Maximal Cohen-Macaulay Modules and Gorenstein Algebras

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Abstract

Let $\mathbf{B}$ be a graded Cohen-Macaulay quotient of a Gorenstein ring, $\mathbf{R}$. It is known that sections of the dual of the canonical module, $\mathbf{K_B}$, can be used to construct Gorenstein quotients of $\mathbf{R}$. The purpose of this paper is to place this method of construction into a broader context. If $\mathbf{M}$ is a maximal Cohen-Macaulay $\mathbf{B}$-module whose sheafified top exterior power is a twist of $\widetilde{\mathbf{K}}_{\mathbf{B}}$ and if $\mathbf{M}$ satisfies certain additional homological conditions then regular sections of $\mathbf{M}^*$ can again be used to construct Gorenstein quotients of $\mathbf{R}$. On Cohen-Macaulay quotients, the normal module, the first Koszul homology module and several other associated modules all have sheafified top exterior power equal to a twist of $\widetilde{\mathbf{K}}_{\mathbf{B}}$. If additional restrictions are placed on the Cohen-Macaulay quotients then these modules will satisfy the required additional homological conditions. This places the canonical module within a broad family of easily manipulated maximal Cohen-Macaulay modules whose sections can be used to construct Gorenstein quotients of $\mathbf{R}$.

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Contents

1 Introduction

Let $\mathbf{R}$ be a Gorenstein ring. If $\mathbf{B}$ is a graded Cohen-Macaulay quotient of $\mathbf{R}$, then degeneracy loci of sections of the dual of the canonical module of $\mathbf{B}$ in $\mathbf{R}$ define Gorenstein algebras. Several authors have made use of the Gorenstein algebras arising from such degeneracy loci in Gorenstein liaison theory as well as in several other contexts. If $\mathbf{M}$ is a free $\mathbf{R}$-module of rank r then the degeneracy loci of regular sections of $\mathbf{M}^*$ again define Gorenstein algebras (in fact complete intersection algebras). Such degeneracy loci as a source of complete intersections is commonplace in complete intersection liaison theory and other areas. A large part of the success of these methods is due to the fact that these modules can be constructed in abundance . In addition, these modules and their sections are particularly easy to manipulate. For instance, by utilizing tensor operations, it is relatively easy to find sections whose degeneracy loci lie within prespecified ideals. There are many other easily manipulated, readily available modules associated to Cohen-Macaulay quotients of a Gorenstein ring. Several of these modules have the same property with respect to degeneracy loci of their regular sections (i.e they can be used to construct Gorenstein algebras). The present paper is an attempt to identify these modules and place them in the more general framework of maximal Cohen-Macaulay modules with "good sections". Section 1 of the paper provides a brief glossary of definitions and recurring notation. Section 2 consists of a theorem on maximal Cohen-Macaulay modules and its proof. Section 3 is filled with several examples and applications identifying important classes of modules which fit into our framework. The examples deal primarily with the conormal module, the first Koszul homology module and several other related modules.

1.1 Notation, definitions and background

Throughout this paper we will use boldface type like $\mathbf{K_B}, \mathbf{B}, \mathbf{R}, \mathbf{M}, \mathbf{I}$ to denote algebraic objects such as modules, ideals and algebras. $\mathbf{R}$ will be a graded Gorenstein quotient of a polynomial ring over an algebraically closed field, k, with the usual grading. $\mathbf{B}$ will be a graded Cohen-Macaulay quotient of $\mathbf{R}$, $\mathbf{F}_i$ will be a graded free $\mathbf{R}$-module and $\mathbf{I}, \mathbf{J}, \mathbf{J}'$ will be homogeneous ideals in $\mathbf{R}$. $\mathbf{M}$ will denote a finitely generated graded module over $\mathbf{B}$ and $\mathbf{A}$ will be a graded quotient of $\mathbf{B}$. $\mathbf{K_B} = \mathrm{Ext}^c_{\mathbf{R}}(\mathbf{B}, \mathbf{R}(e))$ will denote the canonical module of $\mathbf{B}$ in $\mathbf{R}$ (where c denotes the codimension of $\mathbf{B}$ in $\mathbf{R}$ and $\mathbf{R}(e)$ denotes the canonical module of $\mathbf{R}$). $\mathbf{N_B}$ denotes the normal module of $\mathbf{B}$ in $\mathbf{R}$. $\mathbf{H}_i = \mathbf{H}_i(\mathbf{I})$ denotes the i^{th} Koszul homology module of the ideal $\mathbf{I}$ in $\mathbf{R}$. If $\mathbf{B} = \mathbf{R}/\mathbf{I}$ then the $\mathbf{B}$-dual modules, $\mathrm{Hom}_{\mathbf{B}}(-, \mathbf{B})$, to $\mathbf{M}, \mathbf{N_B}, \mathbf{H}_i$ will be denoted by $\mathbf{M}^*, \mathbf{N}^*_{\mathbf{B}}, \mathbf{H}^*_i$ resp. We denote by $\mathbf{F}^*_i$ the $\mathbf{R}$-dual module $\mathrm{Hom}_{\mathbf{R}}(\mathbf{F}_i, \mathbf{R})$. The sheafification of a module will be denoted with a tilde overbar such as $\widetilde{\mathbf{K}}_{\mathbf{B}}, \widetilde{\mathbf{M}}, \widetilde{\mathbf{N}}_{\mathbf{B}}$ and $\widetilde{\mathbf{H}}_i$. Sheaves will be given a calligraphic type such as $\mathcal{M}, \mathcal{Z}, \mathcal{O}_S$ etc.

We will make reference to the notions of a quotient, $\mathbf{R}/\mathbf{J}$, being licci or strongly Cohen-Macaulay. Definitions of these terms are given below.

Definition 1. *Two quotients of* $\mathbf{R}$*,* $\mathbf{R}/\mathbf{J}$ *and* $\mathbf{R}/\mathbf{J}'$ *(equidimensional and without embedded components), are said to be geometrically complete intersection linked (resp. geometrically Gorenstein linked) if* $\mathbf{J}$ *and* $\mathbf{J}'$ *do not share common components and if* $\mathbf{R}/(\mathbf{J} \cap \mathbf{J}')$ *is a complete intersection (resp. Gorenstein). Two quotients of* $\mathbf{R}$*,* $\mathbf{R}/\mathbf{J}$ *and* $\mathbf{R}/\mathbf{J}'$*, are said to be algebraically complete intersection linked (resp. algebraically Gorenstein linked) if there exists a complete intersection (resp. Gorenstein) ideal* $\mathbf{K}$ *(with* $\mathbf{K} \subseteq \mathbf{J} \cap \mathbf{J}'$*) such that* $\mathbf{J} = \mathbf{K} : \mathbf{J}'$ *and* $\mathbf{J}' = \mathbf{K} : \mathbf{J}$*. The relationship of being linked generates an equivalence relation. The resulting equivalence classes are called linkage classes. For more details see [19].*

Definition 2. *The Cohen-Macaulay quotient* $\mathbf{B} = \mathbf{R}/\mathbf{I}$ *is said to be licci if it is in the linkage class of a complete intersection.*

Definition 3. *The quotient* $\mathbf{B} = \mathbf{R}/\mathbf{I}$ *is said to be strongly Cohen-Macaulay if all the non trivial Koszul homology modules* $\mathbf{H}_1(\mathbf{I}), \mathbf{H}_2(\mathbf{I}), \ldots$ *of* $\mathbf{I}$ *are Cohen-Macaulay.*

We will need the following definition, statements and theorems. For a proof of Theorem 5, see [6], Theorem 21.21 and Proposition 21.12. For a proof of Theorem 6, see [14]. For a proof of Theorem 7, see [4], Theorem 4.1, cf. [17], Corollary 6.5 for the case $i = c - 1$.

Definition 4. *If* $\mathbf{M}$ *is a finitely generated* $\mathbf{B}$*-module then*

$$\mathbf{M} \text{ is a maximal Cohen}-\text{Macaulay } \mathbf{B}-\text{module} \iff \operatorname{depth} \mathbf{M} = \dim \mathbf{B}. \tag{1}$$

Here, $\operatorname{depth}_{\mathbf{J}} \mathbf{M}$ *is the length of a maximal* $\mathbf{M}$*-sequence in* $\mathbf{J}$ *and* $\operatorname{depth} \mathbf{M} = \operatorname{depth}_{\mathfrak{m}} \mathbf{M}$ *where* $\mathfrak{m}$ *is the irrelevant maximal ideal. If* $\mathrm{H}^i_{\mathbf{J}}(-)$ *is the right derived functor of* $\Gamma_{\mathbf{J}}(-)$*, sections with support in* $\operatorname{Proj}(\mathbf{R}/\mathbf{J})$*, then for* $\mathbf{M} \neq 0$

$$\mathbf{M} \text{ is a maximal Cohen}-\text{Macaulay } \mathbf{B}-\text{module} \iff \mathrm{H}^i_{\mathfrak{m}}(\mathbf{M}) = 0 \text{ for } i < \dim \mathbf{B}. \tag{2}$$

The previous fact is useful to see when one module in a short exact sequence is maximal Cohen-Macaulay provided the other modules of the short exact sequence are maximal Cohen-Macaulay (or have sufficient depth) (cf. [6], Corollary 18.6 for another description). For maximal Cohen-Macaulay $\mathbf{B}$*-modules, we have*

$$\operatorname{depth}_{\mathbf{J}} \mathbf{M} = \operatorname{depth}_{\mathbf{J}} \mathbf{B}. \tag{3}$$

If $S = \operatorname{Proj}(\mathbf{B})$ *and* Z *is (say closed) in* S *and* $U = S - Z$*, we let* $\mathrm{H}^0_*(U, \widetilde{\mathbf{M}}) = \oplus_v \mathrm{H}^0(U, \widetilde{\mathbf{M}}(v))$*. If* $\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{M} \geq r$*, then* $\mathrm{H}^i_{\mathbf{I}(Z)}(\mathbf{M}) = 0$ *for* $i < r$*, and conversely [11]. This implies*

$$\mathrm{H}^i_*(U, \widetilde{\mathbf{M}}) = 0 \quad \text{for} \quad 1 \leq i \leq r - 2. \tag{4}$$

Moreover, $\mathbf{M} \simeq \mathrm{H}^0_*(U, \widetilde{\mathbf{M}})$ *if* $r \geq 2$*. Since* $\mathbf{B}$ *is Cohen-Macaulay, we remark [6]*

$$\operatorname{depth}_{\mathbf{J}} \mathbf{B} = \dim \mathbf{B} - \dim \mathbf{B}/\mathbf{J}. \tag{5}$$

Theorem 5. *If* $\mathbf{M}$ *is a maximal Cohen-Macaulay* $\mathbf{B}$*-module then* $\operatorname{Hom}_{\mathbf{B}}(\mathbf{M}, \mathbf{K}_{\mathbf{B}})$ *is a maximal Cohen-Macaulay* $\mathbf{B}$*-module and* $\operatorname{Ext}^i_{\mathbf{B}}(\mathbf{M}, \mathbf{K}_{\mathbf{B}}) = 0$ *for* $i > 0$*.*

Theorem 6. *If* $\mathbf{B} = \mathbf{R}/\mathbf{I}$ *is licci, then* $\mathbf{B}$ *is strongly Cohen-Macaulay.*

Theorem 7. *If* $\mathbf{B} = \mathbf{R}/\mathbf{I}$ *is licci of codimension* c*, then every nontrivial* $\operatorname{Tor}^{\mathbf{R}}_i(\mathbf{B}, \mathbf{K}_{\mathbf{B}})$ *is a maximal Cohen-Macaulay* $\mathbf{B}$*-module for* $0 \leq i \leq c$*.*

If $\mathbf{B} = \mathbf{R}/\mathbf{I}$, if $S = \operatorname{Proj}(\mathbf{B})$ is Gorenstein on $U = S - Z$ and if $\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} \geq 2$ (resp. $\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} \geq 1$), then

$$\operatorname{Ext}^i_{\mathbf{B}}(\mathbf{I}/\mathbf{I}^2, \mathbf{B}) \simeq \operatorname{Ext}^i_{\mathbf{B}}(\mathbf{I}/\mathbf{I}^2 \otimes \mathbf{K}_{\mathbf{B}}, \mathbf{K}_{\mathbf{B}}) \quad \text{for} \quad i \leq 2 \quad (\text{resp.} \quad i \leq 1) \tag{6}$$

(this follows from a spectral sequence argument [13]). Since $\operatorname{Tor}_1^{\mathbf{R}}(\mathbf{B},\mathbf{K_B}) \simeq \mathbf{I}/\mathbf{I}^2 \otimes \mathbf{K_B}$, it follows from (6), Theorem 5 and Theorem 7 that

$$\operatorname{Ext}^i_{\mathbf{B}}(\mathbf{I}/\mathbf{I}^2,\mathbf{B}) = 0 \quad \text{for} \quad 1 \le i \le 2 \quad (\text{resp.} \quad i=1) \tag{7}$$

provided $\mathbf{B}$ is licci and $\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} \ge 2$ (resp $\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} \ge 1$), cf [17], Proposition 6.15. Let $\mathbf{F}_1 \twoheadrightarrow \mathbf{I}$ be surjective. Then we have a complex

$$0 \to \mathbf{H}_1 \xrightarrow{\alpha} \mathbf{F}_1 \otimes_{\mathbf{R}} \mathbf{B} \to \mathbf{I}/\mathbf{I}^2 \to 0. \tag{8}$$

$\mathbf{I}$ is called syzygetic if α is injective (or equivalently if (8) is exact). If $\mathbf{B}$ is strongly Cohen-Macaulay (or much weaker, $\mathbf{H}_1(\mathbf{I})$ is $\mathbf{B}$-torsion free) and generically a complete intersection then $\mathbf{I}$ is syzygetic, cf. [15].

