# IDEALS GENERATED BY SUBMAXIMAL MINORS.

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ABSTRACT. The goal of this paper is to study irreducible families $W^{t-1}_{t,t}(\underline{b};\underline{a})$ of codimension 4, arithmetically Gorenstein schemes $X \subset \mathbb{P}^n$ defined by the submaximal minors of a $t \times t$ homogeneous matrix $\mathcal{A}$ whose entries are homogeneous forms of degree $a_j - b_i$. Under some numerical assumption on $a_j$ and $b_i$ we prove that the closure of $W^{t-1}_{t,t}(\underline{b};\underline{a})$ is an irreducible component of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$, we show that $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ is generically smooth along $W^{t-1}_{t,t}(\underline{b};\underline{a})$ and we compute the dimension of $W^{t-1}_{t,t}(\underline{b};\underline{a})$ in terms of $a_j$ and $b_i$. To achieve these results we first prove that $X$ is determined by a regular section of $\mathcal{I}_Y/\mathcal{I}^2_Y(s)$ where $s = \deg(\det(\mathcal{A}))$ and $Y \subset \mathbb{P}^n$ is a codimension 2, arithmetically Cohen-Macaulay scheme defined by the maximal minors of the matrix obtained deleting a suitable row of $\mathcal{A}$.

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## 1. INTRODUCTION

In this paper we deal with determinantal schemes. A scheme $X \subset \mathbb{P}^n$ of codimension $c$ is called *determinantal* if its homogeneous saturated ideal can be generated by the $r \times r$ minors of a homogeneous $p \times q$ matrix with $c = (p-r+1)(q-r+1)$. When $r = min(p,q)$ we say that $X$ is *standard determinantal.* Given integers $r \le p \le q$, $a_1 \le a_2 \le ... \le a_p$ and $b_1 \le b_2 \le ... \le b_q$ we denote by $W^r_{p,q}(\underline{b};\underline{a}) \subset \mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ the locus of determinantal schemes $X \subset \mathbb{P}^n$ of codimension $c = (p-r+1)(q-r+1)$ defined by the $r \times r$ minors of a $p \times q$ matrix $(f_{ji})^{i=1,...,q}_{j=1,...,p}$ where $f_{ji} \in k[x_0, x_1, ..., x_n]$ is a homogeneous polynomial of degree $a_j - b_i$.

The study of determinantal schemes has received considerable attention in the literature (See, for instance, [3], [5], [6] and [23]). Some classical schemes that can be constructed in

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this way are the Segre varieties, rational normal scrolls, and the Veronese varieties. The main goal of this paper is to contribute to the classification of determinantal schemes and we address in the case $p = q = t$, $r = t-1$ the following three fundamental problems:

(1) To determine the dimension of $W^r_{p,q}(\underline{b};\underline{a})$ in terms of $a_j$ and $b_i$,
(2) To determine whether the closure of $W^r_{p,q}(\underline{b};\underline{a})$ is an irreducible component of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$, and
(3) To determine when $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ is generically smooth along $W^r_{p,q}(\underline{b};\underline{a})$.

The first important contribution to these problems was made by G. Ellingsrud [8] in 1975, who proved that every arithmetically Cohen-Macaulay, codimension 2 closed subscheme $X$ of $\mathbb{P}^n$ is unobstructed (i.e. the corresponding point in the Hilbert scheme $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ is smooth) provided $n \geq 3$. He also computed the dimension of the Hilbert scheme at $(X)$. Recall also that the homogeneous ideal of an arithmetically Cohen-Macaulay, codimension 2 closed subscheme $X$ of $\mathbb{P}^n$ is given by the maximal minors of a $(t-1)\times t$ homogeneous matrix, the Hilbert-Burch matrix. That is, such a scheme is standard determinantal. The purpose of this work is to extend Ellingsrud's Theorem, viewed as a statement on standard determinantal schemes of codimension 2, to arbitrary determinantal schemes. The case of codimension 3 standard determinantal schemes, was mainly solved in [19]; Proposition 1.12; and the case of standard determinantal schemes of arbitrary codimension was studied and partially solved in [20]. In [21], we treated the case of codimension 3 determinantal schemes $X \subset \mathbb{P}^n$ defined by the submaximal minors of a symmetric homogeneous matrix. In our opinion, it is difficult to solve the above three questions in full generality and, in this paper, we will focus our attention to the first unsolved case, namely, we will deal with codimension 4 determinantal schemes $X \subset \mathbb{P}^n$, $n \geq 5$, defined by the submaximal minors of a homogeneous square matrix. As in [19], [20] and [21], we prove our results by considering the smoothness of the Hilbert flag scheme of pairs or, more generally, the Hilbert flag scheme of chains of closed subschemes obtained by deleting suitable rows, and its natural projections into the usual Hilbert schemes. We wonder if a similar strategy could facilitate the study of the general case.

Next we outline the structure of the paper. In section 2, we recall the basic facts on local cohomology and deformation theory needed in the sequel. In section 3, we describe the deformations of the codimension 4 arithmetically Gorenstein schemes $X \subset \mathbb{P}^n$ defined as the degeneracy locus of a regular section of the twisted conormal sheaf $\mathcal{I}_Y/\mathcal{I}_Y^2(s)$ of a codimension 2, arithmetically Cohen-Macaulay scheme $Y \subset \mathbb{P}^n$ of dimension $\geq 3$. Section 4 is the heart of the paper. In section 4, we determine the dimension of $W^{t-1}_{t,t}(\underline{b};\underline{a})$ in terms of $b_i$ and $a_j$ provided $a_i \geq b_{i+3}$ for $1 \leq i \leq t-3$ (and $a_1 \geq b_t$ if $t \leq 3$), $a_t > a_{t-1} + a_{t-2} - b_1$ and $\dim X \geq 1$. We also prove that, under this numerical restriction, $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ is generically smooth along $W^{t-1}_{t,t}(\underline{b};\underline{a})$ and the closure of $W^{t-1}_{t,t}(\underline{b};\underline{a})$ is an irreducible component of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ (cf. Theorem 4.6).

The key point for proving our result is the fact that any codimension 4, determinantal scheme $X \subset \mathbb{P}^n$ defined by the submaximal minors of a homogeneous square matrix $\mathcal{A}$ is arithmetically Gorenstein and determined by a regular section of $\mathcal{I}_Y/\mathcal{I}_Y^2(s)$ where $s = \deg(\det(\mathcal{A}))$ and $Y \subset \mathbb{P}^n$ is a codimension 2, arithmetically Cohen-Macaulay scheme defined by the maximal minors of the matrix $\mathcal{N}$ obtained deleting a suitable row of $\mathcal{A}$

(cf. Proposition 4.3). Conversely, any codimension 4, arithmetically Gorenstein scheme $X = \operatorname{Proj}(A) \subset \mathbb{P}^n$ defined by a regular section $\sigma$ of $\mathcal{I}_Y/\mathcal{I}_Y^2(s)$ where $Y = \operatorname{Proj}(B) \subset \mathbb{P}^n$ is a codimension 2, arithmetically Cohen-Macaulay scheme, fits into an exact sequence of the following type

$$0 \longrightarrow K_B(n+1-2s) \longrightarrow N_B(-s) \xrightarrow{\sigma^*} B \longrightarrow A \longrightarrow 0$$

and it is determined by the submaximal minors of a $t \times t$ homogeneous matrix $\mathcal{A}$ obtained by adding a suitable row to the Hilbert-Burch matrix of $Y$ (cf. Proposition 4.3). In the last section, we include some examples which illustrate that the numerical hypothesis in Theorem 4.6, $a_t > a_{t-1} + a_{t-2} - b_1$, cannot be avoided.

**Notation.** Throughout this paper $k$ will be an algebraically closed field $k$, $R = k[x_0, x_1, \dots, x_n]$, $\mathfrak{m} = (x_0, \dots, x_n)$ and $\mathbb{P}^n = \operatorname{Proj}(R)$. As usual, the sheafification of a graded $R$-module $M$ will be denoted by $\widetilde{M}$ and the support of $M$ by $Supp(M)$.

Given a closed subscheme $X$ of $\mathbb{P}^n$ of codimension $c$, we denote by $\mathcal{I}_X$ its ideal sheaf, $\mathcal{N}_X$ its normal sheaf and $I(X) = H^0_*(\mathbb{P}^n, \mathcal{I}_X)$ its saturated homogeneous ideal unless $X = \emptyset$, in which case we let $I(X) = \mathfrak{m}$ . If $X$ is equidimensional and Cohen-Macaulay of codimension $c$, we set $\omega_X = \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n})(-n-1)$ to be its canonical sheaf.

In the sequel, for any graded quotient $A$ of $R$ of codimension $c$, we let $I_A = \ker(R \twoheadrightarrow A)$, $N_A = \operatorname{Hom}_R(I_A, A)$ be the normal module. If $A$ is Cohen-Macaulay of codimension $c$, we let $K_A = \operatorname{Ext}^c_R(A, R)(-n-1)$ be its canonical module. When we write $X = \operatorname{Proj}(A)$, we let $A = R/I(X)$ and $K_X = K_A$. If $M$ is a finitely generated graded $A$-module, let $\operatorname{depth}_J M$ denote the length of a maximal $M$-sequence in a homogeneous ideal $J$ and let $\operatorname{depth} M = \operatorname{depth}_{\mathfrak{m}} M$. If $\Gamma_J(-)$ is the functor of sections with support in $\operatorname{Spec}(A/J)$, we denote by $H^i_J(-)$ the right derived functor of $\Gamma_J(-)$.

Let $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ be the Hilbert scheme parameterizing closed subschemes $X$ of $\mathbb{P}^n$ with Hilbert polynomial $p(x) \in \mathbb{Q}[x]$ (cf. [11]). By abuse of notation we will write $(X) \in \operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ for the $k$-point which corresponds to a closed subscheme $X \subset \mathbb{P}^n$. The Hilbert polynomial of $X$ is sometimes denoted by $p_X$. By definition $X$ is called unobstructed if $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ is smooth at $(X)$.

The pullback of the universal family on $\operatorname{Hilb}^p(\mathbb{P}^n)$ via a morphism $\psi : W \longrightarrow \operatorname{Hilb}^p(\mathbb{P}^n)$ yields a flat family over $W$, and we will write $(X) \in W$ for a member of that family as well. Suppose that $W$ is irreducible. Then, by definition a general $(X) \in W$ has a certain property if there is a non-empty open subset $U$ of $W$ such that all members of $U$ have this property. Moreover, we say that $(X)$ is general in $W$ if it belongs to a sufficiently small open subset $U$ of $W$ (So, any $(X)$ of $U$ has all the openness properties that we want to require).

Finally we let $D = D(p_X, p_Y)$ be the Hilbert flag scheme parameterizing pairs of closed subschemes $(X' \subset Y')$ of $\mathbb{P}^n$ with Hilbert polynomials $p_{X'} = p_X$ and $p_{Y'} = p_Y$, respectively.

## 2. Preliminaries

For convenience of the reader we include in this section the background and basic results on local cohomology and deformation theory needed in the sequel.

