Løysingsforslag,

Obligatorisk innlevering nr. 6

Oppg. 1

$$F = m\frac{d}{dt}\left(\frac{v}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\right)$$

a) Vis at vi får at $F = mv'(t)$ når $c \to \infty$

$$\lim_{c\to\infty} \frac{v}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{v}{\sqrt{1-0}} = v$$

Altså, for $c \to \infty$: $F = m\frac{d}{dt}v = \underline{mv'(t)}$ (q.e.d.)

b) $$\frac{d}{dt}\left(\frac{v}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\right) = \frac{F}{m}$$

$$\frac{v}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \int \frac{F}{m}\,dt = \frac{F}{m}t + C$$

Skal ha: $v(0) = 0$

$$\frac{0}{\sqrt{1-\left(\frac{0}{c}\right)^2}} = \frac{F}{m}\cdot 0 + C \iff C = 0$$

$$\frac{v}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{F}{m}t$$

$$v = \frac{F}{m}t\sqrt{1-\left(\frac{v}{c}\right)^2}$$

$$v^2 = \left(\frac{Ft}{m}\right)^2 \left(1 - \left(\frac{v}{c}\right)^2\right)$$

$$v^2 \left(1 + \left(\frac{Ft}{mc}\right)^2\right) = \left(\frac{Ft}{m}\right)^2$$

$$v^2 = \frac{\left(\frac{Ft}{m}\right)^2}{1 + \left(\frac{Ft}{mc}\right)^2}$$

$$v = \pm \frac{\frac{F}{m} t}{\sqrt{1 + \left(\frac{Ft}{mc}\right)^2}}$$

Skal ha: $v(t) > 0$ for $t > 0$

$$\underline{\underline{v(t) = \frac{\frac{F}{m} t}{\sqrt{1 + \left(\frac{Ft}{mc}\right)^2}}}}$$

c) $$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{\frac{F}{m} t}{\sqrt{1 + \left(\frac{Ft}{mc}\right)^2}} = \lim_{t\to\infty} \frac{\frac{F}{m} t}{\frac{Ft}{mc}\sqrt{\left(\frac{mc}{Ft}\right)^2 + 1}} =$$

$$\lim_{t\to\infty} \frac{c}{\sqrt{\left(\frac{mc}{Ft}\right)^2 + 1}} = \frac{c}{\sqrt{0+1}} = \underline{c} \quad (q.e.d.)$$

Oppg. 2

a) $y'' + y' - 12y = 0$

Antar $y = e^{rx} \Rightarrow y' = re^{rx} \wedge y'' = r^2 e^{rx}$

$r^2 e^{rx} + r e^{rx} - 12 e^{rx} = 0$

$r^2 + r - 12 = 0 \Leftrightarrow r = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-12)}}{2 \cdot 1} = \frac{-1 \pm 7}{2}$

$r = \frac{-1-7}{2} = -4 \vee r = \frac{-1+7}{2} = 3$

Generell løysing: $\underline{\underline{y = Ae^{-4x} + Be^{3x}}}$

b) $xy + y = xy'$

$y(x+1) = x\frac{dy}{dx}$

$\frac{dy}{y} = \frac{x+1}{x}dx = (1+\frac{1}{x})dx$

$\int\frac{dy}{y} = \int(1+\frac{1}{x})dx$

$\ln|y| = x + \ln|x| + C_1'$

$y = \pm e^{x+\ln|x|+C_1'} = Cxe^x \qquad (C = \pm e^{C_1'})$

$\underline{\underline{y = Cxe^x}}$

c) $x^2y' + 2xy + \cos x = 0$

$y' + \frac{2}{x}y = -\frac{1}{x^2}\cos x$

Integrerande faktor: $e^{F(x)}$ der $F'(x) = \frac{2}{x}$

Vel $F = 2\ln|x| = \ln x^2$ slik at

$e^F = e^{\ln x^2} = x^2$

$x^2\cdot(y' + \frac{2}{x}y) = x^2\cdot(-\frac{1}{x^2}\cos x)$

$(x^2y)' = -\cos x$

$x^2y = \int -\cos x\,dx = -\sin x + C$

$\underline{\underline{y = -\frac{\sin x}{x^2} + \frac{C}{x^2}}}$

d) $y'' + 6y' + 9y = 2e^{-3x}$

- Løyser først den homogene differensiallikninga $y_h'' + 6y_h' + 9y_h = 0$

Antar $y_h = e^{rx} \Rightarrow y_h' = re^{rx} \wedge y_h'' = r^2 e^{rx}$

$r^2 e^{rx} + 6r\,e^{rx} + 9e^{rx} = 0$

$r^2 + 6r + 9 = 0 \Leftrightarrow r = \dfrac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 9}}{2 \cdot 1} = \dfrac{-6 \pm 0}{2} = -3$

