Løysingsforslag

Veke 5

Notat om Taylor-polynom

2) a) $\frac{(n+2)!}{n!} = \frac{\cancel{1\cdot 2\cdot 3\cdot\ldots\cdot(n-1)\,n}\,(n+1)\,(n+2)}{\cancel{1\cdot 2\cdot 3\cdot\ldots\cdot(n-1)\,n}} =$

$\underline{(n+1)(n+2)} = \underline{n^2+3n+2}$

c) $(n+1)! - n! = n!\cdot(n+1) - n! =$
$n!\,(n+1-1) = \underline{n!\cdot n}$

e) $\frac{n!}{n} = \frac{1\cdot 2\cdot 3\cdot\ldots\cdot(n-1)\,\cancel{n}}{\cancel{n}} =$

$1\cdot 2\cdot 3\cdot\ldots\cdot(n-1) = \underline{(n-1)!}$

5) $P_n(x) = \sum_{n=0}^{n} \frac{1}{n!} f^{(n)}(a)\,(x-a)^n$

a) $f(x) = \tan x$, $a=0$

$f(0) = 0$

$f'(x) = 1 + \tan^2 x$

$f'(0) = 1 + 0^2 = 1$

$f''(x) = 0 + 2\tan x \cdot (\tan x)' =$

$2\tan x \cdot (1 + \tan^2 x) =$

$2\tan x + 2\tan^3 x$

$f''(0) = 0$

$f'''(x) = 2(1 + \tan^2 x) + 2 \cdot 3\tan^2 x \cdot (1 + \tan^2 x)$

$2 + 8\tan^2 x + 6\tan^4 x$

$f'''(0) = 2$

$P_1(x) = f(0) + f'(0)(x-0) = \underline{x}$

$P_2(x) = P_1 + \frac{1}{2!} f''(0)(x-0)^2 = P_1 = \underline{x}$

$P_3(x) = P_2 + \frac{1}{3!} f'''(0)\,(x-0)^3 = x + \frac{1}{6} \cdot 2x^3 = \underline{x + \frac{1}{3}x^3}$

c) $f(x) = \sqrt{x} = x^{1/2}$, $a = 1$

$f(1) = 1$

$f'(x) = \frac{1}{2} x^{-1/2}$

$f'(1) = \frac{1}{2}$

$f''(x) = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) x^{-3/2} = -\frac{1}{4} x^{-3/2}$

$f''(1) = -\frac{1}{4}$

$f'''(x) = -\frac{1}{4}\left(-\frac{3}{2}\right) x^{-5/2} = \frac{3}{8} x^{-5/2}$

$f'''(1) = \frac{3}{8}$

$P_1(x) = f(1) + f'(1)\,(x-1) = 1 + \frac{1}{2}(x-1) = \underline{\frac{1}{2}x + \frac{1}{2}}$

$P_2(x) = P_1(x) + \frac{1}{2!}\,f''(1)(x-1)^2 = \frac{1}{2}x + \frac{1}{2} + \frac{1}{2}\cdot\left(-\frac{1}{4}\right)(x-1)^2$

$= \underline{\frac{1}{2}x + \frac{1}{2} - \frac{1}{8}(x-1)^2}$

$P_3(x) = P_2 + \frac{1}{3!}\,f'''(1) = \frac{1}{2}x + \frac{1}{2} - \frac{1}{8}(x-1)^2 + \frac{1}{6}\cdot\frac{3}{8}(x-1)^3$

$= \underline{\frac{1}{2}x + \frac{1}{2} - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3}$

e) $f(x) = \sin x, \quad a = \frac{\pi}{4}$

$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x, \quad f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x, \quad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$f'''(x) = -\cos x, \quad f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$P_1(x) = \underline{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)}$

$P_2(x) = P_1(x) + \frac{1}{2!}\,f''\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)^2 =$

$\underline{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2}$

$P_3(x) = P_2 + \frac{1}{3!}\,f'''\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)^3 =$