2.1. **Local cohomology.** Let $B = R/I_B$ be a graded quotient of the polynomial ring $R$, let $M$ and $N$ be finitely generated graded $B$-modules and let $J \subset B$ be an ideal. We say that $0 \neq M$ is a Cohen-Macaulay (resp. maximal Cohen-Macaulay) $B$-module if $\operatorname{depth} M = \dim M$ (resp. $\operatorname{depth} M = \dim B$), or equivalently, $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq \dim M$ (resp. $i < \dim B$) since $\operatorname{depth}_J M \geq r$ is equivalent to $H^i_J(M) = 0$ for $i < r$. If $B$ is Cohen-Macaulay (CM), we know by Gorenstein duality that the $v$-graded piece of $H^i_{\mathfrak{m}}(M)$ satisfies

$$ {}_vH^i_{\mathfrak{m}}(M) \simeq {}_{-v}\operatorname{Ext}^{\dim B - i}_B(M, K_B)^\vee. $$

Let $Z$ be closed in $Y := \operatorname{Proj}(B)$ and let $U = Y - Z$. Then we have an exact sequence

$$ 0 \to H^0_{I(Z)}(M) \to M \to H^0_*(U, \widetilde{M}) \to H^1_{I(Z)}(M) \to 0 $$

and isomorphisms $H^i_{I(Z)}(M) \simeq H^{i-1}_*(U, \widetilde{M})$ for $i \geq 2$ where as usual we write $H^i_*(U, \widetilde{M}) = \oplus_t H^i(U, \widetilde{M}(t))$. More generally, if $\operatorname{depth}_{I(Z)} N \geq i+1$ there is an exact sequence

$$ (2.1) \qquad {}_0\operatorname{Ext}^i_B(M,N) \hookrightarrow \operatorname{Ext}^i_{\mathcal{O}_U}(\widetilde{M}|_U, \widetilde{N}|_U) \to {}_0\operatorname{Hom}_B(M, H^{i+1}_{I(Z)}(N)) \to {}_0\operatorname{Ext}^{i+1}_B(M,N) \to $$

by [14]; exp. VI, where the middle form comes from a spectral sequence also treated in [14].

2.2. **Basic deformation theory.** To use deformation theory, we will need to consider the (co)homology groups of algebras $H_2(R,B,B)$ and $H^2(R,B,B)$. Let us recall their definition. We consider

$$ (2.2) \qquad \ldots \to F_2 := \oplus_{j=1}^{\mu_2} R(-n_{2,j}) \to F_1 := \oplus_{i=1}^{\mu} R(-n_{1,i}) \to R \to B \to 0 $$

a minimal graded free $R$-resolution of $B$ and let $H_1 = H_1(I_B)$ be the first Koszul homology built on a set of minimal generators of $I_B$. Then we may take the exact sequence

$$ (2.3) \qquad 0 \to H_2(R,B,B) \to H_1 \to F_1 \otimes_R B \to I_B/{I_B}^2 \to 0 $$

as a definition of the second algebra homology $H_2(R,B,B)$ (cf. [24]), and the dual sequence,

$$ \to {}_v\operatorname{Hom}_B(F_1 \otimes B, B) \to {}_v\operatorname{Hom}_B(H_1, B) \to {}_vH^2(R,B,B) \to 0, $$

as a definition of graded second algebra cohomology $H^2(R,B,B)$. If $B$ is generically a complete intersection, then it is well known that $\operatorname{Ext}^1_B(I_B/I_B^2, B) \simeq H^2(R,B,B)$ ([1]; Proposition 16.1). We also know that $H^0(Y, \mathcal{N}_Y)$ is the tangent space of $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ in general, while $H^1(Y, \mathcal{N}_Y)$ contains the obstructions of deforming $Y \subset \mathbb{P}^n$ in the case $Y$ is locally a complete intersection (l.c.i.) (cf. [11]). If ${}_0\operatorname{Hom}_R(I_B, H^1_{\mathfrak{m}}(B)) = 0$ (e.g. $\operatorname{depth}_{\mathfrak{m}} B \geq 2$), we have by (2.1) that ${}_0\operatorname{Hom}_B(I_B/I_B^2, B) \simeq H^0(Y, \mathcal{N}_Y)$ and ${}_0H^2(R,B,B) \hookrightarrow H^1(Y, \mathcal{N}_Y)$ is injective in the l.c.i. case, and that ${}_0H^2(R,B,B)$ contains the obstructions of deforming $Y \subset \mathbb{P}^n$ ([16]; Remark 3.7). Thus ${}_0H^2(R,B,B) = 0$ suffices for the unobstructedness of a l.c.i. arithmetically Cohen-Macaulay subscheme $Y$ of $\mathbb{P}^n$ of $\dim Y \geq 1$ (for this conclusion we may even entirely skip "l.c.i." by slightly extending the argument, as done in [16]).

2.3. **Useful exact sequences.** In the last part of this section, we collect some exact sequences frequently used in this paper, in the case that $B = R/I_B$ is a generically complete intersection codimension two CM quotient of $R$. First, applying $\mathrm{Hom}_R(-, R)$ to the minimal graded free $R$-resolution of $B$

$$0 \to F_2 := \oplus_{j=1}^{\mu-1} R(-n_{2,j}) \to F_1 := \oplus_{i=1}^{\mu} R(-n_{1,i}) \to R \to B \to 0 \tag{2.4}$$

we get a minimal graded free $R$-resolution of $K_B$:

$$0 \to R \to \oplus R(n_{1,i}) \to \oplus R(n_{2,j}) \to K_B(n+1) \to 0. \tag{2.5}$$

If we apply $\mathrm{Hom}(-, B)$ to (2.5) we get the exactness to the left in the exact sequence

$$0 \to K_B(n+1)^* \to \oplus B(-n_{2,j}) \to \oplus B(-n_{1,i}) \to I_B/I_B^2 \to 0 \tag{2.6}$$

which splits into two short exact sequences "via $\oplus B(-n_{2,j}) \twoheadrightarrow H_1 \hookrightarrow \oplus B(-n_{1,i})$", one of which is (2.3) with $H_2(R, B, B) = 0$. Indeed since $H_1$ is Cohen-Macaulay by [2], we get $H_2(R, B, B) = 0$ by (2.3). Moreover since $\mathrm{Ext}^1_R(I_B, I_B) \simeq N_B$ we showed in [22]; pg. 788 that there is an exact sequence of the form

$$0 \to F_1^* \otimes_R F_2 \to ((F_1^* \otimes_R F_1) \oplus (F_2^* \otimes_R F_2))/R \to F_2^* \otimes_R F_1 \to N_B \to 0 \tag{2.7}$$

where $F_i^* = \mathrm{Hom}_R(F_i, R)$. Indeed this sequence is deduced from the exact sequence

$$0 \to R \to \oplus I_B(n_{1,i}) \to \oplus I_B(n_{2,j}) \to N_B \to 0$$

which we get by applying $\mathrm{Hom}_R(-, I_B)$ to (2.4), (cf. [22]; (26)). Similarly applying $\mathrm{Hom}_R(-, I_B/I_B^2)$ to (2.4) and noting that $\mathrm{Hom}_R(I_B, I_B/I_B^2) \simeq \mathrm{Hom}_B(I_B/I_B^2, I_B/I_B^2)$ we get the exact sequence

$$0 \to \mathrm{Hom}_B(I_B/I_B^2, I_B/I_B^2) \to \oplus I_B/I_B^2(n_{1,i}) \to \oplus I_B/I_B^2(n_{2,j}) \to N_B \to 0. \tag{2.8}$$

Finally we recall the following frequently used exact sequence (cf. [24])

$$0 \to \wedge^2(\oplus R(-n_{2,j})) \to (\oplus R(-n_{1,i})) \otimes (\oplus R(-n_{2,j})) \to S^2(\oplus R(-n_{1,i})) \to I_B^2 \to 0\ . \tag{2.9}$$

## 3. Deformations of quotients of regular sections

In [18] the first author studied deformations of a scheme $X := \mathrm{Proj}(A)$ defined as the degeneracy locus of a regular section of a "nice" sheaf $\widetilde{M}$ on an arithmetically Cohen-Macaulay (ACM) scheme $Y = \mathrm{Proj}(B)$. Recall that if we take a regular section of the anticanonical sheaf $\widetilde{K}_B^*(s)$ and $Y$ is a l.c.i. of positive dimension, then we get an exact sequence

$$0 \to K_B(-s) \to B \to A \to 0,$$

in which $A$ is Gorenstein. Indeed the mapping cone construction leads to a resolution of $A$ from which we easily see that $A$ is Gorenstein. In [22], we generalized this way of constructing Gorenstein algebras to sheaves of higher rank and, in [18], we studied the deformations of this "construction", notably in the rank two case which we now recall.

Let $M$ be a maximal Cohen-Macaulay $B$-module of rank $r = 2$ such that $\widetilde{M}|_U$ is locally free and $\wedge^2\widetilde{M}|_U \simeq \widetilde{K_B}(t)|_U$ in an open set $U := Y - Z$ of $Y$ satisfying $\mathrm{depth}_{I(Z)} B \geq 2$.

Then a regular section $\sigma$ of $\widetilde{M}^*(s)|_U$ defines an arithmetically Gorenstein scheme $X = \operatorname{Proj}(A)$ given by the exact sequence

$$0 \to K_B(t-2s) \to M(-s) \xrightarrow{\sigma^*} B \to A \to 0 \tag{3.1}$$

and $M \simeq \operatorname{Hom}_B(M, K_B(t))$ ([22]; Theorem 8). In this paper we consider and further develop the case where $M = N_B$ and $\dim B = n-1$ ($n+1 = \dim R$, $n \geq 5$). By [22]; Proposition 13, $N_B$ is a maximal Cohen-Macaulay $B$-module and we have the exact sequence

$$0 \to K_B(n+1-2s) \to N_B(-s) \to I_{A/B} \to 0 \quad , \text{ where } \ I_{A/B} := \ker(B \to A). \tag{3.2}$$

**Example 3.1.** Set $R = k[x_0, \cdots, x_5]$ and let $B = R/I_B$ be a codimension two quotient with minimal resolution

$$0 \to R(-3)^2 \to R(-2)^3 \to R \to B \to 0$$

and suppose $Y = \operatorname{Proj}(B)$ is a l.c.i in $\mathbb{P}^5$. Let $A$ be given by a regular section of $\widetilde{I_B/I_B^2}(s)$, $s \geq 3$. Thanks to the exact sequences (2.5) and (2.7) and the mapping cone construction applied to both (3.2) and $0 \to I_{A/B} \to B \to A \to 0$, we get the following resolution of the Gorenstein algebra $A$,

$$\begin{aligned} &0 \to R(-2s) \to R(2-2s)^3 \oplus R(-1-s)^6 \\ &\to R(3-2s)^2 \oplus R(-s)^{12} \oplus R(-3)^2 \to R(1-s)^6 \oplus R(-2)^3 \to R \to A \to 0 \ . \end{aligned}$$

Indeed $X = \operatorname{Proj}(A)$ is an arithmetically Gorenstein curve of degree $d = 3s^2 - 10s + 9$ and arithmetic genus $g = 1 + d(s-3)$ in $\mathbb{P}^5$ (see [18]; Example 43).