Løysing: $y_{h,1} = e^{-3x}$

Også løysing: $y_{h,2} = x e^{-3x}$

Generell løysing (av homogen likning):

$y_h = A y_{h,2} + B y_{h,1} = e^{-3x}(Ax + B)$

Vi ser at høgresida i den inhomogene likninga er proporsjonal med e^{-3x} – akkurat som løysinga av den homogene likninga. Sidan funksjonar proporsjonale med xe^{-3x} også er inkludert i y_h, prøver vi å finne ei partikulær løysing på forma $y_s = ax^2 e^{-3x}$.

$y_s' = a \cdot 2x\,e^{-3x} + ax^2 e^{-3} \cdot -3 = a e^{-3x}(-3x^2 + 2x)$

$y_s'' = a e^{-3x} \cdot (-3)(-3x^2 + 2x) + a e^{-3x}(-6x + 2) =$

$a e^{-3x}(9x^2 - 12x + 2)$

$y_s'' + 6y_s' + 9y = 2e^{-3x}$

$a e^{-3x}(9x^2 - 12x + 2) + 6a e^{-3x}(-3x^2 + 2x) + 9ax^2 e^{-3x} = 2e^{-3x}$

$$a\,(9x^2-12x+2-18x^2+12x+9x^2)=2$$

$$2a=2\,,\quad a=1$$

$$y_s = x^2e^{-3x}$$

Generell løysing: $y=y_s+y_h=\underline{\underline{e^{-3x}\,(x^2+Ax+B)}}$

<u>Oppg. 3</u>

a) $y'+xy=x,\ y(0)=2$

$$\frac{dy}{dx}=x-xy=x(1-y)$$

$$\frac{dy}{1-y}=x\,dx$$

$$\int\frac{dy}{1-y}=\int x\,dx$$

$$-\ln|1-y|=\tfrac12x^2+C_1'$$

$$|y-1|=e^{-\frac12x^2+C_2'}$$

$$y-1=C_3'\,e^{-\frac12x^2}\qquad (C_3'=\pm e^{C_2})$$

$$\underline{y=1+C'e^{-\frac12x^2}}$$

Alternativt:

e^F der $F'(x)=x$ er integrerande faktor.

Vel $F=\frac12x^2$ slik at $e^F=e^{\frac12x^2}$

$$e^{\frac12x^2}(y'+xy)=e^{\frac12x^2}\cdot x$$

$$(e^{\frac{1}{2}x^2}y)' = e^{\frac{1}{2}x^2}x$$

$$e^{\frac{1}{2}x^2}y = \int e^{\frac{1}{2}x^2}x\,dx$$

$$u = \frac{1}{2}x^2, \quad \frac{du}{dx} = x, \quad du = x\,dx$$

$$\int e^{\frac{1}{2}x^2}x\,dx = \int e^u\,du = e^u + C' = e^{\frac{1}{2}x^2} + C'$$

$$e^{\frac{1}{2}x^2}y = e^{\frac{1}{2}x^2} + C'$$

$$y = e^{-\frac{1}{2}x^2}(e^{\frac{1}{2}x^2} + C') = \underline{1 + C'e^{-\frac{1}{2}x^2}}$$

Initialkrav: $y(0) = 2$

$$1 + C'e^{-0} = 2, \quad C' = 1$$

$$\underline{\underline{y(x) = 1 + e^{-\frac{1}{2}x^2}}}$$

b) $x^2y' + \sqrt{y} = 0, \quad y(1) = 1$

$$x^2\frac{dy}{dx} = -y^{1/2}$$

$$-\frac{dy}{y^{1/2}} = \frac{dx}{x^2}$$

$$-\int y^{-1/2}\,dy = \int x^{-2}\,dx$$

$$-\frac{1}{-\frac{1}{2}+1}y^{-\frac{1}{2}+1} = -x^{-1} + C_1'$$

$$2y^{1/2} = \frac{1}{x} + C$$

$$y(1) = 1 \Rightarrow 2 \cdot 1^{1/2} = \frac{1}{1} + C, \quad C' = 1$$

$$2\sqrt{y} = \frac{1}{x} + 1$$

$$\sqrt{y} = \frac{1}{2x} + \frac{1}{2} = \frac{1}{2}\left(\frac{1}{x} + 1\right)$$

$$y = \frac{1}{4}\left(\frac{1}{x}+1\right)^2$$

Om en brukar initialkravet seinare, når ein har funne eit eksplisitt uttrykk for y, kan ein også få „løysinga" $y = \left(\frac{1}{2x}-\frac{3}{2}\right)^2$, men denne er feil.