2 Sections of Modules over Cohen-Macaulay Quotients

In this section we prove a general theorem on sections of maximal Cohen-Macaulay modules. The theorem will be applied throughout the paper.

Theorem 8. *Let $\mathbf{R}$ be a graded Gorenstein quotient of a polynomial ring and assume $\mathbf{R}$ has canonical module $\mathbf{R}(e)$. Let $\mathbf{B}$ be a codimension c graded Cohen-Macaulay quotient of $\mathbf{R}$. Let $\mathbf{K_B} = \operatorname{Ext}^c_{\mathbf{R}}(\mathbf{B},\mathbf{R}(e))$ be the canonical module of $\mathbf{B}$. Let $\mathbf{M}$ be a finitely generated maximal Cohen-Macaulay $\mathbf{B}$-module. Let $S = \operatorname{Proj}\mathbf{B}$, let Z be a closed scheme such that $\dim(\mathbf{B}) - \dim(\mathbf{B}/\mathbf{I}(Z)) \ge \max(r,2)$ and let $U = S - Z$. Let $\mathbf{M}_i = \mathrm{H}^0_*(U, \wedge^i\widetilde{\mathbf{M}})$ for $i \ge 0$. Suppose the following conditions are satisfied:*

1. *$\widetilde{\mathbf{M}}|_U$ is locally free (of rank r)*
2. *$\wedge^r\widetilde{\mathbf{M}}|_U \simeq \widetilde{\mathbf{K}}_{\mathbf{B}}(t)|_U$ (for some integer t)*
3. *$\mathbf{M}_i$ is a finitely generated maximal Cohen-Macaulay $\mathbf{B}$-module for $2 \le i \le r/2$*

then we can make the following conclusions:

1. *$\mathbf{M} \simeq \mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_r$ are maximal Cohen-Macaulay $\mathbf{B}$-modules*
2. *$\mathbf{M}_{r-i} \simeq \operatorname{Hom}_{\mathbf{B}}(\mathbf{M}_i, \mathbf{K_B})(t)$ for $0 \le i \le r$*
3. *If s is an integer, then any regular section, $\sigma \in \mathrm{H}^0(U, \widetilde{\mathbf{M}}^*(s))$, defines a Gorenstein quotient $\mathbf{R} \twoheadrightarrow \mathbf{A}$ of codimension $r+c$ given by the exact sequence*

$$0 \to \mathbf{M}_r(-rs) \to \mathbf{M}_{r-1}((1-r)s) \to \cdots \to \mathbf{M}_2(-2s) \to \mathbf{M}(-s) \xrightarrow{\sigma^*} \mathbf{B} \to \mathbf{A} \to 0. \tag{9}$$

Proof. Conclusion 2 follows from the isomorphisms

$$\mathbf{M}_{r-i} := \mathrm{H}^0_*(U, \wedge^{r-i}\widetilde{\mathbf{M}}) \simeq \mathrm{H}^0_*(U, (\wedge^i\widetilde{\mathbf{M}})^* \otimes \wedge^r\widetilde{\mathbf{M}}) \simeq \mathrm{H}^0_*(U, (\wedge^i\widetilde{\mathbf{M}})^* \otimes \widetilde{\mathbf{K}}_{\mathbf{B}}(t)) \simeq$$
$$\mathrm{H}^0_*(U, \widetilde{\operatorname{Hom}_{\mathbf{B}}(\mathbf{M}_i, \mathbf{K_B}(t))}) \simeq \operatorname{Hom}_{\mathbf{B}}(\mathbf{M}_i, \mathbf{K_B}(t)).$$

Conclusion 1 follows if it is noted that $\operatorname{Hom}_{\mathbf{B}}(\mathbf{M}_i, \mathbf{K_B})$ is a maximal Cohen-Macaulay module if $\mathbf{M}_i$ is a maximal Cohen-Macaulay module (cf. Theorem 5).

To prove conclusion 3, we start with a regular section $\sigma \in \mathrm{H}^0(U, \widetilde{\mathbf{M}}^*(s))$ i.e. a map

$$\sigma : \mathcal{O}_U \to \widetilde{\mathbf{M}}^*(s)|_U.$$

If we dualize and take global sections then we get

$$\mathbf{M}(-s) = \mathrm{H}^0_*(U, \widetilde{\mathbf{M}}^{**})(-s) \to \mathrm{H}^0_*(U, \mathcal{O}_S) = \mathbf{B}.$$

We define $\mathbf{A}$ to be the cokernel of this map. Since $\widetilde{\mathbf{M}}|_U$ is locally free and since σ is a regular section, the associated Koszul complex

$$0 \to \wedge^r \widetilde{\mathbf{M}}(-rs)|_U \to \wedge^{r-1}\widetilde{\mathbf{M}}(-(r-1)s)|_U \to \cdots \to \wedge^2 \widetilde{\mathbf{M}}(-2s)|_U \to \widetilde{\mathbf{M}}(-s)|_U \xrightarrow{\sigma^*} \widetilde{\mathbf{B}}|_U \quad (10)$$

is exact. We claim that the global section functor, $\mathrm{H}^0_*(U, -)$ is exact on (10). To see this, define

$$\mathcal{Z}_i = \ker(\wedge^{r-i}\widetilde{\mathbf{M}}(-(r-i)s)|_U \to \wedge^{r-i-1}\widetilde{\mathbf{M}}(-(r-i-1)s)|_U) \ \text{ for } \ 1 \le i \le r-1.$$

In particular, $\mathcal{Z}_1 = \wedge^r \widetilde{\mathbf{M}}(-rs)|_U \simeq \widetilde{\mathbf{K}}_{\mathbf{B}}(t-rs)|_U$. Split (10) into short exact sequences and apply the global section functor to each of them. It suffices to show that

$$\mathrm{H}^1_*(U, \mathcal{Z}_i) = 0 \ \text{ for } \ 1 \le i \le r-2.$$

To use an induction on i, we instead prove the more general statement

$$\mathrm{H}^j_*(U, \mathcal{Z}_i) = 0 \ \text{ for } \ 1 \le i \le r-2 \ \text{ and } \ 1 \le j \le r-i-1. \quad (11)$$

Since $\mathcal{Z}_1 = \widetilde{\mathbf{K}}_{\mathbf{B}}(t-rs)|_U$ and since $\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{K_B} = \operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} \ge r$, we obtain (11) for $i = 1$ if we use the fact

$$\mathrm{H}^{j+1}_{\mathbf{I}(Z)}(\mathbf{K_B}) = 0 \ \text{ for } \ j+1 < r.$$

To use induction, we now assume that (11) holds for $i = k$ where $1 \le k < r-2$. Since

$$0 \to \mathcal{Z}_k \to \wedge^{r-k}\widetilde{\mathbf{M}}((-r+k)s)|_U \to \mathcal{Z}_{k+1} \to 0 \quad (12)$$

is exact and since $\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{M}_{r-k} \ge r$ we get

$$\mathrm{H}^j_*(U, \mathcal{Z}_{k+1}) \simeq \mathrm{H}^{j+1}_*(U, \mathcal{Z}_k) \ \text{ for } \ 1 \le j \le r-3$$

and we conclude by the induction assumption. Thus we have proven the exactness of

$$0 \to \mathbf{M}_r(-rs) \to \mathbf{M}_{r-1}((1-r)s) \to \cdots \to \mathbf{M}_2(-2s) \to \mathbf{M}(-s) \xrightarrow{\sigma^*} \mathbf{B} \to \mathbf{A} \to 0. \quad (13)$$

To see that $\mathbf{A}$ is Gorenstein, we first prove that $\mathbf{A}$ is Cohen-Macaulay by showing that $\mathrm{H}^i_{\mathfrak{m}}(\mathbf{A}) = 0$ for $i < \dim \mathbf{A}$. Note that since σ is a regular section on U and $\dim \mathbf{B}/\mathbf{I}(Z) \le$

$\dim \mathbf{B} - r$, it is clear that $\dim \mathbf{A} = \dim \mathbf{B} - r$. Let $\mathbf{Z}_i = \mathrm{H}^0_*(U, \mathcal{Z}_i)$ and $\mathbf{Z}_r = \ker(\mathbf{B} \to \mathbf{A})$. Then the global version of (12) yields

$$\mathrm{H}^i_{\mathfrak{m}}(\mathbf{A}) \simeq \mathrm{H}^{i+1}_{\mathfrak{m}}(\mathbf{Z}_r) \simeq \mathrm{H}^{i+2}_{\mathfrak{m}}(\mathbf{Z}_{r-1}) \simeq \cdots \simeq \mathrm{H}^{i+r}_{\mathfrak{m}}(\mathbf{Z}_1)$$

since $i < \dim \mathbf{A}$ implies $i + r < \dim \mathbf{B} = \operatorname{depth} \mathbf{M}_j$. Since $\mathbf{Z}_1 \simeq \mathbf{K}_{\mathbf{B}}(t - rs)$, we also have $\mathrm{H}^{i+r}_{\mathfrak{m}}(\mathbf{Z}_1) = 0$ and $\mathbf{A}$ is Cohen-Macaulay. It remains to prove

$$\operatorname{Ext}^r_{\mathbf{B}}(\mathbf{A}, \mathbf{K}_{\mathbf{B}}) \simeq \mathbf{A}(\nu)$$

for some integer ν. The isomorphisms in the proof of Conclusion 2 of the theorem are given by the determinant and the section, σ. Since $\mathbf{Z}_1 \simeq \mathbf{M}_r(-rs)$ and $\operatorname{Ext}^1(\mathbf{M}_{r-1}, \mathbf{K}_{\mathbf{B}}) = 0$ (cf. Theorem 5 for the vanishing) we have in particular a commutative diagram

$$\begin{array}{ccccccc}
\operatorname{Hom}_{\mathbf{B}}(\mathbf{M}_{r-1}((-r+1)s), \mathbf{K}_{\mathbf{B}}(t)) & \to & \operatorname{Hom}_{\mathbf{B}}(\mathbf{Z}_1, \mathbf{K}_{\mathbf{B}}(t)) & \to & \operatorname{Ext}^1_{\mathbf{B}}(\mathbf{Z}_2, \mathbf{K}_{\mathbf{B}}(t)) & \to & 0 \\
\downarrow & \circ & \downarrow & & & & \\
\mathbf{M}_1((r-1)s) & \to & \mathbf{B}(rs) & & & &
\end{array}$$

where the down arrows are isomorphisms. The vanishing of Theorem 5 combined with the global version of (12) imply

$$\operatorname{Ext}^r_{\mathbf{B}}(\mathbf{A}, \mathbf{K}_{\mathbf{B}}) \simeq \operatorname{Ext}^{r-1}_{\mathbf{B}}(\mathbf{Z}_r, \mathbf{K}_{\mathbf{B}}) \simeq \operatorname{Ext}^{r-2}_{\mathbf{B}}(\mathbf{Z}_{r-1}, \mathbf{K}_{\mathbf{B}}) \simeq \cdots \simeq \operatorname{Ext}^1_{\mathbf{B}}(\mathbf{Z}_2, \mathbf{K}_{\mathbf{B}}).$$

Hence we get

$$\operatorname{Ext}^r_{\mathbf{B}}(\mathbf{A}, \mathbf{K}_{\mathbf{B}}) \simeq \mathbf{A}(rs - t)$$

as required.