With $M$ and $A$ as above, it turns out that [18]; Theorem 1 and Theorem 25 describe the deformations space, $\operatorname{GradAlg}(R)$, of the graded quotient $A$ and computes the dimension of $\operatorname{GradAlg}(R)$ in terms of a number $\delta := \delta(K_B)_{t-2s} - \delta(M)_{-s}$ where

$$\delta(N)_v := \ _v\!\operatorname{hom}_B(I_B/I_B^2, N) - \ _v\!\operatorname{ext}^1_B(I_B/I_B^2, N). \tag{3.3}$$

Here we have used small letters for the $k$-dimension of $_v\!\operatorname{Ext}^i_B(-,-)$ and of similar groups. If we suppose $M = N_B$, $\operatorname{depth}_{I(Z)} B \geq 4$ and $char(k) \neq 2$, then the conditions of the A (resp. B)-part of [18]; Theorem 25 are satisfied provided $_0\!\operatorname{Ext}^2_B(N_B, N_B) = 0$ (resp. $_{-s}\!\operatorname{Ext}^1_B(I_B/I_B^2, N_B) = 0$). In both cases $X$ is unobstructed and

$$\dim_{(X)} \operatorname{Hilb}^{p(x)}(\mathbb{P}^n) = \dim(N_B)_0 + \dim(I_B/I_B^2)_s - \ _0\!\operatorname{hom}_B(I_B/I_B^2, I_B/I_B^2) + \dim(K_B)_{t-2s} + \delta \tag{3.4}$$

where $t = n+1$ (see [18]; Corollary 41 and its proof, and Remark 42). Using the exact sequence (2.7) we get $_{-s}\!\operatorname{Ext}^1_B(I_B/I_B^2, N_B) = 0$ for $s > 2\max n_{2,j} - \min n_{1,i}$ which led to Corollary 41 of [18] which we slightly generalize in Corollary 3.2 (i) below. The A)-part was considered in Remark 42 of [18]. By the proof of Theorem 25 of [18] we may replace the vanishing of $_0\!\operatorname{Ext}^2_B(N_B, N_B)$ by the vanishing of the subgroup $_t\!\operatorname{Ext}^2_B(S^2(I_{A/B}(s)), K_B)$ and we still get all conclusions of the A)-part. Therefore, we can also prove (ii) of the following corollary to Theorem 25 of [18].

**Corollary 3.2.** *Let $B = R/I_B$ be a codimension two CM quotient of $R$, let $U = \operatorname{Proj}(B) - Z \hookrightarrow \mathbb{P}^n$ be a l.c.i. and suppose* $\operatorname{depth}_{I(Z)} B \geq 4$. *Let $A$ be given by a regular section of $\widetilde{N_B}^*(s)$ on $U$, let $\eta(v) := \dim(I_B/I_B^2)_v$, and put*

$$\epsilon := \eta(s) + \sum_{j=1}^{\mu-1} \eta(n_{2,j}) - \sum_{i=1}^{\mu} \eta(n_{1,i}).$$

*i) Let $j_0$ satisfy $n_{2,j_0} = \max n_{2,j}$. If $s > n_{2,j_0} + \max_{j \neq j_0} n_{2,j} - \min n_{1,i}$ and $char(k) \neq 2$, then $X$ is a $p_Y$-generic unobstructed arithmetically Gorenstein subscheme of $\mathbb{P}^n$ of codimension 4 and* $\dim_{(X)} \operatorname{Hilb}^{p(x)}(\mathbb{P}^n) = \epsilon$.

*ii) If ${}_s\operatorname{Ext}^1_B(N_B, A) = 0$, $char(k) = 0$, $s > \max n_{2,j}/2$ and $(X \subset Y)$ is general, then $X$ is unobstructed,* $\dim_{(X)} \operatorname{Hilb}^{p(x)}(\mathbb{P}^n) = \epsilon + \delta$ *and the codimension of the stratum in* $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ *of subschemes given by (3.1) is* ${}_0\operatorname{ext}^1_B(I_B/I_B^2, I_{A/B})$. *Moreover if* $s > \max n_{2,j} + \max n_{1,i} - \min n_{1,i}$ *(resp.* $s > \max n_{2,j}$*), then*

$${}_0\operatorname{ext}^1_B(I_B/I_B^2, I_{A/B}) = {}_{-s}\operatorname{ext}^1_B(I_B/I_B^2, N_B) = \delta \ ,$$

*(resp.* ${}_0\operatorname{ext}^1_B(I_B/I_B^2, I_{A/B}) = {}_{-s}\operatorname{ext}^1_B(I_B/I_B^2, N_B)$*).*

Here $I_{A/B} = \ker(B \to A)$ and "$X$ is $p_Y$-generic" if there is an open subset of $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ containing $(X)$ whose members $X'$ are subschemes of some closed $Y'$ with Hilbert polynomial $p_Y$. The stratum in $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ of subschemes given by (3.1) around $(X)$ is defined by functorially varying both $B$, $M$ and the regular section around $(B \to A)$ (see [18]; the definition before Theorem 25 for details). Indeed it is proved in [18]; Lemma 29 that pairs of closed subschemes $(X' \subset Y')$ of $\mathbb{P}^n$, $X' = \operatorname{Proj}(A')$ and $Y' = \operatorname{Proj}(B')$, obtained as in (3.1) contain *an open subset* $U \ni (X \subset Y)$ in the Hilbert flag scheme $D$, and taking such a $U$ small enough, we may define the mentioned stratum to be $p(U)$ where $p : D \to \operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ is the projection morphism induced by $(X' \subset Y') \to (X')$. Thus "$X$ is $p_Y$-generic" essentially means that the codimension of the stratum of subschemes given by (3.1) around $(X)$ is zero.

Note also that "$(X \subset Y)$ is general" means that it is the general member of an irreducible (non-embedded) component of the Hilbert flag scheme $D$. Since we in the corollary suppose $\operatorname{depth}_{I(Z)} B \geq 4$ and hence $\operatorname{depth}_{\mathfrak{m}} A \geq 2$, this is equivalent to saying that $(B \to A)$ is the general member of an irreducible (non-embedded) component of the "Hilbert flag scheme" parameterizing pairs of quotients of $R$ with fixed Hilbert functions. Indeed we can replace the schemes $\operatorname{GradAlg}(R)$ of [18] by $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ because we work with algebras of depth *at least* 2 at $\mathfrak{m}$ ([16]; Remark 3.7 or [8]).

*Proof.* By the text before (3.4), to prove (i) it suffices to show that ${}_{-s}\operatorname{Ext}^1_B(I_B/I_B^2, N_B) = 0$. To see it we observe that

$$\operatorname{Ext}^1_B(I_B/I_B^2, N_B) \simeq \operatorname{Ext}^1_B(T_B, K_B(n+1))$$

where $T_B := \operatorname{Hom}_B(I_B/I_B^2, I_B/I_B^2)$ by [18]; Remark 42. We consider the exact sequence (2.8) and we define $F := \ker(\oplus I_B/I_B^2(n_{2,j}) \to N_B)$. Since $N_B$ is a maximal CM $B$-module and $I_B/I_B^2$ has codepth 1 (i.e. $\operatorname{Ext}^i_B(I_B/I_B^2, K_B) = 0$ for $i \geq 2$) by [2] or (2.9), we get

$\operatorname{Ext}^2_B(F,K_B)=0$. It follows that

$$\operatorname{Ext}^1_B(\oplus I_B/I_B^2(n_{1,i}),K_B(n+1)) \to \operatorname{Ext}^1_B(T_B,K_B(n+1))$$

is surjective. Since

$$\operatorname{Ext}^1_B(I_B/I_B^2,K_B(n+1)) \simeq \operatorname{Ext}^3_R(I_B/I_B^2,R) \simeq \operatorname{Ext}^2_R(I_B^2,R)\ ,$$

it suffices to show ${}_{-s}\operatorname{Ext}^2_R(I_B^2(n_{1,i}),R)=0$ for any $i$. Looking to (2.9) it is enough to see ${}_{-s}\operatorname{Hom}(\wedge^2(\oplus R(-n_{2,j}))(n_{1,i}),R)=0$. Since, however, $n_{2,j}+n_{2,j'}-n_{1,i}-s<0$ for any $i,j,j'$, $j\neq j'$ by assumption, we easily get this vanishing for any $i$ and hence ${}_{-s}\operatorname{Ext}^1_B(I_B/I_B^2,N_B)=0$. Finally note that the dimension formula follows from (3.4) and (2.8) since we get $(K_B)_{t-2s}=0$ and $\delta=0$ from the proof of (ii).

(ii) By (2.5) we have $(K_B)_{t-2s}=0$ provided $2s>\max n_{2,j}$. By the discussion before Corollary 3.2 we must prove ${}_t\operatorname{Ext}^2_B(S^2(I_{A/B}(s)),K_B)=0$. Using the proof of [18]; Lemma 28 there is an exact sequence

$$0\to {}_t\operatorname{Ext}^2_B(S^2(I_{A/B}(s)),K_B)\to {}_t\operatorname{Ext}^2_B(S^2(N_B),K_B)\to {}_s\operatorname{Ext}^2_B(N_B,B)$$

induced by (3.2) where we have ${}_t\operatorname{Ext}^2_B(S^2(N_B),K_B)\simeq {}_0\operatorname{Ext}^2_B(N_B,N_B)\simeq {}_0\operatorname{Ext}^2_B(N_B,I_{A/B}(s))$ by (2.1), (3.2) and the fact that $N_B$ is a maximal CM $B$-module. Indeed ${}_t\operatorname{Ext}^2_B(S^2(N_B),K_B)\simeq \operatorname{Ext}^2_{\mathcal{O}_U}(\widetilde{S^2(N_B)}|_U,\widetilde{K_B}|_U(t))\simeq \operatorname{Ext}^2_{\mathcal{O}_U}(\widetilde{N_B}|_U,\widetilde{N_B}^*\otimes\widetilde{K_B}|_U(t))\simeq {}_0\operatorname{Ext}^2_B(N_B,N_B)$ by (2.1). Since $\operatorname{Ext}^1_B(N_B,B)=0$ by (2.1) and (2.9), it follows that

$${}_t\operatorname{Ext}^2_B(S^2(I_{A/B}(s)),K_B)\simeq {}_s\operatorname{Ext}^1_B(N_B,A)$$

which vanishes by assumption.

