Sheaves with Canonical Determinant on Cohen-Macaulay Schemes

Jan O. Kleppe and Chris Peterson

Abstract

This paper studies the class of sheaves which lie on arithmetically Cohen-Macaulay schemes and which have as determinant a twist of the canonical sheaf. Special emphasis is placed on finding minimal, sufficient cohomological conditions which ensure that the top dimensional component of regular sections of the dual of these sheaves vanish along arithmetically Gorenstein schemes. Duals of odd rank Buchsbaum-Rim sheaves, normal sheaves to licci schemes, certain sheafified Koszul homology modules and various other families are shown to satisfy the requisite conditions.
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Contents

1 Introduction

Let $(\phi, \mathcal{A}, \mathcal{B})$ denote a triple where $\mathcal{A}$ and $\mathcal{B}$ are coherent sheaves and ϕ is a morphism of $\mathcal{A}$ to $\mathcal{B}$. To any such triple, there is associated a collection of algebraic varieties (and schemes) which arise as the degeneracy loci of ϕ. By placing restrictions on $\mathcal{A}$, $\mathcal{B}$ or ϕ, one obtains corresponding restrictions on the possible degeneracy loci. It is a wonderful yet wickedly subtle problem to determine what restrictions should be enforced on a triple to ensure that degeneracy loci land in prescribed classes of varieties or schemes. At the same time, it is a worthwhile endeavor. Success allows reformulation of questions about varieties and schemes as questions about sheaves and morphisms.

In this paper, we study the degeneracy loci corresponding to triples, $(\phi, \mathcal{A}, \mathcal{B})$, where $\mathcal{A}$ is a line bundle on an arithmetically Cohen-Macaulay scheme S, $\mathcal{B}$ is a sheaf on S, the determinant of $\mathcal{B}^* \otimes \mathcal{A}$ is a twist of the canonical sheaf, $\mathcal{K}_S$ of S, and ϕ is assumed to be regular. The goal is to describe a class of coherent sheaves on arithmetically Cohen-Macaulay schemes that have the property that regular sections degenerate on arithmetically Gorenstein schemes. The motivation for this problem is as follows. Let $\mathbf{R}$ be a graded Gorenstein k-algebra. In [16], the sheaf $n\mathcal{H} - \mathcal{K}_S$ on an arithmetically Cohen-Macaulay scheme, $S \subset \mathrm{Proj}(\mathbf{R})$ ($\mathcal{H}$ a hyperplane divisor), was utilized to construct divisors on S which were arithmetically Gorenstein. Several applications to Gorenstein liaison theory followed from this approach. Since S is arithmetically Cohen-Macaulay, $\mathbf{K}_S(-n) = \mathrm{H}^0_*(S, \mathcal{K}_S - n\mathcal{H})$, is a maximal Cohen-Macaulay $\mathbf{R}/\mathbf{I}_S$-module. In the same manner that sections of $n\mathcal{H} - \mathcal{K}_S$ can be used to construct arithmetically Gorenstein subschemes of $\mathrm{Proj}(\mathbf{R})$, sections of $\mathbf{K}^*_S(n)$ can be used to construct Gorenstein quotients of $\mathbf{R}$. In [17], many maximal Cohen-Macaulay modules were introduced that could be used to construct Gorenstein quotients of $\mathbf{R}$ (by taking regular sections of their duals). Non maximal Cohen-Macaulay modules can also play a role in the construction of Gorenstein quotients of $\mathbf{R}$. A first instance of these are the duals of Buchsbaum-Rim modules. Buchsbaum-Rim modules are the modules which arise as the kernel of sufficiently general maps between free $\mathbf{R}$-modules (where $\mathbf{R}$ is a Gorenstein k-algebra). Buchsbaum-Rim sheaves are the sheafifications of the Buchsbaum-Rim modules. In the papers [20] and [21], a study was made of the degeneracy loci of sections (and multiple sections) of such sheaves. One consequence of this study was that the top dimensional component of the degeneracy locus of a regular section of an odd rank Buchsbaum-Rim sheaf is always arithmetically Gorenstein. There are several features common to all of the sheaves appearing in [16], [17], [20] and [21]. This led us to search for a single underlying theme that could encompass all of the sheaves appearing in the aforementioned works. We found that if S is an arithmetically Cohen-Macaulay scheme and if $\mathcal{E}$ is a sheaf on S whose determinant is a twist of the canonical sheaf, then one could give weak, sufficient cohomological conditions on $\mathcal{E}$ such that the top dimensional component of the degeneracy locus of any regular section of $\mathcal{E}^*(n)$ is an arithmetically Gorenstein scheme. The dual of Buchsbaum-Rim sheaves, the dual of $n\mathcal{H} - \mathcal{K}_S$, normal sheaves to licci schemes, and all of the various sheaves derived from [17] satisfy these cohomological conditions. In addition, we give examples of large families of sheaves which are not covered by the aforementioned works but which still satisfy the required cohomological conditions. It is our hope that applications will follow in Gorenstein liaison theory, in the study of Hilb(H) and in the study of the strata PGor(H) of Hilb(H).