$\underline{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3}$

f) $f(x) = x^2 - 2x + 1$, $a = 0$

$f(0) = 1$

$f'(x) = 2x - 2$, $f'(0) = -2$

$f''(x) = 2 = f''(0)$

$f'''(x) \equiv 0$

$P_1(x) = f(0) + f'(0)(x-0) = \underline{1 - 2x}$

$P_2(x) = P_1(x) + \frac{1}{2!} f(0)(x-0)^2 = 1 - 2x + \frac{1}{2} \cdot 2 \cdot x^2$

$= \underline{x^2 - 2x + 1 = f(x)}$

$f'''(0) = 0 \Rightarrow \underline{P_3(x) = P_2(x)}$

g) $f(x) = x^2 - 2x + 1$, $a = 1$

- Same $f(x)$ som i f).

$f(1) = 0$

$f'(1) = 2 \cdot 1 - 2 = 0$

$f''(1) = 2$

$P_1(x) = f(1) + f'(1)(x-1) = \underline{0}$

$P_2(x) = P_1 + \frac{1}{2!} f''(1)(x-1)^2 = \underline{(x-1)^2 = f(x)}$

$P_3(x) = P_2(x) + 0 = P_2(x)$

6) c) $f(x) = \ln(1+x^2)$

$$f'(x) = \frac{1}{1+x^2} \cdot 2x, \quad f'(0) = 0$$

$$f''(x) = \frac{2(1+x^2) - 2x(0+2x)}{(1+x^2)^2} = \frac{-2x^2+2}{(1+x^2)^2}$$

$$f''(0) = 2$$

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{1}{2!} f''(0)(x-0)^2 =$$

$$\ln 1 + 0 \cdot x + \frac{1}{2} \cdot 2x^2 = \underline{x^2}$$

$$R_2(x) = \frac{f^{(3)}(c)}{3!}(x-0)^3, \quad c \text{ mellom } 0 \text{ og } x$$

$$f^{(3)}(x) = \frac{(-2x^2+2)'(1+x^2)^2 - (-2x^2+2)((1+x^2)^2)'}{((1+x^2)^2)^2} =$$

$$\frac{-4x(1+x^2)^2 - (-2x^2+2)\,2(1+x^2)\cdot 2x}{(1+x^2)^4} =$$

$$\frac{\cancel{(1+x^2)}\left(-4x(1+x^2) - 8x(1-x^2)\right)}{(1+x^2)^4} =$$

$$\frac{-4x-4x^3-8x+8x^3}{(1+x^2)^3} = \frac{4(x^3-3x)}{(1+x^2)^3}$$

$$R_2(x) = \frac{\frac{4(c^3-3c)}{(1+c^2)^3}}{3!} x^3 = \underline{\frac{2(c^3-3c)}{3(1+c^2)^3} x^3}$$

8) a) $y = f(x)$ definert implisitt ved

$$x^3 + xy - y^5 = 1$$

kring punktet $(a,b) = (1,1)$

Kontroll: $x = y = 1$ gir at $x^3 + xy - y^5 =$
$1^3 + 1 \cdot 1 - 1^5 = 1$ OK.

Implisitt derivasjon:

$$3x^2 + 1 \cdot y + x \cdot y' - 5y^4 \cdot y' = 0$$

$$3x^2 + y + xy' - 5y^4 y' = 0$$

$$(x - 5y^4) y' = -3x^2 - y$$

$$y' = \frac{3x^2 + y}{5y^4 - x}$$

$$\text{I } (1,1): \quad y' = \frac{3 \cdot 1^2 + 1}{5 \cdot 1^4 - 1} = \frac{4}{4} = 1$$

