Obligatorisk innlevering nr. 5

Løysingsforslag

Oppg. 1

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right) \begin{array}{c} \\ \\ \leftarrow \frac{1}{2} \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right) \begin{array}{c} \\ \\ -1 \quad -1 \end{array} \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right) = (I_3 \mid A^{-1})$$

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \underline{\underline{\frac{1}{2} \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}}}$$

Kontroll:

$$A \cdot A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = I_3 \qquad \text{OK}$$

b) $$A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}2\\1\\0\end{pmatrix}$$

$$A^{-1}A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=A^{-1}\begin{pmatrix}2\\1\\0\end{pmatrix}$$

$$I_3\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=A^{-1}\begin{pmatrix}2\\1\\0\end{pmatrix}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=A^{-1}\begin{pmatrix}2\\1\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&-1&-1\\1&1&-1\\1&1&1\end{pmatrix}\begin{pmatrix}2\\1\\0\end{pmatrix}=\frac{1}{2}\left(2\begin{pmatrix}1\\1\\1\end{pmatrix}+\begin{pmatrix}-1\\1\\1\end{pmatrix}\right)=$$

$$\underline{\underline{\frac{1}{2}\begin{pmatrix}1\\3\\3\end{pmatrix}}}$$

c) $$I=\int\frac{2x^2+x}{(x^2+1)(x-1)}\,dx$$

Delbrøksoppspalting:

$$\frac{2x^2+x}{(x^2+1)(x-1)}=\frac{Ax+B}{x^2+1}+\frac{C'}{x-1}=\frac{(Ax+B)(x-1)+C'(x^2+1)}{(x^2+1)(x-1)}=$$

$$\frac{Ax^2-Ax+Bx-B+C'x^2+C'}{(x^2+1)(x-1)}=\frac{(A+C)x^2+(-A+B)x+(-B)+C'}{(x^2+1)(x-1)}$$

Må ha:

$$\begin{cases}A+\quad C'=2\\-A+B\quad=1\\\quad -B+C'=0\end{cases}$$

Eller

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Fann : b): $A = \frac{1}{2}$, $B = \frac{3}{2}$, $C = \frac{3}{2}$

$$I = \int \left(\frac{\frac{1}{2}x + \frac{3}{2}}{x^2+1} + \frac{\frac{3}{2}}{x-1} \right) dx =$$

$$\frac{1}{2} \int \frac{x}{x^2+1}\, dx + \frac{3}{2} \int \frac{1}{x^2+1}\, dx + \frac{3}{2} \int \frac{1}{x-1}\, dx$$

Variabelbyte: $u = x^2+1$, $\frac{du}{dx} = 2x$, $dx = \frac{1}{2x}\, du$

$$\int \frac{x}{x^2+1}\, dx = \int \frac{x}{u} \frac{1}{2x}\, du = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln u + C' =$$

$$\frac{1}{2} \ln (x^2+1) + C'$$

$$\int \frac{1}{x^2+1}\, dx = \arctan x + C''$$

$$\int \frac{1}{x-1}\, dx = \ln |x-1| + C'''$$

$$I = \frac{1}{2} \cdot \frac{1}{2} \ln (x^2+1) + \frac{3}{2} \arctan x + \frac{3}{2} \ln |x-1| + C =$$

$$\underline{\frac{1}{4} \ln (x^2+1) + \frac{3}{2} \arctan x + \frac{3}{2} \ln |x-1| + C}$$

Oppg. 2

a) $\int 3\sin(2t-3)dt = 3\cdot\frac{1}{2}(-\cos(2t-3)) + C' =$

$\underline{-\frac{3}{2}\cos(2t-3) + C'}$

b) $\int \sin^8 x \cos^3 x\, dx = \int \sin^8 x \cdot \cos^2 x \cdot \cos x\, dx =$

$\int \sin^8 x \cdot (1-\sin^2 x)\cos x\, dx =$

$\int \sin^8 x \cos x\, dx - \int \sin^{10} x \cos x\, dx$

Variabel bytte: $u = \sin x$, $\frac{du}{dx} = \cos x$, $dx = \frac{1}{\cos x}du$

$\int \sin^8 x \cos x\, dx = \int u^8 \cos x \frac{1}{\cos x} du = \int u^8 du$

$= \frac{1}{9}u^9 + C' = \frac{1}{9}\sin^9 x + C'$

$\int \sin^{10} \cos x\, dx = \int u^{10} du = \frac{1}{11}u^{11} + C'' = \frac{1}{11}\sin^{11} x + C''$

