



Leaping into Lyapunov Space

While investigating digestion, Mario Markus of the Max Planck Institute for Nutrition discovered beauty in chaos. He and his collaborator Benno Hess have studied several mathematical models in an attempt to simulate how enzymes break down carbohydrates. By adjusting a pair of parameters, they found they could make the simulated enzymes behave in either an orderly or chaotic manner. To illustrate the chaos inherent in the model, Markus created a series of pictures that provided not only food for thought but also a feast for the eyes.

The images are based on a formula named after the Russian mathematician Aleksandr M. Lyapunov. The formula generates a single number for any dynamic system. Known as the Lyapunov exponent, the number indicates how chaotically the system is behaving. When the Lyapunov formula is applied to Markus's model, it produces an exponent for each pair of parameters. By representing each pair as a point on a computer screen and assigning colors

to each point depending on the value of the exponent, Markus created what I call Lyapunov space. In this virtually unexplored territory, order inhabits colored shapes, and chaos lurks in black regions.

Not long after his work appeared in academic journals, Markus rushed several pictures to an art gallery for exhibition. He can hardly be blamed for doing so. The pictures could just as easily have been made by an apprentice to Salvador Dali.

The model developed by Markus is based on a variation of the so-called logistic formula, the simplest-known formula that describes a chaotic dynamic system. The formula contains a variable, x , whose value is always somewhere between 0 and 1. It also involves a system parameter, r . The formula can be written:

$$x \leftarrow rx(1 - x)$$

The arrow indicates that once r , x , and $1 - x$ have all been multiplied together,

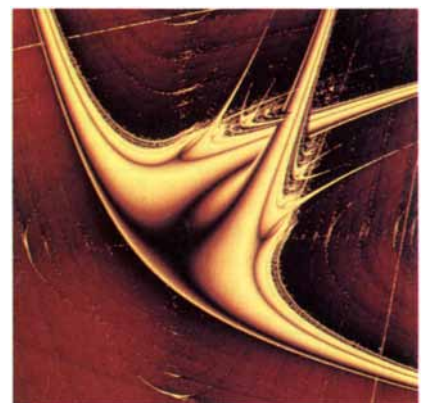
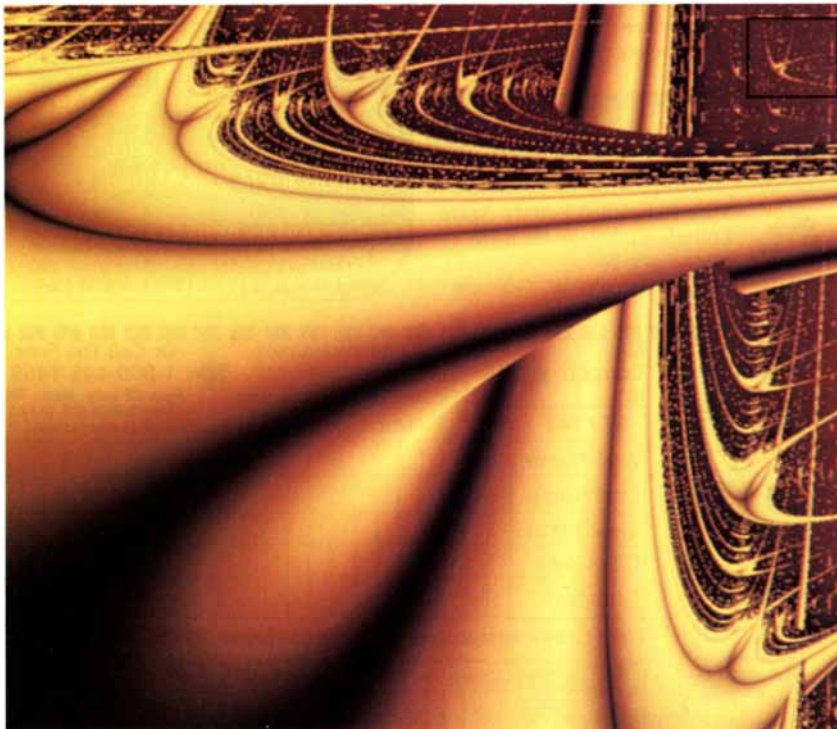
the resulting number becomes the new value for x , that is, it replaces the old value. The process can be repeated so that the formula continually spews out new values for x .