Deriverar igjen:

$$(3x^2 + y + xy' - 5y^4 y')' = 0'$$

$$6x + y' + 1 \cdot y' + xy'' - (5 \cdot 4y^3 \cdot y' \cdot y' + 5y^4 y'') = 0$$

$$6x + 2y' + xy'' - 20y^3 (y')^2 - 5y^4 y'' = 0$$

Set inn $x = y = y' = 1$:

$$6 \cdot 1 + 2 \cdot 1 + 1 \cdot y'' - 20 \cdot 1^3 \cdot 1^2 - 5 \cdot 1^4 \cdot y'' = 0$$

$$-12 - 4y'' = 0 \iff y'' = -\frac{12}{4} = -3$$

Altså:

$$P_2 = 1 + 1\cdot(x-1) + \frac{1}{2!}(-3)(x-1) =$$

$$\underline{x - \frac{3}{2}(x-1)}$$

b) – Same som i a) men i pkt. (0, −1)

Set inn $x=0$, $y=-1$:

$$0^3 + 0\cdot(-1) - (-1)^5) = 1 \quad \text{Ok}$$

$$y' = \frac{3\cdot 0^2 + (-1)}{5\cdot(-1)^4 - 0} = -\frac{1}{5}$$

Set inn $x=0, y=-1$ og $y' = -\frac{1}{5}$ i uttrykket med y'':

$$6\cdot 0 + 2\cdot(-\tfrac{1}{5}) + 0\cdot y'' - 20(-1)^3(-\tfrac{1}{5})^2 - 5(-1)^4\cdot y'' = 0$$

$$-\frac{2}{5} + \frac{20}{25} - 5y'' = 0$$

$$5y'' = \frac{20-10}{25} = \frac{10}{25} = \frac{2}{5}$$

$$y'' = \frac{2}{25}$$

Deriverer igjen:

$$(6x + 2y' + xy'' - 20y^3(y')^2 - 5y^4y'')' = 0'$$

$$6 + 2y'' + 1\cdot y'' + xy''' - (20\cdot 3y^2\cdot y'(y')^2 +$$

$$20y^3\cdot 2y'\cdot y'') - (5\cdot 4y^3\cdot y'\cdot y'' + 5y^4\cdot y''') = 0$$

$$6 + 3y'' + xy''' - 60y^2(y')^3 - 60y^3y'y'' - 5y^4y''' = 0$$

Set inn $x=0$, $y=-1$, $y'=-\frac{1}{5}$ og $y''=\frac{2}{25}$

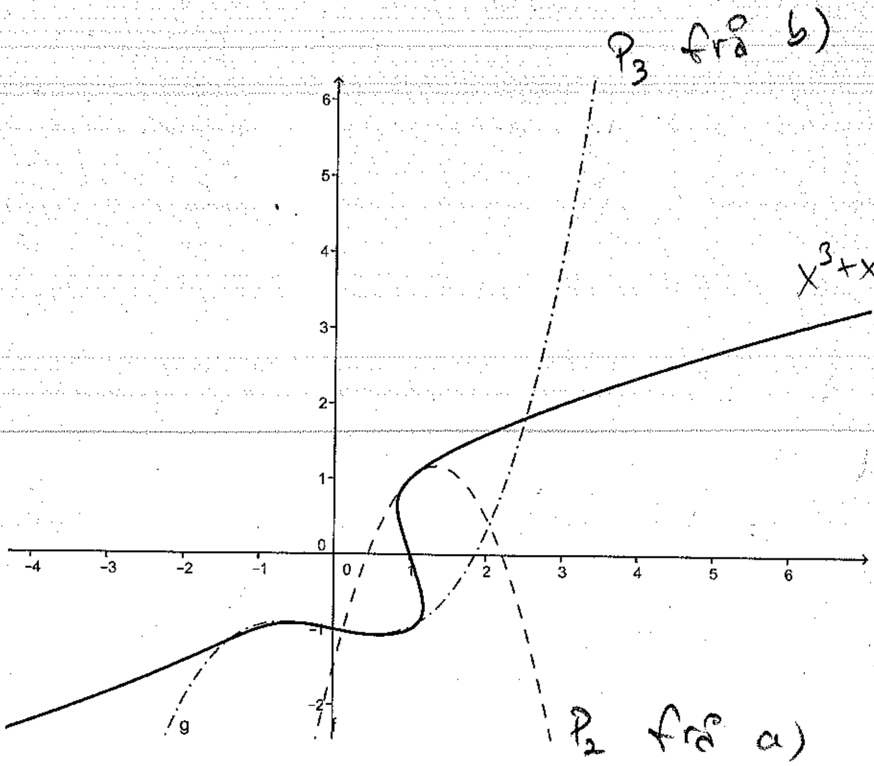
$$6+3\cdot\frac{2}{25}+0-60\cdot(-1)^2\cdot\left(-\frac{1}{5}\right)^3-60(-1)^3\left(-\frac{1}{5}\right)\frac{2}{25}-5y^4y'''=0$$