□

Remark 9. *Conclusion 2 of Theorem 8, slightly extended, imposes a self duality on the long exact sequence* (9) *of Theorem 8 and we get a self duality on* $\mathbf{M}_{r/2}$ *if* r *is even. Note that if* $\widetilde{\mathbf{M}}$ *is locally free (or weaker,* $\mathcal{E}xt^i(\widetilde{\mathbf{M}}_j, \widetilde{\mathbf{K}}_{\mathbf{B}}) = 0$ *for* $i > 0$*, cf. Theorem 5) on* S*, then Serre Duality and* $\wedge^{r-i}\widetilde{\mathbf{M}} \simeq (\wedge^i \widetilde{\mathbf{M}})^* \otimes \widetilde{\mathbf{K}}_{\mathbf{B}}(t)$ *yields*

$$\mathrm{H}^j_*(S, \widetilde{\mathbf{M}}_{r-i}) \simeq \mathrm{H}^{\dim S - j}_*(S, \widetilde{\mathbf{M}}_i)^\vee(t). \tag{14}$$

This explains (and essentially reproves) conclusion 1 of Theorem 8. More importantly, if r *is even, we see immediately from* (14) *that we can weaken the Cohen-Macaulay assumption on* $\mathbf{M}_{r/2}$ *to*

$$\operatorname{depth}_{\mathfrak{m}} \mathbf{M}_{r/2} \geq \frac{1}{2} \dim \mathbf{B} + 1 \tag{15}$$

provided that $\widetilde{\mathbf{M}}$ *is locally free on* S.

Utilizing Serre duality carefully, we can weaken condition 3 of Theorem 8 by replacing the maximal Cohen-Macaulay assumption on $\mathbf{M}_{r/2}$ *with*

$$\operatorname{depth}_{\mathfrak{m}} \mathbf{M}_{r/2} \geq \frac{1}{2} \dim \mathbf{B} + 1 + \frac{1}{2} \dim \mathbf{B}/\mathbf{I}(Z)$$

provided $\widetilde{\mathbf{M}}$ *is locally free on* $U = S - Z$*. Indeed, to conclude as in Theorem 8, it suffices to prove that this implies that* $\mathbf{M}_{r/2}$ *is a maximal Cohen-Macaulay* $\mathbf{B}$*-module. Since*

$$\operatorname{depth}_{\mathfrak{m}} \mathbf{M}_{r/2} \geq \frac{1}{2}\dim \mathbf{B} + 1 + \frac{1}{2}\dim \mathbf{B}/\mathbf{I}(Z) = \dim \mathbf{B} + 1 - \frac{1}{2}\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B},$$

it suffices to prove $\mathrm{H}_*^{\dim S - j}(S, \widetilde{\mathbf{M}}_{r/2}) = 0$ *for* $\dim S - j \geq \dim \mathbf{B} - \frac{1}{2}\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B}$ *(*$j > 0$*). This is equivalent to proving*

$$\mathrm{H}_*^j(U, \widetilde{\mathbf{M}}_{r/2}) = 0 \quad \text{for} \quad j + 1 \leq \frac{1}{2}\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B}$$

since $\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{K_B} \geq j + 2$ *and since* $\operatorname{Ext}^i_{\mathcal{O}_S}(\widetilde{\mathbf{M}}_{r/2}, \widetilde{\mathbf{K}}_{\mathbf{B}}(v))$ *is dual to* $\mathrm{H}^{\dim S - j}(S, \widetilde{\mathbf{M}}_{r/2}(-v))$ *implies*

$$\operatorname{Ext}^j_{\mathcal{O}_S}(\widetilde{\mathbf{M}}_{r/2}, \widetilde{\mathbf{K}}_{\mathbf{B}}(v)) \simeq \operatorname{Ext}^j_{\mathcal{O}_U}(\widetilde{\mathbf{M}}_{r/2}|_U, \widetilde{\mathbf{K}}_{\mathbf{B}}(v)|_U) \simeq \mathrm{H}^j(U, \mathcal{H}om(\widetilde{\mathbf{M}}_{r/2}, \widetilde{\mathbf{K}}_{\mathbf{B}}(v)))$$

by [11], exp. VI. Now, using [22], Lemma 1.8a), it is easy to see that $\operatorname{depth}_{\mathfrak{m}} \mathbf{M}_{r/2} \geq \dim \mathbf{B} + 1 - \frac{1}{2}\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B}$ *implies*

$$\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{M}_{r/2} \geq \operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} + 1 - \frac{1}{2}\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} = 1 + \frac{1}{2}\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B}$$

and we conclude by (4)*.*

Remark 10. *If we, in addition to the assumptions of the theorem, suppose that* $\mathbf{M}_i$ *has finite projective dimension for* $0 \leq i \leq r/2$ *then all* $\mathbf{M}_i$ *have finite projective dimension over* $\mathbf{R}$*. We can then find an* $\mathbf{R}$*-free resolution of* $\mathbf{A}$*. Indeed, we have* $\operatorname{Ext}^i_{\mathbf{R}}(\mathbf{M}_i, \mathbf{R}) = 0$ *for* $i \neq c$ *due to the fact that* $\mathbf{M}_i$ *is a maximal Cohen-Macaulay* $\mathbf{B}$*-module and due to*

$$\operatorname{Ext}^c_{\mathbf{R}}(\mathbf{M}_i, \mathbf{R}(e)) \simeq \operatorname{Hom}_{\mathbf{B}}(\mathbf{M}_i, \mathbf{K_B}) \simeq \mathbf{M}_{r-i}(-t).$$

Applying $(-)^* = \operatorname{Hom}_{\mathbf{R}}(-, \mathbf{R})$ *to the* $\mathbf{R}$*-free resolution*

$$0 \to \mathbf{P}_c \to \mathbf{P}_{c-1} \to \cdots \to \mathbf{P}_1 \to \mathbf{P}_0 \to \mathbf{M}_i \to 0$$

we get the resolution

$$0 \to \mathbf{P}_0^* \to \mathbf{P}_1^* \to \cdots \to \mathbf{P}_{c-1}^* \to \mathbf{P}_c^* \to \operatorname{Ext}^c_{\mathbf{R}}(\mathbf{M}_i, \mathbf{R}) \simeq \mathbf{M}_{r-i}(-e-t) \to 0.$$

We conclude by applying an iterated mapping cone construction to (9)*. Note that the Cohen-Macaulay type of* $\mathbf{A}$ *is the same as the Cohen-Macaulay type of* $\mathbf{M}_r(-rs) = \mathbf{K_B}(t - rs)$*. Hence* $\mathbf{A}$ *is Gorenstein since the Cohen-Macaulay type of* $\mathbf{K_B}$ *is 1.*

Remark 11. *In order to further generalize Theorem 8, we can look precisely at which* $\operatorname{depth}_{\mathfrak{m}}$*-condition is needed to prove the exactness of* (9) *as well as the Gorenstein property of* $\mathbf{A}$*. This leads us to replace assumption 3 of the Theorem with the apparently weaker assumption*

$$3'. \quad \operatorname{depth}_{\mathfrak{m}} \mathbf{M}_{r-i} \geq \dim \mathbf{B} - i + 1 \quad \text{for} \quad 1 \leq i \leq r - 1.$$

A careful use of Serre Duality, along the lines of Remark 9, gives the maximal Cohen-Macaulay property for all modules $\mathbf{M}_i$ *under the assumption of* $3'$*. In a forthcoming paper, where we consider sections of a certain class of sheaves, we have been able to find conditions which guarantees that the saturation,* $\bar{\mathbf{A}}$*, of* $\mathbf{A}$ *is Gorenstein (so* $\operatorname{Proj} \mathbf{A}$ *is arithmetically Gorenstein). Under additional assumptions (but still weaker than the Cohen-Macaulay assumption) we have* $\mathbf{A}$ *is Gorenstein as well.*

3 Families of Maximal Cohen-Macaulay Modules

In this section, we apply Theorem 8 to several families of maximal Cohen-Macaulay modules. There is a unique module, up to twist, with rank $r = 1$ which satisfies the conditions of Theorem 8 (namely $\mathbf{K_B}$). In contrast, there are many modules which have rank $r \geq 2$ on an open set, U, and which satisfy the conditions of Theorem 8. If $\mathbf{M}$ is to be a candidate for such a module, it must certainly satisfy

$$\bigwedge^r \widetilde{\mathbf{M}}|_U \simeq \widetilde{\mathbf{K}}_{\mathbf{B}}(t)|_U. \tag{16}$$

With this in mind, we concentrate on the module of Kaehler differentials $\Omega_{\mathbf{B}}$, the normal module $\mathbf{N_B}$ and the first Koszul homology module $\mathbf{H}_1$ (where the latter is built on some minimal set of generators of $\mathbf{I} = \ker(\mathbf{R} \to \mathbf{B})$). It is well known that these modules satisfy the isomorphism given in (16) on some naturally defined U. For instance, for $\mathbf{N_B}$ and $\mathbf{H}_1$ we let U be the open set of $S = \mathrm{Proj}(\mathbf{B}) \hookrightarrow P = \mathrm{Proj}(\mathbf{R})$ where $U \hookrightarrow P$ is a local complete intersection. Then $\mathbf{N_B}$ satisfies (16) on U by [12], III, Theorem 7.11. Moreover, since the sheafification of (8) is exact on U, the top exterior powers of $(\mathbf{I}/\mathbf{I}^2)^* = \mathbf{N_B}$ and $\mathbf{H}_1$ therefore correspond on U, up to twist ($\mathbf{F}_1$ is a free $\mathbf{R}$-module). As a result, $\mathbf{H}_1$ satisfies (16).

However, Theorem 8 does not apply to $\Omega_{\mathbf{B}}$. Indeed consulting [5], we see that while many of the cohomology groups $\mathrm{H}^p_*(S, \wedge^q \widetilde{\Omega}_{\mathbf{B}})$ vanish (e.g. for $p \neq q$ if S is smooth), not *every* intermediate cohomology group vanishes as the maximal Cohen-Macaulay property of Theorem 8 requires. For the modules $\mathbf{N_B}$ and $\mathbf{H}_1$, we shall see that there are large classes of Cohen-Macaulay quotients over which all of the conditions of Theorem 8 are satisfied.

3.1 Canonical modules

In this subsection we briefly discuss a consequence of Theorem 8 with respect to sections of $\mathbf{K}^*_{\mathbf{B}}(s)$. If S is Gorenstein outside a closed set Z of codimension at least 2 in S, then we easily see that the canonical module satisfies the conditions of Theorem 8. The Gorenstein quotients constructed via these modules are given by an exact sequence of the form

$$0 \to \mathbf{K_B}(-s) \to \mathbf{B} \to \mathbf{A} \to 0. \tag{17}$$

Any quotient, $\mathbf{R}/(\mathbf{J}+\mathbf{J}')$, obtained from two geometrically Gorenstein linked Cohen-Macaulay quotients, $\mathbf{R}/\mathbf{J}$ and $\mathbf{R}/\mathbf{J}'$, is an example of a Gorenstein quotient, $\mathbf{A}$, as given by exact sequence (17). Indeed, since they are geometrically Gorenstein linked, $\mathbf{D} := \mathbf{R}/\mathbf{J} \cap \mathbf{J}'$ is Gorenstein by definition. Hence $\mathbf{K_D} \simeq \mathbf{D}(s)$ and $\mathrm{Hom}_{\mathbf{D}}(\mathbf{R}/\mathbf{J}, \mathbf{D}) \simeq \mathbf{K}_{(\mathbf{R}/\mathbf{J})}(-s)$. Considering the exact sequence

$$0 \to (\mathbf{J} + \mathbf{J}')/\mathbf{J} \to \mathbf{R}/\mathbf{J} \to \mathbf{R}/(\mathbf{J} + \mathbf{J}') \to 0 \tag{18}$$

and observing $(\mathbf{J} + \mathbf{J}')/\mathbf{J} \simeq \mathbf{J}'/\mathbf{J} \cap \mathbf{J}' \simeq \mathrm{Hom}_{\mathbf{D}}(\mathbf{R}/\mathbf{J}, \mathbf{D}) \simeq \mathbf{K}_{(\mathbf{R}/\mathbf{J})}(-s)$, we can transform sequence (18) into the form of sequence (17) by letting $\mathbf{B} = \mathbf{R}/\mathbf{J}$. If we have an $\mathbf{R}$-free resolution of $\mathbf{B}$ then we easily get an $\mathbf{R}$-free resolution of $\mathbf{A}$ by applying the mapping cone construction to sequence (17). The construction of Gorenstein algebras in this manner has been utilized by several authors, cf. [9], [10], [17], [18].

