Maximal families of Gorenstein algebras

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Abstract

The purpose of this paper is to study maximal irreducible families of Gorenstein quotients of a polynomial ring R. Let $\mathrm{GradAlg}^H(R)$ be the scheme parametrizing graded quotients of R with Hilbert function H. We prove there is a close relationship between the irreducible components of $\mathrm{GradAlg}^H(R)$ whose general member is a Gorenstein codimension $(c+1)$ quotient, and the irreducible components of $\mathrm{GradAlg}^{H'}(R)$ whose general member B is a codimension c Cohen-Macaulay algebra of Hilbert function H' related to H. If the Castelnuovo-Mumford regularity of the Gorenstein quotient is large compared to the Castelnuovo-Mumford regularity of B, this relationship actually determines a well-defined injective mapping from such "Cohen-Macaulay" components of $\mathrm{GradAlg}^{H'}(R)$ to "Gorenstein" components of $\mathrm{GradAlg}^H(R)$, in which generically smooth components correspond. Moreover the dimension of the "Gorenstein" components is computed in terms of the dimension of the corresponding "Cohen-Macaulay" component and a sum of two invariants of B. Using linkage by a complete intersection we show how to compute these invariants. Linkage also turns out to be quite effective in verifying the assumptions which appear in a generalization of the main Theorem.
AMS Subject Classification. 14C05, 13D10, 13D03, 13D07, 13C40.
Keywords. Parametrization, Gorenstein algebra, Artinian algebra, liaison, licci, Cohen-Macaulay, canonical module, normal module, Hilbert scheme.

1   Introduction

The main goal of this paper is to contribute to the classification of graded Gorenstein quotients of a polynomial ring R of dimension $n + c$ over an algebraically closed field k. Let $\mathrm{GradAlg}^H(R)$ be the scheme parametrizing graded quotients A of R of depth $A \geq \min(1, \dim A)$ and with Hilbert function H. We let $\mathrm{CM}_c^H(R)$ (resp. $\mathrm{Gor}_c^H(R)$) be the open subscheme parametrizing Cohen-Macaulay (resp. Gorenstein) quotients of codimension c in R. In the Artinian case $\mathrm{Gor}_c^H(R)$ and the scheme $\mathrm{PGor}(H)$ defined by certain catalecticant minors ([21]) are essentially the same (see Theorem 11). We call an irreducible component of $\mathrm{GradAlg}^H(R)$ which has non-empty intersection with $\mathrm{CM}_c^H(R)$ (resp. $\mathrm{Gor}_c^H(R)$) a "Cohen-Macaulay (resp. "Gorenstein) codimension c" component. The main Theorem of this paper establishes a strong connection between "Gorenstein codimension $(c+1)$" components of $\mathrm{GradAlg}^H(R)$ and "Cohen-Macaulay codimension c" components of $\mathrm{GradAlg}^{H'}(R)$ for some H'. Indeed the connection between these components allows us to deduce the generic smoothness and the dimension of the "Gorenstein codimension $(c+1)$" component from the corresponding property and number for the "Cohen-Macaulay codimension c" component, and vica versa. If the socle degree of an Artinian reduction of the Gorenstein quotient is large enough (about twice the Castelnuovo-Mumford regularity, $\mathrm{reg}(B)$, of B), we then prove

that this connection actually determines a well-defined injective mapping between the set of "Cohen-Macaulay codimension c components of $\mathrm{GradAlg}^{H'}(R)$ and the set of "Gorenstein codimension $(c+1)$" components of $\mathrm{GradAlg}^{H}(R)$ whose general members satisfy the Weak Lefschetz property (Theorem 24).

We all know the nice description of $\mathrm{CM}_2^H(R)$, given by the Hilbert-Burch theorem, and of $\mathrm{Gor}_3^H(R)$, proved by Buchsbaum and Eisenbud, in terms of the maximal minors (resp. the Pfaffians) of a suitable matrix. In this case the schemes $\mathrm{CM}_2^H(R)$ and $\mathrm{Gor}_3^H(R)$ are smooth and irreducible. There is, in our opinion, little hope of finding such a single nice description of $\mathrm{Gor}_4^H(R)$ in codimension 4 because this scheme has several irreducible components ([3] and [22]). We think a more promising strategy should consist of classifying the irreducible components of $\mathrm{Gor}_4^H(R)$ together with describing the members of each family concretely. In this context the main theorems show that the corresponding classification of $\mathrm{CM}_3^{H'}(R)$ is central for the classification of $\mathrm{Gor}_4^H(R)$, and they provide a method for answering Iarrobino and Srinivasan's question ([22]) about whether the non-irreducibility of $\mathrm{Gor}_4^H(R)$ is a rare phenomenon, or not. In fact we give several classes of examples of reducible schemes $\mathrm{Gor}_4^H(R)$ whose general members are one dimensional Gorenstein quotients of R (Example 26 and Remark 27).

More precisely let the graded algebra A *be a codimension one quotient of B, well defined by a twist of the canonical module K_B*, by which we mean that there is a graded exact sequence

$$0 \to K_B(-s) \to B \to A \to 0, \tag{1}$$

where B is assumed generically Gorenstein (see the beginning of Section 2). Let K_B^* be the B-dual of the canonical module of $B = R/I_B$, let $_v\mathrm{Ext}_B^i(-,-)$, $i \geq 0$, be the graded piece of $\mathrm{Ext}_B^i(-,-)$ of degree v and let $_v\mathrm{ext}_B^i(-,-)$ be its dimension as a k-vectorspace. Then we show (cf. Theorem 16)

Theorem 1. *Let $B = R/I_B$ be a graded Cohen-Macaulay quotient of a polynomial ring R, and let A be a graded codimension one quotient of B, well defined by a twist of the canonical module K_B.*

A) If B is licci, then A is unobstructed as a graded R-algebra, A is Gorenstein and,

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + \dim(K_B^*)_s - 1 - \delta(B)_{-s}$$

where $\delta(B)_{-s} = {}_{-s}\hom_B(I_B/I_B^2, K_B) - {}_{-s}\mathrm{ext}_B^1(I_B/I_B^2, K_B)$. Moreover if $char(k) = 0$, then the codimension of the H_B-stratum of A at $(B \to A)$ is ${}_{-s}\mathrm{ext}_B^1(I_B/I_B^2, K_B)$.

B) If $\mathrm{Proj}(B)$ is locally licci and $s >> 0$, then A is H_B-generic and Gorenstein, and

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + \dim(K_B^*)_s - 1.$$

Moreover A is unobstructed as a graded R-algebra if and only if B is unobstructed as a graded R-algebra.

If we assume $K_B(-s)_v = 0$ for v less or equal the largest degree of a minimal generator of I_B, then the codimension of the H_B-stratum of A at $(B \to A)$, mentioned in Theorem 1A, is just the codimension at (A) of the stratum of $\mathrm{GradAlg}^{H_A}(R)$ of quotients $R \to A'$ for which there exists some factorization $B' \to A'$ such that B' has Hilbert function H_B. Near (A) this

stratum turns out to be the stratum of quotients A' given by (1), with A', B' replacing A, B in (1) (Lemma 7 and Proposition 13A). A is H_B-*generic* if there is an *open* subset $U \ni (A)$ of $\mathrm{GradAlg}^{H_A}(R)$ contained in this stratum.

Note that if $B = R/I$ and R/J are geometrically linked by a complete intersection, then $A = R/(I + J)$ is given by (1). In this case the unobstructedness of A is known if B is licci ([39]). It is standard to use sums of geometrically linked ideals to construct Gorenstein algebras (e.g. [8], [9], [10], [15]). By the corresponding more general construction of using geometric *Gorenstein* linkage, we also get Gorenstein algebras (e.g. [34], see also [35], Ex. 10.2). Since both constructions are given by (1) (e.g. [34], [30], p. 785), Theorem 1 applies in both cases.

In Theorem 1A, A is in fact shown to be unobstructed in the sense $\mathrm{H}^2(R, A, A) = 0$, a formula which holds also when A is non-graded (Theorem 30). Moreover we prove a more general result that implies Theorem 1 (Proposition 13). To use Proposition 13 we need to verify the vanishing of certain Ext-groups. These vanishings always hold in the licci case. We also prove that the scheme $\mathrm{ZGor}(H)$ consisting of *not necessarily graded* Artinian Gorenstein quotients of R is smooth at a graded quotient (A) given by (1) provided B is licci, and we give a formula for its dimension (Proposition 29). In the final Section 3 we show how we can use linkage to compute the mentioned Ext-groups as well as to find the other dimension invariants of Theorem 1 (Proposition 33 and Corollary 37, see Example 2 below).

In Theorem 1B, we may make "$s >> 0$" quite explicit. For instance if $\mathrm{Proj}(B)$ is a zero dimensional scheme of length d, it may be replaced by "$s \geq 2 \operatorname{reg}(I_B)$" (Remark 22; cf. (7)). In this case $\dim(K_B^*)_s = d$ and the h-vector of A contains a consecutive subsequence of the form (d, d, d). It is precisely in such ranges of s that Theorem 24 applies, *telling that there is a well-defined injective application π from the set of irreducible components of* $\mathrm{GradAlg}^{H_B}(R)$ *whose general members satisfy the assumptions of Theorem 1B, to the set of irreducible "Gorenstein components" of* $\mathrm{GradAlg}^{H_A}(R)$ *whose general members satisfy the Weak Lefschetz property.* This implies in particular that the h-vector of A is an SI-sequence ([15]). In this range of s one knows that any codimension one Gorenstein quotient is given by (1) ([4]). Theorem 24 is a consequence of a thoroughly studied correspondence (Proposition 23). If the codimension of B is $c = 2$, then Theorem 1B, the application π and the mentioned correspondence are well understood by Iarrobino-Kanev's results on "annihilating schemes", in which they construct the algebra B from a given A ([21], Ch. 5). Their approach applies also to $c \geq 3$ and leads e.g. to reducible $\mathrm{Gor}_{c+1}^H(R)$ for $c \geq 4$ ([21], Ch. 6) and to a dimension formula of $\mathrm{Gor}_{c+1}^H(R)$ provided the "first half" of H coincides with the Hilbert function of a complete intersection ([21], Thm. 4.17). The application π is surjective for $c = 2$, while Boij points out that such a surjectivity is not always true in higher codimensions ([3]). Also, results on the codimension of the Betti-strata can be related to the codimension statement in Theorem 1A (see [21], Sect. 5.3.5 and Remark 25(b)). Note that it is known that not every codimension 3 quotient is obtained by (1) ([36], [10]), i.e. that maximal families of quotients given by (1) may correspond to proper strata. This fits nicely with Theorem 1A in which the number $_{-s}\mathrm{ext}_B^1(I_B/I_B^2, K_B))$ essentially measures the codimension of such strata.

In order to prove our results we study in full generality deformations of a quotient A of R which themselves are quotients of some deformation of B. First we recall and further develop some partially known results on the unobstructedness and the "family-dimension" of such a quotient A of R with fixed Hilbert function H_A (Theorem 5). Here we neither assume B

to be Cohen-Macaulay, nor A to be Gorenstein. In this generality the deformation theory developed by Laudal is central ([31]). The Theorem 5 is important for the main application of this paper because it treats simultaneously the two cases of interest: when the *general* member of an irreducible family $R \to A$ factors via some deformation of $B \to A$ (i.e. A is H_B-generic), and when it does not.

To illustrate some of our results we consider the following example. Here the Hilbert function, $H_B(v) := \dim_k B_v$, of B is written as a vector $(H_B(0), H_B(1), H_B(2), ...)$. If A is Artinian, we call $(H_A(0), H_A(1), ..., H_A(s))$ the *h*-vector of A, where s, the socle degree of A, is by definition the smallest number such that $H_A(v) = 0$ for $v > s$.

Example 2. *Let $B = R/I_B$ be a one dimensional almost complete intersection obtained by linking a complete intersection (c.i.) B' of three hypersurfaces of degree 2,2,2 to B via a c.i. D of three hypersurfaces of degree 3,3,3. A standard mapping cone construction yields*

$$0 \to R(-7)^3 \to R(-6)^3 \oplus R(-5)^3 \to R(-3)^4 \to I_B \to 0.$$

Hence the Castelnuovo-Mumford regularity $\mathrm{reg}(I_B) = 5$ and it is easy to see that the Hilbert functions of B and B' are $(1, 4, 10, 16, 19, 19, 19, ...)$ and $(1, 4, 7, 8, 8, 8, ...)$ respectively. Now B is obviously licci and Theorem 1 applies to any quotient A given by (1). Using (1) and duality one shows $H_A(v) = H_B(v) + H_B(s - v) - 19$. It follows that s is the socle degree of A and that the h-vector of A is the $(s + 1)$-tuple $(1, 4, 7, 7, 4, 1)$, $(1, 4, 10, 13, 10, 4, 1)$, $(1, 4, 10, 16, 16, 10, 4, 1)$, $(1, 4, 10, 16, 19, 16, 10, 4, 1)$ and $(1, 4, 10, 16, 19..., 19, 16, 10, 4, 1)$ for $s = 5, 6, 7, 8,$ and ≥ 9 respectively. It is known ([27], Proposition 1.7 or Proposition 33 of this paper) how to compute $\dim(N_B)_0$, the dimension of the tangent space of $\mathrm{GradAlg}^{H_B}(R)$ at (B), from $\dim(N_{B'})_0$. Indeed

$$\dim(N_B)_0 = \dim(N_{B'})_0 + 3H_B(3) - 3H_{B'}(3).$$

Since $N_{B'} \simeq B'(2)^{\oplus 3}$ we get $\dim(N_B)_0 = 45$. Moreover, due to Remark 22, we have $\delta(B)_{-s} = 0$ for $s \geq 2\,\mathrm{reg}(I_B) = 10$ and $\dim(K_B^*)_s = 19$ for $s \geq 2\,\mathrm{reg}(I_B) - 3 = 7$. Hence, for $s \geq 10$ we get that A is unobstructed and H_B-generic and that $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(A)} \mathrm{PGor}(H_A) = 63$. Note that the conclusions so far only use Theorem 1B because we may replace the assumption $s >> 0$ of Theorem 1B by $s \geq 2\,\mathrm{reg}(I_B)$.

However, to deal with $s < 10$ we need Theorem 1A and that B is licci (or that the explicitly mentioned Ext-groups of Proposition 13A vanish). We also have to compute the numbers $\dim(K_B^*)_s$ and $\delta(B)_{-s}$. To do so, we have a nice connection between these invariants using liaison. Indeed by Proposition 33

$$\dim(K_B^*)_v = \dim B_{v-5} + 3\dim(I_{B/D})_{v-2} - \delta(B')_{v-10}$$

and symmetrically, $\delta(B)_{v-10} = \dim B'_{v-5} + 3\dim(I_{B'/D})_{v-2} - \dim(K_{B'}^*)_v$ for any v. Since B' is a c.i., we easily compute $\delta(B')_v$ and $(K_{B'}^*)_v$. Indeed $K_{B'} \simeq B'(2)$ and $\delta(B')_v = {}_v\hom_{B'}(I_{B'}/I_{B'}^2, K_{B'}) = \dim(K_{B'})_{v+2}^{\oplus 3}$. We get that $\dim(K_B^*)_v = H_B(v - 5) + 3H_{B'}(v - 5) - 3H_{B'}(v - 6)$, i.e. $\dim(K_B^*)_s = 4, 13, 19, 19, ...$ for $s = 5, 6, 7, 8...$ respectively. Correspondingly, $\delta(B)_{-s} = 20, 2, -4, -1, 0, 0...$ for $s = 5, 6, 7, 8, 9, 10, ...$ respectively. Hence $\dim_{(A)} \mathrm{PGor}(H_A) =$

$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = 28, 55, 67, 64, 63, 63, ...$ for $s = 5, 6, 7, 8, 9, 10, ...$ respectively.

If $s \geq 7$ we see from the resolution of I_B that $_{-s}\hom_B(I_B/I_B^2, K_B) = 0$, and hence that $_{-s}\ext_B^1(I_B/I_B^2, K_B) = -\delta(B)_{-s} = 4, 1, 0$ for $s = 7, 8, 9$ respectively. By Theorem 1A the numbers 4,1,0 represent the codimensions of the strata of quotients $R \to A'$, $H_{A'} = H_A$, given by (1). It follows that these strata form a 63 dimensional irreducible family at (A) for all $s \geq 7$, while the dimension of the corresponding irreducible component is $63 +_{-s} ext_B^1(I_B/I_B^2, K_B)$. That the conclusion of Theorem 1A holds for $s = 7, 8$, shows the existence of a quotient A with Hilbert function H_A which is not given by (1) for some B with Hilbert function H_B! For a complete such analysis on any codimension 3 almost c.i. B, see Example 40.

In this work the author was inspired by joint work with Chris Peterson where we constructed Gorenstein quotients of R in various ways. We showed for instance that if M is a maximal Cohen-Macaulay B-module of rank r and $\wedge^r \widetilde{M} \simeq \widetilde{K}_B(s)$ as sheaves (in a large enough open set U of $\mathrm{Proj}(B)$), then any regular section of $\widetilde{M}^*(s)$ defines a Gorenstein quotient $R \twoheadrightarrow A$ of codimension $r + c$ provided $\mathrm{H}_*^0(U, \wedge^i \widetilde{M})$ for $2 \leq i \leq r/2$ are maximally Cohen-Macaulay (cf. [30] for details). The idea of this paper is to look at the construction and to use deformation theory to vary every object and morphism in the construction as much as possible, in order to see how large the corresponding stratum in $\mathrm{GradAlg}^H(R)$ will be. This paper looks at the $r = 1$ case because so far it is only when $r = 1$ that the author is also able to show his results for Artinian Gorenstein rings. In a forthcoming paper the author generalizes the results of this paper considerably by considering deformations of regular sections of a maximal Cohen-Macaulay module M of any rank, but, unfortunately, there we must partially restrict to positively dimensional Gorenstein algebras. We thank Olav A. Laudal, Roy Skjelnes and Johannes Kleppe for clarifying discussions. The author heartily thanks the referee for many valuable comments and questions.

For the convenience of the reader we include a section of preliminaries where we give some space to the theory of deformations. Indeed it is not always easy to find a nice reference for the results we use because treatments in the literature are often either too general and concern general cohomology groups of algebras, or too special, assuming certain groups are zero or very manageable.

1.1 Preliminaries

Let B be an n-dimensional graded quotient of a polynomial k-algebra R (k is algebraically closed) in $n + c$ variables (of degree 1), and let M and N be finitely generated graded B-modules. Let $\mathrm{depth}_J M$ denote the length of a maximal M-sequence in a homogeneous ideal J and let $\mathrm{depth}\, M = \mathrm{depth}_{\mathfrak{m}} M$ where $\mathfrak{m}$ is the irrelevant maximal ideal. Let $\mathrm{H}_J^i(-)$ be the right derived functor of the functor, $\Gamma_J(-)$, of sections with support in $\mathrm{Spec}(B/J)$. Recall that $\mathrm{depth}_J M = \inf\{\mathrm{depth}\, M_\wp \mid \wp \supseteq J, \; \wp \text{ graded}\}$ and that $\mathrm{depth}_J M \geq r$ iff $\mathrm{H}_J^i(M) = 0$ for $i < r$, cf. [12] or [16].