It remains to prove the final statement. If we apply $\operatorname{Hom}(-,K_B)$ to the exact sequence (2.2) and we use the exact sequence (2.5), we get ${}_{-2s}\operatorname{Ext}^i_R(I_B,K_B(t))=0$ and hence ${}_{-2s}\operatorname{Ext}^i_B(I_B/I_B^2,K_B(t))=0$ for $i=0,1$ provided $s>\max n_{2,j}$. Similarly we use $\operatorname{Hom}(-,N_B)$ and the exact sequence (2.7) to show that ${}_{-s}\operatorname{Hom}(I_B,N_B)=0$ provided $s>\max n_{2,j}+\max n_{1,i}-\min n_{1,i}$. We conclude by applying $\operatorname{Hom}_B(I_B/I_B^2,-)$ to (3.2). □

**Remark 3.3.** If $\operatorname{depth}_{I(Z)}B\geq 4$ and $char(k)\neq 2$, we showed in [18]; Remark 42 that

$${}_0\operatorname{Ext}^2_B(N_B,N_B)\ \simeq\ {}_0\operatorname{Hom}_B(I_B/I_B^2,H^3_{I(Z)}(I_B/I_B^2))\ \simeq\ {}_0\operatorname{Hom}_B(I_B/I_B^2,H^4_{I(Z)}(I_B^2))\ .$$

Similarly one shows $\operatorname{Ext}^2_B(N_B,B)\ \simeq\ H^4_{I(Z)}(I_B^2)$. Hence the group ${}_s\operatorname{Ext}^1_B(N_B,A)$ of Corollary 3.2 is isomorphic to the kernel of the natural map ${}_0\operatorname{Hom}_B(I_B/I_B^2,H^4_{I(Z)}(I_B^2))\to {}_sH^4_{I(Z)}(I_B^2)$ induced by the regular section $\sigma$. This sometimes allows us to verify ${}_s\operatorname{Ext}^1_B(N_B,A)=0$.

**Remark 3.4.** The first author takes the opportunity to point out a missing assumption in [18] as well as in [17]. In these papers there are several theorems involving the *codimension of a stratum* in which the assumption "$(B\to A)$ is general" or "$(B)$ general" is missing. The main result [17]; Theorem 5 (and hence [18]; Theorem 15) uses generic smoothness in its proof and refers to [19]; Proposition 9.14 where the "general" assumption occurs, as it should. In the proof of [17]; Theorem 5 we need "$(B\to A)$ general" to compute the dimension of the stratum. It is easily seen from the proof that what we really need is that "$(B\to A)$ is general" in the sense that, for a given $(B\to A)$, ${}_0\hom_R(I_B,I_{A/B})$ obtains its least possible value in the irreducible components of $\operatorname{GradAlg}(H_B,H_A)$ to which

$(B \to A)$ belongs. Thus in [17]; Theorem 5, Proposition 13, Theorem 16 (and hence [18]; Theorem 23), for the codimension statement we should assume "$(B)$ general" or at least that $\ _{-s}\hom_R(I_B, K_B)$ obtains its least possible value in the irreducible component of $\mathrm{GradAlg}(H_B)$ to which $(B)$ belongs. If we apply our results in a setting where these hom-numbers vanish (this is what we almost always do), we don't need to assume "$(B)$ or $(B \to A)$ general".

So Remark 3.4 gives the reason for including the assumption "$(X \subset Y)$ is general" in Corollary 3.2 (ii) even though this assumption does not occur in the codimension statements of the A)-part of Theorems 1 and 25 of [18].

## 4. Ideals generated by submaximal minors of square matrices

Let $X = \mathrm{Proj}(A) \subset \mathbb{P}^n$ be a codimension 4, determinantal scheme defined by the submaximal minors of a $t \times t$ homogeneous matrix. The goal of this section is to compute the dimension of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ for $n \geq 5$ at $(X)$ in terms of the corresponding degree matrix. The proof requires a proposition (valid for $n \geq 3$) on how $A$ is determined by a locally regular section of $I_B/I_B^2(s)$ where $B = R/I_B$ is a codimension 2 CM quotient. Let us first fix the notation we will use throughout this section.

Given a homogeneous matrix $\mathcal{A}$, i.e. a matrix representing a degree 0 morphism $\phi$ of free graded $R$-modules, we denote by $I(\mathcal{A})$ (or $I(\phi)$) the ideal of $R$ generated by the maximal minors of $\mathcal{A}$ and by $I_j(\mathcal{A})$ (or $I_j(\phi)$) the ideal generated by the $j \times j$ minors of $\mathcal{A}$.

**Definition 4.1.** A codimension $c$ subscheme $X \subset \mathbb{P}^n$ is called a *determinantal* scheme if there exist integers $r$, $p$ and $q$ such that $c = (p-r+1)(q-r+1)$ and $I(X) = I_r(\mathcal{A})$ for some $p \times q$ homogeneous matrix $\mathcal{A}$. $X \subset \mathbb{P}^n$ is called a *standard determinantal* scheme if $r = min(p, q)$. The corresponding rings $R/I_r(\mathcal{A})$ are called determinantal (resp. standard determinantal) rings.

Let $X \subset \mathbb{P}^n$ be a codimension 4, determinantal scheme defined by the vanishing of the submaximal minors of a $t \times t$ homogeneous matrix $\mathcal{A} = (f_{ji})_{i,j=1,\cdots,t}$ where $f_{ji} \in k[x_0, \cdots, x_n]$ are homogeneous polynomials of degree $a_j - b_i$ with $b_1 \leq b_2 \leq \cdots \leq b_t$ and $a_1 \leq a_2 \leq \cdots \leq a_t$. We assume without loss of generality that $\mathcal{A}$ is minimal; i.e., $f_{ji} = 0$ for all $i, j$ with $b_i = a_j$. If we let $u_{ji} = a_j - b_i$ for all $j = 1, \ldots, t$ and $i = 1, \cdots, t$, the matrix $\mathcal{U} = (u_{ji})_{i,j=1,\cdots,t}$ is called the *degree matrix* associated to $X$.

We denote by $W_{t,t}^{t-1}(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ the locus of determinantal schemes $X \subset \mathbb{P}^n$ of codimension 4 defined by the submaximal minors of a homogeneous square matrix $\mathcal{A} = (f_{ji})_{i,j=1,\cdots,t}$ as above. Notice that $W_{t,t}^{t-1}(\underline{b}; \underline{a}) \neq \emptyset$ if and only if $u_{i-1,i} = a_{i-1} - b_i > 0$ for $i = 2, ..., t$.

Let $\mathcal{N}$ be the matrix obtained by deleting the last row, let $I_B = I_{t-1}(\mathcal{N})$ be the ideal defined by the maximal minors of $\mathcal{N}$ and let $I_A = I_{t-1}(\mathcal{A})$ be the ideal generated by the submaximal minors of $\mathcal{A}$. Set $A = R/I_A = R/I(X)$ and $B = R/I_B$.

**Remark 4.2.** If the entries of $\mathcal{A}$ and $\mathcal{N}$ are sufficiently general polynomials of degree $a_i - b_j$, $1 \leq i, j \leq t$, and $a_{i-1} - b_i > 0$ for $2 \leq i \leq t$, then $B$ is a graded Cohen-Macaulay quotient of codimension 2 and $A$ is a graded Gorenstein quotient of codimension 4.

The goal of this section is to compute, in terms of $a_j$ and $b_i$, the dimension of the determinantal locus $W_{t,t}^{t-1}(\underline{b};\underline{a}) \subset \mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$, where $p(x) \in \mathbb{Q}[x]$ is the Hilbert polynomial of $X$. Note that the Hilbert polynomial of $X$ can be computed explicitly using the minimal free $R$-resolution of $R/I(X)$ given by Gulliksen and Negård in [13], see (4.5). We will also analyse whether the closure of $W_{t,t}^{t-1}(\underline{b};\underline{a})$ in $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ is a generically smooth, irreducible component of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$. To this end, we consider

$$F := \oplus_{i=1}^{t} R(b_i) \xrightarrow{\phi} G := \oplus_{j=1}^{t} R(a_j)$$

the morphism induced by the above matrix $\mathcal{A}$ and

$$F \xrightarrow{\phi_t} G_t := \oplus_{j=1}^{t-1} R(a_j)$$

the morphism induced by the matrix $\mathcal{N}$ obtained by deleting the last row of $\mathcal{A}$. The determinant of $\mathcal{A}$ is a homogeneous polynomial of degree

$$s := \deg(\det(\mathcal{A})) = \sum_{j=1}^{t} a_j - \sum_{i=1}^{t} b_i,$$

and the degrees of the maximal minors of $\mathcal{N}$ are $s + b_i - a_t$, i.e. $I_B$ has the following minimal free $R$-resolution

$$0 \longrightarrow G_t^*(a_t - s) \xrightarrow{{}^t\mathcal{N}} F^*(a_t - s) \xrightarrow{\beta} I_B \longrightarrow 0. \tag{4.1}$$

**Proposition 4.3.** *Suppose char$(k) = 0$.*

*(i) Let $A = R/I_{t-1}(\mathcal{A})$ be a determinantal ring of codimension 4 where $\mathcal{A}$ is a $t \times t$ homogeneous matrix and let $B = R/I_{t-1}(\mathcal{N})$ be the standard determinantal ring associated to $\mathcal{N}$ where $\mathcal{N}$ is the matrix obtained by deleting the last row of $\mathcal{A}$. Moreover, let $Z \subset \mathrm{Proj}(B)$ be a closed subset such that $\mathrm{Proj}(B) - Z \hookrightarrow \mathbb{P}^n$ is a l.c.i. and suppose $\mathrm{depth}_{I(Z)} B \geq 2$. Then, there is a regular section $\sigma$ of $(\widetilde{I_B/I_B^2}(s))_{|\,\mathrm{Proj}(B)-Z}$ where $s = \deg(\det(\mathcal{A}))$ whose zero locus precisely defines $A$ as a quotient of $B$ (i.e. $\sigma$ extends to a map $\sigma : B \longrightarrow I_B/I_B^2(s)$ such that $A = B/(\mathrm{im}\,\sigma^*)$).*

*(ii) Conversely, let $B = R/I_{t-1}(\mathcal{N})$ be a standard determinantal ring of codimension 2, let $Z \subset \mathrm{Proj}(B)$ be a closed subset such that $\mathrm{Proj}(B) - Z \hookrightarrow \mathbb{P}^n$ is a l.c.i. and $\mathrm{depth}_{I(Z)} B \geq 2$ and let $A'$ be defined by a regular section $\sigma$ of $(\widetilde{I_B/I_B^2}(s))_{|\,\mathrm{Proj}(B)-Z}$, i.e. given by*

$$0 \longrightarrow K_B(n+1-2s) \longrightarrow N_B(-s) \xrightarrow{\sigma^*} B \longrightarrow A' \longrightarrow 0 \tag{4.2}$$

*for some integer $s$. Then, there is a $t \times t$ homogeneous matrix $\mathcal{A}'$ obtained by adding a row to $\mathcal{N}$ such that $I_{A'} = I_{t-1}(\mathcal{A}')$.*