Example 12. *Let* $\mathbf{R} = k[a,b,c,d,e,f]$ *and let* $\mathbf{I} = (bc-ad, be-af, de-cf)$ *be the ideal of* 2×2 *minors of the* 2×3 *matrix of indeterminants of* $\mathbf{R}$. *In this case,* $\mathbf{B} = \mathbf{R}/\mathbf{I}$ *is Cohen-Macaulay of codimension 2,* $S = \operatorname{Proj}\mathbf{B}$ *is a local complete intersection (so we can take* $Z = \emptyset$*) and the minimal* $\mathbf{R}$*-free resolution of* $\mathbf{B}$ *is*

$$0 \to \mathbf{R}(-3)^2 \to \mathbf{R}(-2)^3 \to \mathbf{R} \to \mathbf{B} \to 0.$$

A regular section, $\sigma \in \mathrm{H}^0(S, \widetilde{\mathbf{K}}^*_{\mathbf{B}}(-1))$, *defines a Gorenstein algebra,* $\mathbf{A}$, *of codimension 3 with* $\mathbf{R}$*-free resolution*

$$0 \to \mathbf{R}(-5) \to \mathbf{R}(-3)^5 \to \mathbf{R}(-2)^5 \to \mathbf{R} \to \mathbf{A} \to 0.$$

An example of such a Gorenstein algebra is

$$\mathbf{A} = \mathbf{R}/(bc-ad, be-af, de-cf, a^2+ad+ce, ab+bd+cf).$$

3.2 Normal modules

In this subsection, we apply Theorem 8 to the case where $\mathbf{M} = \mathbf{N_B}$ and $\mathbf{B}$ is a codimension c licci quotient of $\mathbf{R}$. Our result overlaps work of Ellingsrud-Peskine [7], which treats the Artin Gorenstein case with $c = 2$ more generally.

Proposition 13. *Let* $\mathbf{R}$ *be a graded Gorenstein quotient of a polynomial ring and assume* $\mathbf{R}$ *has canonical module* $\mathbf{R}(e)$. *Let* $\mathbf{B}$ *be a codimension c graded licci quotient of* $\mathbf{R}$ *(for conclusions 3 and 4, we will assume* $\mathbf{B}$ *has finite projective dimension over* $\mathbf{R}$*). Let* $\mathbf{K_B} = \operatorname{Ext}^c_{\mathbf{R}}(\mathbf{B}, \mathbf{R}(e))$ *be the canonical module of* $\mathbf{B}$ *and let* $\mathbf{N_B}$ *be the normal module of* $\mathbf{B}$. *Let* $S = \operatorname{Proj}(\mathbf{B})$, *let Z be a closed scheme such that* $\dim(\mathbf{B}) - \dim(\mathbf{B}/\mathbf{I}(Z)) \geq \max(c, 2)$ *and let* $U = S - Z$. *If U is a local complete intersection in* $\operatorname{Proj}(\mathbf{R})$ *then*

1. *Every regular section,* $\sigma \in \mathrm{H}^0(U, \widetilde{\mathbf{N}}^*_{\mathbf{B}}(s))$, *defines a Gorenstein quotient* $\mathbf{R} \twoheadrightarrow \mathbf{A}$ *of codimension 2c.*
2. $\mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{N}}_{\mathbf{B}})$ *is a maximal Cohen-Macaulay B-module for every* $0 \leq i \leq c$
3. $\mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{N}}_{\mathbf{B}}) \simeq \operatorname{Ext}^i_{\mathbf{R}}(\mathbf{B}, \mathbf{B}) \simeq \operatorname{Tor}^{\mathbf{R}}_{c-i}(\mathbf{B}, \mathbf{K_B})(-e)$
4. *There is an exact sequence*

$$0 \to \mathbf{K_B}(-e-cs) \to \operatorname{Ext}^{c-1}_{\mathbf{R}}(\mathbf{B},\mathbf{B})((1-c)s) \to \cdots \to \operatorname{Ext}^{i}_{\mathbf{R}}(\mathbf{B},\mathbf{B})(-is)$$
$$\to \cdots \to \operatorname{Ext}^{1}_{\mathbf{R}}(\mathbf{B},\mathbf{B})(-s) \simeq \mathbf{N_B}(-s) \xrightarrow{\sigma^*} \mathbf{B} \to \mathbf{A} \to 0$$

Remark 14. *A consequence of the maximal Cohen-Macaulay property of* $\mathbf{N}_i := \mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{N}}_{\mathbf{B}})$ *of Proposition 13 is that for every* $i \geq 0$, *we have*

$$\mathrm{H}^p_*(U, \wedge^i \widetilde{\mathbf{N}}_{\mathbf{B}}|_U) = 0 \quad \textit{for} \quad 1 \leq p \leq c-2$$
$$\mathrm{H}^p_*(S, \widetilde{\mathbf{N}}_i) = 0 \quad \textit{for} \quad 1 \leq p \leq \dim \mathbf{B} - 2.$$

Proof. If $c = 1$, then Proposition 13 is trivial to prove. Indeed $\mathbf{R} \twoheadrightarrow \mathbf{A}$ is in this case a complete intersection. Suppose $c \geq 2$. We claim that

$$\mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{N}}_{\mathbf{B}}) \simeq \mathrm{Ext}^i_{\mathbf{R}}(\mathbf{B}, \mathbf{B}) \simeq \mathrm{Tor}^{\mathbf{R}}_{c-i}(\mathbf{B}, \mathbf{K}_{\mathbf{B}})(-e).$$

To show it, we consider a minimal $\mathbf{R}$-free resolution

$$0 \to \mathbf{F}_c \to \mathbf{F}_{c-1} \to \cdots \to \mathbf{F}_1 \to \mathbf{F}_0 = \mathbf{R} \to \mathbf{B} \to 0. \tag{19}$$

If we apply $(-)^* = \mathrm{Hom}_{\mathbf{R}}(-, \mathbf{R})$, we get the exact sequence

$$0 \to \mathbf{F}_0^* \to \mathbf{F}_1^* \to \cdots \to \mathbf{F}_{c-1}^* \to \mathbf{F}_c^* \to \mathbf{K}_{\mathbf{B}}(-e) \to 0. \tag{20}$$

It follows that the homology groups of the complex

$$0 \to \mathbf{F}_0^* \otimes_{\mathbf{R}} \mathbf{B} \to \mathbf{F}_1^* \otimes_{\mathbf{R}} \mathbf{B} \to \cdots \to \mathbf{F}_{c-1}^* \otimes_{\mathbf{R}} \mathbf{B} \to \mathbf{F}_c^* \otimes_{\mathbf{R}} \mathbf{B} \to 0 \tag{21}$$

are $\mathrm{Tor}^{\mathbf{R}}_i(\mathbf{B}, \mathbf{K}_{\mathbf{B}})(-e)$. On the other hand since $\mathbf{F}_i^* \otimes_{\mathbf{R}} \mathbf{B} \simeq \mathrm{Hom}_{\mathbf{R}}(\mathbf{F}_i, \mathbf{B})$, we get the same complex (21) by applying $\mathrm{Hom}_{\mathbf{R}}(-, \mathbf{B})$ to (19). The homology groups of (21) are therefore also given by $\mathrm{Ext}^{c-i}_{\mathbf{R}}(\mathbf{B}, \mathbf{B})$. This proves the right isomorphism of the claim.

To see the other isomorphism, we remark that

$$\mathrm{Tor}^{\mathbf{R}}_{c-i}\widetilde{(\mathbf{B}, \mathbf{K}_{\mathbf{B}})}|_U \simeq (\mathrm{Tor}^{\mathbf{R}}_{c-i}\widetilde{(\mathbf{B}, \mathbf{B})} \otimes_{\mathbf{B}} \mathbf{K}_{\mathbf{B}})|_U \simeq (\wedge^{c-i} \widetilde{\mathbf{N}}^*_{\mathbf{B}} \otimes_{\mathcal{O}_S} \widetilde{\mathbf{K}}_{\mathbf{B}})|_U \tag{22}$$

since $\wedge^i \widetilde{\mathbf{N}}^*_{\mathbf{B}}|_U$ is locally free, cf. [1], lem.17, p.169. Moreover, $\wedge^c \widetilde{\mathbf{N}}_{\mathbf{B}}|_U \simeq \widetilde{\mathbf{K}}_{\mathbf{B}}(-e)|_U$ by [12], Chapter III, Theorem 7.11. Combining with

$$\wedge^i \widetilde{\mathbf{N}}_{\mathbf{B}}|_U \simeq ((\wedge^{c-i} \widetilde{\mathbf{N}}_{\mathbf{B}})^* \otimes_{\mathcal{O}_S} \wedge^c \widetilde{\mathbf{N}}_{\mathbf{B}})|_U \simeq (\wedge^{c-i} \widetilde{\mathbf{N}}^*_{\mathbf{B}} \otimes_{\mathcal{O}_S} \widetilde{\mathbf{K}}_{\mathbf{B}}(-e))|_U$$

and the fact that all nontrivial $\mathrm{Tor}^{\mathbf{R}}_{c-i}(\mathbf{B}, \mathbf{K}_{\mathbf{B}})$ are maximal Cohen-Macaulay $\mathbf{B}$-modules (by Theorem 7), we get

$$\mathrm{Tor}^{\mathbf{R}}_{c-i}(\mathbf{B}, \mathbf{K}_{\mathbf{B}})(-e) \simeq \mathrm{H}^0_*(U, \mathrm{Tor}^{\mathbf{R}}_{c-i}\widetilde{(\mathbf{B}, \mathbf{K}_{\mathbf{B}})}(-e)) \simeq \mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{N}}_{\mathbf{B}}) \tag{23}$$

Indeed since the $\mathrm{depth}_{\mathbf{I}(Z)}$ of a maximal Cohen-Macaulay $\mathbf{B}$-module is equal to $\mathrm{depth}_{\mathbf{I}(Z)} \mathbf{B}$ (which is ≥ 2 by assumption), we get the left isomorphism in (23), and the claim is proved.

Moreover (23) shows that $\mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{N}}_{\mathbf{B}})$ is a maximal Cohen-Macaulay module for every $0 \leq i \leq c$ since $\mathrm{Tor}^{\mathbf{R}}_{c-i}(\mathbf{B}, \mathbf{K}_{\mathbf{B}})$ is a maximal Cohen-Macaulay module for every $0 \leq i \leq c$. Finally once having proved this maximal Cohen-Macaulay property and the isomorphism $\wedge^c \widetilde{\mathbf{N}}_{\mathbf{B}}|_U \simeq \widetilde{\mathbf{K}}_{\mathbf{B}}(-e)|_U$, Proposition 13 follows directly from Theorem 8, and we are done. □

Remark 15. *We see from the proof of Proposition 13 that we can weaken the licci assumption slightly. Instead of assuming that* $\mathbf{B}$ *is licci, we can instead assume that* $\mathrm{Tor}^{\mathbf{R}}_i(\mathbf{B}, \mathbf{K}_{\mathbf{B}})$ *is a maximal Cohen-Macaulay* $\mathbf{B}$*-module for each* $0 \leq i \leq c$*. This extends Proposition 13 because it is known that there are nice classes of non-licci algebras* $\mathbf{B}$ *for which every nontrivial* $\mathrm{Tor}^{\mathbf{R}}_i(\mathbf{B}, \mathbf{K}_{\mathbf{B}})$ *is a maximal Cohen-Macaulay module [15], [23].*

Remark 16. *Suppose that the codimension of* $\mathbf{B}$ *is* 2 *or that* $\mathbf{B}$ *is Gorenstein with codimension* 3*. In this case, we can explicitly write down an* $\mathbf{R}$*-free resolution of the Gorenstein algebra* $\mathbf{A}$ *of Proposition 13 in terms of the free* $\mathbf{R}$*-modules,* $\mathbf{F}_i$*, in the resolution of* $\mathbf{I} = \ker(\mathbf{R} \to \mathbf{B})$ *(as given in exact sequence* (19)*).*

If the codimension of $\mathbf{B}$ *is* 2 *then* (19) *is the short exact sequence*

$$0 \to \mathbf{F}_2 \to \mathbf{F}_1 \to \mathbf{I} \to 0. \tag{24}$$

To determine an $\mathbf{R}$*-free resolution of* $\mathbf{A}$ *we apply* $\mathrm{Hom}_{\mathbf{R}}(-, \mathbf{I})$ *to* (24) *(and use* $\mathbf{N_B} \simeq \mathrm{Ext}^1_{\mathbf{R}}(\mathbf{I}, \mathbf{I})$*) to get*

$$0 \to \mathbf{R} \to \mathbf{F}_1^* \otimes_{\mathbf{R}} \mathbf{I} \to \mathbf{F}_2^* \otimes_{\mathbf{R}} \mathbf{I} \to \mathbf{N_B} \to 0. \tag{25}$$