The resulting sequence holds great fascination for dynamicists, but what does it all mean? The logistic equation gets its name from the logistics of animal populations. In the equation, x represents the number of animals in an isolated region divided by the maximum number that the region could ever be expected to support. The amount of food available is therefore proportional to $1 - x$. In other words, as the number of animals (x) approaches the maximum (1), the food supply ($1 - x$) dwindles to nothing (0). The parameter r expresses the proportionality. It may be thought of as the fecundity of the population. The higher the value of r , the more quickly the population will rebound from any disaster. Strangely enough, higher values are precisely the ones that lead most quickly to chaotic populations.

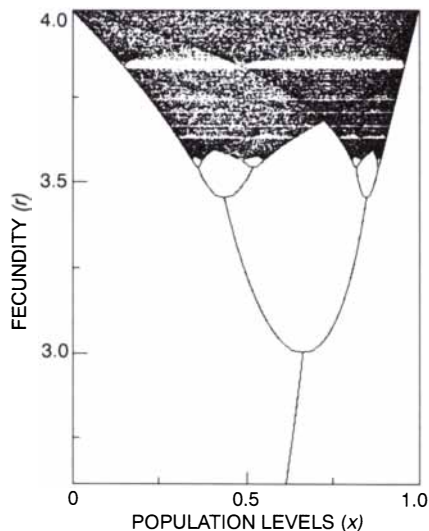
Although the equation is too simple to represent real animal populations, it can serve as a rough approximation to population dynamics.

If the parameter r is less than 2, the sequence of numbers produced quickly homes in on a single value. It makes no difference what number the formula uses for x initially. The population always converges to a stable value. In the jargon of chaos theory, a system whose dynamics stabilize at a single value is said to have a one-point attractor.

If the parameter r is greater than 2 but less than about 2.45, the logistic formula generates numbers that even-



Lyapunov space (left) and a detail (right) showing the "swallow"



Population levels, x , can fluctuate over a wider range of values as the fecundity factor, r , increases.

tually alternate between two values. The system then converges on a two-point attractor. In some sense, when fecundity is high, the population pays a price: its size fluctuates over time.

If the fecundity factor is cranked up to values greater than 2.45, the logistic formula produces numbers that converge on a four-point attractor. Still higher values of r lead very quickly to eight-point attractors, then 16-point ones and so on. But if the value of r is greater than 3.57 (or, to be more precise, 3.56994571869), chaos reigns.

At that level of fecundity, the formula seems to generate values at random, though, to be sure, the formula is deterministic. The reason for this strange behavior lies in the attractor. It happens to be a one-dimensional fractal. Like all fractals, it is self-similar: when any small piece of it is magnified, the enlarged region looks very much like the whole.

The fate of the hypothetical populations is clearly portrayed in the graph above. The diagram is produced by plotting r against the values to which the logistic formula converges. The result is a kind of tree. One-point attractors make up the trunk; two-point attractors, the first pair of branches. At an r value of 3.57, the onset of chaos can be seen clearly: the branches suddenly proliferate.

Chaos can be characterized using the Lyapunov formula. For each dynamic system, the formula produces a single number, the Lyapunov exponent. If the number is less than 0, the system is stable. If the number is greater than 0, the system is capable of chaotic behavior.