c) $y'' - 4y' + 5y = e^{2x}$, $y(0) = 0$, $y'(0) = 2$

- Løyser fyrst den homogene likninga

$$y_h'' - 4y_h + 5y_h = 0$$

Antar $y_h = e^{rx}$ slik at $y_h' = re^{rx}$ og $y_h'' = r^2 e^{rx}$

$$r^2 e^{rx} - 4r e^{rx} + 5 e^{rx} = 0$$

$$r^2 - 4r + 5 = 0 \Leftrightarrow r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{4 \pm \sqrt{-4}}{2}$$

$$r = \frac{4 \pm \sqrt{4} \cdot i}{2} = 2 \pm i$$

Generell løysing av homogen likning:

$$y_h = e^{2x}(A \cos x + B \sin x)$$

Antar partikulær løysing av inhomogen likning på forma $y_s = a e^{2x}$

$$y_s' = 2a e^{2x}, \quad y_s'' = 4a e^{2x}$$

$$y_s'' - 4y_s' + 5y_s = e^{2x}$$

$$4a e^{2x} - 4 \cdot 2a e^{2x} + 5a e^{2x} = e^{2x}$$

$$4a - 8a + 5a = 1, \quad a = 1, \quad y_s = e^{2x}$$

Generell løysing:

$$y = y_s + y_h = 1 \cdot e^{2x} + e^{2x}(A \cos x + B \sin x) =$$

$e^{2x}(1 + A\cos x + B\sin x)$

Skal ha: $y(0)=0$

$e^0(1 + A\cdot\cos 0 + B\cdot\sin 0) = 0$

$A+1=0$, $A=-1$, $y(x) = e^{2x}(1 - 1\cos x + B\sin x)$

$y'(x) = e^{2x}\cdot 2(1 - \cos x + B\sin x) +$

$e^{2x}(0 + \sin x + B\cos x)$

$e^{2x}(2 + (B-2)\cos x + (2B+1)\sin x)$

Skal ha: $y'(0) = 2$

$e^0(2 + (B-2)\cos 0 + (2B+1)\sin 0) = 2$

$2 + B - 2 = 2$, $B = 2$

Løysing:

$$\underline{\underline{y(x) = e^{2x}(1 - \cos x + 2\sin x)}}$$

Oppg. 4

$T' = -k(T - f(t))$, $f(t) = 3.2\cos\left(\frac{\pi}{12}(t-13)\right) + 13.1$

a) Newtons avkjølingslov seier at $T' = -k(T - T_0)$ der T_0 er temperaturen til omgivnaden. Her er altså $T_0 = f(t)$, så funksjonen $f(t)$ er temperaturen til omgivnaden – som varierar med tida sidan vi er utandørs. Gjennomsnittet av ein funksjon $f(x)$ på intervallet $[a,b]$ er definert som $\frac{1}{b-a}\int_a^b f(x)\,dx$. Såleis vert gjennomsnitts-temperaturen denne dagen

$$\frac{1}{24-0}\int_0^{24} f(t)\,dt = \frac{1}{24}\int_0^{24}\left(3.2\cos\left(\frac{\pi}{12}(t-13)\right)+13.1\right)dt =$$

$$\frac{1}{24}\left[3.2\cdot\frac{12}{\pi}\sin\left(\frac{\pi}{12}(t-13)\right)+13.1t\right]_0^{24} =$$

$$\frac{1}{24}\left(\frac{3.2\cdot 12}{\pi}\sin\left(\frac{\pi}{12}(24-13)\right)+13.1\cdot 24 - \frac{3.2\cdot 12}{\pi}\sin\left(\frac{\pi}{12}(0-13)\right)+13.1\cdot 0\right) = 13.1$$

Gjennomsnittstemperaturen var $\underline{13.1\,°C}$.