If $Y = \mathrm{Proj}(B)$ and Z is closed in Y and $U = Y - Z$, we let $\mathrm{H}_*^0(U, \widetilde{M}) = \oplus_v \mathrm{H}^0(U, \widetilde{M}(v))$. Then we have an exact sequence

$$0 \to \mathrm{H}_{I(Z)}^0(M) \to M \to \mathrm{H}_*^0(U, \widetilde{M}) \to \mathrm{H}_{I(Z)}^1(M) \to 0 \qquad (2)$$

and isomorphisms $\mathrm{H}_{I(Z)}^i(M) \simeq \mathrm{H}_*^{i-1}(U, \widetilde{M})$ for $i \geq 2$. In the case $\mathrm{depth}_{I(Z)} N \geq 2$ the graded $\mathrm{Ext}_B^1(M, N)$ injects into the corresponding global $\mathrm{Ext}_{\mathcal{O}_U}^1$-group of sheaves. Indeed we have

in this case an exact sequence in degree zero ([12], exp. VI)

$$_0\mathrm{Ext}^1_B(M, N) \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}_U}(\widetilde{M}|_U, \widetilde{N}|_U) \to {}_0\mathrm{Hom}_B(M, \mathrm{H}^2_{I(Z)}(N)) \to {}_0\mathrm{Ext}^2_B(M, N) \to \mathrm{Ext}^2_{\mathcal{O}_U}(\widetilde{M}|_U, \widetilde{N}|_U) \tag{3}$$

where the form of the middle term comes from a spectral sequence also treated in [12].

A Cohen-Macaulay (resp. maximal Cohen-Macaulay) B-module satisfies $\operatorname{depth} M = \dim M$ (resp. $\operatorname{depth} M = \dim B$) by definition, or equivalently, $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i < \dim M$ (resp. $i < \dim B$). If B is Cohen-Macaulay and $K_B = \mathrm{Ext}^c_R(B, R(-n-c))$ is the canonical module of B, we know by Gorenstein duality that v-graded piece of $\mathrm{H}^i_{\mathfrak{m}}(M)$ satisfies

$$_v\mathrm{H}^i_{\mathfrak{m}}(M) \simeq {}_{-v}\mathrm{Ext}^{n-i}_B(M, K_B)^\vee .$$

Recall that two graded quotients, R/J and R/J', are said to be linked by a complete intersection if there exists a homogeneous complete intersection ideal L such that $J = L : J'$ and $J' = L : J$ (with $L \subseteq J \cap J'$). The relationship of being linked generates the equivalence relation, "linkage". $B = R/I_B$ is said to be licci (and hence Cohen-Macaulay) if it is in the linkage class of a complete intersection (cf. [34] for a survey). We define licci for a quotient of a regular local ring correspondingly. Moreover $\operatorname{Proj}(B)$ is said to be locally licci if each of its local rings are licci.

Even though we in this paper try to avoid the algebraic (co)homology groups $\mathrm{H}_2(R, B, B)$ and $\mathrm{H}^2(R, B, B)$ and use instead the corresponding Tor- or Ext-groups, we will occasionally need them. We recall that the former group is given by an exact sequence

$$0 \to \mathrm{H}_2(R, B, B) \to \mathrm{H}_1 \to G_1 \otimes_R B \to I_B/I_B^2 \to 0. \tag{4}$$

in which G_1 is R-free, $G_1 \twoheadrightarrow I_B$ is surjective and minimal, and $\mathrm{H}_1 = \mathrm{H}_1(I_B)$ is the degree-one Koszul homology of I_B [38]. An ideal I_B of R is called *syzygetic* if $\mathrm{H}_2(R, R/I_B, R/I_B) = 0$. If I_B is syzygetic, then $(I_B)_\wp$ is syzygetic for any prime ideal $\wp$ of R. I_B is called *generically syzygetic* if $(I_B)_\wp$ is syzygetic for any graded prime $\wp$ of $Ass(B)$. We define genericity of other properties similarly. Using (4) one shows that if $R \to B$ is generically a complete intersection and B is licci, then $\mathrm{H}_2(R, B, B) = 0$ because H_1 is a maximal Cohen-Macaulay module in the licci case [20]. Thus, generically complete intersection licci ideals are syzygetic. For the graded group $\mathrm{H}^2(R, B, B)$ we remark that there is an exact sequence

$$0 \to {}_0\mathrm{Ext}^1_B(I_B/I_B^2, N) \to {}_0\mathrm{H}^2(R, B, N) \to {}_0\mathrm{Hom}_B(\mathrm{H}_2(R, B, B), N) \to \tag{5}$$

This comes from the spectral sequence $\mathrm{Ext}^p_B(\mathrm{H}_q(R, B, B), N)$, which converges to $\mathrm{H}^{p+q}(R, B, N)$, and $I_B/I_B^2 \simeq \mathrm{H}_1(R, B, B)$, cf. [1], Prop. 16.1 or [31].

A Cohen-Macaulay quotient $B = R/I_B$ of codimension $c = \dim R - \dim B$ in R has a minimal R-free resolution of the following form (cf. [7])

$$0 \to G_c \to ... \to G_1 \to R \to B \to 0 \qquad , \qquad G_j = \oplus_{i=1}^{r_j} R(-n_{j,i}) \tag{6}$$

In this case the Castelnuovo-Mumford regularity of I_B is given by (cf. [34], p. 8)

$$\operatorname{reg}(I_B) = \max_{j,i} \{n_{j,i} - j + 1\} = \max_i \{n_{c,i}\} - c + 1 . \tag{7}$$

Let G_i^* be the R-dual of G_i. Since $K_B = \operatorname{Ext}_R^c(B, R(-n-c))$, the R-dual complex of (6) yields a minimal R-free resolution of the twisted canonical module $K_B(n+c)$. In particular if $c = 2$, it induces a complex

$$0 \to N_B \to G_1^* \otimes_R B \to G_2^* \otimes_R B \to K_B(n+c) \to 0 \tag{8}$$

which is exact by e.g. [30], p. 795.

Let $\operatorname{Hilb}^p(\mathbb{P})$ be Grothendieck's Hilbert scheme parametrizing closed subschemes Y of $\mathbb{P} = \operatorname{Proj} R$ with Hilbert polynomial $p \in \mathbb{Q}[t]$, cf. [11]. A k-point of $\operatorname{Hilb}^p(\mathbb{P})$ which corresponds to the subscheme Y is usually denoted by $(Y \subset \mathbb{P})$ or just (Y). A closed subscheme Y of $\mathbb{P}$ is called *unobstructed* if $\operatorname{Hilb}^p(\mathbb{P})$ is smooth at $(Y \subset \mathbb{P})$.

Let $\operatorname{GradAlg}^H := \operatorname{GradAlg}^H(R)$ be the stratum of $\operatorname{Hilb}^p(\mathbb{P})$ given by deforming $Y \subset \mathbb{P}$ with constant Hilbert function $H_Y = H$ (i.e. its functor deforms both $Y \subset \mathbb{P}$ and its *homogeneous coordinate ring*, B, flatly), cf. [27]. $\operatorname{GradAlg}^H(R)$ allows a natural scheme structure whose tangent (resp. "obstruction") space at $(Y \subset \mathbb{P})$ is $_0\operatorname{Hom}_B(I_B/I_B^2, B) \simeq {}_0\operatorname{Hom}_R(I_B, B)$ (resp. $_0\operatorname{H}^2(R, B, B)$), i.e. it is given by deforming B as a graded R-algebra [24]. In the case $H(v)$ does not vanish for large v (i.e. B is non-Artinian), we may look upon $\operatorname{GradAlg}^H(R)$ as parametrizing graded R-quotients, $R \to B$, satisfying $\operatorname{depth}_{\mathfrak{m}} B \geq 1$ and with Hilbert function $H_B = H$. If B is Artinian, $\operatorname{GradAlg}^H(R)$ still represents a functor parametrizing graded R-quotients with Hilbert function $H_B = H$ (see Proposition 9), and its open subscheme of Gorenstein quotients coincides, at least topologically and infinitesimally, with $\operatorname{PGor}(H)$ (the corresponding scheme of forms with "catalecticant structure", see Theorem 11 below and the material before it). B is called unobstructed as a graded R-algebra iff $\operatorname{GradAlg}^{H_B}(R)$ is smooth at $(R \to B)$. This definition of unobstructedness coincides with the one given in Remark 3 below by [31] or [37], p. 151. By [24], Remark 3.7,

$$\operatorname{GradAlg}^H(R) \simeq \operatorname{Hilb}^p(\mathbb{P}) \quad \text{at} \quad (Y \subset \mathbb{P}) \tag{9}$$

provided $_0\operatorname{Hom}_R(I_B, \operatorname{H}^1_{\mathfrak{m}}(B)) = 0$. This happens for quotients B of $\operatorname{depth}_{\mathfrak{m}} B \geq 2$.

Similarly we let $\operatorname{GradAlg}(H_B, H_A)$ be the representing object of the functor deforming flags (surjections) $B \to A$ of graded quotients of R of positive depth (for non-Artinian quotients) and with Hilbert functions H_B and H_A of B and A respectively. The tangent space $_0A^1_{B \to A}$ of $\operatorname{GradAlg}(H_B, H_A)$ at $(B \to A)$ and the tangent maps T_p and T_q of the second projection $p : \operatorname{GradAlg}(H_B, H_A) \to \operatorname{GradAlg}^{H_A}(R)$ (essentially given by $p((B' \to A')) = (A')$) and the first projection $q : \operatorname{GradAlg}(H_B, H_A) \to \operatorname{GradAlg}^{H_B}(R)$ are given by the *cartesian* square in the following diagram of exact sequences

$$
\begin{array}{ccccc}
& & & & _0\operatorname{Hom}_R(I_B, I_{A/B}) \\
& & & & \downarrow \\
& _0A^1_{B \to A} & \overset{T_q}{\to} & & _0\operatorname{Hom}_R(I_B, B) \\
& \downarrow & \square & & \downarrow \\
0 \to & _0\operatorname{Hom}_R(I_{A/B}, A) & \to & _0\operatorname{Hom}_R(I_A, A) & \to & _0\operatorname{Hom}_R(I_B, A)
\end{array}
\tag{10}
$$

Here $B \to A \simeq R/I_A$ is a surjection of graded R-algebras with kernel $I_{A/B}$. Since the lower sequence in the diagram above may be continued by $_0\operatorname{H}^2(B, A, A)$ to make a long exact sequence of algebra cohomology groups, and since $_0\operatorname{H}^2(B, A, A)$ injects in general into

$_0\mathrm{Ext}^1_B(I_{A/B}, A)$ ([14], exp. VI), it follows that $_0\mathrm{Hom}_R(I_A, A) \to {}_0\mathrm{Hom}_R(I_B, A)$ is surjective provided $_0\mathrm{Ext}^1_B(I_{A/B}, A) = 0$. One may also prove this surjectivity in a more elementary manner by dualizing the exact sequence $0 \to I_B \otimes_R A \simeq I_B/I_A I_B \to I_A \otimes_R B \to I_{A/B} \to 0$.

Remark 3. *Let $R \to B$ be a graded quotient. Let $(T, \mathfrak{m}_T) \to (S, \mathfrak{m}_S)$ be a small Artinian surjection (i.e. of local Artinian k-algebras with residue fields k whose kernel $\mathfrak{a}$ satisfies $\mathfrak{a} \cdot \mathfrak{m}_T = 0$). A graded deformation B_S of B to S is a graded S-flat quotient of $R \otimes_k S$ satisfying $B_S \otimes_S k \simeq B$. A graded deformation B_T of B_S to T is T-flat and satisfies $B_T \otimes_T S \simeq B_S$.*

A flag (i.e. a surjection) of graded quotients of R, $B \to A$, is said to be unobstructed if for every small Artinian surjection $T \twoheadrightarrow S$ and for every graded deformation $B_S \to A_S$ of $B \to A$ to S, there exists a graded deformation $B_T \to A_T$ (of T-flat quotients) reducing to $B_S \to A_S$ via $(-) \otimes_T S$. Similarly a quotient B of R is unobstructed if every graded deformation B_S deforms further to T

The obstruction, $o_0(B_S)$, to deforming B_S to T sits in $_0\mathrm{H}^2(R, B, B) \otimes_k \mathfrak{a}$ and $_0\mathrm{H}^1(R, B, B) \otimes_k \mathfrak{a}$ corresponds to the set of graded deformations (hence $_0\mathrm{Hom}_R(I_B, B) \simeq {}_0\mathrm{H}^1(R, B, B)$ is the tangent space of $\mathrm{GradAlg}^{H_B}(R)$). More precisely if we fix a graded deformation B_T of B_S to T, then B_T defines a bijection from $_0\mathrm{H}^1(R, B, B) \otimes_k \mathfrak{a}$ onto the set of graded deformations of B_S to T, allowing us to interpret $B_T + \lambda$ as another graded deformation of B_S to T for every non-trivial $\lambda \in {}_0\mathrm{H}^1(R, B, B) \otimes_k \mathfrak{a}$, cf. [31], Thm. 2.2.5 or [24], Thm. 1.5. Similarly the obstruction to deforming $B_S \to A_S$ to a given graded deformation B_T of B_S sits in $_0\mathrm{H}^2(B, A, A) \otimes_k \mathfrak{a}$ and $_0\mathrm{H}^1(B, A, A) \otimes_k \mathfrak{a} \simeq {}_0\mathrm{Hom}_B(I_{A/B}, A) \otimes_k \mathfrak{a}$ corresponds to the set of such deformations. Note that, by definition, the vanishing of the obstruction is equivalent to the existence of the corresponding desired deformation. Hence the vanishing of the obstruction groups ("spaces") leads to the unobstructedness of the corresponding objects.

Finally let $\phi_S : B_S \to A_S$ be a graded deformation of $\phi : B \to A$ to S, and let B_T and A_T be given graded deformations of B_S and A_S respectively to T. By [31], Thm. 2.3.3 or [24], Thm. 1.6 there is an obstruction $o_0(\phi_S; B_T, A_T) \in {}_0\mathrm{H}^1(R, B, A) \otimes_k \mathfrak{a}$ which vanishes if and only if there exists a graded morphism $\phi_T : B_T \to A_T$ such that $\phi_T \otimes_T id_S = \phi_S$. Looking closer to Laudal's proof of [31], Thm. 2.3.3, one may see

$$o_0(\phi_S; B'_T, A_T) - o_0(\phi_S; B_T, A_T) = (\phi_* \otimes id_{\mathfrak{a}})(\lambda)$$

where $B_T + \lambda = B'_T$ and ϕ_* is the morphism $_0\mathrm{H}^1(R, B, B) \to {}_0\mathrm{H}^1(R, B, A)$ induced by ϕ.

One may consider non-graded deformations of e.g. a quotient B of R in which case the whole algebra cohomology group $\mathrm{Hom}_R(I_B, B) \simeq \mathrm{H}^1(R, B, B)$ (resp. $\mathrm{H}^2(R, B, B)$) serves as tangent (resp. "obstruction") space of the deformations of B as an R-algebra. For graded deformations it is the degree zero piece of the cohomology groups which counts. For a generically complete intersection, $R \twoheadrightarrow B$, (or more generally, for a quotient B which satisfies $_0\mathrm{Hom}_B(\mathrm{H}_2(R, B, B), B) = 0$) its obstructions sit in $_0\mathrm{Ext}^1_B(I_B/I_B^2, B) \otimes \mathfrak{a}$ by (5).

The following Proposition is a main result of the author's unpublished thesis. It (as well as the contents of (10)) is just an application, with a few new ideas, of the general deep results of Laudal ([31] and [32], Sect. 2) on deformations of categories as briefly explained in [25], Sect. 2 and its appendix (see also [26], Sect. 1). Since the proofs of [25] or [26] are not in the generality we need in this paper, we include a proof.

Proposition 4. *Let B be a graded quotient of R and let $\phi : B \to A$ be a graded surjection, inducing morphisms $_0\mathrm{Hom}_R(I_B, B) \to {}_0\mathrm{Hom}_R(I_B, A)$ and $_0\mathrm{H}^2(R, B, B) \to {}_0\mathrm{H}^2(R, B, A)$.*

(i) If $_0\mathrm{Ext}^1_B(I_{A/B}, A) = 0$ (or if $_0\mathrm{H}^2(B, A, A) = 0$), then the first projection $q : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}^{H_B}(R)$ is smooth with fiber dimension $_0\hom_B(I_{A/B}, A)$ at $(B \to A)$.

(ii) If $_0\mathrm{Hom}_R(I_B, B) \to {}_0\mathrm{Hom}_R(I_B, A)$ is surjective and $_0\mathrm{H}^2(R, B, B) \to {}_0\mathrm{H}^2(R, B, A)$ is injective, then the second projection $p : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}^{H_A}(R)$ is smooth with fiber dimension $_0\hom_R(I_B, I_{A/B})$ at $(B \to A)$.

(iii) If $_0\mathrm{Hom}_R(I_B, B) \to {}_0\mathrm{Hom}_R(I_B, A)$ is surjective and B is unobstructed as a graded R-algebra, then p is again smooth with fiber dimension $_0\hom_R(I_B, I_{A/B})$ at $(B \to A)$.

(iv) If both $_0\mathrm{Hom}_R(I_B, B) \to {}_0\mathrm{Hom}_R(I_B, A)$ and the connecting homomorphism $_0\mathrm{Hom}_R(I_B, A) \to {}_0\mathrm{H}^2(B, A, A)$ (cf. (10)) are surjective and if B is unobstructed as a graded R-algebra, then A is unobstructed as a graded R-algebra.

Proof. Using the definition of smoothness of a morphism (e.g. [24], p.212) and the results of Remark 3, it is straightforward to get Proposition 4(i) from $_0\mathrm{H}^2(B, A, A) = 0$ and the injection $_0\mathrm{H}^2(B, A, A) \hookrightarrow {}_0\mathrm{Ext}^1_B(I_{A/B}, A)$.

To see (ii) and (iii) let $\phi_S : B_S \to A_S$ be any graded deformation of ϕ to S, and let A_T be a graded deformations of A_S to T. By the definition of smoothness, it suffices to prove the existence of a graded deformation B_T of B_S to T such that $o_0(\phi_S; B_T, A_T) = 0$. In (iii) the existence of some deformation B'_T of B_S follows by assumption. In (ii) we utilize that the two morphisms $_0\mathrm{H}^2(R, B, B) \otimes_k \mathfrak{a} \to {}_0\mathrm{H}^2(R, B, A) \otimes_k \mathfrak{a}$ and $_0\mathrm{H}^2(R, A, A) \otimes_k \mathfrak{a} \to {}_0\mathrm{H}^2(R, B, A) \otimes_k \mathfrak{a}$ induced by ϕ map the obstructions $o_0(B_S)$ and $o_0(A_S)$ onto the same element in $_0\mathrm{H}^2(R, B, A) \otimes_k \mathfrak{a}$ by [31], proof of Thm. 4.1.14. Since $o_0(A_S) = 0$ and the first morphism is injective by assumption, we get $o_0(B_S) = 0$ and hence the existence of some deformation B'_T in this case as well. Now since $o_0(\phi_S; B'_T, A_T) \in {}_0\mathrm{H}^1(R, B, A) \otimes_k \mathfrak{a} \simeq \mathrm{Hom}_R(I_B, A) \otimes_k \mathfrak{a}$ and since ϕ_* is surjective, there is an element such that $(\phi_* \otimes id_{\mathfrak{a}})(\lambda) = o_0(\phi_S; B'_T, A_T)$. We get the smoothness of p by the displayed formula of Remark 3. Since smooth morphisms have smooth fibers, the fiber dimension is as claimed by [24], Thm. 1.6.