$\int \sin^8 x \cos^3 x\, dx = \underline{\frac{1}{9}\sin^9 x - \frac{1}{11}\sin^{11} x + C}$

c) $\int (x^2+1)\ln x\, dx$

Delvis int.: $\int u v' dx = uv - \int u' v\, dx$

$u = \ln x \Rightarrow u' = \frac{1}{x}$

$v' = x^2+1 \Leftarrow v = \frac{1}{3}x^3 + x$

$\int (x^2+1)\ln x\, dx = (\frac{1}{3}x^3 + x)\ln x - \int(\frac{1}{3}x^3+x)\cdot\frac{1}{x}dx =$

$(\frac{1}{3}x^3+x)\ln x - \frac{1}{3}\int x^2 dx - \int 1\, dx =$

$(\frac{1}{3}x^3+x)\ln x - \frac{1}{3}\cdot\frac{1}{3}x^3 - x + C =$

$\underline{(\frac{1}{3}x^3+x)\ln x - \frac{1}{9}x^3 - x + C}$

d) $I = \int \sin(e^x) \cdot e^{2x}\, dx$

Variabelbyte: $z = e^x$, $\frac{dz}{dx} = e^x$, $dx = \frac{1}{e^x} dz$

$I = \int \sin z \cdot e^{2x} \frac{1}{e^x} dz = \int \sin z \cdot e^x dz = \int z \sin z\, dx$

Delvis int.: $\int uv'\, dz = uv - \int u'v\, dz$

$u = z \Rightarrow \frac{du}{dz} = 1$

$\frac{dv}{dz} = \sin z \Leftarrow v = -\cos z$

$I = z \cdot (-\cos z) - \int 1 \cdot (-\cos z)\, dz = -z\cos z + \int \cos z\, dz =$

$-z\cos z + \sin z + C = \underline{\underline{-e^x \cos(e^x) + \sin(e^x) + C}}$

Kontroll:

$\frac{d}{dx}\left(-e^x \cos(e^x) + \sin(e^x) + C\right) =$

$-e^x \cos(e^x) - e^x \cdot (-\sin(e^x) \cdot e^x) + \cos(e^x) \cdot e^x =$

$\sin(e^x)\, e^{2x}$ OK

e) $I = \int \frac{x^4 + 1}{x^2 - x - 2}\, dx$

Polynomdivision:

$$(x^4 + 1) : (x^2 - x - 2) = x^2 + x + 3 + \frac{5x + 7}{x^2 - x - 2}$$

$$\begin{array}{l} -(x^4 - x^3 - 2x^2) \\ \hline \quad x^3 + 2x^2 + 1 \\ \quad -(x^3 - x^2 - 2x) \\ \hline \quad\quad 3x^2 + 2x + 1 \\ \quad\quad -(3x^2 - 3x - 6) \\ \hline \quad\quad\quad 5x + 7 \end{array}$$

Faktoriserar nemnaren:

$$x^2 - x - 2 = 0 \Leftrightarrow x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1} = \frac{1 \pm 3}{2}$$

$$x = \frac{1-3}{2} = -1 \quad \vee \quad x = \frac{1+3}{2} = 2$$

$$x^2 - x - 2 = 1\,(x-2)(x-(-1)) = (x-2)(x+1)$$

Delbrøkesopspalting:

$$\frac{5x+7}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)} =$$

$$\frac{(A+B)x + A - 2B}{x^2-x-2}$$

$$\begin{cases} A+B=5 \\ A-2B=7 \end{cases}$$

$$A = 5 - B$$

$$5 - B - 2B = 7 \Leftrightarrow -3B = 7 - 5 = 2 \Leftrightarrow \underline{B = -\frac{2}{3}}$$

$$A = 5 - \left(-\frac{2}{3}\right) = \frac{17}{3}$$

$$I = \int \left(x^2 + x + 3 + \frac{17/3}{x-2} - \frac{2/3}{x+1}\right) dx =$$

$$\underline{\underline{\frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x + \frac{17}{3}\ln|x-2| - \frac{2}{3}\ln|x+1| + C}}$$

Oppg. 3

a) Kraft nedover: mg

Newtons andre lov: $ma = mg, \quad a = g$

Akselerasjonen a er den deriverte av farten v_u; $a = v_u'(t)$

Indeksen „u" står for „uten luftmotstand".

