -1 \\ \\ \end{matrix} \sim$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} x_1 - x_3 = 0, \quad x_1 = x_3 \\ x_2 = 0 \end{matrix}$$

$$\vec{x} = \begin{pmatrix} x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{eigenvektor: } \underline{\underline{\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}}$$

c) $C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{pmatrix}$

C er triagulær. Difor er eigenverdiane til C gitt ved diagonalelementa;

eigenverdiane er $\underline{\underline{\lambda_1 = 1,\ \lambda_2 = 3 \text{ og } \lambda_3 = -2}}$

Eigenvektor for $\underline{\lambda_1 = 1}$:

$(C - \lambda_1 I_3)\vec{x} = \vec{0}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$C - \lambda_1 I_3 = \begin{pmatrix} 1-1 & 2 & 0 \\ 0 & 3-1 & 2 \\ 0 & 0 & -2-1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & -3 \end{pmatrix} \begin{array}{l} \leftarrow \frac{1}{2} \\ \leftarrow \frac{1}{2} \\ \leftarrow -\frac{1}{3} \end{array} \sim$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} -1 \\ \leftarrow \\ -1 \end{array} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad x_2 = x_3 = 0$$

$\vec{x} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, eigenvektor: $\underline{\underline{\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}}$

Eigenvektor for $\underline{\lambda_2 = 3}$:

$$C - \lambda_2 I_3 = \begin{pmatrix} 1-3 & 2 & 0 \\ 0 & 3-3 & 2 \\ 0 & 0 & -2-3 \end{pmatrix} = \begin{pmatrix} -2 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - x_2 = 0, \quad x_1 = x_2$

$x_3 = 0$

$\vec{x} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, eigenvektor: $\underline{\underline{\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}}$

Eigenvektor for $\underline{\lambda_3 = -2}$:

$$C - \lambda_3 I = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \leftarrow \frac{1}{3} \\ \leftarrow \frac{1}{5} \\ \end{matrix} \sim \begin{pmatrix} 1 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \leftarrow \\ -\frac{2}{3} \\ \end{matrix} \sim \begin{pmatrix} 1 & 0 & -\frac{4}{15} \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_1 - \frac{4}{15} x_3 = 0, \quad x_1 = \frac{4}{15} x_3$$

$$x_2 + \frac{2}{5} x_3 = 0, \quad x_2 = -\frac{2}{5} x_3$$

$$\vec{x} = \begin{pmatrix} \frac{4}{15} x_3 \\ -\frac{2}{5} x_3 \\ x_3 \end{pmatrix} = \frac{1}{15} x_3 \begin{pmatrix} 4 \\ -6 \\ 15 \end{pmatrix}, \text{ eigenvektor: } \underline{\underline{\vec{v}_3 = \begin{pmatrix} 4 \\ -6 \\ 15 \end{pmatrix}}}$$