Finally (iv) follows from (iii), Remark 3 and the fact that $_0\mathrm{H}^2(B, A, A) \otimes_k \mathfrak{a} \to {}_0\mathrm{H}^2(R, A, A) \otimes_k \mathfrak{a}$ maps obstructions to obstructions (by [31], Cor. 4.1.15). We leave the details as an exercise since we don't fully need (iv) in this paper. $\qquad\square$

Let $\phi : B \to A$ be a graded surjection. Using Proposition 4, we get the following Theorem which, at least in the geometric case and with various assumptions on $\mathrm{Proj}(B)$, is not new ([25], Sect.2, [26], Sect.1, [29], Ch.9, e.g. Thm. 9.4 and Prop. 9.14). Note that part B of the Theorem has as a consequence that the generic member of an irreducible family of quotients $R \twoheadrightarrow A$ comes from a member of a family of quotients $R \twoheadrightarrow B \twoheadrightarrow A$. Since it is in general not true that a general member factors via a deformation of B even though a special member does, in part A of the Theorem below we determine the codimension of the stratum of quotients which factors. To be precise, let $U \subset \mathrm{GradAlg}^{H_A}(R)$ be a sufficiently small open subset containing (A). The k-points of the subset $p(p^{-1}(U))$ of U correspond to quotients $R \to A'$ with Hilbert function H_A for which there exists some factorization $B' \to A'$ such that B' has Hilbert function H_B. We will call $p(p^{-1}(U))$ a *stratum of H_B-factorizations around* (A), and $\dim U - \dim p(p^{-1}(U))$ the *codimension of the H_B-stratum of A*. At least if U is smooth, it is the ordinary codimension of $p(p^{-1}(U))$ in U. A is called *H_B-generic* if there

is an *open* subset U_A of $\mathrm{GradAlg}^{H_A}(R)$ such that $(A) \in U_A \subset p(p^{-1}(U))$. The *codimension of the H_B-stratum of A at $(B \to A)$* is defined to be $\dim U - \dim p(U')$ where $U' \subset p^{-1}(U)$ is the intersection of the irreducible components of $p^{-1}(U)$ which contains $(B \to A)$. Using small letters for the k-dimension of ${}_0\mathrm{Hom}(-,-)$ and ${}_0\mathrm{Ext}^1_B(-,-)$ we have

Theorem 5. *Let R be a graded polynomial k-algebra and let $B \simeq R/I_B \twoheadrightarrow A \simeq B/I_{A/B}$ be a graded morphism of quotients of R such that ${}_0\mathrm{Ext}^1_B(I_{A/B}, A) = 0$.*

A) If ${}_0\mathrm{Ext}^1_B(I_B/I_B^2, A) = 0$ and $(I_B)_\wp$ is syzygetic for any graded prime $\wp$ of $\mathrm{Ass}(A)$, then A is unobstructed as a graded R-algebra. Moreover, if $\mathrm{depth}_\mathfrak{m} A \geq \min(1, \dim A)$, then $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) =$

$${}_0\mathrm{hom}_R(I_B, B) + {}_0\mathrm{hom}_B(I_{A/B}, A) - {}_0\mathrm{hom}_R(I_B, I_{A/B}) + {}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B}) - {}_0\mathrm{ext}^1_B(I_B/I_B^2, B).$$

Furthermore let B be unobstructed as a graded R-algebra and let k be of characteristic zero. Then the codimension of the H_B-stratum of A at $(B \to A)$ is

$$ {}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B}) - {}_0\mathrm{ext}^1_B(I_B/I_B^2, B). $$

B) If ${}_0\mathrm{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0$ and $(I_B)_\wp$ is syzygetic for any graded prime $\wp$ of $\mathrm{Ass}(I_{A/B})$, then A is H_B-generic. Moreover A is unobstructed as a graded R-algebra if and only if B is unobstructed as a graded R-algebra. Indeed if $\mathrm{depth}_\mathfrak{m} A \geq \min(1, \dim A)$ and $\mathrm{depth}_\mathfrak{m} B \geq \min(1, \dim B)$, then

$$ {}_0\mathrm{hom}_R(I_A, A) - \dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = {}_0\mathrm{hom}_R(I_B, B) - \dim_{(B)} \mathrm{GradAlg}^{H_B}(R), \quad and $$

$$ \dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + {}_0\mathrm{hom}_B(I_{A/B}, A) - {}_0\mathrm{hom}_R(I_B, I_{A/B}) . $$

Proof. A) By the definition of a syzygetic ideal we have $\mathrm{H}_2(R, B, B)_\wp = 0$ for any $\wp$ of $\mathrm{Ass}(A)$. It follows that ${}_0\mathrm{Hom}_B(\mathrm{H}_2(R, B, B), A) = 0$ and hence that ${}_0\mathrm{H}^2(R, B, A) = 0$ by (5). Since we have in general an injection ${}_0\mathrm{H}^2(B, A, A) \hookrightarrow {}_0\mathrm{Ext}^1_B(I_{A/B}, A)$ ([14], exp. VI), we get ${}_0\mathrm{H}^2(B, A, A) = 0$ by the general assumption of Theorem 5. It follows that ${}_0\mathrm{H}^2(R, A, A) = 0$ by the long exact sequence of algebra cohomology (e.g. [31], Thm. 3.3.4)

$$ \to {}_0\mathrm{H}^2(B, A, A) \to {}_0\mathrm{H}^2(R, A, A) \to {}_0\mathrm{H}^2(R, B, A) \to \tag{11} $$

This proves the unobstructedness of A and $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = {}_0\mathrm{hom}_R(I_A, A)$. Now, looking to (10), we see that the vanishing of ${}_0\mathrm{Ext}^1_B(I_{A/B}, A)$ translates to a certain surjectivity in the cartesian square of (10). Since the vertical sequence of Hom-maps in (10) may be continued by

$$ \to {}_0\mathrm{Ext}^1_B(I_B/I_B^2, I_{A/B}) \to {}_0\mathrm{Ext}^1_B(I_B/I_B^2, B) \to {}_0\mathrm{Ext}^1_B(I_B/I_B^2, A) \to \tag{12} $$

it is straightforward to get the dimension formula from the vanishing assumption of A).

To see that ${}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B}) - {}_0\mathrm{ext}^1_B(I_B/I_B^2, B)$ measures the codimension, we observe that $B \twoheadrightarrow A$ is unobstructed and hence that $\dim_{(B \to A)} \mathrm{GradAlg}(H_A, H_B) = \dim {}_0A^1_{B \to A}$ by Proposition 4(i). It follows that $p : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}^{H_A}(R)$ is a morphism between smooth schemes (i.e. smooth at $(B \to A)$ and (A) respectively) and with fiber dimension ${}_0\mathrm{hom}_B(I_B/I_B^2, I_{A/B})$, from which we may conclude by using the Theorem of generic

smoothness to p at $(B \to A)$, cf. [29], first part of the proof of Prop. 9.14 for more details from a similar proof.

B) Since $H_2(R, B, B)_\wp = 0$ for any $\wp \in \operatorname{Ass}(I_{A/B})$ by assumption, we get $_0H^2(R, B, I_{A/B}) = 0$ from (5). By the long exact sequence of algebra cohomology associated to $0 \to I_{A/B} \to B \to A \to 0$, we get precisely that the assumptions of Proposition 4(ii) are satisfied. By Proposition 4(i) and (ii), both projection morphisms p and q are smooth, and we deduce the H_B-genericity of A and the equivalence of the unobstructedness of A and B. Looking to (10) and the statement about the fiber dimensions of p and q in Proposition 4(i) and (ii), we get the dimension formulas of Theorem 5A as well. $\qquad\square$

Remark 6. *a) Note that the proof of Theorem 5A leads to $_0H^2(R, A, A) = 0$ (and in fact to $_vH^2(R, A, A) = 0$ by twisting the two Ext^1_B vanishing assumptions by v).*

b) It is possible to weaken the assumption $_0\operatorname{Ext}^1_B(I_{A/B}, A) = 0$ in Theorem 5 to " $_0H^2(B, A, A) = 0$" and still conclude as in Theorem 5A and 5B. One may also weaken "$(I_B)_\wp$ is syzygetic for any $\wp$ of $\operatorname{Ass}(A)$" in part A to "$_0\operatorname{Hom}_B(H_2(R, B, B), A) = 0$" and still conclude as in Theorem 5A. Similarly we may in part B replace "$(I_B)_\wp$ is syzygetic for any $\wp$ of $\operatorname{Ass}(I_{A/B})$" by "$_0\operatorname{Hom}_B(H_2(R, B, B), I_{A/B}) = 0$" or the entire assumption "$_0\operatorname{Ext}^1_B(I_B/I_B^2, I_{A/B}) = 0$ and $(I_B)_\wp$ is syzygetic for any graded prime $\wp$ of $Ass(I_{A/B})$" by "$_0\operatorname{Ext}^1_B(I_B/I_B^2, I_{A/B}) \to {_0\operatorname{Ext}^1_B}(I_B/I_B^2, B)$ is injective and B is unobstructed as a graded R-algebra" and still get the conclusion of Theorem 5B. There is no real change in the proofs, except when B is unobstructed in which case we need to use Proposition 4(iii) in the proof instead of Proposition 4(ii). These variations turn out to have an application notably to the case A is Artinian.

c) Continuing (12), we see that the condition $_0\operatorname{Ext}^1_B(I_B/I_B^2, A) = 0$ of part A is satisfied provided $_0\operatorname{Ext}^1_B(I_B/I_B^2, B) = 0$ and $_0\operatorname{Ext}^2_B(I_B/I_B^2, I_{A/B}) \hookrightarrow {_0\operatorname{Ext}^2_B}(I_B/I_B^2, B)$ is injective.

In applying Theorem 5A the codimension of the H_B-stratum of A at $(B \to A)$ is often just the codimension of the H_B-stratum of A, due to

Lemma 7. *Let $B \simeq R/I_B \twoheadrightarrow A \simeq B/I_{A/B}$ be a graded morphism of quotients of R and let $m(B)$ be the largest degree of the minimal generators of I_B.*

a) If $(I_{A/B})_v = 0$ for all $v \le m(B)$, then $_0\operatorname{Hom}_R(I_B, I_{A/B}) = 0$ and $p^{-1}((A))$ consists of a single k-point. (i.e. if $B' \to A$ is a quotient such that $H_{B'} = H_B$, then $B' = B$.)

b) If $_0\operatorname{Ext}^1_B(I_{A/B}, A) = 0$ (or if $_0H^2(B, A, A) = 0$) and $(I_{A/B})_v = 0$ for all $v \le m(B)$, then there is an open subset U of $\operatorname{GradAlg}(H_B, H_A)$ containing $(B \to A)$ such that the restriction of the first projection $q : \operatorname{GradAlg}(H_B, H_A) \to \operatorname{GradAlg}^{H_B}(R)$ to U is smooth, and the restriction of the second projection p to U is unramified and injective. Moreover $U = p^{-1}(p(U))$. In particular if $B \to A$ satisfies the assumptions of Theorem 5A (except for $\operatorname{char}(k) = 0$), then the codimension of the H_B-stratum of A is

$$_0\operatorname{ext}^1_B(I_B/I_B^2, I_{A/B}) - {_0\operatorname{ext}^1_B}(I_B/I_B^2, B).$$

Proof. a) Clearly $_0\operatorname{Hom}_R(I_B, I_{A/B}) = 0$ and $(I_B)_v = (I_A)_v$ for all $v \le m(B)$ by assumption. Since $(I_{B'})_v \subset (I_A)_v$, we get $(I_B)_v = (I_{B'})_v$ for $v \le m(B)$ because $H_B = H_{B'}$. Hence $I_{B'}$ contains all minimal generators of I_B, i.e. $I_B \subset I_{B'}$, and we get equality by using $H_B = H_{B'}$.

b) By Proposition 4(i) there is an open subset U of $\operatorname{GradAlg}(H_B, H_A)$ such that the restriction of q to U is smooth. By possibly shrinking U, we get $m(B') \le m(B)$ for any k-point $(B') \in q(U)$ by semicontinuity. Hence for any $(B' \to A') \in U$ we get $(I_{A'/B'})_v = 0$

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for all $v \le m(B')$ because $H_B = H_{B'}$ and $H_A = H_{A'}$, and we see by (i) that p is unramified on U and injective on the k-points of $p^{-1}(U)$. Now we do not need to use the Theorem of generic smoothness because the codimension of the stratum must be $\dim_{(A)} \mathrm{GradAlg}^{H_A} - \dim_{(B \to A)} \mathrm{GradAlg}(H_A, H_B)$, and we conclude by (10) and (12) and the unobstructedness of A and $B \to A$. $\qquad \square$

We will in the next section need to combine (9) with the following result which in fact is a corollary ([29], Cor. 9.10) of a geometric variation ([29], Thm. 9.4) of Theorem $5B$. Below $s(C/F)$ is the minimal degree of the generators of the homogeneous ideal $I_{C/F}$.

Proposition 8. *Let $C_0 \subset \mathbb{P}^{c+1}$ be an arithmetically Cohen-Macaulay curve of degree d_0 and arithmetic genus g_0, sitting on a smooth rational surface $F \subset \mathbb{P}^{c+1}$ of degree f. Let K be the canonical divisor of F. If t is an integer such that $t \ge 5 - s(C_0/F)$ and $t > (C_0 \cdot K)/d_0$, then any member $C = \mathrm{Proj}(B)$ of the linear system $|C_0 + tH|$ is unobstructed and*

$$\dim_{(C)} \mathrm{Hilb}^p(\mathbb{P}^{c+1}) = h^0(\mathcal{N}_F) + g_0 - C_0 \cdot K + t(d_0 + c + 1 - f/2) + t^2 f/2 - 1$$

Moreover, if F is nondegenerate, then $h^0(\mathcal{N}_F) = c^2 + 4c + 13 - 2K^2$.

We finish this section by looking more closely at deformations of an Artinian algebra A. Firstly we recall the representability of the functor (the sheaf) $\mathrm{GRADALG}^H(-)$, locally given by letting $\mathrm{GRADALG}^H(\mathrm{Spec}(S))$ (S a noetherian k-algebra) be the set of graded S-flat quotients $R_S \to A_S$ such that $H_{A_s} = H$ and $\mathrm{depth}_{\mathfrak{m}} A_s \ge \min(1, \dim A_s)$ for any $s \in \mathrm{Spec}(S)$. Here $R_S = R \otimes_k S$ and A_s is the "fiber" of $\mathrm{Spec}(A_S) \to \mathrm{Spec}(S)$ at $s \in \mathrm{Spec}(S)$. To simplify and clarify [27] (being more careful with the topologies involved), we sketch proofs. Note that the representability below ([27], Thm. 1.1) has later been generalized by Haiman and Sturmfels ([19]).

Proposition 9. *The functor $\mathrm{GRADALG}^H(-)$ is representable. Its representing object, $\mathrm{GradAlg}^H(R)$, is a k-scheme of finite type.*

Proof. Here we sketch a proof which is slightly different from that in [27] in the Artinian case (i.e. when $H(v) = 0$ for $v >> 0$). Indeed the flattening stratification described in [37] and Grothendieck's representability Theorems of the Hilbert functor are the results we need to conclude. In the non-Artinian case, $H(v)$ is a polynomial $p(v)$ for $v >> 0$ and $\mathrm{GradAlg}^H(R)$ is the stratum of $\mathrm{Hilb}^p(\mathbb{P}^{n+c-1})$ of constant postulation (cf. [27], Thm. 1.1 for details). If $H(v) = 0$ for $v >> 0$ and $t = \sum_v H(v)$, then there exists a scheme $\mathrm{Hilb} := \mathrm{Hilb}^t(\mathbb{A}^{n+c})$ of finite type which parametrizes quotients $R \to A$ of length t. The restriction (pullback) $R_S \to A_S$ of the universal quotient to $\mathrm{Spec}(S) \to \mathrm{Hilb}$ defines the subscheme $\mathrm{GradAlg}^H(R)$ of Hilb by requiring $\ker((R_S)_v \to A_S)$ to be flat over S, of rank $\dim R_v - H(v)$ for every $0 \le v \le t$ at each fiber of $\mathrm{Spec}(S)$. Since such a subscheme $\mathrm{GradAlg}^H(R)$ of finite type exists by [37], Lect. 8 or [33], VI, Prop. 1.1, we are done. $\qquad \square$

Remark 10. *We may in the case $H(v) = 0$ for $v >> 0$ use flattening stratification to show the existence of the scheme $\mathrm{Z(H)}$ of not necessarily graded Artinian quotients A of R with Hilbert function H (by requiring $\mathrm{gr}(A_S)$ to be S-flat, of Hilbert function H at each fiber of $\mathrm{Spec}(S)$), to get $\mathrm{Z(H)}$ as a locally closed subscheme (stratum) of $\mathrm{Hilb}^t(\mathbb{A}^{n+c})$. Its local deformation functor at a quotient $A \simeq R/I_A$ (with $\sqrt{I_A} = \mathfrak{m}$) of R, defined on the category of*

local Artinian k-algebras with residue field k, consists of quotients A_S of $R_S = S[X_1...., X_{n+c}]$ (or equivalently, of $S[[X_1...., X_{n+c}]]$) such that A_S and $gr(A_S)$ are deformations of A and $gr(A)$ respectively. If $R \to A$ is graded and satisfies $_v\mathrm{H}^2(R, A, A) = 0$ for $v \geq 0$, then [27], Thm. 1.10 and Rem. 1.11 shows that $\mathrm{Z(H)}$ is smooth at $(R \to A)$ and $\dim_{(A)} \mathrm{Z(H)} = \sum_{v \geq 0} {_v}\hom_R(I_A, A)$.