The map $\mathbf{R} \to \mathbf{F}_1^* \otimes \mathbf{I}$ *in* (25) *lifts to the map* $\mathbf{R} \to \mathbf{F}_1^* \otimes \mathbf{F}_1 \simeq \mathrm{Hom}(\mathbf{F}_1, \mathbf{F}_1)$ *given by* $1 \to id_{\mathbf{F}_1}$ *and further to the map* $\Delta : \mathbf{R} \to (\mathbf{F}_1^* \otimes \mathbf{F}_1) \oplus (\mathbf{F}_2^* \otimes \mathbf{F}_2)$ *defined by* $\Delta(1) = (id_{\mathbf{F}_1}, -id_{\mathbf{F}_2})$*. Using this lifting, the short exact sequences we get by tensoring* (24) *with* $\mathbf{F}_1^*$ *and* $\mathbf{F}_2^*$ *and the mapping cone construction, we get a minimal* $\mathbf{R}$*-free resolution of* $\mathbf{N_B}$

$$0 \to \mathbf{F}_1^* \otimes_{\mathbf{R}} \mathbf{F}_2 \to ((\mathbf{F}_1^* \otimes_{\mathbf{R}} \mathbf{F}_1) \oplus (\mathbf{F}_2^* \otimes_{\mathbf{R}} \mathbf{F}_2))/\Delta(\mathbf{R}) \to \mathbf{F}_2^* \otimes_{\mathbf{R}} \mathbf{F}_1 \to \mathbf{N_B} \to 0. \tag{26}$$

Finally, we use the $\mathbf{R}$*-free resolutions of* $\mathbf{B}$ *and* $\mathbf{K_B}(-e)$ *(as given in* (19) *and* (20)*) together with the exact sequences*

$$0 \to \mathbf{K_B}(-e-2s) \to \mathbf{N_B}(-s) \to \mathbf{I_{B/A}} \to 0$$

and

$$0 \to \mathbf{I_{B/A}} \to \mathbf{B} \to \mathbf{A} \to 0,$$

and apply the mapping cone construction to yield the $\mathbf{R}$*-free resolution*

$$\begin{gathered} 0 \to \mathbf{R}(-2s) \to \mathbf{F}_1^*(-2s) \oplus (\mathbf{F}_1^* \otimes_{\mathbf{R}} \mathbf{F}_2)(-s) \to \\ \mathbf{F}_2^*(-2s) \oplus (((\mathbf{F}_1^* \otimes_{\mathbf{R}} \mathbf{F}_1) \oplus (\mathbf{F}_2^* \otimes_{\mathbf{R}} \mathbf{F}_2))/\Delta(\mathbf{R}))(-s) \oplus \mathbf{F}_2 \to \\ (\mathbf{F}_2^* \otimes_{\mathbf{R}} \mathbf{F}_1)(-s) \oplus \mathbf{F}_1 \to \mathbf{R} \to \mathbf{A} \to 0. \end{gathered}$$

The resolution is minimal for most values of s *and for all values of* $s \gg 0$*.*

If the codimension of $\mathbf{B}$ *is* 3 *and if* $\mathbf{B}$ *is Gorenstein, then* (19) *is the exact sequence*

$$0 \to \mathbf{R}(-f) = \mathbf{F}_3 \to \mathbf{F}_2 \to \mathbf{F}_1 \to \mathbf{I} \to 0.$$

In this case, $\mathbf{N_B} \simeq (\wedge^2 \mathbf{I})(f)$ *provided that* $\mathrm{char} k \neq 2$*, cf. [16], Proposition 2.4. Twisting the exact sequence*

$$0 \to \mathbf{F}_2 \otimes \mathbf{F}_3 \to (\mathbf{F}_1 \otimes \mathbf{F}_3) \oplus S_2\mathbf{F}_2 \to \mathbf{F}_1 \otimes \mathbf{F}_2 \to \wedge^2 \mathbf{F}_1 \to \wedge^2 \mathbf{I} \to 0 \tag{27}$$

of Lebelt-Weyman by f*, we obtain at once a minimal* $\mathbf{R}$*-free resolution of* $\mathbf{N_B}$*. Recalling that the canonical module,* $\mathbf{K_B}$*, satisfies* $\mathbf{K_B}(-e) \simeq \mathbf{B}(f)$ *, cf.* (19) *and* (20)*, and that the canonical module of* $\mathbf{R}$ *is* $\mathbf{R}(e)$*, we get*

$$\mathrm{Ext}^3_{\mathbf{R}}(\mathbf{N_B}, \mathbf{R}) \simeq \mathrm{Ext}^3_{\mathbf{R}}(\mathbf{N_B}, \mathbf{R}(e))(-e) \simeq \mathrm{Hom}_{\mathbf{B}}(\mathbf{N_B}, \mathbf{K_B})(-e) \simeq (\mathbf{I}/\mathbf{I}^2)(f). \tag{28}$$

We have $\mathrm{Ext}^i_{\mathbf{R}}(\mathbf{N_B}, \mathbf{R}) = 0$ *for* $i \neq 3$ *since* $\mathbf{N_B}$ *is a maximal Cohen-Macaulay module. We have* $\mathbf{F}_{3-i}(f)^* \simeq \mathbf{F}_i$ *for* $0 \leq i \leq 3$ *since* $\mathbf{B}$ *is Gorenstein. Noting these isomorphisms, the* $\mathbf{R}$*-dual sequence of* (27) *yields the* $\mathbf{R}$*-free resolution*

$$0 \to \wedge^2 \mathbf{F}_2 \to (\mathbf{F}_1 \otimes \mathbf{F}_2) \to S_2\mathbf{F}_1 \oplus \mathbf{F}_2 \to \mathbf{F}_1 \to \mathbf{I}/\mathbf{I}^2 \to 0. \tag{29}$$

Splitting the exact sequence

$$0 \to \mathbf{B}(f-3s) \to (\mathbf{I}/\mathbf{I}^2)(f-2s) \to \mathbf{N_B}(-s) \to \mathbf{I_{B/A}} \to 0$$

of Proposition 13 into two short exact sequences and using the mapping cone construction, the resolution of $\mathbf{I}/\mathbf{I}^2$ *and the resolution of* $\mathbf{N_B}$*, we obtain the following* $\mathbf{R}$*-free resolution*

$$\begin{gathered} 0 \to \mathbf{R}(-3s) \to \mathbf{F}_2(f-3s) \oplus \wedge^2\mathbf{F}_2(f-2s) \to \\ \mathbf{F}_1(f-3s) \oplus (\mathbf{F}_1 \otimes \mathbf{F}_2)(f-2s) \oplus \mathbf{F}_2(-s) \to \\ \mathbf{R}(f-3s) \oplus S_2\mathbf{F}_1(f-2s) \oplus \mathbf{F}_2(f-2s) \oplus S_2\mathbf{F}_2(f-s) \oplus \mathbf{F}_1(-s) \oplus \mathbf{R}(-f) \\ \to \mathbf{F}_1(f-2s) \oplus (\mathbf{F}_1 \otimes \mathbf{F}_2)(f-s) \oplus \mathbf{F}_2 \to \wedge^2\mathbf{F}_1(f-s) \oplus \mathbf{F}_1 \to \mathbf{R} \to \mathbf{A} \to 0. \end{gathered}$$

Again, the resolution is minimal for most values of s *and for all* $s \gg 0$*.*

Example 17. *Let* $\mathbf{R} = k[a,b,c,d,e,f]$ *and let* $\mathbf{I} = (bc-ad, be-af, de-cf)$ *be the ideal of* 2×2 *minors of the* 2×3 *matrix of indeterminants of* $\mathbf{R}$*. In this case,* $\mathbf{B} = \mathbf{R}/\mathbf{I}$ *is Cohen-Macaulay of codimension 2,* $S = \mathrm{Proj}\,\mathbf{B}$ *is a local complete intersection (so we can take* $Z = \emptyset$*) and the minimal* $\mathbf{R}$*-free resolution of* $\mathbf{B}$ *is*

$$0 \to \mathbf{R}(-3)^2 \to \mathbf{R}(-2)^3 \to \mathbf{R} \to \mathbf{B} \to 0.$$

A regular section, $\sigma \in \mathrm{H}^0(S, \widetilde{\mathbf{N}}^*_{\mathbf{B}}(3))$*, defines a Gorenstein algebra,* $\mathbf{A}$*, of codimension 4 with* $\mathbf{R}$*-free resolution*

$$0 \to \mathbf{R}(-6) \to \mathbf{R}(-4)^9 \to \mathbf{R}(-3)^{16} \to \mathbf{R}(-2)^9 \to \mathbf{R} \to \mathbf{A} \to 0.$$

An example of such a Gorenstein algebra is

$$\mathbf{A} = \mathbf{R}/(bc-ad, be-af, de-cf, ab-c^2, a^2+ce, b^2-cd, ac+af, c^2+cf, ad+bf).$$

3.3 First Koszul homology modules

Huneke [14] has proved that the Koszul homology modules $\mathbf{H}_i(\mathbf{I})$, built on some minimal set of generators of $\mathbf{I}$, are Cohen-Macaulay provided $\mathbf{B} = \mathbf{R}/\mathbf{I}$ is licci. We shall see that this result provides us with a class of Cohen-Macaulay quotients of $\mathbf{R}$ to which we may apply Theorem 8 (with $\mathbf{M} = \mathbf{H}_1(\mathbf{I})$). If all $\mathbf{H}_i(\mathbf{I})$ are Cohen-Macaulay, then $\mathbf{B} = \mathbf{R}/\mathbf{I}$ is called strongly Cohen-Macaulay. We have the following result.

Proposition 18. *Let* $\mathbf{R}$ *be a graded Gorenstein quotient of a polynomial ring and assume* $\mathbf{R}$ *has canonical module* $\mathbf{R}(e)$*. Let* $\mathbf{B} = \mathbf{R}/\mathbf{I}$ *be a codimension* c *graded strongly Cohen-Macaulay quotient of* $\mathbf{R}$*. Let* $\mathbf{K_B} = \mathrm{Ext}^c_{\mathbf{R}}(\mathbf{B}, \mathbf{R}(e))$ *be the canonical module of* $\mathbf{B}$ *and let* $\mathbf{H}_1 = \mathbf{H}_1(\mathbf{I}), \mathbf{H}_2 = \mathbf{H}_2(\mathbf{I}), \ldots$ *be the Koszul homology modules of* $\mathbf{I}$*. Let* $\mu(\mathbf{I})$ *denote the number of minimal generators of* $\mathbf{I}$ *and let* $r = \mu(\mathbf{I}) - c$*. Let* $d = \Sigma n_{1i}$ *where* $\mathbf{F}_1 = \oplus\mathbf{R}(-n_{1i})$ *minimally surjects to* $\mathbf{I}$*. Let* $S = \mathrm{Proj}(\mathbf{B})$*, let* Z *be a closed scheme such that* $\dim(\mathbf{B}) - \dim(\mathbf{B}/\mathbf{I}(Z)) \geq \max(r, 2)$ *and let* $U = S - Z$*. If* U *is a local complete intersection in* $\mathrm{Proj}(\mathbf{R})$ *then*

1. *Every regular section, $\sigma \in \mathrm{H}^0(U, \widetilde{\mathbf{H}}_1^*(s))$, defines a Gorenstein quotient $\mathbf{R} \twoheadrightarrow \mathbf{A}$ of codimension $\mu(\mathbf{I})$.*

2. $\mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{H}}_1)$ *is a maximal Cohen-Macaulay* $\mathbf{B}$*-module for every* $1 \le i \le r$

3. $\mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{H}}_1) \simeq \mathbf{H}_i$

4. *Letting $a = e + d + rs$, there is an exact sequence*

$$0 \to \mathbf{K}_{\mathbf{B}}(-a) \to \mathbf{H}_{r-1}((1-r)s) \to \cdots \to \mathbf{H}_2(-2s) \to \mathbf{H}_1(-s) \to \mathbf{B} \to \mathbf{A} \to 0.$$

Proof. Looking to (8), and taking maximal exterior powers, we see that

$$\wedge^r \widetilde{\mathbf{H}}_1|_U \simeq (\wedge^c \widetilde{\mathbf{N}}_{\mathbf{B}} \otimes_{\mathcal{O}_S} \wedge^{\mu(\mathbf{I})}(\oplus \widetilde{\mathbf{B}}(-n_{1i})))|_U \simeq \widetilde{\mathbf{K}}_{\mathbf{B}}(-e-d)|_U \tag{30}$$

since $(\wedge^c \widetilde{\mathbf{N}}_{\mathbf{B}})|_U \simeq \widetilde{\mathbf{K}}_{\mathbf{B}}(-e)|_U$ and $d = \Sigma n_{1i}$. To apply Theorem 8 to $\mathbf{M} = \mathbf{H}_1$, it suffices to prove that $\mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{H}}_1)$ is a maximal Cohen-Macaulay $\mathbf{B}$-module for each $1 \le i \le r$. Since $\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_r$ are maximal Cohen-Macaulay modules (cf. [14], Remark 1.3 to see $\dim \mathbf{H}_i(\mathbf{I}) = \dim \mathbf{B}$), it is enough to prove

$$\mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{H}}_1) \simeq \mathbf{H}_i.$$