*Proof.* First of all, to define $\sigma$, we consider the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & G_t^*(a_t-s) & \stackrel{{}^t\mathcal{N}}{\longrightarrow} & F^*(a_t-s) & \stackrel{\beta}{\longrightarrow} & I_B & \longrightarrow & 0 \\
 & & \downarrow\alpha & & \| & & \downarrow & & \\
 & & G^*(a_t-s) & \stackrel{\phi^*(a_t-s)}{\longrightarrow} & F^*(a_t-s) & \longrightarrow & (\operatorname{coker}\phi^*)(a_t-s) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & R(-s) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

where $\alpha : G_t^*(a_t-s) \hookrightarrow G^*(a_t-s)$ is the natural inclusion defined by $\alpha\begin{pmatrix} f_1 \\ \vdots \\ f_{t-1}\end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_{t-1} \\ 0\end{pmatrix}$ and $\beta$ is given by multiplication with the maximal minors of the matrix $\mathcal{N}$. The snake Lemma yields the exact sequence

$$
R(-s) \stackrel{\cdot\det(\phi)}{\longrightarrow} I_B \longrightarrow (\operatorname{coker}\phi^*)(a_t-s) \longrightarrow 0 \tag{4.3}
$$

and hence

$$
(\operatorname{coker}\phi^*)(a_t) \simeq I_B(s)/\det(\phi). \tag{4.4}
$$

If we tensor $R(-s) \stackrel{\cdot\det(\phi)}{\longrightarrow} I_B$ with $B(s)$, we get a section $\sigma$ of $I_B/I_B^2(s)$. Before proving that the zero locus of $\sigma$ defines precisely $A$ as a quotient of $B$ via $\operatorname{im}(\sigma^*) = I_{A/B}$, we claim that any locally regular section $\sigma'$ of $I_B/I_B^2(s)$ defining $A'$ via $A' = B/\operatorname{im}(\sigma'^*)$ gives rise to a homogeneous matrix $\mathcal{A}'$ and a corresponding map $\phi'$ such that (4.3) and (4.4) hold with $\phi'$ instead of $\phi$. Indeed, given a section $\sigma'$ of $I_B/I_B^2(s)$, there exists a map $\sigma''$ fitting into a commutative diagram

$$
\begin{array}{ccc}
 & & F^*(a_t)\otimes B \\
 & \stackrel{\sigma''}{\nearrow} & \downarrow \\
B & \stackrel{\longrightarrow}{\sigma'} & I_B/I_B^2(s)
\end{array}
$$

and we denote by $\sigma_R \in \operatorname{Hom}_R(F, R(a_t))$ the map which corresponds to $\sigma''(1)$. Since $\operatorname{Hom}_R(F, R(a_t)) = \operatorname{Hom}(\oplus_{i=1}^t R(b_i), R(a_t))$, the morphism $\sigma_R$ determines a $1\times t$ row $\underline{g} = (g_1, \cdots, g_t)$ where $g_i$ is a homogeneous form of degree $a_t - b_i$, $1 \le i \le t$ and we define $\mathcal{A}' = \begin{pmatrix} \mathcal{N} \\ \underline{g}\end{pmatrix}$. Since the vertical map in the above diagram is induced by $\beta$ described above, we may assume that $\det(\phi') = \sigma'(1)$ modulo $I_B^2(s)$ and we get the claim.

It remains to show that $\operatorname{im}(\sigma^*) = I_{A/B}$ where $I_A = I_{t-1}(\mathcal{A})$ and that $\sigma$ is a locally regular section. Note that this will also show that $\operatorname{im}(\sigma'^*) = I_{A'/B}$ where $I_{A'} = I_{t-1}(\mathcal{A}')$, i.e. we get the converse. Moreover looking at the exact sequence (4.2) with $A$ instead of $A'$, and recalling that

$$
N_B \simeq K_B(n+1) \otimes I_B/I_B^2,
$$

we see that $\mathrm{im}(\sigma^*) = \mathrm{coker}(\sigma(-2s) \otimes id)$ where $id : K_B(n+1) \longrightarrow K_B(n+1)$ is the identity map and $\sigma$ is induced by $\det(\phi)$. Since we get

$$F(s-a_t) \longrightarrow G_t(s-a_t) \longrightarrow K_B(n+1) \longrightarrow 0$$

by dualizing the exact sequence (4.1), we see that the cokernel above is the same as the twisted cokernel of the composition

$$\gamma : G_t(-a_t) \longrightarrow K_B(n+1-s) \stackrel{\sigma(-s)\otimes id}{\longrightarrow} N_B.$$

Hence, we must prove that $\mathrm{coker}(\gamma) = I_{A/B}(s)$ where $I_A = I_{t-1}(\mathcal{A})$.

By [13]; Théorème 2 (see also [15]; Theorem 2), we have an exact sequence:

$$\mathrm{ker}[\mathrm{Hom}(F,F) \oplus \mathrm{Hom}(G,G) \stackrel{j}{\longrightarrow} R] \longrightarrow \mathrm{Hom}(F,G) \longrightarrow I_A(s) \longrightarrow 0 \tag{4.5}$$

where $j(\rho_0,\rho_1) = tr(\rho_0) - tr(\rho_1)$ and $tr$ is the trace map. The map $\mathrm{Hom}(F,G) \longrightarrow I_A(s)$ is given by $\gamma \longrightarrow tr(\gamma\psi)$ where $\psi$ is the matrix of cofactors, i.e. this map is given by the submaximal minors of $\mathcal{A}$ while the map $\mathrm{Hom}(F,F) \oplus \mathrm{Hom}(G,G) \stackrel{\eta}{\longrightarrow} \mathrm{Hom}(F,G)$ is given as a difference of the obvious compositions with $\phi$, i.e., $\eta(\rho_0,\rho_1) = \rho_1\phi - \phi\rho_0$. Since we have

$$\begin{array}{ccccc} \mathrm{Hom}(F,F) \oplus \mathrm{Hom}(G,G) & \stackrel{\eta}{\longrightarrow} & \mathrm{Hom}(F,G) & \longrightarrow & I_A(s) \\ (id,0) & & \mapsto & & t\cdot\det(\phi) \end{array}$$

and since there is a commutative diagram

$$\begin{array}{ccccccc} 0 & & & & & & \\ \downarrow & & & & & & \\ \mathrm{ker}(j) & \longrightarrow & \mathrm{Hom}(F,G) & \longrightarrow & I_A(s) & \longrightarrow & 0 \\ \downarrow & & \| & & \downarrow & & \\ \mathrm{Hom}(F,F)\oplus\mathrm{Hom}(G,G) & \stackrel{\eta}{\longrightarrow} & \mathrm{Hom}(F,G) & \longrightarrow & \mathrm{coker}\,\eta & \longrightarrow & 0 \\ \downarrow & & & & & & \\ R & & & & & & \end{array}$$

we get an exact sequence

$$R \stackrel{\cdot t\det(\phi)}{\longrightarrow} I_A(s) \longrightarrow \mathrm{coker}\,\eta \longrightarrow 0.$$

Hence, $\mathrm{coker}(\eta) \simeq I_A(s)/\det(\phi)$ ($char(k) = 0$) and the following sequence is exact:

$$\mathrm{Hom}(F,F) \oplus \mathrm{Hom}(G,G) \stackrel{\eta}{\longrightarrow} \mathrm{Hom}(F,G) \longrightarrow I_A(s)/\det(\phi) \longrightarrow 0.$$

Now we look at the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}(R(-a_t),G^*) & \longrightarrow & \mathrm{Hom}(R(-a_t),F^*) & \longrightarrow & I_B(s)/\det(\phi) & \longrightarrow & 0 \\
\simeq\downarrow & & \simeq\downarrow & & \downarrow & & \\
\mathrm{Hom}(G,R(a_t)) & \longrightarrow & \mathrm{Hom}(F,R(a_t)) & & & & \\
\downarrow{\scriptstyle(0,\cdot)} & & \downarrow & & & & \\
\mathrm{Hom}(F,F)\oplus\mathrm{Hom}(G,G) & \xrightarrow{\eta} & \mathrm{Hom}(F,G) & \longrightarrow & I_A(s)/\det(\phi) & \longrightarrow & 0 \\
\downarrow{\scriptstyle(id,\alpha_1^*)} & & \downarrow{\scriptstyle\alpha_2^*} & & \downarrow & & \\
\mathrm{Hom}(F,F)\oplus\mathrm{Hom}(G,G_t) & \xrightarrow{\eta_t} & \mathrm{Hom}(F,G_t) & \longrightarrow & \mathrm{coker}(\eta_t) & \longrightarrow & 0
\end{array}$$

where $\alpha_1^*$ and $\alpha_2^*$ are induced by $\alpha$ in a natural way and $\eta_t$ is a difference of the obvious compositions, i.e., $\eta_t(\rho_0,\rho_1') = \rho_1'\phi - \phi_t\rho_0$. We see, in particular, that the ideal $I_{A/B} = I_A/I_B$ is given by an exact sequence

$$\mathrm{Hom}(F,F)\oplus\mathrm{Hom}(G,G_t) \xrightarrow{\eta_t} \mathrm{Hom}(F,G_t) \longrightarrow I_{A/B}(s) \longrightarrow 0$$

where the rightmost map is given by the submaximal minors of the matrix $\mathcal{A}$ which do not belong to $I_B$.

On the other hand, by (2.7), there is an exact sequence

$$\mathrm{Hom}(G_t^*,G_t^*)\oplus\mathrm{Hom}(F^*,F^*) \longrightarrow \mathrm{Hom}(G_t^*,F^*) \longrightarrow N_B \longrightarrow 0$$

or, equivalently,

$$\mathrm{Hom}(F,F)\oplus\mathrm{Hom}(G_t,G_t) \xrightarrow{\eta'} \mathrm{Hom}(F,G_t) \longrightarrow N_B \longrightarrow 0$$

where $\eta'$ is given by $\eta'(\rho_0,\rho_2) = \rho_2\phi_t - \phi_t\rho_0$. Using again the exact sequence $0 \longrightarrow R(a_t) \longrightarrow G \xrightarrow{\alpha^*} G_t \longrightarrow 0$ we get a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}(F,F)\oplus\mathrm{Hom}(G_t,G_t) & \xrightarrow{\eta'} & \mathrm{Hom}(F,G_t) & \longrightarrow & N_B & \longrightarrow & 0 \\
\downarrow{\scriptstyle(id,\alpha_3)} & & \| & & \downarrow & & \\
\mathrm{Hom}(F,F)\oplus\mathrm{Hom}(G,G_t) & \xrightarrow{\eta_t} & \mathrm{Hom}(F,G_t) & \longrightarrow & I_{A/B}(s) & \longrightarrow & 0 \\
\downarrow & & & & & & \\
\mathrm{Hom}(R(a_t),G_t) & & & & & &
\end{array}$$

where $\alpha_3$ is induced by $\alpha$. Hence we get an exact sequence

$$\mathrm{Hom}(R(a_t),G_t) \xrightarrow{\gamma} N_B \longrightarrow I_{A/B}(s) \longrightarrow 0.$$

This proves that $\mathrm{coker}(\gamma) = I_{A/B}(s)$, i.e. $\mathrm{im}(\sigma^*) = I_{A/B}$ as required. Finally note that the above codimension and depth relations imply that $\sigma$ is a *regular* section on $U := \mathrm{Proj}(B) - Z$ because $(\mathrm{im}\,\tilde{\sigma}^*)_{|U}$ must locally on $U$ be generated by two regular elements (to get that $(B/\mathrm{im}\,\tilde{\sigma}^*)_{|U}$ is a codimension 2 Cohen-Macaulay quotient of $\widetilde{B}|U$). This completes the proof of Proposition 4.3. $\square$

The above proposition seems to be known in special cases. We have for instance noticed that Ellingsrud and Peskine claim that the Artinian Gorenstein ring associated to an invertible sheaf $\mathcal{O}_S(C)$ on a surface $S$ in $\mathbb{P}^3$, where $C$ is an arithmetically CM curve, is given by the submaximal minors of a square matrix which extends the Hilbert-Burch matrix associated to $C$ in $\mathbb{P}^3$ (see the text of [9] before Proposition 6). Since, we get (3.1) with $M = N_B$ by applying $H^0_*(-)$ to the exact sequence

$$0 \to \mathcal{N}_{C/S}(-s) \to \mathcal{N}_C(-s) \to \mathcal{N}_S|_C(-s) \simeq \mathcal{O}_C \to 0$$

of normal sheaves, it is clear that their Gorenstein ring (see their construction 2) is essentially the same as ours in the Artinian case. However, we have given a proof of the above proposition suited to our applications.