Inside $\mathrm{GradAlg}^H(R)$ there is an open set (and hence a scheme $\mathrm{Gor}_c^H(R)$ with induced scheme structure) consisting of graded Gorenstein quotients $R \to A$ with Hilbert function H. $\mathrm{Gor}_c^H(R)$ is empty unless the difference $\Delta^{\dim A} H$, i.e. the h-vector of A, is a symmetric Gorenstein sequence. In the Artinian case there is another known scheme parametrizing graded Gorenstein quotients with suitable Hilbert function H, namely the determinantal loci $\mathrm{PGor}(H)$ parametrizing forms F of degree s in $\mathbb{P}^N$, $N = \binom{s+n+c-1}{n+c-1} - 1$, given by requiring the "catalecticant matrices" of F to have ranks given by the Hilbert function (see Iarrobino-Kanev's book [21], Sect. 1.1). We denote the restriction of the generic s-form of $\mathbb{P}^N$ to $\mathrm{PGor}(H)$ by $F_{\mathrm{PGor}(H)}$ ([21], Def. 1.3). Then the underlying sets of closed points of $\mathrm{Gor}_c^H(R)$ and $\mathrm{PGor}(H)$ are the same by apolarity (the Macaulay correspondence), and one may see from the proof of our next result that they are in fact isomorphic as topological spaces. Moreover since Iarrobino and Kanev have proved that the tangent space of $\mathrm{PGor}(H)$ at A is isomorphic to $_0\mathrm{Hom}_A(I_A/I_A^2, A)$ ([21], Thm. 3.9) and it is well known that the tangent space of $\mathrm{GradAlg}^H(R)$ is the same group [24], we get

Theorem 11. *Let $R \to A$ be a graded Artinian Gorenstein quotient with Hilbert function H. Then $\dim_{(A)} \mathrm{GradAlg}^H(R) = \dim_{(A)} \mathrm{PGor}(H)$. Hence $\mathrm{PGor}(H)$ is smooth at $(R \to A)$ if and only if $\mathrm{GradAlg}^H(R)$ is smooth at $(R \to A)$. In particular $\mathrm{PGor}(H)$ is smooth at $(R \to A)$ provided $_0\mathrm{H}^2(R, A, A) = 0$, i.e. provided the natural surjection $(S_2 I_A)_s \to (I_A{}^2)_s$ from the second symmetric power to the second power of I_A is an isomorphism in the socle degree s of A.*

Proof. By duality, $_0\mathrm{H}^2(R, A, A) = 0$ is equivalent to $_s\mathrm{H}_2(R, A, A) = 0$ which again is equivalent to $(S_2 I_A)_s \simeq (I_A{}^2)_s$ by (28) below. Hence we only need to prove $\dim_{(A)} \mathrm{GradAlg}^H(R) = \dim_{(A)} \mathrm{PGor}(H)$, because we know their tangent spaces are isomorphic. Let $V \subset \mathrm{PGor}(H)$ be a closed irreducible subset (e.g. of the form $V = \mathrm{Spec}(T)$), and let V have the reduced scheme structure. By the definition of $\mathrm{PGor}(H)$, the restriction F_V of the "universal" s-form $F_{\mathrm{PGor}(H)}$ to V defines via apolarity a family of graded Artinian Gorenstein quotients over V (e.g. the family $A_T \simeq R_T \circ F_V$ over $V = \mathrm{Spec}(T)$ where R_T is interpreted as the T-algebra of contractions, "partial derivations without coefficients") with constant Hilbert function H. Since V is integral, it follows that the family (e.g. the morphism $\mathrm{Spec}(A_T) \to \mathrm{Spec}(T)$) is flat ([37], Lect. 6). Hence we have a morphism $\pi : V \to \mathrm{GradAlg}^H(R)$ by the universal property of $\mathrm{GradAlg}^H(R)$. $\pi(V)$ is irreducible and closed in $\mathrm{Gor}_c^H(R)$ (it is closed because an "inverse" $(\mathrm{Gor}_c^H(R))_{red} \to \mathrm{PGor}(H)$ on closed points exists by [21], Ch. 8). So chains of closed irreducible subsets in $\mathrm{PGor}(H)$ and $\mathrm{Gor}_c^H(R)$ correspond, and we are done. $\square$

Problem 12. *It would be interesting to prove that $\mathrm{PGor}(H)$ and $\mathrm{Gor}_c^H(R)$ are isomorphic as schemes, cf. [27], Rem. 1.9. The arguments in the proof above, which easily lead to an isomorphism of $\mathrm{PGor}(H)$ and $\mathrm{Gor}_c^H(R)$ as topological spaces, are mainly contained in [27], Rem. 1.9. In that Remark we also claimed that these arguments showed the existence of a*

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natural morphism $\Pi : \mathrm{PGor}(H) \to \mathrm{Gor}_c^H(R)$. *Unfortunately an index "red" is missing in Rem. 1.9, i.e. we can from the arguments only be sure of the existence of a natural morphism* $\pi : (\mathrm{PGor}(H))_{red} \to \mathrm{Gor}_c^H(R)$, *and the existence of* Π *above remains to be proved.*

2 Families of codimension one Gorenstein quotients of Cohen-Macaulay algebras

The main purpose of this paper is to study families of graded Gorenstein quotients A obtained by taking a regular section σ of the anticanonical system $(K_B^*)_s$, i.e. quotients A given by

$$0 \to K_B(-s) \xrightarrow{\sigma^*} B \to A \to 0 \tag{13}$$

for some integer s. Our main results apply to Artinian Gorenstein algebras as well as to Gorenstein quotients of higher dimension. In what follows B is Cohen-Macaulay. In order to make the description of A in (13) meaningful, we must suppose that $B = R/I_B$ is generically Gorenstein (cf. [29], Lemma 5.2 and 5.4). Then we say *"A is a codimension one quotient of B, well-defined by a twist of the canonical module"*. If A is such a quotient of B, then a stratum of quotients given by (13) around (A) is just a maximal (flat) family consisting of quotients (A') of $\mathrm{GradAlg}^{H_A}(R)$ for which there exists a factorization $B' \to A'$ as in (13) where (B') (resp. the sections) belong to some open neighbourhood of (B) in $\mathrm{GradAlg}^{H_B}(R)$ (resp. of σ in the linear system $(K_{B'}^*)_s \subset \mathrm{H}^0(\mathrm{Proj}(B'), \widetilde{K_{B'}}^*(s)))$, making each A' well-defined in the sense above. Looking to the proof of Proposition 13, one may see that such a stratum is just the image $p(U)$ of some open U of $\mathrm{GradAlg}(H_B, H_A)$ containing $(B \to A)$ via the second projection p, provided $_s\mathrm{Ext}_B^1(S_2(K_B), K_B) = 0$.

The following result determines $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R)$ in terms of invariants of B, such as $N_B := \mathrm{Hom}_B(I_B/I_B^2, B)$ and $K_B^* := \mathrm{Hom}_B(K_B, B) \simeq \mathrm{Hom}_B(S_2(K_B), K_B)$ (here $S_2(K_B)$ is the second symmetric power of K_B), and leads to the main theorems of this section. Since Proposition 13 is a variation of Theorem 5 with $I_{A/B} = K_B(-s)$, Remark 6 (except the first sentence of Remark 6(b)) will apply as a remark to Proposition 13 as well.

Proposition 13. *Let B be a generically Gorenstein, graded Cohen-Macaulay quotient of a polynomial ring R, let A be a codimension one quotient of B, defined by (13) for some s, and let $_s\mathrm{Ext}_B^1(S_2(K_B), K_B) = 0$.*

A) If $_0\mathrm{Ext}_B^1(I_B/I_B^2, A) = 0$ and $(I_B)_\wp$ is syzygetic for any graded prime $\wp$ of $\mathrm{Ass}(A)$, then A is unobstructed as a graded R-algebra, A is Gorenstein and

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim(N_B)_0 + \dim(K_B^*)_s - 1 - \delta(B)_{-s} - {}_0\mathrm{ext}_B^1(I_B/I_B^2, B)$$

where $\delta(B)_{-s} = {}_{-s}\mathrm{hom}_R(I_B, K_B) - {}_{-s}\mathrm{ext}_B^1(I_B/I_B^2, K_B)$. Moreover if B is unobstructed as a graded R-algebra and $char(k) = 0$, then the codimension of the H_B-stratum of A at $(B \to A)$ is

$$_{-s}\mathrm{ext}_B^1(I_B/I_B^2, K_B) - {}_0\mathrm{ext}_B^1(I_B/I_B^2, B).$$

This number also equals the codimension of the stratum of quotients given by (13) at (A).

B) If $\quad _{-s}\mathrm{Ext}^1_B(I_B/I_B^2, K_B) = 0$ *and* I_B *is generically syzygetic,*
then A *is Gorenstein and* H_B*-generic. Moreover* A *is unobstructed as a graded* R*-algebra if and only if* B *is unobstructed as a graded* R*-algebra. Indeed*

$$_0\mathrm{hom}_R(I_A, A) - \dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \ _0\mathrm{hom}_R(I_B, B) - \dim_{(B)} \mathrm{GradAlg}^{H_B}(R), \ and$$

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \ \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + \ \dim(K_B^*)_s - 1 - \ _{-s}\mathrm{hom}_R(I_B, K_B) \ .$$

Remark 14. *(a) With the notations,* $0 \to \oplus_i R(-n_{c,i}) \to ... \to \oplus_i R(-n_{1,i}) \to I_B \to 0$, *of (6), recalling* $\dim R = n + c$ *and* $\dim B = n$, *we claim that*

$$\delta(B)_{-s} = 0 \ provided \ s > 2\mathrm{reg}(I_B) - n$$

By (7) it suffices to prove $_{-s}\mathrm{Ext}^i_B(I_B/I_B^2, K_B) = 0$ *for* $i = 0$ *and* 1 *provided* $s > \max n_{c,i} + \max n_{2,i} - n - c$. *Indeed dualizing (6) we get a surjection* $\oplus_i R(n_{c,i}) \twoheadrightarrow K_B(n+c)$, *and it follows that* $(K_B)_{-v} = 0$ *for* $v > \max n_{c,i} - n - c$. *We conclude by applying* $_0\mathrm{Hom}_B(-, K_B)$ *to the right-exact sequence* $\oplus_i B(-n_{2,i}) \to \oplus_i B(-n_{1,i}) \to I_B/I_B^2 \to 0$ *and the definiton of* $\mathrm{Ext}^i_B(I_B/I_B^2, K_B)$.

(b) If $\dim B = 1$, *we have* $\mathrm{Ext}^i_B(-, K_B) = 0$ *for* $i \geq 2$ *by Gorenstein duality. In this case* $_0\mathrm{Ext}^1_B(I_B/I_B^2, B) = 0$ *implies* $_0\mathrm{Ext}^1_B(I_B/I_B^2, A) = 0$ *by Remark 6(c). Moreover looking to Remark 6(b) and using that* $A \simeq K_A(-s)$ *and Gorenstein duality, we may weaken "*I_B *is syzygetic for any graded prime* $\wp$ *of* $Ass(A)$*" to "*$(\mathrm{H}_2(R, B, B) \otimes_B A)_s = 0$*". Hence if we suppose*

$$_0\mathrm{Ext}^1_B(I_B/I_B^2, B) = \ (\mathrm{H}_2(R, B, B) \otimes_B A)_s = 0,$$

we still have the conclusion concerning the unobstructedness of A *and the formula for* $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R)$ *of Proposition 13A. Furthermore if* I_B *is generically syzygetic (this is always true if* $c \leq 3$ *since* B *is generically Gorenstein, cf. the proof of Theorem 16), then* B *is unobstructed as a graded algebra and the codimension formula of Proposition 13A holds as well (in the characteristic zero case).*

(c) Slightly reformulating Proposition 13A (by replacing the vanishing of the two $_0\mathrm{Ext}^1_B$ *groups by the corresponding vanishing of* Ext^1_B*) one may use the proof below to show that* A *is unobstructed as an* R*-algebra in the sense* $\mathrm{H}^2(R, A, A) = 0$, *cf. Remark 6. A further slight reformulation (replacing* Proj *by* Spec *and skipping the twists) allows us to prove a corresponding result for non-graded objects as well. If* B *is licci then the three* Ext_B *groups,* $\mathrm{Ext}^1_B(S_2(K_B), K_B)$, $\mathrm{Ext}^1_B(I_B/I_B^2, B)$ *and* $\mathrm{Ext}^2_B(I_B/I_B^2, K_B)$, *vanish ([5], [20], [38] Thm. 4.2.6, or Corollary 37 of this paper), and we get the unobstructedness of the Gorenstein quotient* A *by Remark 6(c). Indeed* $S_2(K_B)$ *is Cohen-Macaulay if* B *is licci. In the interesting codimension* $c = 3$ *case, the Cohen-Macaulayness of* $S_2(K_B)$ *and of the Koszul homology group,* $\mathrm{H}_1 = \mathrm{H}_1(I_B)$, *are equivalent [38], Thm. 4.2.9. This Cohen-Macaulayness holds in particular if the Cohen-Macaulay type of* B *is 2 [38], Cor. 4.2.11.*

(d) If I_B *is generically syzygetic, then* $\mathrm{Hom}_B(\mathrm{H}_2(R, B, B), K_B) = 0$. *Hence applying* $Hom_B(-, K_B)$ *to (4), we see that* $_{-s}\mathrm{Ext}^2_B(I_B/I_B^2, K_B) \hookrightarrow _{-s}\mathrm{Ext}^1_B(\mathrm{H}_1, K_B)$ *is injective and that*

$$\delta(B)_{-s} = \dim(G_1^* \otimes_R K_B)_{-s} - \ _{-s}\mathrm{hom}_B(\mathrm{H}_1, K_B)$$

allowing us to restate the dimension formula in terms of H_1 *and* G_1. *If* B *is licci, then* $_{-s}\mathrm{Hom}_B(\mathrm{H}_1, K_B) \simeq \ (\mathrm{H}_{r-1})_{-t-s}$ *by [30], Prop. 18, where* r *and* t *are given by* $\wedge^r \widetilde{\mathrm{H}_1}|_U \simeq \widetilde{K}_B(t)|_U$.

Proof. Note that we can show $A \simeq K_A(-s)$ and hence that A is Gorenstein by applying the mapping cone construction to (13), (or see [29], Lemma 5.2 and 5.4). We *claim* that Proposition 13 above follows from Theorem 5 and Remark 6 by letting $I_{A/B} = K_B(-s)$ and by paying some extra attention to the codimension of the strata. Indeed it is straightforward to obtain the assumptions of Theorem 5 from the assumptions of Proposition 13, except possibly the assumption $_0\mathrm{Ext}^1_B(I_{A/B}, A) = 0$ of Theorem 5. Suppose temporarily that B is Gorenstein in codimension ≤ 1. Then $_0\mathrm{Ext}^1_B(K_B(-s), B) \simeq {}_0\mathrm{Ext}^1_B(K_B \otimes K_B(-s), K_B) \simeq {}_0\mathrm{Ext}^1_B(S_2(K_B)(-s), K_B) = 0$ by a well known spectral sequence ([18], Satz 1.2). Moreover, since K_B is a maximal Cohen-Macaulay B-module, one knows $_0\mathrm{Ext}^i_B(K_B, K_B) = 0$ for $i > 0$ and we get $_0\mathrm{Ext}^1_B(K_B(-s), A) = 0$ by applying $_0\mathrm{Hom}(K_B(-s), -)$ to (13), i.e. we get the claim. In the general case where B is generically Gorenstein we get the conclusion of Theorem 5 by combining Lemma 15 and Remark 6(b), recalling $K_B(-s) \simeq I_{A/B}$. Furthermore to get the dimension formulas we note that $_0\mathrm{Hom}(K_B(-s), K_B(-s))$ is one-dimensional. Hence by again applying $_0\mathrm{Hom}(K_B(-s), -)$ to (13), we get the dimension formulas from the corresponding formulas of Theorem 5.

It remains to see that the codimension may also be computed from the stratum of quotients given by (13). Now looking to the definition of the codimension of the H_B-stratum of A at $(B \to A)$ before Theorem 5, it suffices to show that any $(B' \to A')$ in a small enough open neighbourhood of $t := (B \to A)$ in $G := \mathrm{GradAlg}(H_B, H_A)$ is given by (13). Let (S, m_S) be the local ring of G at t and let $S_i = S/m_S^i$. Since G is a scheme of finite type which represents a certain functor of graded flat quotients, there exists a universal quotient whose pullback to $\mathrm{Spec}(S)$ is denoted by $B_S \to A_S$. Its kernel I_{A_S/B_S} is a flat deformation of the B-module $I_{A/B} \simeq K_B(-s)$ to S. By the theory of deformations of the module $I_{A/B}$ its tangent and obstruction spaces are given by $_0\mathrm{Ext}^i_B(K_B(-s), K_B(-s))$ for $i = 1$ and 2 respectively. Since both these Ext-groups vanish, there is a unique deformation of $I_{A/B}$ to B_{S_i}, i.e. we get isomorphisms $l_i : I_{A_S/B_S} \otimes_S S_i \simeq K_{B_S}(-s) \otimes_S S_i$ which commute with l_{i-1} because $K_{B_S} := \mathrm{Ext}^c_{R_S}(B_S, R_S(-n-c))$ where $R_S := R \otimes_k S$, is S-flat. (cf. [23], Prop. A1). It suffices to show that there exists a morphism $l : I_{A_S/B_S} \to K_{B_S}(-s)$ which reduces to l_i mod m_S^i because l must then be an isomorphism which extends to an isomorphism in an open neighbourhood of t in G. To show the existence of l, we *claim* that $K^*_{B_S}(s) \to K^*_B(s)$ is surjective. By the proof of [26], Prop. 2.4, it suffices to show that $\eta_i : K^*_{B_{S_i}}(s) \to K^*_{B_{S_{i-1}}}(s)$ is surjective. If $_0\mathrm{Ext}^1_B(K_B(-s), B) = 0$ (e.g. B is Gorenstein in codimension ≤ 1, cf. the first part of the proof), this follows by applying $_0\mathrm{Hom}(K_{B_{S_i}}(-s), -)$ onto $0 \to B \otimes_k \mathfrak{a} \to B_{S_i} \to B_{S_{i-1}} \to 0$ where $\mathfrak{a} := m_S^{i-1}/m_S^i$. In the general case, replacing every B by A in the latter sequence and applying $_0\mathrm{Hom}(K_{B_{S_i}}(-s), -) \simeq {}_0\mathrm{Hom}(I_{A_{S_i}/B_{S_i}} \otimes_{B_{S_i}} A_{S_i}, -)$ to that sequence, we get the surjectivity of η_i from $_0\mathrm{Ext}^1_A(I_{A/B}/I^2_{A/B}, A) = 0$ and $_0\mathrm{Ext}^1_B(K_B(-s), K_B(-s)) = 0$. Note that we have the last mentioned vanishings by (5), Lemma 15 and $_s\mathrm{Ext}^1(S_2(K_B), K_B) = 0$, and the claim is proved. Now using the proven claim we get a morphism $K_{B_S}(-s) \to B_S$ whose cokernel is an S-flat deformation of A. By the universal property of G, there is a morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(S)$ whose pullback induces the desired map $I_{A_S/B_S} \to K_{B_S}(-s)$ and the proof is complete. $\square$

Lemma 15. *Let B be a graded Cohen-Macaulay quotient of R and let $A \simeq B/I_{A/B}$ be a graded Gorenstein quotient such that $A \simeq K_A(-s)$ and $\dim B - \dim A = r$. If $r > 0$ then $_0\mathrm{H}^2(B, A, A)$ and $_{-s}\mathrm{Ext}^r_B(S_2(I_{A/B}), K_B)$ vanish simultaneously.*

Proof. Using (5) we see that the vanishing of $_0\mathrm{H}^2(B,A,A)$ is equivalent to $_0\mathrm{Ext}^1_A(I_{A/B}/I^2_{A/B},A) = 0$ *and* the injectivity of the natural map $_0\mathrm{Hom}_A(\mathrm{H}_2(B,A,A),A) \hookrightarrow\ _0\mathrm{Ext}^2_A(I_{A/B}/I^2_{A/B},A)$. Moreover since $A \simeq K_A(-s)$ we have by Gorenstein duality (applied to both A and B) that

$$_0\mathrm{Ext}^i_A(I_{A/B}/I^2_{A/B},A) \simeq\ _s\mathrm{H}^{\dim A-i}_{\mathfrak{m}}(I_{A/B}/I^2_{A/B})^\vee \simeq\ _{-s}\mathrm{Ext}^{i+r-1}_B(I^2_{A/B},K_B)$$

for $i \geq 1$ because the middle term is isomorphic to $_s\mathrm{H}^{\dim A+1-i}_{\mathfrak{m}}(I^2_{A/B})$ by the long exact sequence of local cohomology applied to $0 \to I^2_{A/B} \to I_{A/B} \to I_{A/B}/I^2_{A/B} \to 0$. Correspondingly $_0\mathrm{Hom}_A(\mathrm{H}_2(B,A,A),A) \simeq\ _{-s}\mathrm{Ext}^r_B(\mathrm{H}_2(B,A,A),K_B)$. Hence $_0\mathrm{H}^2(B,A,A) = 0$ is equivalent to $_{-s}\mathrm{Ext}^r_B(I^2_{A/B},K_B) = 0$ *and* the injectivity of of the natural map $_{-s}\mathrm{Ext}^r_B(\mathrm{H}_2(B,A,A),K_B) \hookrightarrow\ _{-s}\mathrm{Ext}^{r+1}_B(I^2_{A/B},K_B)$. Recalling the exact sequence

$$0 \to \mathrm{H}_2(B,A,A) \to S_2(I_{A/B}) \to I^2_{A/B} \to 0 \tag{14}$$

(cf. [1], p.106 or [38], Sect. 2.1) and that $_{-s}\mathrm{Ext}^{r-1}_B(\mathrm{H}_2(B,A,A),K_B) = 0$ since $\mathrm{H}_2(B,A,A)$ is supported at the codimension r quotient A of B, we get the conclusion of Lemma 15 by the long exact sequence of $_{-s}\mathrm{Ext}^i_B(-,K_B)$ applied to (14). $\qquad\square$

Theorem 16. *Let $B = R/I_B$ be a graded Cohen-Macaulay quotient of a polynomial ring R, and let A be a graded codimension one quotient of B, well-defined by a twist of the canonical module K_B.*

A) If B is licci, then A is unobstructed as a graded R-algebra, A is Gorenstein and,

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + \dim(K_B^*)_s - 1 - \delta(B)_{-s}$$

where $\delta(B)_{-s} = {}_{-s}\hom_B(I_B/I^2_B,K_B) - {}_{-s}\mathrm{ext}^1_B(I_B/I^2_B,K_B)$. Moreover if $char(k) = 0$, then the codimension of the H_B-stratum of A at $(B \to A)$ is $_{-s}\mathrm{ext}^1_B(I_B/I^2_B,K_B)$. This number also equals the codimension of the stratum of quotients given by (13) around (A).