However, since $\widetilde{\mathbf{H}}_1$ is locally free on U, one knows that $\wedge^i \widetilde{\mathbf{H}}_1|_U \simeq \widetilde{\mathbf{H}}_i|_U$, cf. [20] (the inaccuracy of [20] does not effect our conclusion because $\mathbf{B}$ contains a field, see [21]). We conclude the proof using $\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{H}_i = \operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} \ge 2$. $\square$

To construct Gorenstein algebras of small codimension, we use Proposition 13 and Proposition 18 for small c and small $r = \mu(\mathbf{I}) - c$. We have in particular

Corollary 19. *Let $\mathbf{R} \twoheadrightarrow \mathbf{B} = \mathbf{R}/\mathbf{I}$ be a graded Cohen-Macaulay quotient of codimension $c \ge 1$ and a local complete intersection outside a set Z of codimension $\ge \max(\mu(\mathbf{I}) - c, 2)$ in $S = \operatorname{Proj} \mathbf{B}$.*

1. *If $\mu(\mathbf{I}) \le c + 2$ or*

2. *if $\mathbf{B}$ is licci or*

3. *if $c = 2$ or*

4. *if $c = 3$ and $\mathbf{B}$ is Gorenstein*

then any regular section, $\sigma \in \mathrm{H}^0(S - Z, \widetilde{\mathbf{H}}_1^*(s))$, defines a Gorenstein quotient $\mathbf{R} \twoheadrightarrow \mathbf{A}$ of codimension $\mu(\mathbf{I})$.

Proof. $\mathbf{B}$ is strongly Cohen-Macaulay if $\mu(\mathbf{I}) - c \le 2$ [24], Corollary 3.3.14. If $\mathbf{B}$ is licci then $\mathbf{B}$ is strongly Cohen-Macaulay [14]. If $c = 2$ and $\mathbf{B}$ is Cohen-Macaulay then $\mathbf{B}$ is licci [8]. If $c = 3$ and $\mathbf{B}$ is Gorenstein then $\mathbf{B}$ is licci [25]. $\square$

Remark 20. *If the codimension of* $\mathbf{B}$ *is* 2 *or if* $\mathbf{B}$ *is Gorenstein with codimension* 3 *and* $r \leq 3$, *we can explicitly write down an* $\mathbf{R}$*-free resolution of the Gorenstein algebra* $\mathbf{A}$ *of Proposition 18 in terms of the free modules,* $\mathbf{F}_i$, *in the resolution of* $\mathbf{I} = \ker(\mathbf{R} \to \mathbf{B})$ *as given in* (19).

If the codimension of $\mathbf{B}$ *is* 2 *then exact sequence* (19) *becomes the short exact sequence*

$$0 \to \mathbf{F}_2 \to \mathbf{F}_1 \to \mathbf{I} \to 0. \tag{31}$$

By [2], we have an exact sequence

$$0 \to \wedge^{i+1}\mathbf{F}_2 \to \wedge^{i+1}\mathbf{F}_1 \to \wedge^{i}\mathbf{F}_2 \to \mathbf{H}_i \to 0$$

for every $i \geq 1$. *Splitting the exact sequence*

$$0 \to \mathbf{K}_{\mathbf{B}}(-a) \to \cdots \to \mathbf{H}_2(-2s) \to \mathbf{H}_1(-s) \to \mathbf{B} \to \mathbf{A} \to 0 \tag{32}$$

of Proposition 18 into short exact sequences and applying the mapping cone construction, we obtain an $\mathbf{R}$*-free resolution of* $\mathbf{A}$.

If $\mathbf{B}$ *is Gorenstein with codimension* 3 *and if* $1 \leq r \leq 3$ *then we remark that* $r = 2$ *since* $\mu(I) = r + c$ *must be odd. In view of* (32) *it suffices to find an* $\mathbf{R}$*-free resolution of* $\mathbf{H}_1$. *If we combine the isomorphism of conclusion (2) of Theorem 8 with Proposition 18, noting that* $t = -e - d$ *by* (30) *and* $\mathbf{K}_{\mathbf{B}}(-e) \simeq \mathbf{B}(f)$, *we get* $\mathbf{H}_i \simeq \mathbf{H}^*_{r-i}(f-d)$. *Hence it suffices to find an* $\mathbf{R}$*-free resolution of (the maximal Cohen-Macaulay module)* $\mathbf{H}^*_1$. *Now since* $\mathbf{B}$ *is Gorenstein, sequence* (8) *is exact (cf. Section 1). In addition, its* $\mathbf{B}$*-dual*

$$0 \to \mathbf{N}_{\mathbf{B}} \to \mathbf{F}^*_1 \otimes \mathbf{B} \to \mathbf{H}^*_1 \to 0 \tag{33}$$

is exact since $\operatorname{Ext}^1_{\mathbf{B}}(\mathbf{I}/\mathbf{I}^2, \mathbf{B}) = 0$ *by* (7). *The* $\mathbf{R}$*-free resolution of* $\mathbf{N}_{\mathbf{B}}$ *is given by twisting sequence* (27) *by* f *(noting that* $\mathbf{N}_{\mathbf{B}} \simeq (\wedge^2\mathbf{I})(f)$ *if char* $k \neq 2$*). The mapping cone construction applied to* (33) *yields*

$$\begin{gathered} 0 \to \mathbf{F}_1 \oplus S_2\mathbf{F}_2(f) \to (\mathbf{F}_2 \otimes \mathbf{F}_2(f)) \oplus (\mathbf{F}_1 \otimes \mathbf{F}_2(f)) \to \\ \wedge^2\mathbf{F}_1(f) \oplus (\mathbf{F}_1 \otimes \mathbf{F}_2(f)) \to \mathbf{F}_2(f) \to \mathbf{H}^*_1 \to 0. \end{gathered}$$

This reduces to

$$0 \to \mathbf{F}_1 \to \wedge^2\mathbf{F}_2(f) \to \wedge^2\mathbf{F}_1(f) \to \mathbf{F}_2(f) \to \mathbf{H}^*_1 \to 0.$$

Finally, combining with $\mathbf{H}_1 \simeq \mathbf{H}^*_1(f-d)$ *and* (32) *and using that* $2f = d$ *by [3] (since* $\mu(\mathbf{I}) = 5$*) we readily obtain the following* $\mathbf{R}$*-free resolution of* $\mathbf{A}$

$$\begin{gathered} 0 \to \mathbf{R}(-2f-2s) \to \mathbf{F}_2(-f-2s) \oplus \mathbf{F}_1(-f-s) \to \mathbf{F}_1(-f-2s) \oplus \wedge^2\mathbf{F}_2(-s) \oplus \mathbf{R}(-f) \to \\ \mathbf{R}(-f-2s) \oplus \wedge^2\mathbf{F}_1(-s) \oplus \mathbf{F}_2 \to \mathbf{F}_2(-s) \oplus \mathbf{F}_1 \to \mathbf{R} \to \mathbf{A} \to 0. \end{gathered}$$

Example 21. *Let* $\mathbf{R} = k[a,b,c,d,e,f]$ *and let* $\mathbf{I} = (bc-ad, be-af, de-cf, a^2+ad+ce, ab+bd+cf)$ *be the ideal defining the Gorenstein algebra constructed in Example 12. In this case,* $\mathbf{B} = \mathbf{R}/\mathbf{I}$ *is Gorenstein of codimension 3,* $S = \operatorname{Proj}\mathbf{B}$ *is a local complete intersection (but not smooth) and the minimal* $\mathbf{R}$*-free resolution of* $\mathbf{B}$ *is*

$$0 \to \mathbf{R}(-5) \to \mathbf{R}(-3)^5 \to \mathbf{R}(-2)^5 \to \mathbf{R} \to \mathbf{B} \to 0.$$

A regular section, $\sigma \in \mathrm{H}^0(S, \widetilde{\mathbf{H}}_1^*(-1))$ *defines a Gorenstein algebra,* $\mathbf{A}$*, of codimension* $\mu(\mathbf{I}) = 5$ *with* $\mathbf{R}$*-free resolution*

$$0 \to \mathbf{R}(-8) \to \mathbf{R}(-6)^{10} \to \mathbf{R}(-5)^{16} \to \mathbf{R}(-3)^{16} \to \mathbf{R}(-2)^{10} \to \mathbf{R} \to \mathbf{A} \to 0.$$

An example of such a Gorenstein algebra is

$$\mathbf{A} = \mathbf{R}/(\mathbf{I} + (b^2 - cd + df, c^2 + bd, ac + cd - e^2 + f^2, bd - ae + bf, d^2 - ce + df)).$$

3.4 Further applications

In this subsection we give some ideas and examples of how we can use Theorem 8 to construct other Gorenstein algebras, especially when the rank of $\mathbf{M}$ is small. This will lead to Gorenstein quotients $\mathbf{R} \twoheadrightarrow \mathbf{B} = \mathbf{R}/\mathbf{I}$ of small codimension. Throughout this subsection, we assume that $\mathbf{B}$ is of finite projective dimension over $\mathbf{R}$.

Indeed, one may hope to construct Gorenstein quotients by applying Theorem 8 to the left term of a short exact sequence provided the (dual of the) right term is "good" for the Theorem (as we did in (30) using (8)). For instance looking also to (19), we have a surjection $\mathbf{F}_2 \otimes \mathbf{B} \twoheadrightarrow \mathbf{H}_1$ if $\mathbf{H}_1 \to \mathbf{F}_1 \otimes \mathbf{B}$ is injective (cf. (8)). If we define $\mathbf{G}_2$ by the exact sequence

$$0 \to \mathbf{G}_2 \to \mathbf{F}_2 \otimes \mathbf{B} \to \mathbf{H}_1 \to 0 \tag{34}$$

and if U is a local complete intersection in $\mathrm{Proj}(\mathbf{R})$, then a regular section $\sigma \in \mathrm{H}^0(U, \widetilde{\mathbf{G}}_2(s))$ defines a Gorenstein quotient in good cases. Indeed, sheafifying (34) and taking maximal exterior powers, we get (16). The module $\mathbf{G}_2^*$ is maximal Cohen-Macaulay in the licci case by the next lemma. However, we have not been able to to show the maximal Cohen-Macaulayness of all $\mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{G}}_2^*)$ except in the low rank case of Remark 27.

To analyze this idea further, it seems more natural to consider the dual of (19), or more precisely the complex

$$0 \to \mathbf{F}_0^* \otimes_{\mathbf{R}} \mathbf{B} \xrightarrow{\psi_0} \mathbf{F}_1^* \otimes_{\mathbf{R}} \mathbf{B} \xrightarrow{\psi_1} \cdots \to \mathbf{F}_{c-1}^* \otimes_{\mathbf{R}} \mathbf{B} \xrightarrow{\psi_{c-1}} \mathbf{F}_c^* \otimes_{\mathbf{R}} \mathbf{B} \to 0 \tag{35}$$

where $\operatorname{coker} \psi_{c-1} = \mathbf{K}_{\mathbf{B}}(-e)$ and $\mathbf{F}_0^* \otimes_{\mathbf{R}} \mathbf{B} = \mathbf{B}$ leads to $\psi_0 = 0$. As we have seen when we considered (21), the homology of (35) is given by the modules $\mathrm{Tor}_i^{\mathbf{R}}(\mathbf{B}, \mathbf{K}_{\mathbf{B}})(-e)$, and $\mathbf{N}_{\mathbf{B}} \simeq \mathrm{Tor}_{c-1}^{\mathbf{R}}(\mathbf{B}, \mathbf{K}_{\mathbf{B}})(-e) = \ker \psi_1$.