1.1 Preliminaries

Throughout the paper, $\mathbf{R}$ is a finitely generated Gorenstein k-algebra, generated by $\mathbf{R}_1$, over an algebraically closed field k, and $\mathbf{B}$ is a graded quotient of $\mathbf{R}$. Note that such a ring, $\mathbf{B}$, is *Catenarian* [5]. If $c = \dim \mathbf{R} - \dim \mathbf{B}$, the canonical module of $\mathbf{B}$ is given by

$$\mathbf{K}_{\mathbf{B}} := \mathrm{Ext}^c_{\mathbf{R}}(\mathbf{B}, \mathbf{R}(e))$$

where $\mathbf{R}(e)$ is the canonical module of $\mathbf{R}$.

If $\mathbf{M}$ is a finitely generated graded $\mathbf{B}$-module, we denote by $\mathrm{depth}_{\mathbf{J}} \mathbf{M}$ the length of a maximal $\mathbf{M}$-sequence in a homogeneous ideal $\mathbf{J}$. Let $\mathrm{depth}\, \mathbf{M} = \mathrm{depth}_{\mathfrak{m}} \mathbf{M}$ where $\mathfrak{m}$ is the irrelevant maximal ideal. Let $\mathrm{H}^i_{\mathbf{J}}(-)$ be the i^{th} right derived functor of the functor, $\Gamma_{\mathbf{J}}(-)$, of sections with support in $\mathrm{Spec}(\mathbf{B}/\mathbf{J})$. Recall that $\mathrm{depth}_{\mathbf{J}}\,\mathbf{M} \geq r$ iff $\mathrm{H}^i_{\mathbf{J}}(\mathbf{M}) = 0$ for $i < r$, cf. [8], Thm 3.8.

If $S = \mathrm{Proj}(\mathbf{B})$ and $U = S - \mathrm{Proj}(\mathbf{B}/\mathbf{J})$, we let $\mathrm{H}^i_*(U, \widetilde{\mathbf{M}}) = \oplus_v \mathrm{H}^i(U, \widetilde{\mathbf{M}}(v))$. Then we have an exact sequence

$$0 \to \mathrm{H}^0_{\mathbf{J}}(\mathbf{M}) \to \mathbf{M} \to \mathrm{H}^0_*(U, \widetilde{\mathbf{M}}) \to \mathrm{H}^1_{\mathbf{J}}(\mathbf{M}) \to 0 \tag{1}$$

and isomorphisms $\mathrm{H}^i_{\mathbf{J}}(\mathbf{M}) \simeq \mathrm{H}^{i-1}_*(U, \widetilde{\mathbf{M}})$ for $i \geq 2$ cf. [8], Prop 2.2 or [7]. Hence the condition $\mathrm{depth}_{\mathbf{J}}\,\mathbf{M} \geq r$ (for $r \geq 2$) is equivalent to

$$\mathbf{M} \simeq \mathrm{H}^0_*(U, \widetilde{\mathbf{M}}) \quad \text{and} \quad \mathrm{H}^i_*(U, \widetilde{\mathbf{M}}) = 0 \ \text{ for } \ 1 \leq i \leq r-2.$$

Cohen-Macaulay modules satisfy $\mathrm{depth}\,\mathbf{M} = \dim \mathbf{M}$ by definition, or equivalently, $\mathrm{H}^i_{\mathfrak{m}}(\mathbf{M}) = 0$ for $i < \dim \mathbf{M}$.