$$6+\frac{6}{25}+\frac{60}{125}-\frac{120}{125}-5y'''=0$$

$$5y'''=\frac{144}{25},\quad y'''=\frac{144}{125}$$

$$P_3(x)=-1+\left(-\frac{1}{5}\right)(x-0)+\frac{1}{2!}\cdot\frac{2}{25}(x-0)^2+\frac{1}{3!}\frac{144}{125}(x-0)^3=$$

$$\underline{-1-\frac{x}{5}+\frac{x^2}{25}+\frac{24}{125}x^3}$$



9) $f(x) = \ln x, \quad a = 1$

$f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1}, \quad f'(1) = 1$

$f''(x) = -x^{-2}, \quad f''(1) = -1$

$f'''(x) = 2x^{-3}, \quad f'''(1) = 2$

$f^{(4)}(x) = -6x^{-4}, \quad f^{(4)}(1) = -6$

$f^{(5)}(x) = 24x^{-5}, \quad f^{(5)}(1) = 24$

$P_1(x) = 0 + 1(x-1) = x-1$

$P_2(x) = P_1 + \frac{1}{2!} \cdot (-1)(x-1)^2 = P_1(x) - \frac{1}{2}(x-1)^2$

$P_3(x) = P_2 + \frac{1}{3!} \cdot 2(x-1)^3 = P_2(x) + \frac{1}{3}(x-1)^3$

$P_4(x) = P_3 + \frac{1}{4!}(-6)(x-1)^4 = P_3(x) - \frac{1}{4}(x-1)^4$

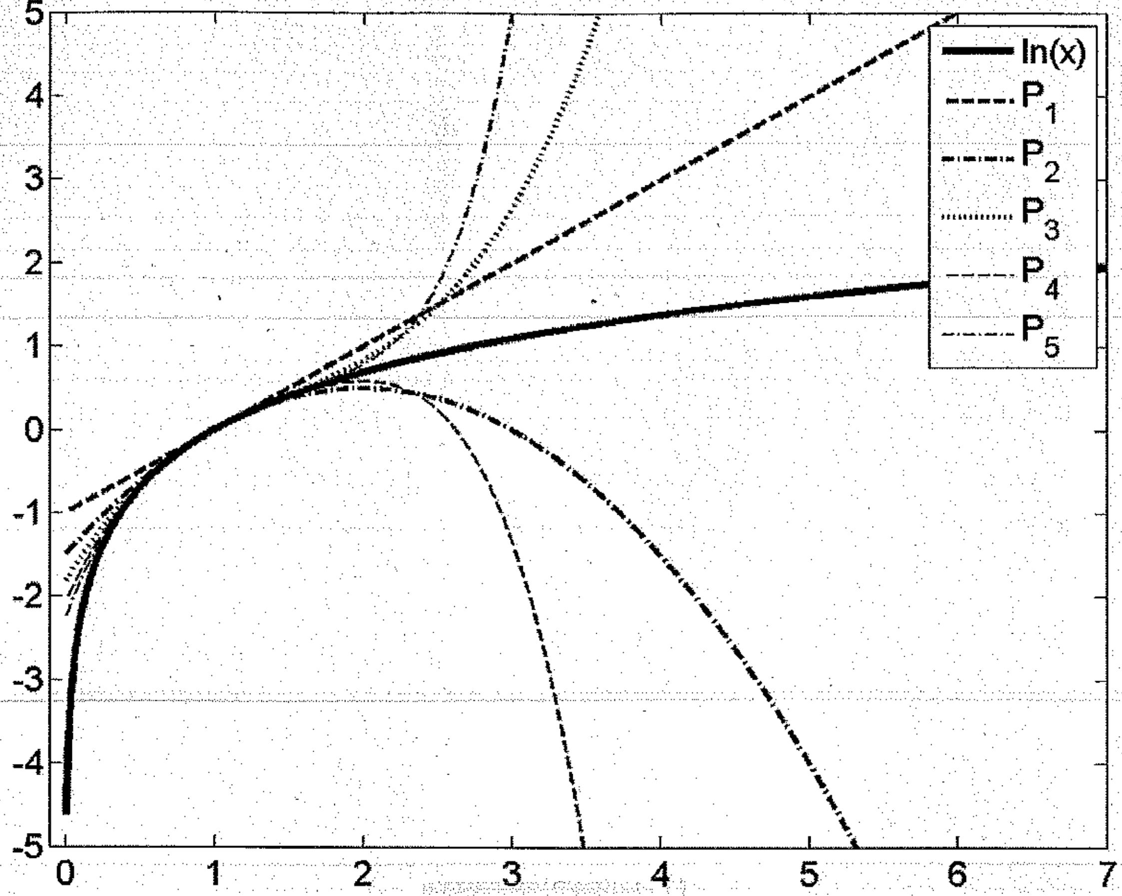
$P_5(x) = P_4 + \frac{1}{5!} \cdot 24(x-1)^5 = P_4(x) + \frac{1}{5}(x-1)^5$

Skript:

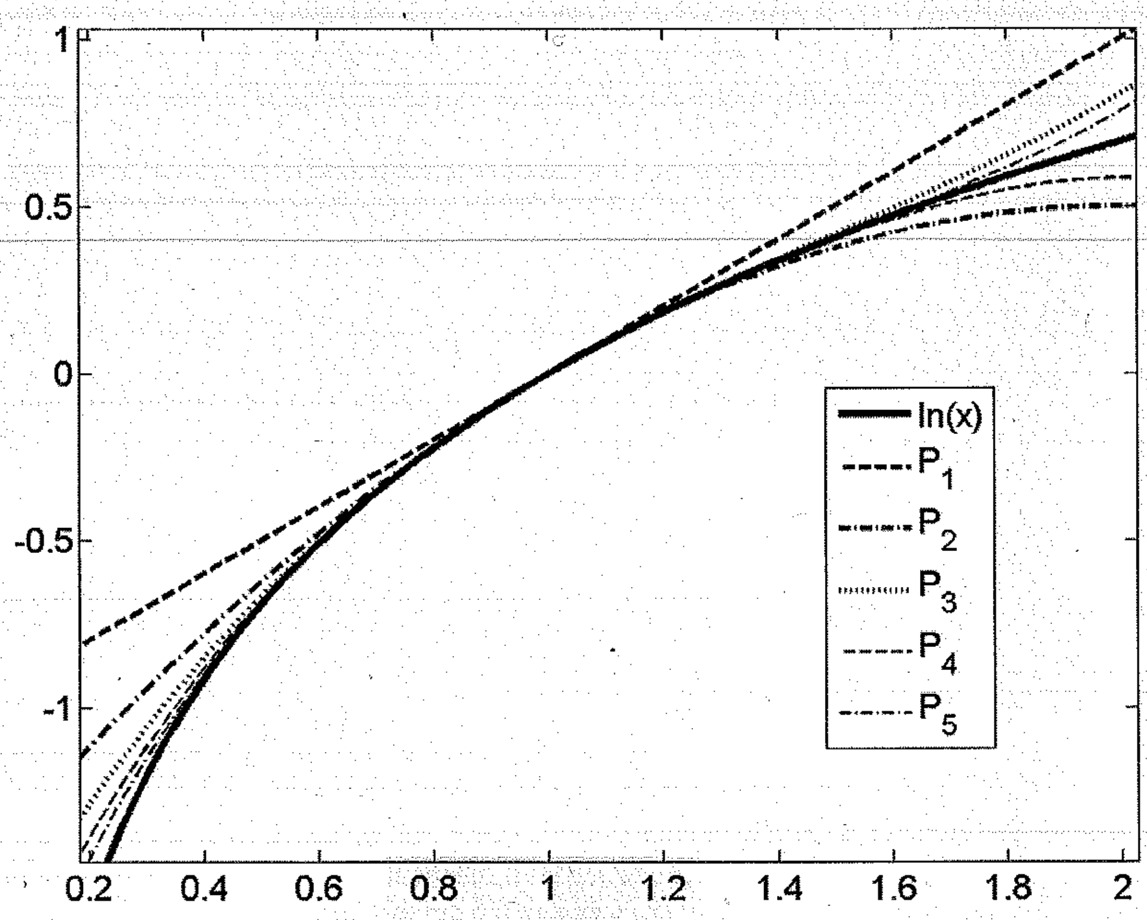
```
% Vektor med x-verdiar
x=1e-2:1e-2:7;

% Lagar Taylor-polynoma
P1=x-1;
P2=P1-1/2*(x-1).^2;
P3=P2+1/3*(x-1).^3;
P4=P3-1/4*(x-1).^4;
P5=P4+1/5*(x-1).^5;

% Plottar
plot(x,log(x),'k-','linewidth',3)
hold on
plot(x,P1,'k--','linewidth',2)
plot(x,P2,'k-.','linewidth',2)
plot(x,P3,'k:','linewidth',2)
plot(x,P4,'k--','linewidth',1)
plot(x,P5,'k-.','linewidth',1)
hold off
set(gca,'fontsize',15)
legend('ln(x)','P_1','P_2','P_3','P_4','P_5')
axis([-.1 7 -5 5])
```



ln(x)
P1
P2
P3
P4
P5
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
1
2
3
4
5
6
7



ln(x)
P1
P2
P3
P4
P5
1
0.5
0
-0.5
-1
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

18) Frå teksten har vi sett at

$$P_5 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

$$\sin\left(\tfrac{1}{2}\right) \approx P_5\left(\tfrac{1}{2}\right) = \frac{1}{2} - \frac{1}{3!}\cdot\left(\frac{1}{2}\right)^3 + \frac{1}{5!}\left(\frac{1}{2}\right)^5$$

$$= \frac{1}{2} - \frac{1}{48} + \frac{1}{3840} = \frac{1841}{3840} \approx 0.4794271$$

Feil

Med $f(x) = \sin x$:

$f^{(6)}(x) = -\sin x$, $f^{(7)}(x) = -\cos x$

i) $$R_5(x) = \frac{-\sin c}{6!}(x-0)^6 = -\frac{\sin c}{720}x^6$$

$$R_5\left(\tfrac{1}{2}\right) = -\frac{\sin c}{720}\cdot\left(\frac{1}{2}\right)^6$$

$$\left|R_5\left(\tfrac{1}{2}\right)\right| = \frac{|\sin c|}{720\cdot 2^6} \leq \frac{1}{46080} \approx 0.00002170$$

Her har vi brukt at $|\sin c| \leq 1$

Men: Sidan $f^{(6)}(0) = -\sin 0 = 0$, er

$P_6(x) = P_5(x)$.