Altså:

$$v_u'(t) = g$$

$$v_u(t) = \int g \, dt = g \cdot t + C$$

Skal ha: $v_u(0) = 0$ (initialkrav)

$$g \cdot 0 + C = 0$$

$$C = 0$$

$$\underline{v_u(t) = gt}$$

b) Nå: Også luftmotstand, $-kv$

Newtons andre lov:

$$ma = mg - kv$$

$$a = g - \frac{k}{m} v$$

Med $a = v'(t)$:

$$\underline{\underline{v'(t) = g - \frac{k}{m} v(t)}}$$

Dette er ei fyrste ordens linear differensial likning.

c) $v' = g - \frac{k}{m} v$

$v' + \frac{k}{m} v = g$

Integrerande faktor: $e^{F(t)}$ der $F'(t) = \frac{k}{m}$

Vel $F(x) = \frac{k}{m} t$

$e^{\frac{k}{m}t} \left(v' + \frac{k}{m} v\right) = e^{\frac{k}{m}t} \cdot g$

$\left(e^{\frac{k}{m}t} v\right)' = e^{\frac{k}{m}t} \cdot g$

$e^{\frac{k}{m}t} v = \int e^{\frac{k}{m}t} \cdot g \, dx = g \int e^{\frac{k}{m}t} dx =$

$g \frac{m}{k} \cdot e^{\frac{k}{m}t} + C$

$v = \frac{1}{e^{\frac{k}{m}t}} \cdot \left(g \frac{m}{k} e^{\frac{k}{m}t} + C\right) = \frac{gm}{k} + \frac{C}{e^{k/m t}} =$

$\frac{gm}{k} + C e^{-\frac{k}{m}t}$

Generell løysing:

$$\underline{\underline{v(t) = \frac{gm}{k} + C e^{-\frac{k}{m}t}}}$$

Likninga er også separabel og kan alternativt løysast slik:

$$v' = g - \frac{k}{m} v$$

$$\frac{dv}{dt} = g - \frac{k}{m} v$$

$$\frac{dv}{g - \frac{k}{m} v} = 1 \cdot dt$$

$$\int \frac{dv}{g - \frac{k}{m} v} = \int 1 \, dt = t + C_1$$

$$\int \frac{dv}{g - \frac{k}{m} v} = \frac{1}{-\frac{k}{m}} \ln \left| g - \frac{k}{m} v \right| + C_2 =$$

$$-\frac{m}{k} \ln \left| g - \frac{k}{m} v \right| + C_2$$

Altså:

$$-\frac{m}{k} \ln \left| g - \frac{k}{m} v \right| = t + C_3$$

$$\ln \left| g - \frac{k}{m} v \right| = -\frac{k}{m} t + C_4$$

$$\left| g - \frac{k}{m} v \right| = e^{-\frac{k}{m} t + C_4} = e^{C_4} e^{-\frac{k}{m} t}$$

$$g - \frac{k}{m} v = \pm e^{C_4} e^{-\frac{k}{m} t} = C_5 \, e^{-\frac{k}{m} t}$$

$$\frac{k}{m} v = g - C_5 \, e^{-\frac{k}{m} t}$$

$$\underline{v(t) = \frac{gm}{k} + C e^{-\frac{k}{m} t}} \qquad \left(C = -\frac{m}{k} C_5 \right)$$

d) Initialkrav: $v(0) = 0$

$$\frac{gm}{k} + Ce^0 = 0$$

$$C = -\frac{gm}{k}$$

Altså: $v(t) = \frac{gm}{k} + \left(-\frac{gm}{k}\right)e^{-\frac{k}{m}t} = \frac{gm}{k}\left(1 - e^{-\frac{k}{m}t}\right)$

$$\underline{\underline{v(t) = \frac{gm}{k}\left(1 - e^{-\frac{k}{m}t}\right)}}$$

e) $e^{-\frac{k}{m}t}$ er alltid positiv og avtakande.
Når $t \to \infty$, vil $e^{-\frac{k}{m}t} \to 0$, og v blir maksimal (dess lenger steinen fell, dess større fart får han).