Definition 1. *A finitely generated* $\mathbf{B}$*-module,* $\mathbf{M}$*, satisfies Serre's condition* S_k *if*

$$\mathrm{depth}\,\mathbf{M}_{\mathfrak{p}} \geq \min\{k, \dim \mathbf{M}_{\mathfrak{p}}\} \quad \text{for all} \quad \mathfrak{p} \in \mathrm{Supp}(\mathbf{M}).$$

$\mathbf{M}$ is Cohen-Macaulay if and only if it satisfies S_q for any q. Moreover, $\mathbf{K_B}$ satisfies S_2 by [22], Lemma 1.8. By [22], Corollary 1.11 and Corollary 2.17 we have

Proposition 2. *Suppose* $\mathbf{B}$ *satisfies* S_2*. Then* $\mathbf{B}$ *is equidimensional. Moreover for any integer* $k \geq 2$*, the following conditions are equivalent:*

(i) $\mathbf{K_B}$ *satisfies condition* S_k*.*

(ii) $\mathrm{H}^i_{\mathfrak{m}}(\mathbf{B}) = 0$ *for all* $\dim \mathbf{B} - k + 2 \leq i < \dim \mathbf{B}$.

Definition 3. $\mathbf{B}$ *is called quasi-Gorenstein if* $\mathbf{K_B} \simeq \mathbf{B}(m)$ *for some integer* m*.*

If $\mathbf{B}$ is quasi-Gorenstein, it satisfies S_2. $\mathbf{B}$ is therefore equidimensional. Moreover, if $\mathbf{B}$ is "half-way" Cohen-Macaulay, then it is Cohen-Macaulay and hence Gorenstein. Indeed using Proposition 2 we easily prove the following theorem of Hartshorne-Ogus [22], Theorem 1.12.

Theorem 4. *If* $\mathbf{B}$ *is quasi-Gorenstein and if* $\mathrm{depth}\,\mathbf{B}_{\mathfrak{p}} \geq \min\{\dim \mathbf{B}_{\mathfrak{p}}, \frac{1}{2}\dim \mathbf{B} + 1\}$ *for all graded primes* $\mathfrak{p}$*, then* $\mathbf{B}$ *is Gorenstein.*

Lemma 5. *Let* $\mathbf{B}$ *be Cohen-Macaulay. Let* r *and* t *be integers. Let* $\mathbf{J} \subseteq \mathbf{B}$ *be an ideal satisfying* $\mathrm{depth}_{\mathbf{J}}\,\mathbf{B} \geq r$ *and let* $\mathbf{M}$ *be a finitely generated* $\mathbf{B}$*-module satisfying* $\mathrm{depth}_{\mathfrak{m}}\,\mathbf{M} \geq \dim \mathbf{B} - t$*. Then* $\mathrm{depth}_{\mathbf{J}}\,\mathbf{M} \geq r - t$*.*

Proof. By the assumption $\mathrm{depth}_{\mathfrak{m}}\,\mathbf{M} \geq \dim \mathbf{B} - t$ and by Gorenstein duality, we have $\mathrm{Ext}^j_{\mathbf{B}}(\mathbf{M}, \mathbf{K_B}) = 0$ for $j > t$. If $\mathfrak{p} \supseteq \mathbf{J}$, we get $\mathrm{Ext}^j_{\mathbf{B}_{\mathfrak{p}}}(\mathbf{M}_{\mathfrak{p}}, \mathbf{K}_{(\mathbf{B}_{\mathfrak{p}})}) = 0$ because $(\mathbf{K_B})_{\mathfrak{p}} \simeq \mathbf{K}_{(\mathbf{B}_{\mathfrak{p}})}$. Hence $\mathrm{depth}_{\mathfrak{p}}\,\mathbf{M}_{\mathfrak{p}} \geq \dim \mathbf{B}_{\mathfrak{p}} - t \geq r - t$ and $\mathrm{depth}_{\mathbf{J}}\,\mathbf{M} \geq r - t$. $\square$