b) $T + kT' = kf(t)$

Integrerande faktor: $e^{F(t)}$ der $F'(t) = k$ (konstant).
Vel $F = kt$ slik at $e^F = e^{kt}$

$$e^{kt}(T+kT') = e^{kt}\cdot kf(t)$$

$$(e^{kt}T)' = k\,e^{kt}f(t)$$

$$e^{kt}T = \int k\,e^{kt}f(t)\,dt = \int\left(ke^{kt}3.2\cos\left(\frac{\pi}{12}(t-13)\right)+ke^{kt}\cdot 13.1\right)dt =$$

$$\int k\,e^{kt}\cdot 3.2\cos\left(\frac{\pi}{12}(t-13)\right)dt + 13.1\,e^{kt}$$

Delvis integrasjon: $\int uv'\,dt = uv - \int u'v\,dt$

$u = 3.2\cos\left(\frac{\pi}{12}(t-13)\right) \Rightarrow u' = -3.2\cdot\frac{\pi}{12}\sin\left(\frac{\pi}{12}(t-13)\right)$

$v' = ke^{kt} \Leftarrow v = e^{kt}$

$$\int ke^{kt}3.2\cos\left(\frac{\pi}{12}(t-13)\right)dt = e^{kt}\cdot 3.2\cos\left(\frac{\pi}{12}(t-13)\right)+\frac{\pi}{12}\int e^{kt}\,3.2\sin\left(\frac{\pi}{12}(t-13)\right)dt$$

Ny delvis int.:

$u = 3.2\sin\left(\frac{\pi}{12}(t-13)\right) \Rightarrow u' = \frac{\pi}{12}\,3.2\cos\left(\frac{\pi}{12}(t-13)\right)$

$v' = e^{kt} \Leftarrow v = \frac{1}{k}e^{kt}$

$\int e^{kt}\,3.2\sin\left(\frac{\pi}{12}(t-13)\right)dt = \frac{1}{k}e^{kt}\cdot 3.2\sin\left(\frac{\pi}{12}(t-13)\right) - \frac{1}{k}\cdot\frac{\pi}{12}\int e^{kt}\cdot 3.2\cos\left(\frac{\pi}{12}(t-13)\right)dt$

Altså:

$$\int k e^{kt}\,3.2\cos\left(\frac{\pi}{12}(t-13)\right)dt =$$

$$3.2\,e^{kt}\cos\left(\frac{\pi}{12}(t-13)\right) + \frac{\pi}{12}\left(\frac{1}{k}e^{kt}\cdot 3.2\sin\left(\frac{\pi}{12}(t-13)\right) - \frac{1}{k^2}\frac{\pi}{12}\int k e^{kt}\cdot 3.2\cos\left(\frac{\pi}{12}(t-13)\right)dt\right)$$

Med $k = 0.1 = \frac{1}{10}$

$$\int k e^{kt}\,3.2\cos\left(\frac{\pi}{12}(t-13)\right)dt = 3.2\,e^{kt}\cos\left(\frac{\pi}{12}(t-13)\right) + 3.2\cdot\frac{5\pi}{6}\,e^{kt}\sin\left(\frac{\pi}{12}(t-13)\right) - \left(\frac{5\pi}{6}\right)^2\int k e^{kt}\cdot 3.2\cos\left(\frac{\pi}{12}(t-13)\right)dt \Leftrightarrow$$

$$\left(1+\left(\frac{5\pi}{6}\right)^2\right)\int k e^{kt}\cdot 3.2\cos\left(\frac{\pi}{12}(t-13)\right)dt =$$

$$3.2\,e^{kt}\left(\cos\left(\frac{\pi}{12}(t-13)\right) + \frac{5\pi}{6}\sin\left(\frac{\pi}{12}(t-13)\right)\right) + C' \Leftrightarrow$$

$$\int k e^{kt}\cdot 3.2\cos\left(\frac{\pi}{12}(t-13)\right)dt = \frac{3.2}{1+\left(\frac{5\pi}{6}\right)^2}\left(\cos\left(\frac{\pi}{12}(t-13)\right) + \frac{5\pi}{6}\sin\left(\frac{\pi}{12}(t-13)\right)\right) + C$$