The Lyapunov formula is complicated,



A Lyapunov "jellyfish"

ed, but it can be translated into a series of simple steps. In the case of the logistic system, we start with one particular value of r . The logistic formula is iterated, say, 600 times, so that the numbers converge to whatever attractor is present in the system. After the numbers settle down, it is safe to compute the Lyapunov exponent. The following recipe outlines the computation:

```
total ← 0
for n ← 1 to 4,000
  x ← rx(1 - x)
  total ← total + ((log |r - 2rx|) / log 2)
lyap ← total / 4,000
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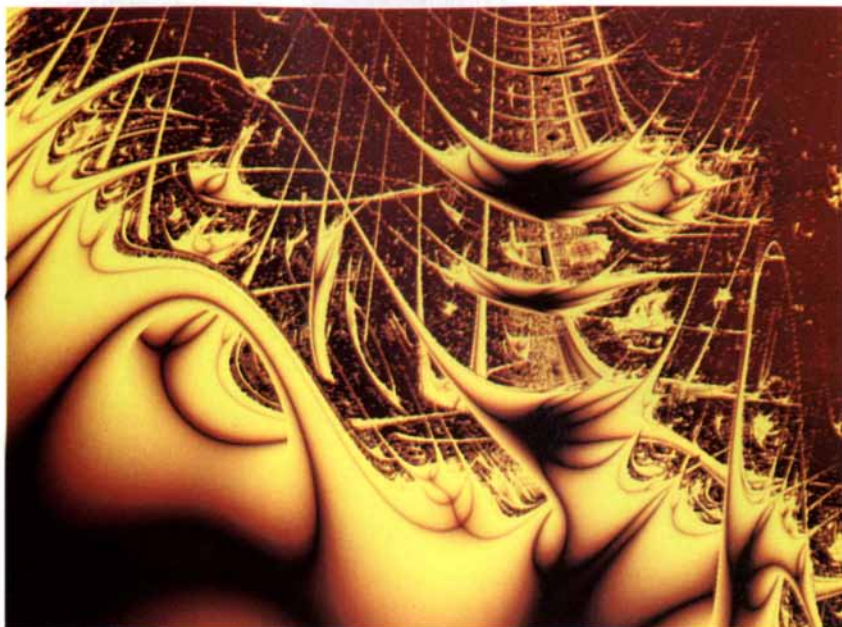
The algorithm first sets $total = 0$ and then iterates the logistic formula 4,000 times. On each iteration it computes a new value for $total$ by adding the old value of $total$ to the logarithm of $|r - 2rx|$ divided by the logarithm of 2. (The vertical bars indicate the absolute value of $r - 2rx$.) The quantity $|r - 2rx|$ represents the rate at which the mag-

nitude of successive values is growing or shrinking. When it has added up all 4,000 logarithms, the algorithm divides the sum by 4,000. The result, which has been assigned to the variable *lyap* above, is something like an average logarithm of change. The result closely approximates the Lyapunov exponent.

Readers who demand precision can more accurately estimate the Lyapunov exponent by increasing the number of iterations and, at the end of the procedure, by dividing the sum of logarithms by the number of iterations.

I encourage readers to use the algorithm above to calculate the Lyapunov exponent for r equal to 2. Then compare the result with that obtained when r equals 3. The first number should be negative, indicating a stable system, and the second number should be positive, a warning of chaos.

The pictures accompanying this article are all based on the logistic equation. Markus merely adds one twist of his own. To produce his pictures, Mar-



Zircon Zity

kus used periodic forcing. This means that r systematically changes its value, alternating between two fixed numbers, a and b . In other words, the logistic equation is iterated with r values of a , then b , a , b , a , b and so on. The resulting system may or may not develop

chaotic behavior. The issue can be settled only by calculating the Lyapunov exponent.

For that reason, Markus plotted the value of the exponent for each possible combination of a and b values. A two-dimensional image in Lyapunov space emerges when the points (a, b) are colored in some consistent fashion. Markus assigned black to all points (a, b) that produce a non-negative value for the Lyapunov exponent. Chaos is black. Shades of a single color, such as yellow, appear everywhere else in Lyapunov space. From a Lyapunov exponent of zero down to minus infinity, the shade ranges continuously from light to dark. At zero, there is an abrupt discontinuity as the color suddenly turns from bright yellow to black.

The resulting images are maps of chaos in the forced logistic system. In particular, the map on page 178 depicts the straightforward system just described. The parameter r alternates in completely regular fashion between a and b .

The crossing point of two or more spikes in any of the accompanying images reveals the coexistence of periodic attractors. This means that, at a point (a, b) where such a crossing occurs, the corresponding dynamic system in which r alternates between a and b will have two attractors. Which attractor operates depends, strangely enough, on the initial value that one chooses for x before iteration.