The top dimensional component of $\mathbf{B} = \mathbf{R}/\mathbf{I}$ is given as the intersection of those primary ideals in a Noetherian decomposition of $\mathbf{I}$ which are of dimension $\dim \mathbf{B}$. Since such primary ideals belong to the minimal primes of $\mathbf{I}$, they are uniquely determined. Letting $V(\mathbf{J})$ denote the zero set of the ideal $\mathbf{J}$ in $\mathrm{Proj}(\mathbf{B})$, we characterize the top dimensional component of $\mathbf{B}$ in the following way.

Proposition 6. *Let* $\mathbf{J} \subseteq \mathbf{B}$ *be a homogeneous ideal such that* $\dim \mathbf{B}/\mathbf{J} < \dim \mathbf{B}$*. Let* $\overline{\mathbf{B}} = \mathbf{B}/\operatorname{H}^0_{\mathbf{J}}(\mathbf{B})$ *and let* $U = \operatorname{Proj}(\mathbf{B}) - V(\mathbf{J})$ *be equidimensional and without embedded components (resp. let* U *satisfy* S_2 *and suppose* $\operatorname{H}^1_{\mathbf{J}}(\mathbf{B}) = 0$*). Then* $\overline{\mathbf{B}}$ *satisfies* S_1 *(resp.* S_2*). Moreover* $\overline{\mathbf{B}}$ *is the top dimensional component of* $\mathbf{B}$*.*

Proof. First, we suppose $\dim \mathbf{B}/\mathbf{J} < \dim \mathbf{B}$ and U is equidimensional and without embedded components. By [5], Proposition 3.13, we have

$$\operatorname{H}^0_{\mathbf{J}}(\mathbf{B}) = \bigcap_{\mathfrak{p}_i \not\supseteq \mathbf{J}} \mathfrak{a}_i$$

where $(0) = \cap \mathfrak{a}_i$ is a primary decomposition in $\mathbf{B}$ and $\mathfrak{a}_i$ is $\mathfrak{p}_i$-primary. Since $\dim \mathbf{B}/\mathbf{J} < \dim \mathbf{B}$ and U is equidimensional and without embedded components, the $\mathfrak{p}_i$-primary ideals in the intersection above are precisely the minimal primes of $\mathbf{B}$. This proves conclusion 2 and the S_1-statement of conclusion 1 of Proposition 6.

Now we suppose that $\dim \mathbf{B}/\mathbf{J} < \dim \mathbf{B}$, that U satisfies S_2, and that $\operatorname{H}^1_{\mathbf{J}}(\mathbf{B}) = 0$. Since $\operatorname{H}^0_{\mathbf{J}}(\mathbf{B})$ is supported at $V(\mathbf{J})$, we have $\widetilde{\mathbf{B}}|_U \simeq \widetilde{\overline{\mathbf{B}}}|_U$, i.e. $\mathbf{B}_{\mathfrak{p}} \simeq \overline{\mathbf{B}}_{\mathfrak{p}}$ satisfies S_2 for any graded prime $\mathfrak{p} \not\supseteq \mathbf{J}$. Comparing the exact sequence (1) for $\mathbf{M} = \mathbf{B}$ with the corresponding sequence with $\mathbf{M} = \overline{\mathbf{B}}$ and using $\operatorname{H}^0_*(U, \widetilde{\mathbf{B}}) \simeq \operatorname{H}^0_*(U, \widetilde{\overline{\mathbf{B}}})$, we get $\operatorname{H}^0_{\mathbf{J}}(\overline{\mathbf{B}}) = 0$ and $\operatorname{H}^1_{\mathbf{J}}(\mathbf{B}) \simeq \operatorname{H}^1_{\mathbf{J}}(\overline{\mathbf{B}})$. Since $\operatorname{H}^1_{\mathbf{J}}(\mathbf{B}) = 0$ by assumption, we have $\operatorname{H}^0_{\mathbf{J}}(\overline{\mathbf{B}}) = \operatorname{H}^1_{\mathbf{J}}(\overline{\mathbf{B}}) = 0$. Hence

$$\operatorname{depth}_{\mathbf{J}} \overline{\mathbf{B}} = \inf_{\mathfrak{p} \supset \mathbf{J}} \operatorname{depth} \overline{\mathbf{B}}_{\mathfrak{p}} \geq 2.$$

It follows that $\overline{\mathbf{B}}$ satisfies S_2. By Proposition 2, $\overline{\mathbf{B}}$ is equidimensional and hence is the top dimensional component of $\mathbf{B}$ by the arguments of the first part of the proof.