Difor kan vi bruke at feilen er avgrensa av R_6 – noko som gir ein mindre feil; svaret over er meir nøyaktig enn $2.170\cdot 10^{-5}$.

ii) $R_6(\frac{1}{2}) = \frac{-\cos c}{7!} \cdot \left(\frac{1}{2}\right)^7$, $c \in [0, \frac{1}{2}]$

Sidan $\cos c > 0$ for $c \in [0, \frac{1}{2}]$, kan vi sei at $R_6(\frac{1}{2})$ er negativ. Difor vil $P_5(\frac{1}{2}) = P_6(\frac{1}{2})$ vere ei ~~øvre~~ grense for $\sin \frac{1}{2}$.

Feilen, som altså er negativ, er avgrensa av at

$$|R_6(\tfrac{1}{2})| = \left| \frac{-\cos c}{7!} \left(\frac{1}{2}\right)^7 \right| \leq \frac{1}{7! \cdot 2^7} =$$

$$\frac{1}{645120} \approx 1.5501 \cdot 10^{-6}$$

Altså:

$$\sin \frac{1}{2} \in \left[P_5(\tfrac{1}{2}) - \frac{1}{645120},\ P_5(\tfrac{1}{2})\right] =$$

$$\left[\frac{1841}{3840} - \frac{1}{645120},\ \frac{1841}{3840}\right]$$

Med desimaltal:

$$0.47942553 \leq \sin \frac{1}{2} \leq 0.47942708$$

201) $f(x)=\sqrt{x}$, $n=2$

Vi lar $a=100$

$f(100)=10$

$f'(x)=(x^{1/2})'=\frac{1}{2}x^{-1/2}$, $f'(100)=\frac{1}{2\cdot\sqrt{100}}=\frac{1}{20}$

$f''(x)=-\frac{1}{4}x^{-3/2}$, $f''(100)=-\frac{1}{4000}$

$f'''(x)=-\frac{1}{4}\cdot(-\frac{3}{2})x^{-5/2}=\frac{3}{8(\sqrt{x})^5}$

$P_2(x)=10+\frac{1}{20}(x-100)+\frac{1}{2!}\cdot(-\frac{1}{4000})(x-100)^2$

$P_2(101)=10+\frac{1}{20}-\frac{1}{8000}=$

$$\frac{80000+400-1}{8000}=\frac{80399}{8000}=10.049875$$

Feil:

$R_2(x)=\frac{f^{(3)}(c)}{3!}(x-100)^3$, $c\in[100,101]$

$$0<R_2(101)=\frac{\frac{3}{8(\sqrt{c})^5}}{6}\cdot 1^3=\frac{3}{48(\sqrt{c})^5}\leq$$

$$\frac{3}{48(\sqrt{100})^5}=\frac{3}{4800000}=6.25\cdot 10^{-7}$$

Altså:

$10.049875000\leq\sqrt{101}\leq 10.0498756 25$

23) Har: $\sqrt{b} \approx \frac{a}{2} + \frac{b}{2a}$

der a er det største naturlege talet som er slik at $a^2 < b$.

a) Med $b = 83$:

$a = 9$ gir $a^2 = 81 < b$, $10^2 > b$

Altså: $\sqrt{83} \approx \frac{9}{2} + \frac{83}{2 \cdot 9} = \frac{82}{9} \approx 9.1111$

b) Med $f(x) = \sqrt{x}$

Finn Taylor-polynomet av grad 1 omkring 81:

$$P_1(x) = f(81) + f'(81)(x - 81)$$

$$= \sqrt{81} + \frac{1}{2 \cdot \sqrt{81}}(x - 81) = 9 + \frac{x - 81}{2 \cdot 9}$$

$$P_1(83) = 9 + \frac{83 - 81}{2 \cdot 9} = \frac{82}{9}$$

Meir generelt: Omkring a^2

$$P_1(b) = \sqrt{a^2} + \frac{1}{2\sqrt{a^2}}(b - a^2) =$$

$$a + \frac{1}{2a}(b - a^2) = \underline{\frac{a}{2} + \frac{b}{2a}}$$