If the Lyapunov exponent is plotted for a succession of initial x values, it

may take on a specific value, say, 0.015 for a number of these initial values. Then the exponent may suddenly switch to another value, 0.142, to which it may stick for several more successive initial x values before reverting to the first value. The switching back and forth can become quite frequent.

Lyapunov space often contains darkish bands that run along the spikes. These represent superstable regions in which the underlying forced logistic systems exhibit the most regular behavior.

The Lyapunov space generated by alternating a and b values contains a tiny fleck that resembles a swallow. The fleck is enlarged in the illustration on page 178. There, just off the swallow's tail, lies another little fleck. Readers are free to guess just what it might turn out to be when it is similarly enlarged.

The appearance of self-similarity in the figure should not surprise students of chaos. Structures that exhibit self-similarity are more often than not produced by chaotic processes.

The methods used to create the mother swallow and its offspring can be varied slightly to generate a host of different creatures. The images on these pages differ only in the a and b value sequences that were used. For example, a jellyfish—the yellow tentacled blob shown on the preceding page—is spawned from a sequence that begins b, b, a, b, a and that repeats over and over again.

The scene above resembles the cover of a science fiction magazine from the 1950s. I call it Zircon Zity because it is obviously the futuristic metropolis of the Zirconites (whoever they are). The underlying sequence of the zity is $bbb-bbbbaaaaaa$. By repeating this sequence while calculating the Lyapunov exponent, a computer can build the zity with all its delicate bridges, golden space-ships and interplanetary walkways.

What does all this have to do with enzymes, carbohydrates and nutrition? At best, a small region in some Lyapunov map might actually describe the dynamics of enzymes breaking down carbohydrates. But perhaps more to the point, Markus's work makes it possible to visualize the dynamics of periodic forcing. One might say he has made chaos easier to digest.

The Lyapunov Program

For those readers who wish to explore Lyapunov space, I can give a few hints about how they might create the appropriate program. The heart of the program should be a double loop that runs through values of a and b . To compute the Lyapunov exponent for each combination of a and b , the program should include a routine based on the algorithm given on page 179. The routine should allow various sequences of a 's and b 's to be stored in an array. The routine could enable the user to specify the sequence by filling the array with 0's (representing a) and 1's (representing b).

With each iteration of the Lyapunov loop, a counter called *index* may be used to step through the array. Each time the inner loop produces a new value for *index*, it will look up the current value of either a or b , depending on whether it finds a 0 or a 1 in the array at the value of *index*. Finally, the program should include a table of logarithms to help speed the computations. Indeed, it may take hours for a personal computer to generate a single image in Lyapunov space.

FURTHER READING

CHAOS IN MAPS WITH CONTINUOUS AND DISCONTINUOUS MAXIMA. Mario Markus in *Computers in Physics*, pages 481-493; September/October 1990.
THE MAGIC MACHINE: A HANDBOOK OF COMPUTER SECRECY. A. K. Dewdney. W. H. Freeman and Company, 1990.

FIRE & ICE

A TOUGH CHOICE

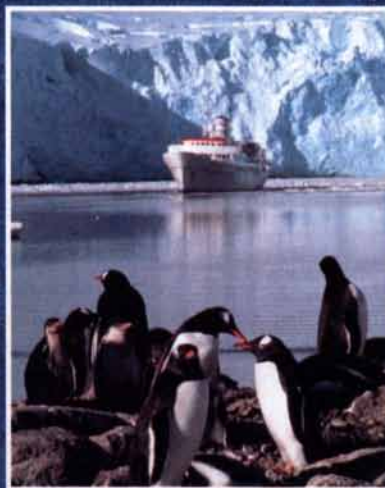
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