B) If $s >> 0$ and the sheaf $\mathcal{E}xt^1_{O_{\mathrm{Proj}(B)}}(\widetilde{S_2(K_B)},\widetilde{K_B}) = 0$, then A is H_B-generic and Gorenstein, and

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + \dim(K_B^*)_s - 1.$$

Moreover A is unobstructed as a graded R-algebra if and only if B is unobstructed as a graded R-algebra.

Note that if $Y = \mathrm{Proj}(B)$ is locally licci, then the sheaf $\widetilde{S_2(K_B)}$ is maximally Cohen-Macaulay by Remark 14(c), i.e. $\mathcal{E}xt^1_{O_Y}(\widetilde{S_2(K_B)},\widetilde{K_B}) = 0$, and we get Theorem 1. The sheaf also vanishes if Y is (locally) Gorenstein, e.g. if $\dim Y = 0$.

Proof. A) Since B is licci and generically Gorenstein, I_B is generically syzygetic by [29], Prop. 6.17 which implies $\mathrm{H}^2(R,B,B)_\wp = 0$ provided $\dim B_\wp = 0$. This in turn is equivalent to $\mathrm{H}_2(R,B,B)_\wp = 0$ by Gorenstein duality and the spectral sequence relating algebra-homology to algebra-cohomology, i.e. I_B is generically syzygetic. Then I_B is in fact syzygetic by (4) because H_1 is a maximal Cohen-Macaulay module in the licci case. Since all Ext^1_B groups of Proposition 13A vanish by Remark 14(c) and Remark 6(c), we conclude by Proposition 13.

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B) If $\mathcal{E}xt^1_{O_Y}(\widetilde{S_2(K_B)}, \widetilde{K_B}) = 0$, we see that $_s\mathrm{Ext}^1_B(S_2(K_B), K_B)$ vanishes for large s because of the exact sequence

$$\to\ _s\mathrm{Hom}_B(S_2(K_B), \mathrm{H}^1_{\mathfrak{m}}(K_B)) \to\ _s\mathrm{Ext}^1_B(S_2(K_B), K_B) \to \mathrm{H}^1(Y, \mathcal{H}om_B(\widetilde{S_2(K_B)}, \widetilde{K_B})(s))$$

(which is analogous to (3)). Since the vanishing of $_{-s}\mathrm{Hom}_B(\mathrm{H}_2(R, B, B), K_B) = 0$ and $_{-s}\mathrm{Ext}^i_B(I_B/I_B^2, K_B)$, $i = 0, 1$, are obviously true for $s >> 0$, we conclude by Proposition 13B and Remark 6(b). $\qquad\square$

To apply Proposition 13 and Theorem 16, we need to compute $\dim(K_B^*)_s$ and verify its assumptions. In several cases it is easy to compute $\dim(K_B^*)_s$ at least for large s. For instance if B is 2-dimensional and $C = \mathrm{Proj}(B)$ is *smooth*, then $\widetilde{K_B^*}$ is just the tangent sheaf (sheaf of derivations), θ_C, of C, and we can get the information we need from the vanishing of cohomology and the Riemann-Roch theorem. In this case the most difficult assumption to verify is perhaps the unobstructedness of B (or $_0\mathrm{Ext}^1_B(I_B/I_B^2, B) = 0$), for which we point out a couple of important cases where the unobstructedness is known;

1. $_0\mathrm{Ext}^1_B(I_B/I_B^2, B) = 0$ provided C is linked (in possibly several steps) to a generically Gorenstein curve $C' = \mathrm{Proj}(B')$ satisfying $_0\mathrm{Ext}^1_{B'}(I_{B'}/I_{B'}^2, B') = 0$ (cf. next section).

2. B is unobstructed if $C = \mathrm{Proj}(B)$ sits on a smooth rational surface in $\mathbb{P}^4$ and the degree of C is ≥ 24 ([29], Prop. 9.13; the statement follows also from Proposition 8).

By 1) we get $_0\mathrm{Ext}^1_B(I_B/I_B^2, B) = 0$ if B is licci (e.g. if C' above is a complete intersection), or if C' is a local complete intersection curve and $\mathrm{H}^1(\mathcal{N}_{C'}) = 0$ (because $_0\mathrm{Ext}^1_{B'}(I_{B'}/I_{B'}^2, B') \hookrightarrow \mathrm{H}^1(\mathcal{N}_{C'})$ is injective by (3)). Indeed for a local complete intersection curve $C = \mathrm{Proj}(B)$ we have $\mathrm{H}^1(\mathcal{N}_C(v)) \simeq \mathrm{Ext}^1_{\mathcal{O}_C}(\widetilde{I_B/I_B^2}, \widetilde{B}(v))$ and by (3) an exact sequence

$$_v\mathrm{Ext}^1_B(I_B/I_B^2, B) \hookrightarrow \mathrm{H}^1(\mathcal{N}_C(v)) \to\ _v\mathrm{Hom}_B(I_B/I_B^2, \mathrm{H}^2_{\mathfrak{m}}(B)) \to\ _v\mathrm{Ext}^2_B(I_B/I_B^2, B) \to 0 \quad (15)$$

We illustrate Theorem 16 by taking a relatively simple example, remarking that far more complex examples can be given by the same methods, by increasing s.

Example 17. *Let B be the homogeneous coordinate ring of a smooth rational arithmetically Cohen-Macaulay curve C of degree $d = 4$ in $\mathbb{P}^4 = \mathrm{Proj}(R)$, with minimal resolution*

$$0 \to R(-4)^3 \to R(-3)^8 \to R(-2)^6 \to R \to B \to 0, \quad (16)$$

and let A be defined by a regular section of $(K_B^*)_s$ for $s \geq 0$. Then C is the determinantal curve given by the maximal minors of a 2 by 4 matrix with linear entries, and it is straightforward to see that the Hilbert function H_A of A is given by $\dim A_v = 4v + 1$ for $0 \leq v \leq s$ and $\dim A_v = 4s + 2$ for $v > s$. Applying $\mathrm{Hom}_R(-, R(-5))$ onto (16) we get the minimal resolution of K_B which combined with (13) and the mapping cone construction yields the resolution

$$0 \to R(-s-5) \to R(-s-3)^6 \oplus R(-4)^3 \to R(-s-2)^8 \oplus R(-3)^8 \to R(-s-1)^3 \oplus R(-2)^6 \to I_A \to 0.$$

Even though B is not licci, Theorem 16 applies for $s >> 0$. Indeed, looking to Proposition 13 and Remark 14(a), we claim that the conclusions of Theorem 16B hold for $s \geq 3$ (and those

of Theorem 16A for $s \geq 0$, see Example 35 in the next section). Indeed we need only to verify its assumptions. Since C is smooth and $\theta_C := \widetilde{K_B^*}$ is of degree $\deg \theta_C = -\deg \widetilde{K_B} = 2 - 2g$ (g = the genus of C), we get $\mathrm{H}^1(\theta_C(v)) = 0$ and $h^0(\theta_C(v)) = dv + 3 - 3g = 4v + 3$ for $v \geq 0$ ([37], Lect. 11). Since we have an exact sequence

$$0 \to \theta_C \to \theta_{\mathbb{P}}|_C \to \mathcal{N}_C \to 0$$

where $\theta_{\mathbb{P}}|_C = \mathrm{coker}(\mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus 5})$ is the tangent sheaf of $\mathbb{P} = \mathbb{P}^4$ restricted to C, we get $_0\mathrm{Ext}_B^1(I_B/I_B^2, B) = 0$ from $\mathrm{H}^1(\mathcal{N}_C) = 0$ and (15). Moreover $_v\mathrm{Ext}_B^1(S_2(K_B), K_B)^\vee \simeq {}_{-v}\mathrm{H}_{\mathfrak{m}}^1(S_2(K_B))$ and since $\mathrm{H}_*^0(C, \widetilde{S_2(K_B)}) \to \mathrm{H}_{\mathfrak{m}}^1(S_2(K_B))$ is surjective and $\mathrm{H}^0(C, \widetilde{S_2(K_B)}(-v))^\vee \simeq \mathrm{H}^1(\theta_C(v))$ we get

$$_v\mathrm{Ext}_B^1(S_2(K_B), K_B) = 0 \qquad for \quad v \geq 0.$$

By Remark 14(a), we know $_{-s}\mathrm{Ext}_B^1(I_B/I_B^2, K_B) = 0$ for $s > 2$ and the claim is proved for $s \geq 3$. In conclusion the quotient A is an unobstructed Gorenstein algebra for $s \geq 3$ and

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = h^0(\mathcal{N}_C) + h^0(\theta_C(s)) - 1 = 5d + 1 - g + ds + 3 - 3g - 1 = 4s + 23$$

Remark 18. *This example shows that if $\mathrm{Proj}(B)$ is a smooth curve of degree d and genus g, and if we let v be such that $2 - 2g + vd > 2g - 2$, i.e. $v > (4g - 4)/d$, then $\mathrm{H}^1(\theta_C(v)) = 0$ and we get both $_v\mathrm{Ext}_B^1(S_2(K_B), K_B) = 0$ and $\dim(K_B^*)_v = h^0(\theta_C(v)) = dv + 3 - 3g$. In particular if $s > \max\{2\,\mathrm{reg}(I_B) - 2, (4g - 4)/d\}$ and B is unobstructed, then A is unobstructed and H_B-generic by Proposition 13B and Remark 14(a) and*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim(N_B)_0 + ds + 2 - 3g.$$

Hence we may replace "$s >> 0$" in Theorem 1 and Theorem 16 by "$s > \max\{2\,\mathrm{reg}(I_B) - 2, (4g - 4)/d\}$" in this case.

Now we consider Artinian Gorenstein quotients. In particular we will concentrate on Artinian reductions obtained from algebras A similar to those in Example 17, through modding out by a general linear form L (cf. Example 21 below). More generally let $B \simeq R/I_B$ be an n-dimensional Cohen-Macaulay algebra, let $A \simeq B/K_B(-s)$, and let $R_L := R/(L) \to B_L = B/(L) \to A_L = A/(L)$ be their quotients modulo (L). The Hilbert functions of A_L and B_L are given by the first differences of the Hilbert functions of A and B respectively, e.g. $H_{B_L}(v) = H_B(v) - H_B(v - 1) = \Delta H_B(v)$. Moreover an R_L-free minimal resolution of B_L has the same graded Betti numbers as the R-free minimal resolution of B ([34], p.28), i.e. their numbers $n_{j,i}$ given in (6) coincide. Dualizing (6) we see that the minimal resolutions of $K_B(n + c)$ and $K_{B_L}(n + c - 1)$ also coincide, i.e. $K_B \otimes B_L \simeq K_{B_L}(-1)$. Looking to (13) it follows that A_L is given by

$$0 \to K_{B_L}(-s - 1) \to B_L \to A_L \to 0, \tag{17}$$

(note $-s - 1$ instead of $-s$!) and A_L is an Gorenstein quotient, well defined by a twist of the canonical module. Hence we may apply Proposition 13 and Theorem 16. To compute $\dim(K_{B_L}^*)_{s+1}$ and verify its assumptions, we see, thanks to the arguments of Remark 14(a), that $\delta(B_L)_{-s-1} = 0$ and $_{-s-1}\mathrm{Ext}_{B_L}^1(I_{B_L}/I_{B_L}^2, K_{B_L}) = 0$ (where $I_{B_L} = \ker(R_L \to B_L)$), provided $s + 1 > \max n_{c,i} + \max n_{2,i} - (n - 1) - c$ which is equivalent to the inequality of that Remark, used to show $\delta(B)_{-s} = 0$. To check the other assumptions, we claim that

Lemma 19. *(i) If $_{v-1}\mathrm{Ext}^1_B(S_2(K_B), K_B) = 0$, then $\dim(K^*_{B_L})_{v+1} = \dim(K^*_B)_v - \dim(K^*_B)_{v-1}$.*
(ii) If $_v\mathrm{Ext}^1_B(S_2(K_B), K_B) = {}_{v-1}\mathrm{Ext}^2_B(S_2(K_B), K_B) = 0$, then

$$_{v+1}\mathrm{Ext}^1_{B_L}(S_2(K_{B_L}), K_{B_L}) = 0.$$

(iii) If B is an unobstructed graded R-algebra and the map $_{-1}\mathrm{Ext}^1_B(I_B/I_B^2, B) \hookrightarrow {}_0\mathrm{Ext}^1_B(I_B/I_B^2, B)$ induced by L is injective, then B_L is an unobstructed graded R_L-algebra and

$$\dim(N_{B_L})_0 = \dim(N_B)_0 - \dim(N_B)_{-1}.$$

(iv) If $_0\mathrm{Ext}^1_B(I_B/I_B^2, B) = {}_{-1}\mathrm{Ext}^2_B(I_B/I_B^2, B) = 0$, then $_0\mathrm{Ext}^1_{B_L}(I_{B_L}/I_{B_L}^2, B_L) = 0$.

Proof. (*i*) and (*ii*). Apply $\mathrm{Hom}_B(S_2(K_B), -)$ to the exact sequence $0 \to K_B \to K_B(1) \to K_{B_L} \to 0$ and use that $S_2(K_B) \otimes B_L \simeq S_2(K_{B_L})(-2)$ and that $K^*_B = \mathrm{Hom}_B(S_2(K_B), K_B)$.

(*iii*) The unobstructedness is just a very particular case of Theorem 5B and Remark 6(*b*) letting $I_{A/B} \simeq B(-1)$. To compute N_{B_L}, just apply $_0\mathrm{Hom}_B(I_B/I_B^2, -)$ to $0 \to B(-1) \to B \to B_L \to 0$. Continuing this sequence into a long exact sequence of Ext-groups, we prove (*iv*) as well because $_0\mathrm{Ext}^1_{B_L}(I_{B_L}/I_{B_L}^2, B_L) \simeq {}_0\mathrm{Ext}^1_B(I_B/I_B^2, B_L)$. $\qquad\square$

Remark 20. *(a) In the case that $\mathrm{Proj}(B)$ is a curve ($n = 2$), we have by duality*
$_v\mathrm{Ext}^2_B(S_2(K_B), K_B)^\vee \simeq {}_{-v}\mathrm{H}^0_{\mathfrak{m}}(S_2(K_B)) \hookrightarrow S_2(K_B)_{-v}$. *By Remark 14(a) there is a surjection $S_2(\otimes R(n_{c,i})) \twoheadrightarrow S_2(K_B(n+c))$. Hence $S_2(K_B)_{-v}$ vanishes if $v > 2 \max n_{c,i} - 2(n+c)$, i.e.*

$$_v\mathrm{Ext}^2_B(S_2(K_B), K_B) = 0 \text{ provided } v > 2\,\mathrm{reg}(I_B) - 2n - 2 = 2\,\mathrm{reg}(I_B) - 6$$

(b) If the curve $C = \mathrm{Proj}(B)$ is a local complete intersection, then (15) shows that $_{-1}\mathrm{Ext}^2_B(I_B/I_B^2, B) = 0$ provided $\mathrm{H}^1(\mathcal{O}_C(n_{1,i} - 1)) \simeq \mathrm{H}^0(\widetilde{K}_B(-n_{1,i} + 1))^\vee = 0$, i.e.

$$_{-1}\mathrm{Ext}^2_B(I_B/I_B^2, B) = 0 \text{ provided } \min n_{1,i} > \mathrm{reg}(I_B) - n = \mathrm{reg}(I_B) - 2.$$

Example 21. *We consider the Artinian reduction A_L of the Gorenstein algebra of Example 17. Since B is 2-dimensional we get the vanishing of $_0\mathrm{Ext}^1_{B_L}(I_{B_L}/I_{B_L}^2, B_L)$, and hence the unobstructedness of B_L, from Lemma 19(iii) and Remark 20(b). Moreover since we have seen that Remark 14(a) also applies to the Artinian reduction, we get $_{-s-1}\mathrm{Ext}^1_{B_L}(I_{B_L}/I_{B_L}^2, K_{B_L}) = 0$ for $s > 2$. Finally Lemma 19(ii) and Remark 20(a) shows that $_{v+1}\mathrm{Ext}^1_{B_L}(S_2(K_{B_L}), K_{B_L}) = 0$ for $v \geq 0$. By Remark 14(b) all assumptions of Proposition 13B concerning B_L are satisfied. It follows that A_L is an unobstructed Artinian Gorenstein algebra for $s \geq 3$, i.e. $\mathrm{PGor}(H_L)$ is smooth at (A_L) by Theorem 11 and*

$$\dim_{(A_L)} \mathrm{PGor}(H_L) = \dim(N_{B_L})_0 + \dim(K^*_{B_L})_{s+1} - 1 = 3d + d - 1 = 15$$

Note that the Hilbert function of B_L is given by $(1, 4, 4, 4, ..)$. Since the h-vector of A_L is the $(s+2)$-tuple $(1, 4, 4, ..., 4, 1)$, we get that the value of $\dim_{(A_L)} \mathrm{PGor}(H_L)$ above coincides with the dimension given in [21], Thm. 2.6 and Thm. 4.10A.