Altså:

$$e^{kt}\cdot T(t) = \frac{3.2\,e^{kt}}{1+\left(\frac{5\pi}{6}\right)^2}\left(\cos\left(\frac{\pi}{12}(t-13)\right) + \frac{5\pi}{6}\sin\left(\frac{\pi}{12}(t-13)\right)\right) + 13.1\,e^{kt} + C \Leftrightarrow$$

$$\underline{T(t) = \frac{3.2}{1+\left(\frac{5\pi}{6}\right)^2}\left(\cos\left(\frac{\pi}{12}(t-13)\right) + \frac{5\pi}{6}\sin\left(\frac{\pi}{12}(t-13)\right)\right) + 13.1 + C\,e^{-0.1t}}$$

$(k = 0.1)$

c) Ledd proporsjonalt med vilkårleg konstant:

$$Ce^{-0.1t} = Ce^{-kt}$$

Homogen likning:

$$T' + kT = 0$$

$$\frac{dT}{dt} = -kT, \quad \frac{dT}{T} = -k\,dt, \quad \int \frac{dT}{T} = -\int k\,dt$$

$$\ln|T| = -kt + C_1$$

$$T = \pm e^{-kt + C_1} = \pm e^{C_1} e^{-kt} = \underline{Ce^{-kt}} \quad \text{(q.e.d.)}$$

$$(C = \pm e^{C_1})$$

d) Initialkrav: $T(16.25) = 21$

$$\frac{3.2}{1+\left(\frac{5\pi}{6}\right)^2}\left(\cos\left(\frac{\pi}{12}(16.25-13)\right) + \frac{5\pi}{6}\sin\left(\frac{\pi}{12}(16.25-13)\right)\right) + 13.1 +$$

$$Ce^{-0.1\cdot 16.25} = 21$$

$$C = e^{+1.625}\cdot\left[21 - 13.1 - \frac{3.2}{1+\left(\frac{5\pi}{6}\right)^2}\left(\cos\left(\frac{\pi}{12}\cdot 3.25\right) + \frac{5\pi}{6}\sin\left(\frac{\pi}{12}\cdot 3.25\right)\right)\right]$$

$$\approx 34.68$$

Altså:

$$\underline{\underline{T(t) = \frac{3.2}{1+\left(\frac{5\pi}{6}\right)^2}\left(\cos\left(\frac{\pi}{12}(t-13)\right) + \frac{5\pi}{6}\sin\left(\frac{\pi}{12}(t-13)\right)\right) + 13.1 + 34.68\,e^{-0.1\cdot t}}}$$

e) Ved dødstidspunktet var temperaturen 37°C. Vi kan altså finne ut når personen døydde ved å løyse likninga $T(t) = 37$.

$T(t) = 37$

$$\frac{3.2}{1+\left(\frac{5\pi}{6}\right)^2}\left(\cos\left(\frac{\pi}{12}(t-13)\right)+\frac{5\pi}{6}\sin\left(\frac{\pi}{12}(t-13)\right)\right)+13.1+34.68\,e^{-0.1t}=37$$

$$\underline{\frac{3.2}{1+\left(\frac{5\pi}{6}\right)^2}\left(\cos\left(\frac{\pi}{12}(t-13)\right)+\frac{5\pi}{6}\sin\left(\frac{\pi}{12}(t-13)\right)\right)+34.68\,e^{-0.1t}=23.9}$$

f) Vi flyttar over slik at vi får 0 på høgre side:

$$\frac{3.2}{1+\left(\frac{5\pi}{6}\right)^2}\left(\cos\left(\frac{\pi}{12}(t-13)\right)+\frac{5\pi}{6}\sin\left(\frac{\pi}{12}(t-13)\right)\right)+34.68\,e^{-0.1t}-23.9=0$$

Vi kallar venstresida for $g(t)$.

Newtons metode går ut på å velge ein t_0 og så iterere på uttrykket

$$t_{n+1}=t_n-\frac{g(t_n)}{g'(t_n)}$$

$$g'(t)=\frac{3.2}{1+\left(\frac{5\pi}{6}\right)^2}\cdot\frac{\pi}{12}\left(-\sin\left(\frac{\pi}{12}(t-13)\right)+\frac{5\pi}{6}\cos\left(\frac{\pi}{12}(t-13)\right)\right)-3.468\,e^{-0.1t}$$