□

Let $S = \operatorname{Proj}(\mathbf{B})$ be Cohen-Macaulay, let $U \subseteq S$ be an open subset and let $\mathcal{K}_S = \widetilde{\mathbf{K}_{\mathbf{B}}}$. A coherent non-trivial $\mathcal{O}_S$-module, $\mathcal{M}$, is called a maximal Cohen-Macaulay sheaf on U if $\mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{M}, \mathcal{K}_S)|_U = 0$ for $i > 0$. For a finitely generated module $\mathbf{M} \neq 0$ over a locally Cohen-Macaulay ring $(\mathbf{B}, \mathfrak{m})$ the corresponding vanishing is, by Gorenstein duality, equivalent to the vanishing of $\operatorname{H}^i_{\mathfrak{m}}(\mathbf{M})$ for $i < \dim \mathbf{B}$, i.e. to $\mathbf{M}$ being a maximal Cohen-Macaulay $\mathbf{B}$-module. If $\dim \mathbf{M} = \dim \mathbf{B}$, the notions "maximal Cohen-Macaulay" and "Cohen-Macaulay" coincide. In particular if $\mathcal{M}$ is locally free on some open dense subset of U, then $\mathcal{M}$ is maximal Cohen-Macaulay on U if it is Cohen-Macaulay on U. Clearly, locally free $\mathcal{O}_S$-modules are Cohen-Macaulay sheaves on S.

Finally, we recall a main result of [17] in a scheme theoretic format. Below, $\operatorname{Hom}_{\mathbf{B}}(\mathbf{M}, \mathbf{B})$ is denoted by $\mathbf{M}^*$.

Theorem 7. *We fix the following notation and conditions.*

(i) Let $\mathbf{B}$ *be a graded Cohen-Macaulay quotient of a finitely generated graded Gorenstein* k*-algebra* $\mathbf{R}$*. Let* $\mathbf{M}$ *be a finitely generated* $\mathbf{B}$*-module. Let* $S = \operatorname{Proj}(\mathbf{B})$*.*

(ii) Let $U = S - Z$ *be an open set such that* $\widetilde{\mathbf{M}}|_U$ *is locally free of rank* r *and such that* $\dim \mathbf{B} - \dim \mathbf{B}/\mathbf{I}(Z) \geq \max(r, 2)$*.*

(iii) Let $\mathbf{M}_i = \mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{M}})$ *for* $0 \leq i \leq r$ *and let* U' *be an open subset of* S *such that for each* $1 \leq i \leq \frac{r}{2}$*,* $\widetilde{\mathbf{M}}_i|_{U'}$ *is Cohen-Macaulay.*

(iv) Let s *be any integer and let* $\sigma \in \mathrm{H}^0(U, \widetilde{\mathbf{M}}^*(s))$*.*

If σ *is a regular section on* U *and if there exists an integer* t *such that* $\wedge^r \widetilde{\mathbf{M}}|_U \simeq \mathcal{K}_S(t)|_U$ *then we can conclude the following*

1. *The zero locus of* σ *defines a closed subscheme* $C = \mathrm{Proj}(\mathbf{A})$

   (given by $\mathbf{A} := \mathrm{coker}(\mathrm{H}^0_*(U, \widetilde{\mathbf{M}}^{**}(-s)) \xrightarrow{\sigma^*} \mathbf{B}))$ *of codimension* r *in* S*.*