Lemma 22. *If* $\mathbf{R} \twoheadrightarrow \mathbf{B} = \mathbf{R}/\mathbf{I}$ *is licci of codimension at least 2 and if* $S \hookrightarrow P$ *is a local complete intersection on a non-empty open set* $U = S - Z$*, then* $\operatorname{coker} \psi_i$*,* $\ker \psi_i$ *and nontrivial* $\operatorname{im} \psi_i$ *are maximal Cohen-Macaulay* $\mathbf{B}$*-modules for* $1 \le i \le c-1$*. Moreover, when* $\dim \mathbf{B} - \dim \mathbf{B}/\mathbf{I}(Z) \ge 2$*, we have* $\mathbf{H}_1^* \simeq \operatorname{im} \psi_1$*,* $\mathbf{G}_2^* \simeq \operatorname{coker} \psi_1$ *and* $\mathbf{H}_1^*$ *(if nontrivial) and* $\mathbf{G}_2^*$ *are maximal Cohen-Macaulay* $\mathbf{B}$*-modules.*

Proof. Since $\operatorname{coker} \psi_{c-1} = \mathbf{K}_{\mathbf{B}}(-e)$, we suppose by induction that $\operatorname{coker} \psi_j$ are maximal Cohen-Macaulay modules for $j > i$. Looking to the exactness of the horizontal sequences in

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{im} \psi_i & \to & F_{i+1}^* \otimes_R B & \to & \operatorname{coker} \psi_i & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \ker \psi_{i+1} & \to & F_{i+1}^* \otimes_R B & \to & \operatorname{im} \psi_{i+1} & \to & 0 \end{array}$$

we see by (2) that nontrivial $\operatorname{im}\psi_j$ and $\ker\psi_j$ are maximal Cohen-Macaulay modules for $j > i$. Now $\ker\psi_{i+1}/\operatorname{im}\psi_i \simeq \operatorname{Tor}^{\mathbf{R}}_{c-i-1}(\mathbf{B},\mathbf{K_B})(-e)$ is a maximal Cohen-Macaulay module (cf. Theorem 7 and note that $U \neq \emptyset$ shows that the Tor-group is non-vanishing by (22)). Hence, $\operatorname{im}\psi_i$ (if non-vanishing), $\ker\psi_i$ and by the snake lemma, $\operatorname{coker}\psi_i$, are maximal Cohen-Macaulay modules.

To finish the proof of the lemma, it suffices to prove

$$\mathbf{H}_1^* = \operatorname{im}\psi_1 \quad \text{and} \quad \mathbf{G}_2^* \simeq \operatorname{coker}\psi_1.$$

Indeed, for $i = 1$ and 2 we get $\operatorname{Ext}^i_{\mathbf{B}}(\mathbf{I}/\mathbf{I}^2,\mathbf{B}) = 0$ by (7). Hence we have the short exact sequence (33) in this case as well (i.e. we get $\mathbf{H}_1^* = \operatorname{im}\psi_1$). Since $\operatorname{Ext}^1_{\mathbf{B}}(\mathbf{H}_1,\mathbf{B}) \simeq \operatorname{Ext}^2_{\mathbf{B}}(\mathbf{I}/\mathbf{I}^2,\mathbf{B})$ by (8), the conclusion follows by applying $(-)^*$ to (34). □

Proposition 23. *Let $\mathbf{R} \twoheadrightarrow \mathbf{B} = \mathbf{R}/\mathbf{I}$ be licci of codimension $c \geq 2$ and $S = \mathit{Proj}(\mathbf{B}) \hookrightarrow \mathit{Proj}(\mathbf{R})$ a local complete intersection on a non-empty open $U = S - Z$. Then $\mathbf{N_B}, \mathbf{H}_1 \neq 0, \ker\psi_{c-1}$, and $\mathbf{K_B}$ are each maximal Cohen-Macaulay $\mathbf{B}$-modules whose sheafified maximal exterior power is a twist of $\widetilde{\mathbf{K}}_{\mathbf{B}}$ on U. The same statement holds for $\mathbf{G}_2^*$ and a nontrivial $(\operatorname{im}\psi_{c-1})^*$ provided $\dim\mathbf{B} - \dim\mathbf{B}/\mathbf{I}(Z) \geq 2$.*

Proof. By (17) and the proofs of Proposition 13 and Proposition 18, $\mathbf{K_B}$, $\mathbf{N_B}$ and $\mathbf{H}_1$ are as claimed. Next, splitting the right part of (35) into short exact sequences (resp. using (34)), we see that the sheafified maximal exterior powers of a nontrivial $(\operatorname{im}\psi_{c-1})^*$ and $\ker\psi_{c-1}$ (resp. $\mathbf{G}_2^*$) are a twist of $\widetilde{\mathbf{K}}_{\mathbf{B}}$ on U. Moreover, $\ker\psi_{c-1}$ and $\mathbf{G}_2^*$ are maximal Cohen-Macaulay modules by Lemma 22. To prove that a nontrivial $(\operatorname{im}\psi_{c-1})^*$ is a maximal Cohen-Macaulay module, we remark that there is an exact sequence

$$0 \to \mathbf{K_B}(-e)^* \to \mathbf{F}_c^* \otimes_{\mathbf{R}} \mathbf{B} \to (\operatorname{im}\psi_{c-1})^* \to 0. \tag{36}$$

Indeed $\operatorname{Ext}^1_{\mathbf{B}}(\mathbf{K_B},\mathbf{B}) \simeq \operatorname{Ext}^1_{\mathbf{B}}(\mathbf{K_B}\otimes\mathbf{K_B},\mathbf{K_B}) = 0$ because $\operatorname{Ext}^1_{\mathbf{B}}(\wedge^2\mathbf{K_B},\mathbf{K_B}) = 0$ by the the codimension assumption and $\operatorname{Ext}^1_{\mathbf{B}}(S_2(\mathbf{K_B}),\mathbf{K_B}) = 0$ by the Cohen-Macaulayness of $S_2(\mathbf{K_B})$ [24], thm 4.2.6 (if the characteristic of k is 2 then the argument above works if we replace $\wedge^2\mathbf{K_B}$ by $\ker(\mathbf{K_B}\otimes\mathbf{K_B} \to S_2(\mathbf{K_B}))$) . The Cohen-Macaulayness implies also that $\mathbf{K_B^*} \simeq \operatorname{Hom}_{\mathbf{B}}(\mathbf{K_B}\otimes\mathbf{K_B},\mathbf{K_B}) \simeq \operatorname{Hom}_{\mathbf{B}}(S_2(\mathbf{K_B}),\mathbf{K_B})$ is a maximal Cohen-Macaulay module. Moreover (36) above and (2) imply that a nontrivial $(\operatorname{im}\psi_{c-1})^*$ is a maximal Cohen-Macaulay module provided we can prove $\operatorname{Ext}^1_{\mathbf{B}}((\operatorname{im}\psi_{c-1})^*,\mathbf{K_B}) = 0$. Since the dual of (36) is short exact, and since we at least have right exactness after applying $(-)\otimes_{\mathbf{R}}\mathbf{K_B}$ to this dual sequence, we get precisely this last mentioned Ext-group to vanish, and we are done. □

In the codimension 2 case ($c = 2$), then (35) leads to the exact sequence

$$0 \to \mathbf{N_B} \to \mathbf{F}_1^* \otimes_{\mathbf{R}} \mathbf{B} \xrightarrow{\psi_1} \mathbf{F}_2^* \otimes_{\mathbf{R}} \mathbf{B} \to \mathbf{K_B}(-e) \to 0 \tag{37}$$

and $\operatorname{im}\psi_1 \simeq \mathbf{H}_1^*$ and $\mathbf{G}_2^* \simeq \mathbf{K_B}(-e)$ by Lemma 22. Hence (17), Proposition 13 and Proposition 18 take care of all Gorenstein quotients to be constructed by this idea of applying Theorem 8 to the "syzygy" modules of (35) and its dual.

If the codimension $c = 3$, then we have a complex

$$0 \to \mathbf{N_B} \to \mathbf{F}_1^* \otimes_{\mathbf{R}} \mathbf{B} \xrightarrow{\psi_1} \mathbf{F}_2^* \otimes_{\mathbf{R}} \mathbf{B} \xrightarrow{\psi_2} \mathbf{F}_3^* \otimes_{\mathbf{R}} \mathbf{B} \to \mathbf{K_B}(-e) \to 0 \tag{38}$$

which is exact except in the middle. Indeed $\ker\psi_2/\operatorname{im}\psi_1 \simeq \operatorname{Tor}_1^{\mathbf{R}}(\mathbf{B},\mathbf{K_B})(-e) \simeq \mathbf{I}/\mathbf{I}^2 \otimes \mathbf{K_B}(-e)$. By Proposition 23 we see that, in addition to the modules $\mathbf{K_B}$, $\mathbf{N_B}$ and $\mathbf{H}_1$, there are precisely 3 more modules to which we might apply Theorem 8, using this idea.

To apply Theorem 8 to these 3 modules, we must prove the maximal Cohen-Macaulayness of $\mathrm{H}^0_*(U,\wedge^i(-))$ of the 3 modules for all $2 \le i \le rank/2$. We have not been able to prove this extra assumption in general, and we leave it as an interesting question for further investigations.

Question 24. *Let $\mathbf{R}$ be a Gorenstein ring. Let $\mathbf{B}$ be a codimension 3 graded licci quotient of $\mathbf{R}$. Let $S = \operatorname{Proj}(\mathbf{B})$, let Z be a scheme such that $\dim(\mathbf{B}) - \dim(\mathbf{B}/\mathbf{I}(Z)) \ge \max(r,2)$ and let $U = S - Z$. If S is a local complete intersection on U, do regular sections on U of twists of $\widetilde{(\ker\psi_2)^*}$, $\widetilde{\mathbf{G}}_2$ and $\widetilde{\operatorname{im}\psi_2}$ define Gorenstein quotients of $\mathbf{R}$ (where r denotes the rank of each of these modules)?*

The extra assumption above is fulfilled if the rank of the module is at most 3. Remarking that the part of the proof of Proposition 23 which is concerned with the module $\operatorname{im}\psi_{c-1}$, works with a local Gorenstein assumption instead of the local complete intersection assumption, we have in particular

Corollary 25. *Let $\mathbf{R} \twoheadrightarrow \mathbf{B} = \mathbf{R}/\mathbf{I}$ be licci of codimension $c \ge 2$ and Cohen-Macaulay type g and suppose it is locally Gorenstein outside some closed set Z of codimension $\ge max(g-1,2)$ in $S = \operatorname{Proj}(\mathbf{B})$. If $g \le 4$, then any regular section of $\mathrm{H}^0(S - Z, \widetilde{\operatorname{im}\psi_{c-1}}(s))$ defines a Gorenstein quotient $\mathbf{R} \twoheadrightarrow \mathbf{A}$ of codimension $c+g-1$.*

Remark 26. *Under reasonable additional assumptions (which we expect a licci quotient, $\mathbf{B}$, to have) we can extend Corollary 25 further. For instance, if*

$$\operatorname{depth}_{\mathfrak{m}} S_3(\mathbf{K_B}) \ge \dim\mathbf{B} - 1 \tag{39}$$

and $S = \operatorname{Proj}(\mathbf{B})$ is locally Gorenstein (so $U = S$ and $\mathbf{I}(Z) = \mathfrak{m}$) and licci, then the conclusion of Corollary 25 holds also for $g = 5$. This follows from Theorem 8 if we can prove that (39) and the assumption $\dim\mathbf{B} \ge 4$ implies that $\mathrm{H}^0_*(S, \wedge^2(\widetilde{\operatorname{im}\psi_{c-1}})^*)$ is a maximal Cohen-Macaulay $\mathbf{B}$-module. To indicate a proof, let $\mathbf{N} = \operatorname{im}\psi_{c-1}$. We have the exact sequence (36) and its $\mathbf{B}$-dual

$$0 \to \mathbf{N} \to \mathbf{F}_c \otimes_{\mathbf{R}} \mathbf{B} \xrightarrow{\beta} \mathbf{K_B}(-e) \to 0. \tag{40}$$

Let $\overline{\mathbf{N}}$ be the kernel of the composition of $\beta \otimes id : \mathbf{F}_c \otimes_{\mathbf{R}} S_2(\mathbf{K_B}) \to \mathbf{K_B}(-e) \otimes S_2(\mathbf{K_B})$ and the natural surjection $\mathbf{K_B}(-e) \otimes S_2(\mathbf{K_B}) \twoheadrightarrow S_3(\mathbf{K_B})(-e)$. By (39), $\overline{\mathbf{N}}$ is a maximal Cohen-Macaulay $\mathbf{B}$-module as is $(\mathbf{N} \otimes \mathbf{K_B})^*$ (by Theorem 5). Indeed, since the sheafification of $\overline{\mathbf{N}}$ and $\mathbf{N} \otimes \mathbf{K_B} \otimes \mathbf{K_B}$ are isomorphic, we get

$$(\mathbf{N} \otimes \mathbf{K_B})^* \simeq \operatorname{Hom}(\mathbf{N} \otimes \mathbf{K_B} \otimes \mathbf{K_B}, \mathbf{K_B}) \simeq \operatorname{Hom}(\overline{\mathbf{N}}, \mathbf{K_B}).$$

Hence, we get $\mathrm{H}^1_*(S, (\widetilde{\mathbf{N} \otimes \mathbf{K_B}})^*) = 0$. Now using exact sequence (36) and [12], II, exc 5.16, we have the exact sequence

$$0 \to (\widetilde{\mathbf{K}}_{\mathbf{B}}(-e) \otimes \widetilde{\mathbf{N}})^* \to \wedge^2 \widetilde{\mathbf{F}_c^* \otimes \mathbf{B}} \to \wedge^2 \widetilde{\mathbf{N}}^* \to 0$$

which is still exact after applying $\mathrm{H}^0_*(S,-)$. The maximal Cohen-Macaulayness of $(\mathbf{K_B} \otimes \mathbf{N})^*$ implies $\operatorname{depth}_{\mathfrak{m}} \mathrm{H}^0_*(S, \wedge^2\widetilde{\mathbf{N}}^*) \ge \dim\mathbf{B} - 1$ since the rank of $\mathbf{N}^*$ is 4. We conclude by (15) of Remark 9.