As a nice application of Proposition 4.3 we have

**Proposition 4.4.** *Let $X \subset \mathbb{P}^n$, $n \geq 4$, be a codimension 4 scheme defined by the submaximal minors of a $t \times t$ homogeneous square matrix $\mathcal{A}$. Then $X$ is in the Gorenstein liaison class of a complete intersection, i.e. $X$ is glicci.*

*Proof.* By [13]; Théorème 2 (see also Proposition 4.3), $X$ is arithmetically Gorenstein and hence glicci ([4]; Theorem 7.1). □

**Remark 4.5.** The above proposition has been recently generalized by Gorla. In [10]; Theorem 3.1, she has proved that any codimension $(t-r+1)^2$ ACM scheme $X \subset \mathbb{P}^n$ defined by the $r \times r$ minors of a $t \times t$ homogeneous square matrix $\mathcal{A}$ is glicci.

For an introduction to glicciness, see [19].

We are now ready to compute $\dim W^{t-1}_{t,t}(\underline{b};\underline{a})$ and $\dim_{(X)} \operatorname{Hilb}^{p(x)} \mathbb{P}^n$, $n \geq 5$, in terms of $a_1, \cdots, a_t$ and $b_1, \cdots, b_t$. Note that if $t = 2$ then a general $X$ is a complete intersection in which case these dimensions are well known.

**Theorem 4.6.** *($char(k) = 0$) Fix integers $a_1 \leq a_2 \leq \cdots \leq a_t$ and $b_1 \leq b_2 \leq \cdots \leq b_t$. Assume $t > 2$, $a_i \geq b_{i+3}$ for $1 \leq i \leq t-3$ (and $a_1 \geq b_t$ if $t = 3$), $a_t > a_{t-1} + a_{t-2} - b_1$ and $n \geq 5$. Then $W^{t-1}_{t,t}(\underline{b};\underline{a})$ is irreducible. Moreover, if $(X)$ is general in $W^{t-1}_{t,t}(\underline{b};\underline{a})$, then $X$ is unobstructed, and*

$$\dim W^{t-1}_{t,t}(\underline{b};\underline{a}) = \dim_{(X)} \operatorname{Hilb}^{p(x)}(\mathbb{P}^n) =$$

$$\sum_{1\leq i,j\leq t} \binom{a_j - b_i + n}{n} - \sum_{\substack{1\leq i\leq t-1\\ 1\leq j\leq t}} \binom{a_j - a_i + n}{n} - \sum_{1\leq i,j\leq t} \binom{b_i - b_j + n}{n} + \sum_{\substack{1\leq i\leq t\\ 1\leq j\leq t-1}} \binom{b_i - a_j + n}{n}$$

$$- \sum_{\substack{1\leq j\leq t\\ 1\leq i\leq k\leq t}} \binom{a_t - s - b_i - b_k + a_j + n}{n} + \sum_{\substack{1\leq i,j\leq t\\ 1\leq k\leq t-1}} \binom{a_t - s - b_i - a_k + a_j + n}{n} -$$

$$\sum_{\substack{1\leq i<k\leq t-1\\ 1\leq j\leq t}} \binom{a_t - s - a_i - a_k + a_j + n}{n} + \sum_{2\leq i\leq t} \binom{a_t - s + b_i - 2b_1 + n}{n}.$$

*Proof.* Let $X \subset \mathbb{P}^n$ be an arithmetically Gorenstein scheme of codimension 4 defined by the submaximal minors of a homogeneous square matrix $\mathcal{A} = (f_{ji})^{i=1,\dots,t}_{j=1,\dots,t}$ where $f_{ji} \in k[x_0, ..., x_n]$ is a sufficiently general homogeneous polynomial of degree $a_j - b_i$ and let $Y \subset \mathbb{P}^n$ be a codimension 2 subscheme defined by the maximal minors of the matrix

$\mathcal{N}$ obtained deleting the last row of $\mathcal{A}$ (see Remark 4.2). So, the homogeneous ideal $I_B = I(Y)$ of $Y$ has the following minimal free $R$-resolution

$$(4.6) \qquad 0 \longrightarrow F_2 = \oplus_{j=1}^{t-1} R(a_t - s - a_j) \xrightarrow{{}^t\mathcal{N}} F_1 = \oplus_{i=1}^{t} R(a_t - s - b_i) \longrightarrow I_B \longrightarrow 0.$$

By Proposition 4.3, $X$ is the zero locus of a suitable regular section of $\widetilde{I_B}/\widetilde{I_B^2}(s)$ where $s = \deg(\det(\mathcal{A}))$ and $W_{t,t}^{t-1}(\underline{b};\underline{a})$ is irreducible by [18]; Corollary 41. Since the hypothesis $a_t > a_{t-1} + a_{t-2} - b_1$ is equivalent to

$$s > s + a_{j_0} - a_t + \max_{\substack{1\le j\le t-1\\ j\neq j_0}} (s + a_j - a_t) - \min_{1\le i\le t}(s + b_i - a_t),$$

where $s + a_{j_0} - a_t = \max_{1\le j\le t-1}(s + a_j - a_t)$; and since $a_i \ge b_{i+3}$ for $1 \le i \le t-3$ (and $a_1 \ge b_t$ if $t = 3$) implies that $B := R/I_B$ given by (4.6) satisfies $\operatorname{depth}_{I(Z)} B \ge 4$ (cf. [20]; Remark 2.7), we can apply Corollary 3.2 and we get that $X$ is unobstructed and

$$\dim W_{t,t}^{t-1}(\underline{b};\underline{a}) = \dim_{(X)} \operatorname{Hilb}^{p(x)}(\mathbb{P}^n) = \eta(s) + \sum_{j=1}^{t-1} \eta(n_{2,j}) - \sum_{i=1}^{t} \eta(n_{1,i})$$

where $\eta(t) = \dim(I(Y)/I(Y)^2)_t = \dim I(Y)_t - \dim I(Y)^2_t$, $n_{2,j} = s + a_j - a_t$, $1 \le j \le t-1$, and $n_{1,i} = s + b_i - a_t$, $1 \le i \le t$. By (2.9), $I(Y)^2$ has a minimal free $R$-resolution of the following type:

$$(4.7) \qquad \begin{gathered} 0 \longrightarrow \wedge^2 F_2 = \oplus_{1\le i<j\le t-1} R(-a_i - a_j + 2a_t - 2s) \longrightarrow \\ F_1 \otimes F_2 = \oplus_{\substack{1\le i\le t\\ 1\le j\le t-1}} R(-b_i - a_j + 2a_t - 2s) \longrightarrow \\ S^2 F_1 = \oplus_{1\le i\le j\le t} R(-b_i - b_j + 2a_t - 2s) \longrightarrow I(Y)^2 \longrightarrow 0. \end{gathered}$$

Using (4.6) and (4.7), we obtain

$$\eta(s) = \sum_{1\le i\le t} \binom{a_t - b_i + n}{n} - \sum_{1\le i\le t-1} \binom{a_t - a_i + n}{n} - \sum_{1\le i\le j\le t} \binom{2a_t - s - b_i - b_j + n}{n} + \sum_{\substack{1\le i\le t\\ 1\le j\le t-1}} \binom{2a_t - s - b_i - a_j + n}{n} - \sum_{1\le i<j\le t-1} \binom{2a_t - s - a_i - a_j + n}{n}.$$

Using again (4.6) and (4.7), we get

$$\begin{aligned} \sum_{j=1}^{t-1} \eta(n_{2,j}) - \sum_{i=1}^{t} \eta(n_{1,i}) = {} & \sum_{\substack{1\le i\le t\\ 1\le j\le t-1}} \binom{a_j - b_i + n}{n} - \sum_{\substack{1\le i\le t-1\\ 1\le j\le t-1}} \binom{a_j - a_i + n}{n} - \\ & \sum_{\substack{1\le j\le t-1\\ 1\le i\le k\le t}} \binom{a_t - s - b_i - b_k + a_j + n}{n} + \sum_{\substack{1\le i\le t\\ 1\le j,k\le t-1}} \binom{a_t - s - b_i - a_k + a_j + n}{n} - \\ & \sum_{\substack{1\le i<k\le t-1\\ 1\le j\le t-1}} \binom{a_t - s - a_i - a_k + a_j + n}{n} - \sum_{\substack{1\le i\le t\\ 1\le j\le t}} \binom{b_i - b_j + n}{n} + \sum_{\substack{1\le i\le t\\ 1\le j\le t-1}} \binom{b_i - a_j + n}{n} + \end{aligned}$$

$$\sum_{\substack{1\le i\le t\\ 1\le j\le k\le t}} \binom{a_t-s+b_i-b_j-b_k+n}{n} - \sum_{\substack{1\le i,k\le t\\ 1\le j\le t-1}} \binom{a_t-s+b_i-b_k-a_j+n}{n} +$$
$$\sum_{\substack{1\le k<j\le t-1\\ 1\le i\le t}} \binom{a_t-s+b_i-a_k-a_j+n}{n}.$$

Since $a_{i-1} > b_i$ and $a_i \ge b_{i+3}$ for $1 \le i \le t-3$ (and $a_1 \ge b_t$ if $t=3$), by hypothesis, the last two sums of binomials vanish. Indeed, to see that $a_t - s + b_i - b_k - a_j < 0$ (resp. $a_t - s + b_i - a_k - a_j < 0$) for $1 \le i, k \le t$ and $1 \le j \le t-1$ (resp. $1 \le i \le t$ and $1 \le k < j \le t-1$), it suffices to show that $b_t - b_1 - a_1 < s - a_t = a_1 + a_2 + \cdots + a_{t-1} - b_1 - b_2 - \cdots - b_t$ (resp. $b_t - a_1 - a_1 < s - a_t = a_1 + a_2 + \cdots + a_{t-1} - b_1 - b_2 - \cdots - b_t$) which is rather straightforward to prove.