$$\lim_{t\to\infty} v(t) = \frac{gm}{k} \lim_{t\to\infty}\left(1 - e^{-\frac{k}{m}t}\right) = \frac{gm}{k}(1-0) = \frac{gm}{k}$$

Maksimal fart (horisontal asymptote):

$$v_{max} = \underline{\frac{gm}{k}} = \frac{9.8\,m/s^2 \cdot 1.0\,kg}{0.10\,kg/s} = \underline{\underline{98\,m/s}} \approx 350\,km/h$$

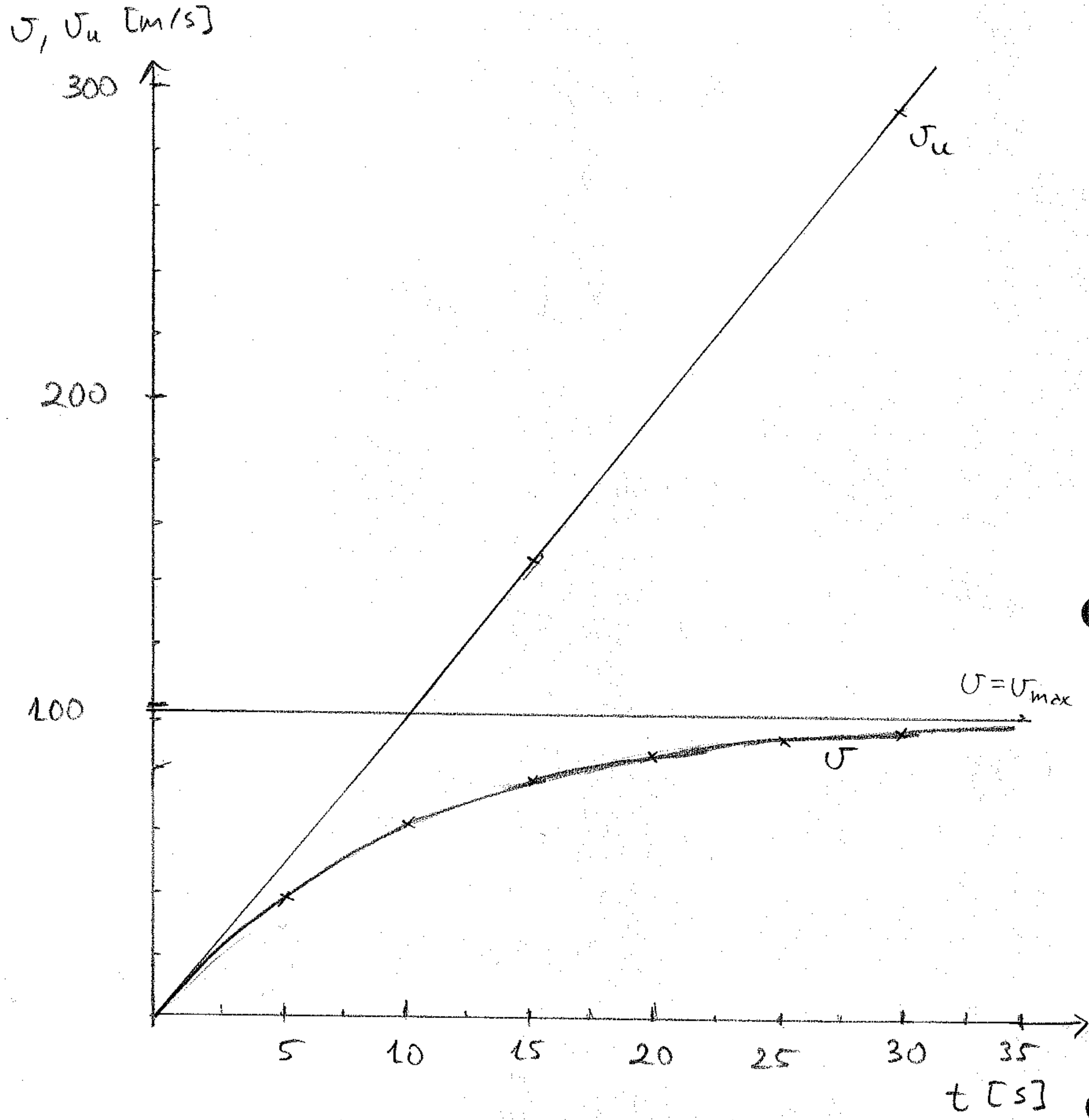
(Dette kan vi også sjå direkte av diff.-likninga; når $v'(t) = 0$, er $0 = g - \frac{k}{m}v \Leftrightarrow v = \frac{gm}{k}$)

f)

t [s]	0	15	30
v_u [m/s]	0	147	294

Tabellar

t [s]	0	5	10	15	20	25	30
v [m/s]	0	39	62	76	85	90	93



9) Vi lar $h(t)$ vere høgda som funksjon av tida; $h_u(t)$ er tilsvarande høgd utan luftmotstand.