Remark 27. *Suppose that* $\mathbf{B}$ *has codimension* 3. *If* $\mathbf{B}$ *is an almost complete intersection (i.e.* $\mu(\mathbf{I_B}) = \operatorname{rank}\mathbf{F}_1 = 4$*) and if the Cohen-Macaulay type of* $\mathbf{B}$ *is* $g = 2$*, we see from* (8) *and* (34) *that* $\operatorname{rank}\widetilde{\mathbf{G}}_2 = 4$*. If* (39) *holds and if* $S \hookrightarrow \operatorname{Proj}\mathbf{R}$ *is a local complete intersection of dimension* $\dim S \geq 3$*, we can argue exactly as in Remark 26 to see that* $\mathrm{H}^0_*(S, \wedge^2\widetilde{\mathbf{G}}_2^*)$ *is a maximal Cohen-Macaulay* $\mathbf{B}$*-module. Indeed,* (8) *shows that* $\operatorname{rank}\widetilde{\mathbf{H}}_1 = 1$ *and we have* $\widetilde{\mathbf{H}}_1 = \widetilde{\mathbf{K}}_\mathbf{B}(t)$ *by* (30)*. Hence,* $\mathbf{H}_1 \simeq \mathbf{K_B}(t)$ *and we conclude by the arguments of Remark 26 by replacing exact sequence* (40) *(resp. exact sequence* (36)*) by the sequence* (34) *(resp. its short exact dual sequence, cf. Lemma 22). Such almost complete intersections are licci [3]. Hence, by Proposition 23 and Theorem 8, each regular section of* $\mathrm{H}^0(S, \widetilde{\mathbf{G}}_2(s))$ *defines a Gorenstein quotient* $\mathbf{R} \to \mathbf{A}$ *of codimension 7.*

We finish this subsection by giving a few more examples of low rank modules where we can apply Theorem 8. If $\mathbf{B} = \mathbf{R}/\mathbf{I}$ is licci, it is known that $S_2(\mathbf{K_B})$ is a maximal Cohen-Macaulay $\mathbf{B}$-module, as is $\mathbf{K}^*_\mathbf{B} \simeq \operatorname{Hom}_\mathbf{B}(S_2(\mathbf{K_B}), \mathbf{K_B})$ (cf. the proof of Proposition 23). Theorem 8 therefore immediately applies to $\mathbf{M} = \mathrm{H}^0_*(U, S_2(\widetilde{\mathbf{K}}_\mathbf{B}) \oplus \widetilde{\mathbf{K}}^*_\mathbf{B}(t_1))$ and to $\mathbf{M} = \mathrm{H}^0_*(U, \widetilde{\mathbf{K}}_\mathbf{B} \oplus \widetilde{\mathbf{K}}_\mathbf{B}(t_2) \oplus \widetilde{\mathbf{K}}^*_\mathbf{B}(t_3))$, i.e.

Proposition 28. *Let* $S = \operatorname{Proj}\mathbf{B}$ *and let* $\mathbf{B} = \mathbf{R}/\mathbf{I}$ *be licci of codimension* c *and locally Gorenstein outside some closed set* $Z \subseteq S$.

i) If $\dim\mathbf{B} - \dim\mathbf{B}/\mathbf{I}(Z) \geq 2$*, then any regular section of* $\mathrm{H}^0(S - Z, S_2(\widetilde{\mathbf{K}}_\mathbf{B})^*(s_1) \oplus \widetilde{\mathbf{K}}_\mathbf{B}(s_2))$ *defines a Gorenstein quotient* $\mathbf{R} \twoheadrightarrow \mathbf{A}$ *of codimension* $c + 2$.

ii) If $\dim\mathbf{B} - \dim\mathbf{B}/\mathbf{I}(Z) \geq 3$*, then any regular section of* $\mathrm{H}^0(S - Z, \widetilde{\mathbf{K}}^*_\mathbf{B}(s_1) \oplus \widetilde{\mathbf{K}}^*_\mathbf{B}(s_2) \oplus \widetilde{\mathbf{K}}_\mathbf{B}(s_3))$ *defines a Gorenstein quotient* $\mathbf{R} \twoheadrightarrow \mathbf{A}$ *of codimension* $c + 3$.

Example 29. *Let* $\mathbf{R} = k[a, b, c, d, e, f]$ *and let* $\mathbf{I} = (bc - ad, be - af, de - cf)$ *be the ideal of* 2×2 *minors of the* 2×3 *matrix of indeterminants of* $\mathbf{R}$ *(as in Example 12). In this case,* $\mathbf{B} = \mathbf{R}/\mathbf{I}$ *is Cohen-Macaulay of codimension 2,* $S = \operatorname{Proj}\mathbf{B}$ *is a local complete intersection and the minimal* $\mathbf{R}$*-free resolution of* $\mathbf{B}$ *is*

$$0 \to \mathbf{R}(-3)^2 \to \mathbf{R}(-2)^3 \to \mathbf{R} \to \mathbf{B} \to 0.$$

A regular section, $\sigma \in \mathrm{H}^0(S, \widetilde{\mathbf{K}}^*_\mathbf{B}(-1) \oplus \widetilde{\mathbf{K}}^*_\mathbf{B}(-1) \oplus \widetilde{\mathbf{K}}_\mathbf{B}(4))$ *defines a Gorenstein algebra,* $\mathbf{A}$*, of codimension* 5*. An example of such a Gorenstein algebra is*

$$\mathbf{A} = \mathbf{R}/(\mathbf{I} + (a^2 + ad + ce, ab + bd + cf, ad + ce + ef, bd + cf + f^2, ce + 2ef, ac - cd, ae - cf))$$

with $\mathbf{R}$*-free resolution*

$$0 \to \mathbf{R}(-8) \to \mathbf{R}(-6)^{10} \to \mathbf{R}(-4)^3 \oplus \mathbf{R}(-5)^{16} \to \mathbf{R}(-3)^{16} \oplus \mathbf{R}(-4)^3 \to \mathbf{R}(-2)^{10} \to \mathbf{R} \twoheadrightarrow \mathbf{A}.$$

This should be compared with the Gorenstein algebra constructed in Example 21.

Remark 30. *Assume* $\mathcal{L}$ *is an* $\mathcal{O}_S$*-module.*

i) If $\mathcal{L}$ *is invertible on the set* $U = S - Z$ *of Proposition 28 (or some subset of* U*, no licci assumption on* $\mathbf{B}$ *is needed) and* $\mathrm{H}^0_*(U, \mathcal{L}^*)$ *(resp.* $\mathrm{H}^0_*(U, \mathcal{L} \oplus \mathcal{L}^*)$*) is a maximal Cohen-Macaulay module, then Theorem 8 also applies to* $\mathbf{M} = \mathrm{H}^0_*(U, (\mathcal{L} \otimes \widetilde{\mathbf{K}}_\mathbf{B}) \oplus \mathcal{L}^*(t_1))$ *(resp. to* $\mathbf{M} = \mathrm{H}^0_*(U, \mathcal{L} \oplus \widetilde{\mathbf{K}}_\mathbf{B}(t_2) \oplus \mathcal{L}^*(t_3))$*) provided the codimension of* Z *is* ≥ 2 *(resp.* ≥ 3*) in*

$S = \mathrm{Proj}(\mathbf{B})$. *Indeed,* $\mathrm{H}^0_*(U, \mathcal{L} \otimes \widetilde{\mathbf{K}}_{\mathbf{B}}) \simeq \mathrm{H}^0_*(U, \mathcal{H}om(\mathcal{L}^*, \widetilde{\mathbf{K}}_{\mathbf{B}})) \simeq \mathrm{Hom}_{\mathbf{B}}(\mathrm{H}^0_*(U, \mathcal{L}^*), \mathbf{K}_{\mathbf{B}})$ *is a maximal Cohen-Macaulay* $\mathbf{B}$*-module by Theorem 5.*

ii) If $\mathcal{L}$ *is an* n^{th} *root of a twist of the canonical bundle and* $\mathbf{L} = \mathrm{H}^0_*(S, \mathcal{L}), S_2\mathbf{L}, \ldots, S_n\mathbf{L} \simeq \mathbf{K}_{\mathbf{B}}(t)$ *are all maximal Cohen-Macaulay modules then there are many possible choices for* $\mathbf{M}$ *in terms of direct sums of various symmetric powers of* $\mathbf{L}$*. This occurs in the case of determinantal ideals of codimension* $n+1$ *(see [6] page 595 and the discussion of the Buchsbaum-Rim complex).*

Example 31. *Let* $\mathbf{R} = k[a, b, c, d, e, f, g, h, i, j]$ *and let*

$$\mathbf{I} = (bc - ad, be - af, bg - ah, bi - aj, de - cf, dg - ch, di - cj, fg - eh, fi - ej, hi - gj)$$

be the ideal of 2×2 *minors of the* 2×5 *matrix of indeterminants,* $\mathbf{N}$*, of* $\mathbf{R}$*. In this case,* $\mathbf{B} = \mathbf{R}/\mathbf{I}$ *is Cohen-Macaulay of codimension 4,* $S = \mathrm{Proj}\,\mathbf{B}$ *is a local complete intersection and the minimal* $\mathbf{R}$*-free resolution of* $\mathbf{B}$ *is*

$$0 \to \mathbf{R}(-5)^4 \to \mathbf{R}(-4)^{15} \to \mathbf{R}(-3)^{20} \to \mathbf{R}(-2)^{10} \to \mathbf{R} \to \mathbf{B} \to 0.$$

The cokernel of the map defined by the matrix $\mathbf{N}$ *is a* $\mathbf{B}$*-module,* $\mathbf{L}$*. The minimal* $\mathbf{R}$*-free resolution of* $\mathbf{L}$ *is*

$$0 \to \mathbf{R}(-5)^3 \to \mathbf{R}(-4)^{10} \to \mathbf{R}(-3)^{10} \to \mathbf{R}(-1)^5 \to \mathbf{R}^2 \to \mathbf{L} \to 0.$$

The sheafification of $\mathbf{L}$ *is a line bundle,* $\mathcal{L}$*, on* $\mathrm{Proj}\,\mathbf{B}$*. Exactness of the complexes given in [6], page 595, show that* $\mathbf{L}, S^2\mathbf{L}$ *and* $S^3\mathbf{L} = \mathbf{K}_{\mathbf{B}}(t)$ *are all maximal Cohen-Macaulay modules. Therefore, a regular section,* $\sigma \in \mathrm{H}^0(S, \mathcal{L}^*(2) \oplus \mathcal{L}^*(2) \oplus \mathcal{L}^*(2))$ *defines a Gorenstein algebra,* $\mathbf{A}$*, of codimension 7. An example of such a Gorenstein algebra is*

$$\mathbf{A} = \mathbf{R}/(\mathbf{I} + (bh + aj + dj, ah + ai + cj, cf + aj, ce + ai, d^2 + gh + hj, cd + g^2 + gj))$$

with $\mathbf{R}$*-free resolution*

$$\begin{aligned} 0 \to \mathbf{R}(-11) \to \mathbf{R}(-9)^{16} \to \mathbf{R}(-7)^9 \oplus \mathbf{R}(-8)^{35} \to \mathbf{R}(-5)^4 \oplus \mathbf{R}(-6)^{60} \oplus \mathbf{R}(-7)^{15} \to \\ \mathbf{R}(-4)^{15} \oplus \mathbf{R}(-5)^{60} \oplus \mathbf{R}(-6)^4 \to \mathbf{R}(-3)^{35} \oplus \mathbf{R}(-4)^9 \to \mathbf{R}(-2)^{16} \to \mathbf{R} \to \mathbf{A} \to 0. \end{aligned}$$

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