Moreover, the same type of argument applies to see that $a_t - s + b_i - b_j - b_k < 0$ for all $1 \le i \le t$ and $1 \le j < k \le t$ and we can replace the summand $\sum_{\substack{1\le i\le t\\ 1\le j\le k\le t}} \binom{a_t-s+b_i-b_j-b_k+n}{n}$ by $\sum_{2\le i\le t} \binom{a_t-s+b_i-2b_1+n}{n}$. Putting all together we get

$$\dim W^{t-1}_{t,t}(\underline{b};\underline{a}) = \dim_{(X)} \operatorname{Hilb}^{p(x)}(\mathbb{P}^n) =$$
$$\sum_{\substack{1\le i\le t\\ 1\le j\le t}} \binom{a_j-b_i+n}{n} - \sum_{\substack{1\le i\le t-1\\ 1\le j\le t}} \binom{a_j-a_i+n}{n} - \sum_{\substack{1\le i\le t\\ 1\le j\le t}} \binom{b_i-b_j+n}{n} +$$
$$\sum_{\substack{1\le i\le t\\ 1\le j\le t-1}} \binom{b_i-a_j+n}{n} - \sum_{\substack{1\le j\le t\\ 1\le i\le k\le t}} \binom{a_t-s-b_i-b_k+a_j+n}{n} +$$
$$\sum_{\substack{1\le i,j\le t\\ 1\le k\le t-1}} \binom{a_t-s-b_i-a_k+a_j+n}{n} - \sum_{\substack{1\le i<k\le t-1\\ 1\le j\le t}} \binom{a_t-s-a_i-a_k+a_j+n}{n} +$$
$$\sum_{2\le i\le t} \binom{a_t-s+b_i-2b_1+n}{n}.$$

□

## 5. Examples

We will end this work with some examples where we use Theorem 4.6. Moreover, these examples show that the hypothesis $a_t > a_{t-1} + a_{t-2} - b_1$ cannot be avoided! To handle such cases, we state a proposition which estimates the codimension of the stratum in $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ of subschemes given by the exact sequence (3.1).

**Example 5.1.** Let $R = k[x_0, \cdots, x_5]$ and let $X = \operatorname{Proj}(A) \subset \mathbb{P}^5 = \operatorname{Proj}(R)$ be a general arithmetically Gorenstein curve defined by the submaximal minors of a $4\times 4$ matrix whose first 3 rows are linear forms and whose last row are forms of degree $s-3$ ($s \ge 4$), i.e. $b_i = 0$ for $1 \le i \le 4$, $a_j = 1$ for $1 \le j \le 3$ and $a_4 = s-3$. Then, Theorem 4.6 applies provided $s > 5$ and we get that $X$ is unobstructed and

$$\dim W^3_{4,4}(\underline{0}; 1,1,1,s-3) = \dim_{(X)} \operatorname{Hilb}^{p(x)}(\mathbb{P}^5) = 12\binom{6}{5} + 4\binom{s+2}{5} - 9\binom{5}{5} - 3\binom{s+1}{5} -$$

$$16\binom{5}{5} - 10\binom{s-1}{5} + 12\binom{s-2}{5} - 3\binom{s-2}{5} = 2s^3 - 10s^2 + 13s + 48.$$

Moreover deleting the last row and taking maximal minors, we get a threefold $Y = \mathrm{Proj}(B)$ with resolution

$$(5.1) \qquad 0 \longrightarrow R(-4)^3 \longrightarrow R(-3)^4 \longrightarrow R \longrightarrow B \longrightarrow 0,$$

leading to

$$H_B(\nu) = \binom{\nu+3}{3} + 2\binom{\nu+2}{3} + 3\binom{\nu+1}{3} = p_Y(\nu) \text{ for } \nu \geq 0.$$

Since $A$ is given by (3.1) with $t = 6$ and $M = N_B$, we get $\mathcal{O}_X \simeq \omega_X(2s-6)$. Hence $h^1(\mathcal{O}_X(s-3)) = h^0(\mathcal{O}_X(s-3))$ and the Hilbert polynomial of $X$ must be of the form $p_X(\nu) = d\nu + 1 - g = d(\nu - s + 3)$. Looking to (5.1) we get

$$p_X(s-2) = h^0(\mathcal{O}_X(s-2)) - h^0(\mathcal{O}_X(s-4)) =$$
$$= h^0(\mathcal{O}_Y(s-2)) - h^0(\mathcal{O}_Y(s-4)) = 6s^2 - 28s + 36,$$

i.e. $d = deg(X) = 6s^2 - 28s + 36$ and $g = 1 + d(s-3)$.

Note that Theorem 4.6 takes care of all cases except for $s = 4$ and $s = 5$. For these two values of $s$, we can, however, use Corollary 3.2 (ii) to find $\dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^5)$ because

$${}_0\mathrm{Ext}^2_B(N_B, N_B) \simeq {}_0\mathrm{Hom}(I_B/I_B^2, H^4_{\mathfrak{m}}(I_B^2)) = 0$$

by (5.1) and Remark 3.3. Indeed ${}_3H^4_{\mathfrak{m}}(I_B^2) \hookrightarrow {}_3H^6_{\mathfrak{m}}(R(-8)^3) = 0$ by (2.9). Hence $X$ is unobstructed,

$$\dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^5) = 2s^3 - 10s^2 + 13s + 48 + \delta$$

where $\delta = \delta(K_B)_{6-2s} - \delta(N_B)_{-s}$, and moreover, if $s = 5$, then $\delta$ is the codimension of the closure of $W^3_{4,4} := W^3_{4,4}(\underline{0}; 1, 1, 1, s-3)$ in $\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)$. We claim that

$$(\delta(K_B)_{6-2s}, \delta(N_B)_{-s}) = \begin{cases} (-3, -15) & \text{for } s = 4 \\ (0, -12) & \text{for } s = 5, \end{cases}$$

i.e., $\delta = 12$ in both cases.

To find $\delta(K_B)_{6-2s}$ we apply $\mathrm{Hom}_B(-, K_B(6))$ to (2.3) and we get ${}_{-2s}\mathrm{Hom}_B(I_B/I_B^2, K_B(6)) = 0$ and ${}_{-2s}\mathrm{Ext}^1_B(I_B/I_B^2, K_B(6)) = {}_{-2s}\mathrm{Hom}(H_1, K_B(6))$. Since the rank of $H_1$ is 2, we have

$$(5.2) \qquad \mathrm{Hom}(H_1, K_B(6)) \simeq H_1(\sum_i n_{1,i}) = H_1(12)$$

by [2] or [22]; Theorem 8, see the isomorphism accompanying (3.1). Using (2.6) or more precisely the exactness of

$$(5.3) \qquad \wedge^2(R(-3)^4) \longrightarrow R(-4)^3 \longrightarrow H_1 \longrightarrow 0$$

(cf. [2]), we get

$$\delta(K_B)_{6-2s} = -\dim H_1(12)_{-2s} = \begin{cases} -3 & \text{for } s = 4 \\ 0 & \text{for } s = 5. \end{cases}$$

It remains to compute $\delta(N_B)_{-s}$. If we dualize the exact sequence (2.3) we get

$$0 \longrightarrow N_B \longrightarrow B(3)^4 \longrightarrow H_1^* \longrightarrow 0$$

to which we apply ${}_{-s}\mathrm{Hom}(I_B/I_B^2,-)$. Combining with

$$ {}_{-s}\mathrm{Hom}(I_B/I_B^2,H_1^*)\simeq {}_{-s}\mathrm{Hom}(I_B/I_B^2\otimes K_B(6),H_1^*\otimes K_B(6))\simeq {}_{-s}\mathrm{Hom}(N_B,H_1(12)) $$

where again we have used (5.2), we get

$$ \delta(N_B)_{-s}=4\dim(N_B)_{3-s}-\dim({}_{-s}\mathrm{Hom}(N_B(-12),H_1)). $$

Using (2.6), we see that

$$ 0\to {}_{-s}\mathrm{Hom}(N_B(-12),K_B(6)^*)\to {}_{-s}\mathrm{Hom}(N_B(-12),B(-4)^3)\to {}_{-s}\mathrm{Hom}(N_B(-12),H_1)\to 0 $$

is exact because we have $\mathrm{Ext}^1_B(I_B/I_B^2\otimes K_B,K_B^*)=0$ by [20]; Lemma 4.9. Using [20]; (4.17) we also get the surjectivity of the natural map $K_B^*\otimes B(-4)^4\longrightarrow \mathrm{Hom}_B(I_B/I_B^2\otimes K_B,K_B^*)$. Since we may use (5.3) to see that $(H_1)_\nu\simeq R(-4)^3_\nu\simeq B(-4)^3_\nu$ for $\nu\le 5$, we get $K_B(6)^*_\nu=0$ for $\nu\le 5$ by (2.6) and hence

$$ {}_{-s}\mathrm{Hom}(N_B(-12),K_B^*(-6))\simeq {}_{-s}\mathrm{Hom}(I_B/I_B^2\otimes K_B(6),K_B^*(6))=0 $$

for $s\ge 4$. It follows that

$$ {}_{-s}\mathrm{Hom}(N_B(-12),H_1)\simeq (I_B/I_B^2)^3_{8-s} $$

for $s\ge 4$ which implies (cf. (2.7) and (2.9)) that

$$ \delta(N_B)_{-s}=\begin{cases}4\dim(N_B)_{-2}-3\dim(I_B)_3=-12 & \text{for } s=5\\ 4\dim(N_B)_{-1}-3\dim(I_B)_4=-15 & \text{for } s=4.\end{cases} $$

Putting all together we get

$$ \dim_{(X)}\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)=\begin{cases}2s^3-10s^2+13s+48=\dim W^3_{4,4} & \text{for } s>5\\ 125 & \text{for } s=5\\ 80 & \text{for } s=4.\end{cases} $$

Moreover, applying Corollary 3.2 (ii), we get $\mathrm{codim}_{\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)}W^3_{4,4}(\underline{0};1,1,1,2)=12$ in the case $s=5$. Finally, for $s=4$, using Macaulay 2 program [12] we have computed the dimension ${}_0\mathrm{hom}(I_B,I_{A/B})=3$ for $(B\to A)$ general and hence $\mathrm{codim}_{\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)}W^3_{4,4}(\underline{0};\underline{1})={}_0\mathrm{hom}(I_B,I_{A/B})+\delta=15$.

If $a_t\le a_{t-1}+a_{t-2}-b_1$ we see in the example above that $W^{t-1}_{t,t}(\underline{b};\underline{a})$ is a proper closed irreducible subset, i.e. the generic curve of the component of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)$ to which $W^{t-1}_{t,t}(\underline{b};\underline{a})$ belongs is not defined by submaximal minors of a matrix of forms of degree $a_j-b_i$. The converse inequality always implies $\dim W^{t-1}_{t,t}(\underline{b};\underline{a})=\dim_{(X)}\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ by Theorem 4.6. The pattern above for small $a_t$ may be typical, but is in general rather difficult to prove. We illustrate this by two more examples.