Finally we claim that this conclusion about A_L holds for any Artinian Gorenstein quotient A (well-defined by a twist of the canonical module of B) with the same graded Betti numbers as A_L. Indeed to use Remark 14(a) we only need the graded Betti numbers of the minimal resolution. Since we may prove Remark 20 for a one-dimensional B by replacing $\mathrm{Ext}^2_B(-, -)$

by the corresponding $\mathrm{Ext}_B^1(-,-)$ *(in (b) we use* $_0\mathrm{Ext}_B^1(I_B/I_B^2, B) \simeq \ _0\mathrm{Ext}_B^1(I_B \otimes_R K_B, K_B) \simeq$ $_0\mathrm{H}_{\mathfrak{m}}^0(I_B \otimes_R K_B)^\vee$ *in order to see that this group vanishes for* $\min n_{1,i} > \mathrm{reg}(I_B) - 1$*), we get the claim.*

Remark 22. *Pushing the arguments of this example a little further, we get that if* $Y = \mathrm{Proj}(B)$ *is a zero-scheme of degree* d *and if* $v \geq 2\,\mathrm{reg}(I_B) - 3$*, then* $_v\mathrm{Ext}_B^1(S_2(K_B), K_B) = 0$ *and* $\dim(K_B^*)_v = d$*. Indeed with* v *as above we have* $S_2(K_B)_{-v} = 0$ *and* $_{-v}\mathrm{H}_{\mathfrak{m}}^0(S_2(K_B)) = 0$ *(cf. Remark 20(a)) and it follows that* $_v\mathrm{Ext}_B^1(S_2(K_B), K_B) = 0$ *by Gorenstein duality. Hence it suffices to show* $(K_B^*)_v \simeq \mathrm{H}^0(Y, \widetilde{K_B^*}(v))$*, i.e. to show* $_v\mathrm{H}_{\mathfrak{m}}^1(K_B^*) = 0$*. By Gorenstein duality this group is dual to* $_{-v}\mathrm{Hom}_B(K_B^*, K_B) = \ _{-v}\mathrm{Hom}_B(\mathrm{Hom}_B(S_2(K_B), K_B), K_B) \simeq S_2(K_B)_{-v} = 0$*. In this case note that* $(K_B^*)_v$ *contains a regular section. In particular if* $s \geq 2\,\mathrm{reg}(I_B)$ *and* B *is unobstructed, we have by Proposition 13B and Remark 14(a) that* A *is unobstructed and* H_B*-generic and that*

$$\dim_{(A)} \mathrm{PGor}(H_A) = \ \dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim(N_B)_0 + d - 1$$

cf. Theorem 11. Hence we may replace "$s >> 0$*" in Theorem 1 (and in Theorem 16 provided* B *is generically syzygetic) by "*$s \geq 2\,\mathrm{reg}(I_B)$*".*

Now we will study the case $s >> 0$ of Theorem 16 to see that (13) actually determines a well-defined injective application from the set of irreducible components of $\mathrm{GradAlg}^{H_B}(R)$ containing locally licci, Cohen-Macaulay codimension c quotients of R (not necessarily unobstructed !), to the set of irreducible components of $\mathrm{GradAlg}^{H_A}(R)$ containing Gorenstein codimension $c+1$ quotients of R. To be precise let $U_s \subset \mathrm{GradAlg}^{H'}(R)$ *be an open (e.g. the largest open) subscheme whose corresponding* k*-points* $(R \to B)$ *are such that* B *is Cohen-Macaulay and generically Gorenstein,* $\mathrm{Proj}(B)$ *is locally licci and satisfies* $H_B = H'$*,* $_s\mathrm{Ext}_B^1(S_2(K_B), K_B) = 0$ *and* $(K_B^*)_s$ *contains a regular section.* When we apply our results we usually take U_s to be the open set in which "B is generically Gorenstein and $\mathrm{Proj}(B)$ is locally licci" is replaced by "$\mathrm{Proj}(B)$ is locally a complete intersection" or " $\mathrm{Proj}(B)$ is smooth" or " $\mathrm{Proj}(B)$ is locally Gorenstein and the codimension of B is $c = 3$", and with s so large such that every such Cohen-Macaulay quotient B satisfies $_s\mathrm{Ext}_B^1(S_2(K_B), K_B) = 0$ and $(K_B^*)_s$ contains a regular section (cf. Remark 18 and 22). If we take a point $(R \to B)$ of U_s and two particular quotients $B \to A_1$ and $B \to A_2$ (corresponding to two regular sections σ_i of $(K_B^*)_s$), we have $I_{A_1/B} \simeq I_{A_2/B}$ as graded B-modules. Recalling that the $_0\mathrm{H}^2(B, A, A)$-group of Lemma 7 is given by $_s\mathrm{Ext}_B^1(S_2(K_B), K_B)$ (Lemma 15), we can take the open set U of Lemma 7 to contain $(B \to A_2)$ provided it contains $(B \to A_1)$. Hence we can let U be $q^{-1}(U_s)$ intersected by the space of those quotients $(B \to A)$ which correspond to the regular sections of $(K_B^*)_s$ where $q : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}^{H_B}$ is the first projection. Calling this intersection $q^{-1}(U_s)_{reg}$, we have an incidence correspondence (cf. the text before (10))

$$q^{-1}(U_s)_{reg} \xrightarrow{q_{res}} U_s \subset \mathrm{GradAlg}^{H_B}(R)$$
$$\downarrow^{p_{res}} \tag{18}$$
$$\mathrm{GradAlg}^{H_A}(R)$$

of restricted projection morphisms in which q_{res} and p_{res} have nice properties. Note that $v = n - \mathrm{reg}(I_B)$, $n = \dim B$, is the largest integer such that $(K_B)_v = 0$ by Remark 14(a). Letting $n_{j,i}$ be the number appearing in the minimal resolution (6) of B, we get

Proposition 23. *With notations as above, we have*

i) The morphism q_{res} in (18) is smooth and its fibers are geometrically connected, of fiber dimension $\dim(K_B^*)_s - 1$.

ii) If $U_{n_{1i}}$ is the open subset of U_s whose corresponding k-points (B) satisfy $\max n_{1,i} \leq n - \mathrm{reg}(I_B) + s$, then $p^{-1}(p_{res}(q_{res}^{-1}(U_{n_{1i}}))) = q_{res}^{-1}(U_{n_{1i}})$ and p_{res} is unramified and injective on $q_{res}^{-1}(U_{n_{1i}})$. Moreover if $s \geq 2\,\mathrm{reg}(I_B) - n$, then $U_{n_{1i}} = U_s$.

iii) If $U_{n_{2i}}$ is the open subset of U_s whose k-points (B) satisfy $\max n_{2,i} \leq n - \mathrm{reg}(I_B) + s$, then $p^{-1}(p_{res}(q_{res}^{-1}(U_{n_{2i}}))) = q_{res}^{-1}(U_{n_{2i}})$ and the restriction of p_{res} to $q_{res}^{-1}(U_{n_{2i}})$ is an isomorphism onto an open subscheme of $\mathrm{GradAlg}^{H_A}(R)$. In particular if $s \geq 2\,\mathrm{reg}(I_B) - n + 1$, then $U_{n_{2i}} = U_s$ and the morphism p_{res} in (18) is an isomorphism onto an open subscheme of $\mathrm{GradAlg}^{H_A}(R)$.

Proof. i) To see the smoothness of q_{res}, we combine Proposition 4(i) and Lemma 15 (see also the proof of Proposition 13). To show that the fibers of q_{res} are (geometrically) connected, it suffices to see that two quotients $B \to A_1$ and $B \to A_2$, given as in (13) by two regular sections σ_i of $(K_B^*)_s$, correspond to two k-points of a flat irreducible family $U' \subset \mathbb{A}^1(k) = \mathrm{Spec}(k[t])$ contained in the fiber $q_{res}^{-1}((B))$. This is, however, quite trivial because we can easily show that $t\sigma_1 + (1-t)\sigma_2$ is a regular section of $(K_{B[t]}^*)_s$ for any t in some open set of $\mathbb{A}^1(k)$ containing $t = 0$ and $t = 1$.

ii) It is straightforward to get ii) from Lemma 7(b) because $K_B(-s)_{\max n_{1,i}} = 0$ by assumption.

iii) Note that p is universally injective ("radiciel") on the set U of Lemma 7 because the proof of the injectivity of $p|_U$ in Lemma 7(b) extends to F-valued points where F is any field extension of k. Hence by [13], Thm. 17.9.1, it suffices to prove that p_{res} is etale on $q_{res}^{-1}(U_{n_{2i}})$. By (ii) above, it suffices to prove smoothness. Looking to Remark 14(a), we see that $_{-s}\mathrm{Ext}_B^1(I_B/I_B^2, K_B) = 0$ by assumption. The first sentence of the proof of Theorem 16 shows that I_B is generically syzygetic. Hence we get the smoothness of p_{res} by Proposition 4(ii) and the proof of Theorem 5B. $\qquad\square$

Recall that a quotient A of R has the Weak Lefschetz property if there is a linear form l and an Artinian reduction $A_L := A/(L)$ of A such that the multiplication $l : (A_L)_v \to (A_L)_{v+1}$ is either injective or surjective for every v (cf. the text before (17)). Letting "general member or k-point" mean any member or k-point in a suitable open dense subset, we get

Theorem 24. *Let $s \geq 2\,\mathrm{reg}(I_B) - n + 1$. Then the correspondence (18) determines a well-defined injective application π from the set of irreducible components W of U_s (whose members B are codimension c Cohen-Macaulay quotients of R satisfying the defining conditions of U_s above), to the set of irreducible components V of $\mathrm{GradAlg}^{H_A}(R)$ whose general members are Gorenstein codimension $c+1$ quotients A of R satisfying the Weak Lefschetz property. In this application the generically smooth components correspond. Indeed if m_W (resp. m_V) is the maximal ideal of the local ring of a general k-point of W (resp. of $V = \pi(W)$), then $\dim m_V/m_V^2 - \dim V = \dim m_W/m_W^2 - \dim W$, and we have*

$$\dim V = \dim W + \dim(K_B^*)_s - 1.$$

Proof. By Proposition 23(i) and [17], Prop. 1.8, $q_{res}^{-1}(W)$ is an irreducible component of $q^{-1}(U_s)_{reg}$. The application is therefore well-defined, and it is injective by Proposition 23(iii). Since q_{res} is smooth and p_{res} is an open embedding, we easily get the dimension formulas.

It remains to see that A has the Weak Lefschetz property. Indeed we *claim* that any A' which corresponds to a point in the image $p_{res}(q^{-1}(U_s)_{reg})$ has the Weak Lefschetz property. First suppose $\dim B' = 1$, with $(B' \to A') \in q^{-1}(U_s)_{reg}$. Since $\operatorname{reg}(I_{B'} = \sigma(B')$ where $\sigma(B')$ is defined by $min\{i|\Delta H_{B'}(i) = 0\}$ (e.g. [35], Sect. 2), it follows that the multiplication by a regular element of B' of degree one induces a bijection $(B')_{t-1} \to (B')_t$ for $t \geq \operatorname{reg}(I_{B'}$, hence $(A')_{t-1} \to (A')_t$ is surjective for $t \geq \operatorname{reg}(I_{B'}$. Let $t < \operatorname{reg}(I_{B'}$. Since the assumption $s \geq 2\operatorname{reg}(I_B) - n + 1$ leads to $K_{B'}(-s)_v = 0$ for $v \leq \operatorname{reg}(I_{B'} + 1$, we get $(B')_v \simeq (A')_v$ for $v \leq \operatorname{reg}(I_{B'} + 1$ and certainly for $v < \operatorname{reg}(I_{B'}$. The latter is precisely what we need to get the injectivity of $(A')_{t-1} \to (A')_t$ from the injectivity of $(B')_{t-1} \to (B')_t$. Hence A' has the Weak Lefschetz property. Moreover if $n = \dim B' > 1$ we know that the graded Betti numbers of B' and of $K_{B'}(-n)$ are unchanged by modding out by a $(n-1)$-dimensional space L' of general linear forms (cf. the explanation to (17)). Hence we may apply the argument above to the one dimensional quotient $B'/(L')$, and we are done. $\qquad\square$

Remark 25. *(a) Let $\dim B = 1$, i.e. let $\operatorname{Proj}(B)$ be a zero-scheme of degree d and let $s \geq 2\operatorname{reg}(I_B)$. Then we automatically get the vanishing of the Ext^1-group appearing in the definition of U_s by Remark 22. Since the quotients of U_s satisfy $K_B(-s)_{\operatorname{reg}(I_B)+1} = 0$, we get $\dim B_{\operatorname{reg}(I_B)+1} = \dim A_{\operatorname{reg}(I_B)+1}$. Hence we see that the Hilbert function of each A in Theorem 24 is an unimodal SI-sequence with at least three consecutive peaks, that is, $H_A \supset (d,d,d)$ (because A has the Weak Lefschetz property, see e.g. [35]). Moreover*

$$\dim V = \dim W + d - 1.$$

Indeed we really have by Proposition 23(iii) much more, namely a smooth morphism $q_{res} \circ p_{res}^{-1}$ of schemes defined on $p_{res}(q^{-1}(U_s)_{reg})$ with values in $\operatorname{GradAlg}^{H_B}(R)$, allowing us to compare properties of $\operatorname{GradAlg}^{H_B}(R)$ at (B) with the corresponding properties of $\operatorname{GradAlg}^{H_A}(R)$ at (A) beyond that done in Theorem 24. Note that this correspondence is well studiet by Iarrobino-Kanev if the codimension of B is $c = 2$ ([21], Sect. 5.3.2), in which case they get a topological fibration defined on the whole parameter space $\operatorname{PGor}(H_A)$. In that case they also get the dimension formula above ([21], Cor. 5.50). Since the points of $p_{res}(q^{-1}(U_s)_{reg}) \subset \operatorname{PGor}(H_A)$ satisfy the Weak Lefschetz property, we can not in general extend the morphism $q_{res} \circ p_{res}^{-1}$ to the whole of $\operatorname{PGor}(H_A)$ (cf.[3])!

(b) Let $\dim B = 1$ and suppose $s \geq 2\operatorname{reg}(I_B) - 1$. Using the arguments of (a) it follows that the h-vector of A is an SI-sequence with at least two peaks. By Proposition 23(ii) the correspondence (18) applies provided the points (B) of U_s (or of $U_{n_{1i}}$ if we only assume $\max n_{1,i} \leq 1 - \operatorname{reg}(I_B) + s)$ satisfy the assumptions of Proposition 13A. These assumptions are satisfied if B is licci (Remark 14(c)), or if I_B is generically syzygetic (e.g. if $c \leq 3$) and satisfies

$$(\operatorname{H}_2(R,B,B) \otimes_B A)_s = 0 \text{ and } {}_0\operatorname{H}^0_{\mathfrak{m}}(I_B \otimes_R K_B) = 0$$

(Remark 14(b)) because the latter vanishing is equivalent to ${}_0\operatorname{Ext}^1_B(I_B/I_B^2, B) = 0$ by Gorenstein duality. Hence we get by Proposition 13A and Theorem 11 that to each irreducible component W of say U_s whose general member is licci or as above, then (18) determines an irreducible closed subset V of codimension ${}_{-s}\operatorname{ext}^1_B(I_B/I_B^2, K_B)$ in $\operatorname{PGor}(H_A)$ such that $\operatorname{PGor}(H_A)$ is generically smooth along V, and $\dim V = \dim W + d - 1$. If $c = 2$ this result is known ([21], Sect. 5.3.3 and Cor. 5.50).

Example 26. *Here we will use Theorem 24 and Proposition 8 to give classes of examples in which the scheme* $\text{GradAlg}^H(R)$ *has several irreducible components. To simplify the analysis, we will use the classification of ACM (arithmetically Cohen-Macaulay) curves on a general rational ACM surface appearing in [29]. So our examples will yield irreducible components of* $\text{GradAlg}^H(R)$ *of Gorenstein codimension 4 quotients of dimension one. It is not obvious that the corresponding Artinian reductions lead to irreducible components of* $\text{PGor}(H)$ *of Artinian Gorenstein codimension 4 quotients.*

First we explicitly describe three irreducible components of $\text{Hilb}^p(\mathbb{P}^4)$ *with the same Hilbert function (in fact their graded Betti numbers are the same, and are the minimum possible consistent with the Hilbert function). Recalling that a general Castelnuovo surface* $F \subset \mathbb{P}^4$ *is obtained by blowing up 8 general points in* $\mathbb{P}^2$, *thus that* $\text{Pic}(F) \simeq \mathbb{Z}^9$, *we write the hyperplane section and the canonical divisor as* $H = (4; 2, 1^7)$ *and* $K = (-3; -1^8)$ *respectively. Consider the three families of curves,* $C \in |C_0 + tH|$, *of [29], p.73 given by*

$$(4t + 1; 2t, t^7), (4t + 2; 2t, t + 1^4, t^3), (4t + 3; 2t, t + 2, t + 1^6)$$

having $C_0 \cdot K = (-3, -2, -1)$ *respectively. We always have* $(d_0, g_0) := (deg(C_0), gen(C_0)) = (4, 0)$ *and* $H^2 = 5$. *By Proposition 8 any such* C *is unobstructed provided* $t \geq 3$ *and*

$$\dim_{(C)} \text{Hilb}^p(\mathbb{P}^4) = 31 - C_0 \cdot K + (11t + 5t^2)/2 \ ,$$

where p *is the Hilbert polynomial of* C. *Since the three families have different* $C_0 \cdot K$, *they must correspond to three different components of* $\text{Hilb}^p(\mathbb{P}^4)$.

We claim that all curves $C = \text{Proj}(B)$ *of the three families have the same Hilbert function, and hence that their quotients* B *belong to the same* $\text{GradAlg}^H(R)$. *Indeed looking to the sequence*

$$0 \to \mathcal{J}_{C/F} \simeq \mathcal{O}_F(-C) \to \mathcal{O}_F \to \mathcal{O}_C \to 0,$$

and the corresponding one for C_0, *we get* $h^1(\mathcal{O}_C(v)) = h^1(\mathcal{O}_{C_0}(v - t)) = 0$ *for* $v \geq t$. *It is well known that the homogeneous ideal of* F *has three minimal generators of degrees* $2, 3, 3$ *and two relations of degrees* $4, 4$. *Combining* $h^0(\mathcal{J}_{C/F}(v)) = h^0(\mathcal{J}_{C_0/F}(v - t))$ *and* $s(C_0/F) = 2$ *with* $h^1(\mathcal{O}_C(t)) = 0$, *we see that the additional minimal generators of* I_B *sit all in degree* $t + 2$. *Hence we get that all* C *of the three families have the same Hilbert function and the same regularity,* $\text{reg}(I_B) = t + 2$, *i.e. we get the claim. Indeed the members of the three families satisfy* $H_B(v) = h^0(\mathcal{O}_C(v)) = dv + 1 - g$ *for* $v \geq t$ *and* $H_B(v) = h^0(\mathcal{O}_F(v))$ *for* $v < t$, *and we easily find the minimal resolution to be*

$$0 \to R(-t-4)^5 \to R(-t-3)^{10} \oplus R(-4)^2 \to R(-t-2)^5 \oplus R(-3)^2 \oplus R(-2) \to I_B \to 0.$$

Now, thanks to (9), the corresponding Cohen-Macaulay algebras B *are unobstructed as graded algebras and* $\dim_{(B)} \text{GradAlg}^H(R) = \dim_C \text{Hilb}^p(\mathbb{P}^4)$. *The three families must therefore correspond to three generically smooth irreducible components of* $\text{GradAlg}^{H_B}(R)$ *of different dimensions whose general members* B *are two-dimensional Cohen-Macaulay codimension 3 quotients of* R.