$v(t)$ er farten høgda <u>avtar</u> med, altså:

$$v(t) = -h'(t)$$

$$h(t) = -\int v(t)\,dt$$

Utan luftmotstånd:

$$h_u(t) = -\int v_u(t)\,dt = -\int gt\,dt =$$

$$-g\cdot \tfrac{1}{2}t^2 + C' = -\tfrac{g}{2}t^2 + C'$$

Initialleran: $h_u(0) = 1000\text{ m}$

$$-\tfrac{g}{2}\cdot 0^2 + C' = 1000\text{ m}$$

$$C' = 1000\text{ m}$$

$$\underline{h_u(t) = -\tfrac{g}{2}t^2 + 1000\text{ m} = -4.9\,\text{m/s}^2\cdot t^2 + 1000\text{ m}}$$

Med luftmotstånd:

$$h(t) = -\int v(t)\,dt = -\int \frac{gm}{k}\left(1 - e^{-\frac{k}{m}t}\right)dt =$$

$$-\frac{gm}{k}\int\left(1 - e^{-\frac{k}{m}t}\right)dt = -\frac{gm}{k}\left(t - \left(-\frac{m}{k}\right)e^{-\frac{k}{m}t}\right) + C' =$$

$$-\frac{gm}{k}\left(t + \frac{m}{k}e^{-\frac{k}{m}t}\right) + C'$$

$h(0) = 1000\text{ m}$

$$-\frac{gm}{k}\left(0 + \frac{m}{k}e^{0}\right) + C' = 1000\text{ m}$$

$$-\frac{gm^2}{k^2} + C' = 1000\text{ m}$$

$$C' = 1000\text{ m} + \frac{gm^2}{k^2} = 1000\text{ m} + \frac{9.8\,\text{m/s}^2\cdot(1.0\text{ kg})^2}{(0.1\text{ kg/s})^2} = 1980\text{ m}$$

Altså:

$$\underline{h(t) = -\frac{gm}{k}\left(t + \frac{m}{k}e^{-\frac{k}{m}t}\right) + 1980\text{ m} = -98\,\text{m/s}\left(t + 10\,\text{s}\,e^{-\frac{t}{10\,\text{s}}}\right) + 1980\text{ m}}$$

Desse uttrykka er berre meiningsfulle for tider mellom tidspunktet steinen blir sluppen og tidspunktet han treff bakken. Steinen blir sluppen ved $t=0$. Når han treff bakken, er høgda 0.

Utan luftmotstand:

$$h_u(t)=0$$

$$-4.9\,m/s^2\,t^2+1000\,m=0$$

$$t=+\sqrt{\frac{1\,000\,m}{4.9\,m/s^2}}=\sqrt{\frac{1000}{4.9}}\,s\approx 14.29\,s$$

Funksjonsuttrykket er gyldig for $t\in[0s, 14.29s]$

Med luftmotstand:

$$h(t)=0$$

$$-98\,m/s\,(t+10s\,e^{-\frac{t}{10s}})+1980\,m=0$$

Dette er ei transcendental likning som ikkje kan løysast analytisk. Vi brukar Newtons metode:

$$t_{n+1}=t_n-\frac{h(t_n)}{h'(t_n)}=t_n-\frac{h(t_n)}{-v(t_n)}=t_n+\frac{h(t_n)}{v(t_n)}=$$

$$t_n+\frac{-98\,m/s\,(t_n+10s\,e^{-\frac{t_n}{10s}})+1980\,m}{98\,m/s\cdot(1-e^{-\frac{t_n}{10s}})}$$

Vel $t_0=15\,s$

$t_1\approx 18.827\,s$

$t_2\approx 18.656\,s$

$t_3 \approx 18.656\,s \approx t_2$

Funksjonsuttrykket er gyldig for $\underline{t \in [0s, 18.66s]}$