**Example 5.2.** Let $X=\mathrm{Proj}(A)\subset\mathbb{P}^5$ be a general arithmetically Gorenstein curve defined by the submaximal minors of a $3\times 3$ matrix whose first 2 rows are linear forms and whose last row are forms of degree $s-2$ ($s\ge 3$), i.e. $b_i=0$ for $1\le i\le 3$, $a_j=1$ for $1\le j\le 2$ and $a_3=s-2$. Thanks to Proposition 4.3 the analysis of [18]; Example 43, immediately transfers to our case. Hence, for $s>4$ (i.e., $a_t>a_{t-1}+a_{t-2}-b_1$), we see that $X$ is unobstructed and

$$ \dim W^2_{3,3}(\underline{0};1,1,s-2)=\dim_{(X)}\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)=(s+1)(s-1)^2+23. $$

Since by deleting the last row and taking maximal minors we get a threefold $Y = \mathrm{Proj}(B)$ for which ${}_0\mathrm{Ext}^2_B(N_B, N_B) = 0$, we have the unobstructedness of $X$ also for $s = 3, 4$, and

$$(\delta(K_B)_{6-2s}, \delta(N_B)_{-s}) = \begin{cases} (-1, 2) & \text{for } s = 3 \\ (0, -3) & \text{for } s = 4 \end{cases}.$$

That is, $\delta = -3$ when $s = 3$, and $\delta = 3$ when $s = 4$. In both cases,

$$\dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^5) = (s+1)(s-1)^2 + 23 + \delta.$$

Thus

$$\dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^5) = \begin{cases} 36 & \text{for } s = 3 \\ 71 & \text{for } s = 4, \end{cases}$$

see [18]; Example 43 for the computations. Now, applying Corollary 3.2 (ii), we get $\mathrm{codim}_{\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)} W^2_{3,3}(\underline{0}; 1, 1, 2) = 3$ in the case $s = 4$. Finally, for $s = 3$, a Macaulay 2 computation shows ${}_0\hom(I_B, I_{A/B}) = 3$ and hence

$$\mathrm{codim}_{\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)} W^2_{3,3}(\underline{0}; \underline{1}) = {}_0\hom(I_B, I_{A/B}) + \delta = 0\ !$$

In the above examples we were able to analyse the case $a_t \le a_{t-1} + a_{t-2} - b_1$ through Corollary 3.2 (ii) because ${}_s\mathrm{Ext}^1_B(N_B, A) = 0$. Since this vanishing may be rare, we want to improve upon Corollary 3.2 (ii), at least to get estimates of the codimension of the stratum. We prefer to do it in the generality of Theorem 25 of [18] to extend Theorem 25 in this direction. This leads to the proposition below. Indeed with assumptions as in Proposition 5.3, one knows that the projection morphism $q : D \to \mathrm{Hilb}^{p_Y}(\mathbb{P}^n)$ induced by $(X' \subset Y') \to (Y')$ is smooth at $(X \subset Y)$ ([18]; Theorem 47). Using that the corresponding tangent map is surjective, we get Proposition 5.3 and Remark 5.4 (a). Since we only use these results in Example 5.6 and Remark 5.5, we skip the details of the proof which are rather straightforward once having the results and proofs of [18]. Put

$$c(I_{A/B}) := {}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B}) - {}_t\mathrm{ext}^2_B(S^2(I_{A/B}(s)), K_B).$$

**Proposition 5.3.** *Let $B = R/I_B$ be a graded licci quotient of $R$, let $M$ be a graded maximal Cohen-Macaulay $B$-module, and suppose $\widetilde{M}$ is locally free of rank 2 in $U := \mathrm{Proj}(B) - Z$, that $\dim B - \dim B/I(Z) \ge 2$ and $\wedge^2 \widetilde{M}|_U \simeq \widetilde{K_B}(t)|_U$. Let $A$ be defined by a regular section $\sigma$ of $\widetilde{M}^*(s)$ on $U$, i.e. given by (3.1), let $X = \mathrm{Proj}(A)$ and suppose ${}_s\mathrm{Ext}^1_B(M, B) = 0$ and $\dim B \ge 4$. Moreover let char$(k) = 0$, let $(B \to A)$ be general and suppose $(M, B)$ is unobstructed along any graded deformation of $B$ and ${}_{-s}\mathrm{Ext}^2_B(I_B/I_B^2, M) = 0$ . Then the codimension, codi, of the stratum in $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ of subschemes given by (3.1) around $(X)$ satisfies*

$$c(I_{A/B}) \le codi \le c(I_{A/B}) + {}_0h^2(R, A, A) \le {}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B}),$$

*and $codi = c(I_{A/B}) + {}_0h^2(R, A, A)$ if and only if $X$ is unobstructed.*

Here "$(M, B)$ unobstructed along any graded deformation of $B$" means that for every graded deformation $(M_S, B_S)$ of $(M, B)$, $S$ local and Artinian with residue field $k$, there is a graded deformation of $M_S$ to any graded deformation $B_T$ of $B_S$ for any small Artin surjection $T \to S$ ([18]; Definition 11). The important remark for our application $M = N_B$

where the codimension 2 CM quotient $B$ satisfies $\operatorname{depth}_{I(Z)} B \geq 4$, is that *all* assumptions of the proposition are satisfied provided $char(k) = 0$ and $(B \to A)$ is general (see proof of Corollary 41 and Remark 42 of [18]).

Moreover recall that if we put $\delta := -\delta(I_{A/B})_0 = {}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B}) - {}_0\hom_R(I_B, I_{A/B})$ and we use the exact sequence (3.2), we get $\delta = \delta(K_B)_{t-2s} - \delta(N_B)_{-s}$ , as previously.

**Remark 5.4.** a) With assumptions as in Proposition 5.3, except for $(B \to A)$ being general we can also show

$$\delta - {}_t\mathrm{ext}^2_B(S^2(I_{A/B}(s)), K_B) \leq codi \ .$$

b) Moreover if $\operatorname{depth}_{I(Z)} B \geq 4$, then we show

$${}_t\mathrm{Ext}^2_B(S^2(I_{A/B}(s)), K_B) \simeq {}_s\mathrm{Ext}^1_B(M, A)$$

exactly as we did for $M = N_B$ in the proof of Corollary 3.2. Hence if ${}_s\mathrm{Ext}^1_B(M, A) = 0$, then the lower bound $c(I_{A/B})$ of Proposition 5.3 is equal to the upper bound and we essentially get Corollary 3.2 (ii)! Moreover since $codi \geq 0$, Corollary 3.2 (i) corresponds to the case where the upper bound is zero!

**Remark 5.5.** In the case $s > \max n_{2,j}/2$, $\operatorname{depth}_{I(Z)} B \geq 4$ and $char(k) = 0$, the inequalities of Proposition 5.3 lead to

$$\epsilon + \delta - {}_0\mathrm{ext}^1_B(N_B, A) \leq \dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^n) \leq \epsilon + \delta$$

with $\epsilon$ as in Corollary 3.2.

**Example 5.6.** Now let $X = \mathrm{Proj}(A) \subset \mathbb{P}^5$ be a general arithmetically Gorenstein curve defined by the submaximal minors of a $3 \times 3$ matrix whose first (resp. second) row consists of linear (resp. quadratic) forms and whose last row are forms of degree $s - 3$ ($s \geq 5$), i.e. $b_i = 0$ for $1 \leq i \leq 3$, $a_1 = 1, a_2 = 2$ and $a_3 = s - 3$. In the following we skip a few details which we leave to the reader. Note that the case $a_t > a_{t-1} + a_{t-2} - b_1$ or equivalently $s > 6$, is taken care of by Theorem 4.6. So we concentrate on the cases $s = 5$ and 6, which we analyse by using Proposition 5.3 and Remark 5.4. First, we use Remark 3.3 to compute ${}_0\mathrm{ext}^2_B(N_B, N_B)$ where $B$ is obtained by deleting the last row and taking maximal minors. We easily get ${}_0\mathrm{ext}^2_B(N_B, N_B) = {}_s\mathrm{ext}^1_B(N_B, A) = 3$ by using

$$0 \longrightarrow R(-5) \oplus R(-4) \longrightarrow R(-3)^3 \longrightarrow R \longrightarrow B \longrightarrow 0,$$

(2.9) and ${}_0\mathrm{Ext}^2_B(N_B, N_B) \simeq {}_0\mathrm{Hom}_B(I_B/I_B^2, H^4_{\mathfrak{m}}(I_B^2))$. Moreover $\dim(K_B)_{6-2s} = 0$ by (2.5). Now if we apply ${}_{-2s}\mathrm{Hom}(-, K_B(6))$ to (2.3) we get $\delta(K_B)_{6-2s} = 0$ and ${}_{-2s}\mathrm{Ext}^1_B(I_B/I_B^2, K_B(6)) = 0$ for $s \geq 5$ provided we can show ${}_{-2s}\mathrm{Hom}(H_1, K_B(6)) = 0$. Using (2.3) we get that $H_1$ has rank 1 and $H_1 \simeq K_B(-3)$. Hence ${}_{-2s}\mathrm{Hom}(H_1, K_B(6)) \simeq B(9)_{-2s} = 0$ for $s \geq 5$.

It remains to compute $\delta(N_B)_{-s}$. We claim that $\delta(N_B)_{-s} = -8$ (resp. $\delta(N_B)_{-s} = -3$) for $s = 5$ (resp. $s = 6$). Indeed dualizing the exact sequence (2.3) we get

$$0 \longrightarrow N_B \longrightarrow B(3)^3 \longrightarrow H_1^* \longrightarrow 0 \ .$$

If we apply ${}_{-s}\mathrm{Hom}(I_B/I_B^2, -)$ to this sequence, recalling $H_1 \simeq K_B(-3)$ and hence ${}_{-s}\mathrm{Hom}(I_B/I_B^2, H_1^*) \simeq (I_B/I_B^2)_{9-s}$, we get an exact sequence which rather easily proves the claim. It follows that the numbers $\delta = \delta(K_B)_{6-2s} - \delta(N_B)_{-s}$ and ${}_s\mathrm{ext}^1_B(N_B, A)$ appearing in Remark 5.4 are computed. We conclude, for $s = 5$, that the codimension, *codi*, of the stratum in $\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)$ of subschemes given by (3.1) around $(X)$ is at least 5-dimensional. In fact a Macaulay 2 computation shows ${}_0h^2(R,A,A) = 0$ and ${}_0\mathrm{hom}(I_B, I_{A/B}) = 1$ and hence we have $codi = c(I_{A/B}) + \ {}_0h^2(R,A,A) = 6$ by Proposition 5.3. For $s = 6$ the lower bound for *codi* of Remark 5.4 (a) is 0. Since a Macaulay 2 computation shows ${}_0\mathrm{hom}(I_B, I_{A/B}) = 0$ the better lower bound of Proposition 5.3 is also 0 while the smallest upper bound of Proposition 5.3 is 3. The latter is the correct bound for the codimension of the stratum provided $X$ is unobstructed. In conclusion if $X$ belongs to a reduced component $V$ of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)$, then $codi = 3$, but $codi = 0$ is possible in which case $V$ is non-reduced. We have not been able to fully tell what happens, but we expect $V$ to be reduced and $codi = 3$.

The last case of the preceding example illustrates how difficult the analysis of when *codi* is positive could be. Especially cases where $a_t$ is close to $a_{t-1} + a_{t-2} - b_1$ seem difficult to handle. Since it turns out that the lower bounds of Proposition 5.3 and Remark 5.4 (a) are often negative (also in the case $a_t > a_{t-1} + a_{t-2} - b_1$ treated in Theorem 4.6), they are not very helpful. This, however, also indicates that the conclusions of Theorem 4.6 are rather strong.

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