By Theorem 24, we get three generically smooth irreducible components V *of* $\text{GradAlg}^{H_A}(R)$ *whose general members are one-dimensional Gorenstein codimension 4 quotients* A *of* R. *Indeed Theorem 24 applies for* $s > 2\text{reg}(I_B) - \dim B = 2t + 2$ *provided we can show*

$_s\mathrm{Ext}_B^1(S_2(K_B), K_B) = 0$ *for such* s. *Looking to the arguments of Remark 18 it suffices to prove* $s > (4g-4)/d$ *which is straightforward since* $(d_0, g_0) = (4, 0)$, $d = d_0 + 5t$ *and* $2g = 2g_0 + 2td_0 + 5t^2 - 3t$ *by the adjunction formula* $2g - 2 = C \cdot (C + K)$. *Now, inserting these values for* d *and* g *into the expression* $\dim(K_B^*)_s = ds + 3 - 3g$ *of Remark 18, we get, for every* $t \geq 3$ *and every* $s > 2t + 2$, *three irreducible components* V *of* $\mathrm{GradAlg}^{H_A}(R)$ *satisfying*

$$\dim V = \dim_C \mathrm{Hilb}^p(\mathbb{P}^4) + \dim(K_B^*)_s - 1 = 33 - 2t - 5t^2 - C_0 \cdot K + s(4 + 5t).$$

It is not difficult to compute the Hilbert function H_A *from the Hilbert function* H_B. *Indeed the h-vector of* A, *i.e. the h-vector of an Artinian reduction of* A, *is just*

$$(1, 4, 9, 14, 19, ..., 4 + 5t, 4 + 5t, 4 + 5t, ..., 19, 14, 9, 4, 1).$$

The simplest case $t = 3$ *and* $s = 9$ *yields the h-vector* $(1, 4, 9, 14, 19, 19, 19, 14, 9, 4, 1)$.

Remark 27. *We can use the classification of ACM curves on a general rational ACM surface in* $\mathbb{P}^4$ *in [29] to get many more reducible schemes* $\mathrm{GradAlg}^{H_A}(R)$ *whose general members are Gorenstein codimension 4 algebras. For instance on a Castelnuovo surface, look to the linear systems,* $C \in |C_0 + tH|$, *of [29], p.73 where* C_0 *is a minimal curve and* $t \geq 0$. *For both* $(d_0, g_0) = (5, 1)$ *and* $(6, 2)$ *there are at least two linear systems with different* $C_0 \cdot K$. *Here* $s(C_0/F) = 2$ *and Proposition 8 applies for* $t \geq 3$. *Since* $\mathrm{H}^1(\mathcal{O}_{C_0}(1)) = 0$ *one may see that* $\mathrm{reg}(I_B) = t + 3$ *(* $C = \mathrm{Proj}(B)$ *) and that Theorem 24 applies for every* $s > 2t + 4$. *The h-vectors of the simplest case* $t = 3$ *and* $s = 11$ *of the two families are*

$$(1, 4, 9, 14, 19, 20, 20, 20, 19, 14, 9, 4, 1) \text{ and } (1, 4, 9, 14, 19, 21, 21, 21, 19, 14, 9, 4, 1).$$

For both $(d_0, g_0) = (2, 0)$ *and* $(3, 0)$ *there are also two linear systems with different* $C_0 \cdot K$. *Now* $s(C_0/F) = 1$ *and* $\mathrm{H}^1(\mathcal{O}_{C_0}) = 0$ *and Proposition 8 applies for* $t \geq 4$ *and Theorem 24 for* $s > 2\mathrm{reg}(I_B) - 2 = 2t + 2$. *The h-vectors of the simplest case* $t = 4$ *and* $s = 11$ *are*

$$(1, 4, 9, 14, 19, 22, 22, 22, 19, 14, 9, 4, 1) \text{ and } (1, 4, 9, 14, 19, 23, 23, 23, 19, 14, 9, 4, 1).$$

Putting the analysis on a Castelnuovo surface together we get that $\mathrm{GradAlg}^{H_A}(R)$ *has at least two generically smooth irreducible components whose general members are one-dimensional Gorenstein codimension* 4 *quotients* A *of* R *in the following range of the h-vector;*

$$(1, 4, 9, 14, ..., 5\alpha - 1, \beta, \beta, ..., \beta, 5\alpha - 1, 5\alpha - 6, ..., 9, 4, 1)$$

where $5\alpha - 1 \leq \beta \leq 5\alpha + 4$, $\alpha \geq 3$, $\beta \geq 19$, *and with at least three consecutive* $\beta's$ *(three peaks).*

Of course the part $(1, 4, 9, 14, ..., 5\alpha - 1)$ *of the h-vector comes from the Castelnuovo surface. On a cubic surface there are two linear systems with* $(d_0, g_0) = (1, 0)$ *and with different* $C_0 \cdot K$, *giving rise to reducible* $\mathrm{GradAlg}^{H_A}(R)$ *for every* $t \geq 4$ *and* $s > 2t$. *For* $(t, s) = (3, 7)$ *the h-vector is* $(1, 4, 7, 10, 10, 10, 7, 4, 1)$. *Increasing* t *to* 4 *and* s *to 10 we get precisely the h-vector of Boij's example [3], which is different from ours because our algebras have the Weak Lefschetz property (see also [22], Thm. 3.9). There are also quite a lot of linear systems on a Bordiga surface with which we could make a similar analysis. All examples we get in this case have a symmetric h-vector starting with* $(1, 4, 10, 16, 22, ...)$.

We end this section by using Theorem 16 to essentially reprove the smoothness and give a formula for the dimension of $\mathrm{GradAlg}^H(R)$ for (e.g. Artinian) Gorenstein codimension 3 quotients equivalent to those in [27], [28] or [6].

Example 28. *Recall that if $A \simeq R/I_A$ is Gorenstein of codimension 3 with minimal resolution*

$$0 \to R(-f) \simeq F_3 \to F_2 \to F_1 \to I_A \to 0. \tag{19}$$

one knows that A is unobstructed and that $N_A \simeq \wedge^2 I_A(f)$ (char(k) $\neq$ 2). It follows that

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim(\wedge^2 F_1)_f - \dim(F_1 \otimes F_2)_f + \dim(S_2(F_2))_f \tag{20}$$

cf. [28], Thm. 2.6 and its Remark. In (20) it is not really necessary to suppose (19) to be minimal because we easily see that (20) is invariant under adding a common free factor to both F_1 and F_2.

If $B \simeq R/I_B$ is Cohen-Macaulay of codimension 2 (and hence licci), and A is given as in Theorem 16, then we claim *the dimension formula for $\dim_{(A)} \mathrm{GradAlg}^{H_A}(R)$ in Theorem 16 coincides with (20). Note that since every irreducible component of the Hilbert scheme of Gorenstein codimension 3 quotients of R contains a quotient constructed via (13) by [10], this example proves the generic smoothness of $\mathrm{GradAlg}^H(R)$ and $\mathrm{PGor}(H)$, as well as (20). The argument holds in the case char(k) = 2 as well.*

First we compute $\dim(K_B^*)_s$. Indeed the dual of (8) (for $c = 2$) yields the exact sequence

$$0 \to K_B^*(-n-c) \to G_2 \otimes B \to G_1 \otimes B \to I_B/I_B^2 \to 0 \tag{21}$$

in which H_1 is the kernel of $G_1 \otimes B \to I_B/I_B^2$ by (4). By [2] one knows that

$$0 \to \wedge^{i+1} G_2 \to \wedge^{i+1} G_1 \to \wedge^i G_2 \to \mathrm{H}_i \to 0 \tag{22}$$

is exact for any i. Let $f = s + n + c$. Using the last mentioned sequence for $i = 1$, (21) and the R-free resolution of $G_2 \otimes B$ which we easily deduce from (6), we get

$$\dim(K_B^*)_s = \dim(\wedge^2 G_1)_f - \dim(G_1 \otimes G_2)_f + \dim(S_2(G_2))_f. \tag{23}$$

Next we will compute $\delta = \delta(B)_{-s} = \dim(G_1^* \otimes_R K_B)_{-s} - {}_{-s}\mathrm{hom}_B(\mathrm{H}_1, K_B)$, cf. Remark 14(d). If $r = \mathrm{rank}\,\mathrm{H}_1$, we have by (22) a resolution of H_{r-1} leading to

$$0 \to G_2^* \to (\wedge^2 G_1)^* \to (\wedge^2 G_2)^* \to \mathrm{H}_{r-1} \otimes (\wedge^{r+1} G_2)^* \simeq \mathrm{Hom}(\mathrm{H}_1, K_B(n+c)) \to 0 \tag{24}$$

We have an R-free resolution of K_B by dualizing (6) and hence a resolution of $G_1^* \otimes K_B$. Combining with (24), we get

$$-\delta = \dim(G_1 \otimes G_2)^*_{-f} - \dim(S_2(G_1)^*)_{-f} - \dim(\wedge^2 G_2)^*_{-f} + \dim(G_1)^*_{-f} - \dim(G_2)^*_{-f}. \tag{25}$$

Note that $\dim(N_B)_0$ is well known, e.g. one may easily deduce it from the exact sequence

$$0 \to G_1^* \otimes G_2 \to ((G_1^* \otimes G_1) \oplus (G_2^* \otimes G_2))/k \to G_2^* \otimes G_1 \to N_B \to 0 \tag{26}$$

of [30], Remark 16 (k is a 1-dimensional subspace). Putting these formulas together, using Theorem 16, we get exactly (20) because we find the following resolution of I_A,

$$0 \to R(-f) \to G_1^*(-f) \oplus G_2 \to G_2^*(-f) \oplus G_1 \to I_A \to 0. \tag{27}$$

by applying the mapping cone construction onto (13).

Inside the scheme Z(H) of Remark 10 we have the open subscheme ZGor(H) consisting of *not necessarily graded* Gorenstein quotients $R \to A$ with Hilbert function H. ZGor(H) has been studied in different contexts (see [21], p.126 for references). Since we by Remark 14(c) easily see that $_v\mathrm{H}^2(R, A, A) = 0$ for $v \geq 0$ provided B is licci, we can use Remark 10 to prove the smoothness and find the dimension of the scheme ZGor(H) at (A). Indeed $_v\hom_R(I_A, A)$ is found using (10) as in the proof of Theorem 5. And if we apply $_v\mathrm{Hom}(K_B(-s), -)$ to (13), we get $_v\hom(K_B(-s), A) = \dim(K_B^*)_{s+v} - \dim B_v$. Since $A_v = 0$ for $v > s$, we have

Proposition 29. *Let $B \simeq R/I_B$ be licci and generically Gorenstein, and let A be a graded codimension one quotient of B, defined by (13) for some s. Then* ZGor(H) *is smooth at* $(R \to A)$ *and*

$$\dim_{(A)} \mathrm{Z(H)} = \sum_{v \geq 0}^{s} \left(\dim(N_B)_v - \delta(B)_{-s+v} \right) + \sum_{v \geq 0}^{s} \left(\dim(K_B^*)_{s+v} - \dim B_v \right).$$

Invoking Remark 14(c), we take the opportunity to state the main result of this section in the non graded case. The proof is omitted because it is the same as in the graded case.

Theorem 30. *Let $B \simeq R/I_B$ be Cohen-Macaulay, generically Gorenstein and generically syzygetic of dimension ≥ 1 and suppose B satisfies*

$$\mathrm{Ext}_B^1(I_B/I_B^2, B) = 0 \ , \quad \mathrm{Ext}_B^1(S_2(K_B), K_B) = 0 \quad and \quad \mathrm{Ext}_B^2(I_B/I_B^2, K_B) = 0.$$

Consider K_B as an ideal of B and let $A := B/K_B$. Then A is Gorenstein and strongly unobstructed in the sense $\mathrm{H}^2(R, A, A) = 0$.

In particular let $B \simeq R/I_B$ be licci and generically Gorenstein. Then A is Gorenstein and strongly unobstructed in the sense $\mathrm{H}^2(R, A, A) = 0$. Indeed licci and generically Gorenstein imply Cohen-Macaulay and generically syzygetic as well as the vanishing of the three Ext-groups above.

3   Computing the dimension via liaison

In this section we show how we can compute $\delta(B)_{-s} = \, _{-s}\hom_R(I_B, K_B) - \, _{-s}\mathrm{ext}_B^1(I_B/I_B^2, K_B)$ and $\dim(K_B^*)_s$. For instance it turns out that the difference $\dim(K_B^*)_s - \delta(B)_{-s}$ (which appears in Theorem 16) is quite easy to compute because, after adding a simple term to it, it becomes a liaison invariant among syzygetic ideals I_B. Moreover $\delta(B)_{-s}$ is computable in terms of $\dim(K_{B'}^*)_s$ of the linked algebra B'. To allow any complete intersection linkage of Cohen-Macaulay quotients and still get the liaison invariance described above, we need to define $\delta(B)$ also for Cohen-Macaulay quotients $R \to B$ which satisfy $\mathrm{H}_2(R, B, B) \neq 0$ generically. To do this we consider the exact sequence ([38], Sect. 2.1)

$$0 \to \mathrm{H}_2(R, B, B) \to S_2 I_B \to I_B \to I_B/I_B^2 \to 0. \tag{28}$$

Let $\psi : \mathrm{H}_\mathfrak{m}^n(\mathrm{H}_2(R, B, B)) \to \mathrm{H}_\mathfrak{m}^n(S_2 I_B)$, $\dim B = n$, be the induced map.

Definition 31. *Let $B = R/I_B$ be Cohen-Macaulay of dimension n, not necessarily generically syzygetic. Then*

$$\delta^*(B)_v = {}_v\mathrm{hom}_R(I_B, K_B) - {}_v\mathrm{ext}^1_B(I_B/I_B^2, K_B) - \dim(\mathrm{im}\,\psi)_{-v}$$

Remark 32. *If B is generically syzygetic, then ${}_v\mathrm{Hom}_B(\mathrm{H}_2(R, B, B), K_B) = 0$, and we get $_{-v}\mathrm{H}^n_\mathfrak{m}(\mathrm{H}_2(R, B, B)) = 0$ by duality and hence $\delta^*(B)_v = \delta(B)_v$, i.e. the "old and the new" definition of δ coincides.*

Using (28) and ${}_v\mathrm{Ext}^i_B(I_B/I_B^2, K_B) \simeq {}_{-v}\mathrm{H}^{n-i}_\mathfrak{m}(I_B/I_B^2)^\vee$ and letting $h^j_\mathfrak{m}(-)_v = \dim({}_v\mathrm{H}^j_\mathfrak{m}(-))$, one shows

$$\delta^*(B)_v = h^{n+1}_\mathfrak{m}(S_2I_B)_{-v} - h^n_\mathfrak{m}(S_2I_B)_{-v} - h^{n+1}_\mathfrak{m}(I_B)_{-v} \tag{29}$$

Indeed splitting (28) into two short exact sequences we see that $h^{n-1}_\mathfrak{m}(I_B/I_B^2)_{-v} = h^n_\mathfrak{m}(I_B^2)_{-v}$ and $h^n_\mathfrak{m}(I_B/I_B^2)_{-v} = h^{n+1}_\mathfrak{m}(I_B^2)_{-v} - h^{n+1}_\mathfrak{m}(I_B)_{-v}$. Moreover $h^{n+1}_\mathfrak{m}(I_B^2)_{-v} = h^{n+1}_\mathfrak{m}(S_2I_B)_{-v}$ and $h^n_\mathfrak{m}(I_B^2)_{-v} = h^n_\mathfrak{m}(S_2I_B)_{-v} - \dim(\mathrm{im}\,\psi)_{-v}$; thus we get (29) from Definition 31.

Proposition 33. *Let R be a finitely generated polynomial k-algebra with canonical module $R(e)$ (so $e = -n - c$), and let B and B' be graded Cohen-Macaulay quotients, algebraically linked by a complete intersection $R \twoheadrightarrow D$ of dimension $n \geq 1$, and with minimal resolution*

$$0 \to R(-a) \to \ldots \to \oplus_{i=1}^c R(-c_i) \to I_D \to 0.$$

Let $\beta(B, D)_v := \dim B_v + \sum_{i=1}^c \dim(I_{B/D})_{v+c_i}$. Then $I_{B/D} \simeq K_{B'}(-e-a)$ and

(i) $\dim(N_B)_0 - \sum_{i=1}^c \dim B_{c_i} = \dim(N_{B'})_0 - \sum_{i=1}^c \dim B'_{c_i}$

(ii) If B and B' are generically Gorenstein (resp. Gorenstein in codimension ≤ 1), then ${}_v\mathrm{Ext}^i_B(I_B/I_B^2, B)$ and ${}_v\mathrm{Ext}^i_{B'}(I_{B'}/I_{B'}^2, B')$ are isomorphic for $i = 1$ (resp. $i = 2$).

(iii) $\delta^*(B)_{v-2e-2a} = \beta(B', D)_{v-e-a} - \dim(K^*_{B'})_v$ for any v. Moreover if $\beta(B', D)_{v-e-a} = 0$ and I_B is generically syzygetic, then ${}_{v-2e-2a}\mathrm{ext}^1_B(I_B/I_B^2, K_B) = \dim(K^*_{B'})_v$.

(iv) $\dim(K^*_B)_v - \delta^*(B)_{-v} - \beta(B, D)_{v-e-a} = \dim(K^*_{B'})_{-v+2e+2a} - \delta^*(B')_{v-2e-2a} - \beta(B', D)_{-v+e+a}.$

(v) If I_B is generically syzygetic (resp. syzygetic in codimension ≤ 1), then ${}_v\mathrm{Ext}^2_B(I_B/I_B^2, K_B) \hookrightarrow {}_{v+2e+2a}\mathrm{Ext}^1_{B'}(S_2(K_{B'}), K_{B'})$ is injective (resp. an isomorphism).

(vi) The conclusion of (ii) holds if B is linked to B' in possibly several steps. Here the algebras of the intermediate steps need not necessarily be generically Gorenstein.

Proof. *(i)* is proved in [27], Prop. 1.7. *(ii)* Also this part, i.e. the liaison invariance of $I_B/I_B^2 \otimes K_B$, is known ([5], or [29], Ch. 6). Here the Gorenstein assumption is required for having ${}_v\mathrm{Ext}^i_B(I_B/I_B^2, B) \simeq {}_v\mathrm{Ext}^i_B(I_B/I_B^2 \otimes K_B, K_B)$. Note that the liaison invariance of $I_B/I_B^2 \otimes K_B$ takes care of *(vi)* as well.

To prove *(iii)*, recall that $K_D \simeq D(e+a)$ and hence, $I_{B/D} \simeq \mathrm{Hom}_D(B, D) \simeq K_{B'}(-e-a)$, i.e. we get exact sequences

$$0 \to I_D \cdot I_B \to S_2I_B \to S_2(K_{B'})(-2e - 2a) \to 0 \tag{30}$$

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$$0 \to I_D \cdot I_B \to I_D \to I_D \otimes_R R/I_B \simeq I_D/I_D^2 \otimes_D B \simeq \oplus_{i=1}^c B(-c_i) \to 0, \qquad (31)$$

cf. [38], Lemma 4.2.7 to see (30). Since $\beta(B', D)_v := \dim B'_v + \sum_{i=1}^c \dim(I_{B'/D})_{v+c_i}$ and $I_{B'/D} \simeq K_B(-e-a)$, we get $\beta(B', D)_v = h_{\mathfrak{m}}^n(K_{B'})_{-v} + \sum_i h_{\mathfrak{m}}^n(B)_{e-v+a-c_i}$ by Gorenstein duality. Using (29) and (30), we have $\dim(K_{B'}^*)_v = h_{\mathfrak{m}}^n(S_2 K_{B'})_{-v} = -\delta^*(B)_{-2e+v-2a} - h_{\mathfrak{m}}^{n+1}(I_B)_{2e-v+2a} + h_{\mathfrak{m}}^{n+1}(I_D I_B)_{2e-v+2a}$ while (31) and $h_{\mathfrak{m}}^n(K_{B'})_{-v+e+a} = h_{\mathfrak{m}}^{n+1}(I_D)_{2e-v+2a} - h_{\mathfrak{m}}^{n+1}(I_B)_{2e-v+2a}$ show $-h_{\mathfrak{m}}^{n+1}(I_B)_{2e-v+2a} + h_{\mathfrak{m}}^{n+1}(I_D I_B)_{2e-v+2a} = \beta(B', D)_{v-e-a}$. Hence we have proved

$$\dim(K_{B'}^*)_v = -\delta^*(B)_{-2e+v-2a} + \beta(B', D)_{v-e-a} \qquad (32)$$

Moreover if $\beta(B', D)_{v-e-a} = 0$, then $\dim B'_{v-e-a} = 0$, i.e. $v - e - a < 0$ and hence $H_{\mathfrak{m}}^{n+1}(I_D)_{2e-v+2a} = 0$ by duality (e.g. if $\dim R > n+1$, then $h_{\mathfrak{m}}^{n+1}(I_D)_{2e-v+2a} = h_{\mathfrak{m}}^n(D)_{2e-v+2a} = h_{\mathfrak{m}}^n(K_D)_{-v+e+a} = 0$ by duality). Since $h_{\mathfrak{m}}^n(K_{B'})_{-v+e+a} = 0$, it follows that $h_{\mathfrak{m}}^{n+1}(I_D I_B)_{2e-v+2a} = \beta(B', D)_{v-e-a} = 0$ which easily leads to the final conclusion of (iii).