Vi vel $t_0=0$ og får

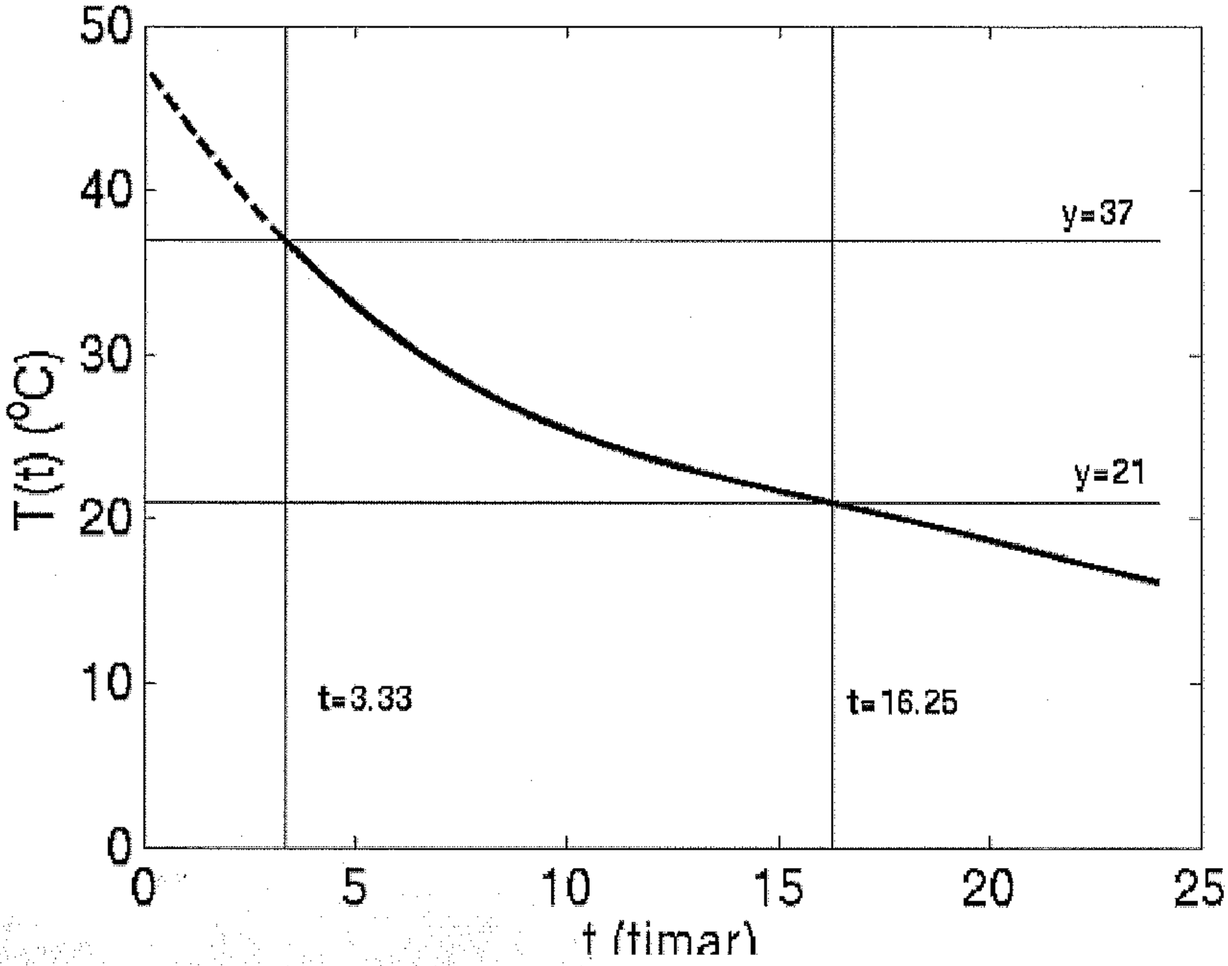
$$t_1=0-\frac{\frac{3.2}{1+\left(\frac{5\pi}{6}\right)^2}\left(\cos\left(-13\cdot\frac{\pi}{12}\right)+\frac{5\pi}{6}\sin\left(-13\cdot\frac{\pi}{12}\right)\right)+34.68-23.9}{\frac{3.2}{1+\left(\frac{5\pi}{6}\right)^2}\cdot\frac{\pi}{12}\left(-\sin\left(-13\cdot\frac{\pi}{12}\right)+\frac{5\pi}{6}\cos\left(-13\cdot\frac{\pi}{12}\right)\right)-3.468}\approx 2.83$$

$t_2 \approx 3.32$

$t_3 \approx 3.33$

$t_4 \approx 3.33 \approx 3+\frac{20}{60}$

I følge modellen døydde personen klokka $\underline{03{:}20}$.


y=37
y=21
t=3.33
t=16.25
T(t) (°C)
t (timar)