Note that (32) also shows $\dim(K_B^*)_v = -\delta^*(B')_{v-2e-2a} + \beta(B, D)_{v-e-a}$. Moreover if we replace v by $-v + 2e + 2a$ in (32), we get (iv) by combining these formulas.

To prove (v), we use Gorenstein duality. Indeed

$$_v\mathrm{Ext}_B^2(I_B/I_B^2, K_B) \simeq {}_{-v}\mathrm{H}_{\mathfrak{m}}^{n-2}(I_B/I_B^2)^\vee \simeq {}_{-v}\mathrm{H}_{\mathfrak{m}}^{n-1}(I_B^2)^\vee \hookrightarrow {}_{-v}\mathrm{H}_{\mathfrak{m}}^{n-1}(S_2 I_B)^\vee,$$

where the final injection follows from Gorenstein duality and $_v\mathrm{Hom}_B(\mathrm{H}_2(R, B, B), K_B) = 0$ because I_B is generically syzygetic. Note that the injection is an isomorphism if I_B is syzygetic in codimension ≤ 1. Now combining (30) and (31) we get

$$_{-v}\mathrm{H}_{\mathfrak{m}}^{n-1}(S_2 I_B)^\vee \simeq {}_{-v}\mathrm{H}_{\mathfrak{m}}^{n-1}(S_2(K_{B'})(-2e-2a))^\vee \simeq {}_{v+2e+2a}\mathrm{Ext}_{B'}^1(S_2(K_{B'}), K_{B'}) \qquad (33)$$

and we conclude easily. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 34. *With notations as in Proposition 33, allowing also $n = \dim B \geq 0$, we have*

$$\dim(N_B)_v - \sum_{i=1}^c \dim B_{v+c_i} = \dim(N_{B'})_v - \sum_{i=1}^c \dim B'_{v+c_i} \quad \text{for every integer } v .$$

This follows from [27], Prop. 1.7 for $v = 0$, and from [26], Cor. 2.12 and (2.19) or [29], Prop. 9.20 for $n \geq 2$ and every v. Since the arguments of the proof of [27], Prop. 1.7 hold for the linkage appearing in the non-graded deformation functors, we may use the proof to get the displayed dimension formula by showing that the isomorphism of their tangent spaces, $A_{D \to B}^1 \simeq A_{D \to B'}^1$ (whose degree zero parts are given by (10)), is graded. This isomorphism is, however, concretely studied in [29], pp. 38-39, without using deformations. There, if we take the diagrams (6.11) and (6.12) without sheafification, we get a graded isomorphism $A_{D \to B}^1 \simeq A_{D \to B'}^1$ from which we get the displayed formula by e.g. the proof of [29], Prop.9.20.

Example 35. *We now complete Example 17 for the remaining cases $(0 \leq s \leq 2)$ by verifying the assumptions of Proposition 13A, using liaison. Since we in Example 17 proved $_0\mathrm{Ext}_B^1(I_B/I_B^2, B) = 0$, it suffices by Remark 6(c) to show $_0\mathrm{Ext}_B^2(I_B/I_B^2, K_B(-s)) = 0$. Now if we link B to a B' via a complete intersection of three hypersurfaces, all of degree 2, the linked curve $C' = \mathrm{Proj}(B')$ has degree $d' = 8 - 4 = 4$ and genus $g' = 0$, i.e. we see that C and C' are essentially the same curve. Hence $_v\mathrm{Ext}_{B'}^1(S_2(K_{B'}), K_{B'}) = 0$ for $v \geq 0$*

since B satisfies this vanishing. By Proposition 33 it follows that $_{-s}\text{Ext}_B^2(I_B/I_B^2, K_B) \hookrightarrow$ $_{-s+2}\text{Ext}_{B'}^1(S_2(K_{B'}), K_{B'}) = 0$ for $s \leq 2$, which precisely covers the cases of Example 17 which are not already analyzed. So, for $0 \leq s \leq 2$, Proposition 13A applies! In Example 17 we saw that $\dim(K_B^*)_s = 4s + 3$ for any $s \geq 0$. It remains to compute $\delta(B)_{-s}$. Since $\delta(B)_{v-2} = \beta(B', D)_{v-1} - \dim(K_{B'}^*)_v$ and $\beta(B', D)_v = \dim B_v' + 3\dim(I_{B'/D})_{v+2}$ by Proposition 33, we get $(\delta(B)_0, \delta(B)_{-1}, \delta(B)_{-2})$ to be equal to $(15, 3, -3)$, and as previously

$$\dim_{(A)} \text{GradAlg}^H(R) = h^0(\mathcal{N}_C) + h^0(\theta_C(s)) - 1 - \delta(B)_{-s} = 4s + 23 - \delta(B)_{-s}$$

Hence $\dim_{(A)} \text{GradAlg}^H(R)$ equals $(8, 24, 34)$ for $s = (0, 1, 2)$ respectively. Note that it is easy, for $s = 2$, to see that $(K_B)_0 = 0$ and hence that $_{-2}\text{Hom}(I_B/I_B^2, K_B) = 0$ and $_{-2}ext_B^1(I_B/I_B^2, K_B) = -\delta(B)_{-2} = 3$ by looking to the minimal resolution of I_B (or use Proposition 33(iii)). By the interpretation of $_{-s}ext_B^1(I_B/I_B^2, K_B)$ of Proposition 13A we get that the codimension of the Gorenstein quotients given by (13) is 3.

Example 36. *Now we consider the Artinian reduction A_L of the Example 21 of socle degree $s + 1$, to treat the remaining cases of interest $(0 \leq s \leq 2)$. Since $s \leq 2$ and I_B has no linear generator, $_{s+1}\text{H}_2(R_L, B_L, B_L) \hookrightarrow (S_2 I_{B_L})_{s+1} = 0$. Looking to Example 21 and Remark 14(b) we see that all assumptions of Proposition 13A are satisfied. Now examining $\text{H}^i(\theta_C(v))$ of Example 17 for $v = -1$ a little further and invoking $(S_2 K_B)_1 = 0$ we get $_{-1}ext_B^1(S_2(K_B), K_B) = 1$ in addition to $_v\text{Ext}_B^1(S_2(K_B), K_B) = 0$ for $v \geq 0$. By Lemma 19(i) we get $\dim(K_{B_L}^*)_v = 4$ for $v \geq 1$. Moreover if we apply $\text{Hom}_B(I_B/I_B^2, -)$ to the exact sequence $0 \to K_B \to K_B(1) \to K_{B_L} \to 0$, we get*

$$\delta(B_L)_{v-1} = \delta(B)_v - \delta(B)_{v-1} + {}_v ext_B^2(I_B/I_B^2, K_B) - {}_{v-1} ext_B^2(I_B/I_B^2, K_B).$$

Since $_{-3}ext_B^2(I_B/I_B^2, K_B) = {}_{-1}ext_{B'}^1(S_2(K_{B'}), K_{B'}) = 1$ and $_{-s}\text{Ext}_B^2(I_B/I_B^2, K_B) = 0$ for $s \leq 2$, we get that $(\delta(B_L)_{-1}, \delta(B_L)_{-2}, \delta(B_L)_{-3})$ is equal to $(12, 6, -4)$ from the values of $\delta(B)_v$ in Example 35. It follows that A_L is an unobstructed Artinian Gorenstein algebra for $0 \leq s \leq 2$, i.e. $\text{PGor}(H_L)$ is smooth at (A_L) by Theorem 11 and

$$\dim_{(A_L)} \text{PGor}(H_L) = \dim(N_{B_L})_0 + \dim(K_{B_L}^*)_{s+1} - 1 - \delta(B_L)_{-s-1} = 15 - \delta(B_L)_{-s-1} .$$

Moreover, for $s = 2$, we see that $-\delta(B_L)_{-3} = {}_{-3}ext_{B_L}^1(I_{B_L}/I_{B_L}^2, K_{B_L}) = 4$ is the codimension of the stratum of Gorenstein quotients given by (13) around (A_L).

Finally we remark that the conclusions (about A_L) also hold for any Artinian Gorenstein quotient A, well-defined by a twist of the canonical module of B, with the same graded Betti numbers as A_L. In this case we can compute the numbers $\delta(B)_v$ of a one-dimensional B as we did in Example 35 by essentially the same linkage argument.

Mainly using Proposition 33 twice (for (i) below, we utilize (33)), we get

Corollary 37. *Let B and B' be Cohen-Macaulay quotients of R, algebraically linked by a complete intersection D of type $(c_1, c_2, .., c_c)$, let B' and B'' be algebraically linked by a complete intersection D' of type $(c_1', c_2', .., c_c')$, and let $a = \sum c_i$, $a' = \sum c_i'$ and $\dim B > 0$. Then*

(i) $_{v+2a'}\text{Ext}_{B''}^1(S_2(K_{B''}), K_{B''}) \simeq {}_{v+2a}\text{Ext}_B^1(S_2(K_B), K_B)$ for any v.

(ii) $\quad _{v-2a'}\mathrm{Ext}^2_{B''}(I_{B''}/I^2_{B''}, K_{B''}) \simeq {}_{v-2a}\mathrm{Ext}^2_B(I_B/I^2_B, K_B)$
provided I_B and $I_{B''}$ are syzygetic in codimension ≤ 1.

(iii) $\quad \delta^*(B'')_{v-2e-2a'} = \delta^*(B)_{v-2e-2a} - \beta(B', D)_{v-e-a} + \beta(B', D')_{v-e-a'}.$

(iv) $\quad \dim(K^*_{B''})_{v+2e+2a'} = \dim(K^*_B)_{v+2e+2a} - \beta(B', D)_{v+e+a} + \beta(B'', D')_{v+e+a'}$ or equivalently,
$\dim(K^*_{B''})_{v+2e+2a'} = \dim(K^*_B)_{v+2e+2a} + \dim B''_{v+e+a'} - \dim B_{v+e+a} + \sum^c_{i=1}(\dim(K_{B'})_{v+c'_i} - \dim(K_{B'})_{v+c_i}).$

We include one example of biliaison starting with the smooth rational curve of Example 17. Of course starting with a licci curve, it is immediate to verify the assumptions of Proposition 13 (see Theorem 16). However, some part of the argument below is still needed to find $\dim(K^*_B)_s - \delta^*(B)_{-s}$ and to see how small we can let the number s in Theorem 16B be.

Example 38. *If we link the homogeneous coordinate ring B of the smooth rational quartic curve of Example 17 having minimal resolution*

$$0 \to R(-4)^3 \to R(-3)^8 \to R(-2)^6 \to R \to B \to 0,$$

to a B' using a complete intersection of type $(2, 2, 3)$ and further link B' to a B'' via a complete intersection of type $(2, 3, 4)$ (using the same hypersurface of degree 2 and 3 as in the first link) one may, notably due to $I_{B'/D} \simeq K_B(-2)$ and the mapping cone construction used twice, see that the minimal resolution of B'' is

$$0 \to R(-7) \oplus R(-6)^3 \to R(-5)^{10} \to R(-4)^5 \oplus R(-3) \oplus R(-2) \to R \to B'' \to 0.$$

If A is defined by a regular section of $(K^*_{B''})_s$, we can easily find the minimal resolution of A and its Hilbert function by using (13). For instance $\dim A_v = 16v - 19$ for $3 \leq v \leq s - 3$ and $\dim A_v = 16s - 38$ for $v > s$. The h-vector of A is in fact

$$(1, 4, 8, 8, 4, 1), \ (1, 4, 9, 14, 9, 4, 1), \ (1, 4, 9, 15, 15, 9, 4, 1), \ (1, 4, 9, 15, 16, 15, 9, 4, 1)$$

for $s = 4, 5, 6, 7$ and $(1, 4, 9, 15, 16, ..., 16, 15, 9, 4, 1)$ for $s \geq 8$ respectively. Let us suppose $s > 3$ since otherwise $\mathrm{Proj}(A)$ is degenerate. To use Proposition 13, we may argue as in Example 17 to see that $\delta(B'')_{-s} = 0$ for $s > 7$ by Remark 14(a), and that $\mathrm{H}^1(\theta_{C''}(v)) = 0$ and hence $\dim(K^*_{B''})_v = h^0(\theta_{C''}(v)) = 16v - 57$ for $v \geq 5$ because $C'' := \mathrm{Proj}(B'')$ is smooth curve of degree $d'' = 16$ and genus $g'' = 20$. Corollary 37 leads, however, to more precise information. Indeed it is rather straightforward to combine the Corollary and the computations of Example 17 and Example 35 to see that the conclusions of Theorem 16 hold for $s \geq 4$, that $\dim(K^*_{B''})_4 = 8$, and that $(\delta(B'')_{-4}, \delta(B'')_{-5}, \delta(B'')_{-6})$ is equal to $(7, -6, -6)$ and $\delta(B'')_{-s} = 0$ for $s > 6$. Finally patiently using Proposition 33(i) several times, one shows $\dim(N_{B''})_0 = 62$ (It is, however, faster to show $_0\mathrm{Ext}^i_{B''}(I_{B''}/I^2_{B''}, B'') = 0$ for $i = 1$ and 2 by Proposition 33(ii), using the computations of Example 17 and (15), and then use (15) once more to see $h^1(\mathcal{N}_{C''}) = 1$ and hence $\dim(N_{B''})_0 = 5d'' + 1 - g'' + h^1(\mathcal{N}_{C''}) = 62$). Putting this together we have by Proposition 13 that A is an unobstructed Gorenstein algebra for $s \geq 4$ and

$$\dim_{(A)} \mathrm{GradAlg}^H(R) = \dim(N_{B''})_0 + \dim(K^*_{B''})_s - 1 - \delta(B'')_{-s} = 16s + 4 \quad for \quad s \geq 7,$$

$\dim_{(A)} \mathrm{GradAlg}^H(R) = 16s + 10$ *for* $5 \leq s \leq 6$ *and* $\dim_{(A)} \mathrm{GradAlg}^H(R) = 62$ *for* $s = 4$. *By the interpretation of* $_{-s}\mathrm{ext}^1_{B''}(I_{B''}/I^2_{B''}, K_{B''})$ *of Proposition 13, we get by Proposition 33(iii) that the codimension of the stratum of Gorenstein quotients given by* (13) *is 6 for* $5 \leq s \leq 6$.

We may treat the Artinian reduction of $A \simeq B''/K_{B''}(-s)$ in Example 38 satisfactorily by remarking that $_{s+1}\mathrm{H}_2(R_L, B''_L, B''_L) = 0$ for $4 \leq s \leq 6$ (since one may show $(S_2 I_{B''_L})_{s+1} \simeq (I^2_{B''_L})_{s+1}$ by using that the generators of $I_{B''_L}$ of degree 2 and 3 form a regular sequence), and that B''_L is unobstructed by Lemma 19(*iii*) for any $s > 3$. For the latter, we use the liaison invariance of $_v\mathrm{Ext}^1_B(I_B/I^2_B, B)$ and the fact that $\mathrm{H}^1(\mathcal{N}_C(v)) = 0$ for $v \geq -1$ for a smooth rational curve C. Instead of working out these details, we will finish this subsection by considering two general classes of examples where B is licci. We concentrate on licci codimension 3 quotients, leading to unobstructed Gorenstein codimension 4 quotients by Theorem 16. In one of the cases the use of Proposition 33 is quite effective.

Example 39. *Let B be a Gorenstein quotient of R of codimension 3 given by*

$$0 \to R(-f) \simeq F_3 \to F_2 \to F_1 \to I_B \to 0. \tag{34}$$

(as A in Example 28). In this case $e = -n - 3$ and $K_B \simeq B(-f + n + 3)$ and since B is licci ([38]) and therefore satisfies $_v\mathrm{Ext}^1_B(I_B/I^2_B, B) = 0$ for any v by Proposition 33, we get $\delta(B)_{-s} = \dim(N_B)_{-(f+s-n-3)}$. If A is defined by a section of $(K^*_B)_s$ as in Theorem 16, then A is Gorenstein and unobstructed and

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim(N_B)_0 + \dim B_{f+s-n-3} - 1 - \dim(N_B)_{-f-s+n+3}$$

where $\dim(N_B)_v$ is given by [28], Thm. 2.6, i.e. by modifying (20) *to*

$$\dim(N_B)_v = \dim(\wedge^2 F_1)_{f+v} - \dim(F_1 \otimes F_2)_{f+v} + \dim(S_2(F_2))_{f+v} + \dim(F_1)_v - \dim(F_2)_v$$

Example 40. *Let B be an almost complete intersection of R of codimension 3 with a minimal resolution*

$$0 \to G_3 \to G_2 \to G_1 = \oplus^3_{i=1} R(-c_i) \oplus R(-h) \to I_B \to 0 \tag{35}$$

and let B' be the Gorenstein quotient obtained by linking B to B' via a complete intersection whose minimal generators (of degree c_i) corresponds to three of those of I_B. A standard mapping cone construction and duality yields

$$0 \to R(h - a) \to G^*_2(-a) \to G^*_3(-a) \oplus (\oplus^3_{i=1} R(-c_i)) \to I_{B'} \to 0 \tag{36}$$

and we see that $K_{B'} \simeq B'(a - h - n - 3)$ where $a = \sum c_i$. If A is defined by the zero locus of a section of $(K^*_B)_s$ as in Theorem 16, then A is Gorenstein and unobstructed and invoking Proposition 33 and $\dim(K^*_{B'})_{-s+2a-2n-6} - \delta(B')_{s-2a+2n+6} = \dim B'_{-s+h+a-n-3} - \dim(N_{B'})_{s-h-a+n+3}$ we get

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim(N_{B'})_0 + \dim B'_{-s+h+a-n-3} - \dim(N_{B'})_{s-h-a+n+3} - 1 + \gamma$$

where $\gamma := \sum^3_{i=1}(\dim B_{c_i} - \dim B'_{c_i} + \dim(I_{B/D})_{s-a+n+3+c_i} - \dim(I_{B'/D})_{-s+a-n-3+c_i}) + \dim B_{s-a+n+3} - \dim B'_{-s-n-3+a}$. Using $I_{B/D} \simeq K_{B'}(n + 3 - a) \simeq B'(-h)$, we easily express γ in terms of the Hilbert function of B' (or of B) and D. Moreover $\dim(N_{B'})_v$ is given by the final formula of Example 39. See Example 2 for a more explicit example.

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