2. $C \cap U'$ *is equidimensional and locally Gorenstein.*

3. $\mathcal{K}_C|_{U'} \simeq \mathcal{O}_C(rs - t)|_{U'}$*.*

4. $C \subseteq S$ *is given by a complex*

   $$0 \to \mathcal{K}_S(t - rs) \to \widetilde{\mathbf{M}}_{r-1}((1-r)s) \to \cdots \to \widetilde{\mathbf{M}}_2(-2s) \to \widetilde{\mathbf{M}}_1(-s) \xrightarrow{\sigma^*} \mathcal{O}_S \to \mathcal{O}_C \to 0.$$

5. *The restriction of this complex to* U' *is exact.*

6. $\mathbf{M}_{r-i} \simeq \mathrm{Hom}_{\mathbf{B}}(\mathbf{M}_i, \mathbf{K}_{\mathbf{B}})(t)$ *for* $0 \leq i \leq r$*.*

Remark 8. *(a) Note that* $\mathbf{M}_1 \simeq \mathbf{M}^{**}$ *and* $\mathbf{M}_i \simeq (\wedge^i \mathbf{M})^{**}$*. It follows that all* $\mathbf{M}_i$ *are finitely generated. Moreover, if* $\mathrm{depth}_{\mathbf{I}(Z)} \mathbf{M} \geq 2$*, then* $\mathbf{M}_1 \simeq \mathbf{M}$*.*

(b) Theorem 7 is true even for "$U' = \mathrm{Spec}(\mathbf{B})$*". More precisely if we replace condition (iii) by the assumption "the graded modules* $\mathbf{M}_i$ *are Cohen-Macaulay for each* $1 \leq i \leq \frac{r}{2}$*", then all conclusions hold for the graded modules/algebras, e.g.* $\mathbf{A}$ *is Gorenstein and* $\mathrm{H}^0_*(U, -)$ *is exact on the sequence of conclusion 4 of Theorem 7. This is proved in [17]. A well known and special case of this is obtained by choosing* $\mathbf{M}$ *to be* $\mathbf{K}_{\mathbf{B}}$ *in which case* $\mathbf{A}$ *is Gorenstein for any regular section of* $\mathbf{K}^*_{\mathbf{B}}(s)$*. Note that, in this graded form, Theorem 7 also holds for the case "*$\dim C = -1$*", i.e. in the case where* $\mathbf{A}$ *is Artinian.*

2 Sections of sheaves with nice cohomology

The main purpose of this paper is to understand when the top dimensional component of the degeneracy locus of a regular section of a coherent $\mathcal{O}_S$-module (of rank r) defines an arithmetically Gorenstein subscheme of S. The following result will provide a basic tool for such investigations. Since the result represents no generalization (compared to Theorem 7 and Remark 8) for the case $r = 1$, we will suppose $r \geq 2$.

Theorem 9. *We fix the following notation and conditions.*

(i) Let $n \geq r \geq 2$*. Let* $P = \mathrm{Proj}(\mathbf{R})$ *be an arithmetically Gorenstein scheme of finite type over* k*. Let* $S = \mathrm{Proj}(\mathbf{B}) \subseteq P$ *be an arithmetically Cohen-Macaulay closed subscheme of* P *of dimension* n*, where* $\mathbf{B}$ *is the homogeneous coordinate ring in* $\mathbf{R}$*.*

(ii) *Let $U = S - Z$ be an open subset of S such that* $\dim \mathbf{B} - \dim \mathbf{B}/\mathbf{I}(Z) \geq r$. *Let $\mathcal{M}$ be a coherent $\mathcal{O}_S$-module, locally free on U such that*

$$\wedge^r \mathcal{M}|_U \simeq \mathcal{K}_S(t)|_U, \quad \text{for some } t.$$

(iii) *Let $U' = S - Z'$ be open in S such that* $\dim \mathbf{B} - \dim \mathbf{B}/\mathbf{I}(Z') \geq r+1$. *Let* $\mathbf{M}_i := \mathrm{H}^0_*(U, \wedge^i \mathcal{M})$ *and $\mathcal{M}_i = \widetilde{\mathbf{M}}_i$ for $0 \leq i \leq r$ and suppose, for each $1 \leq i \leq \frac{r}{2}$, that $\mathcal{M}_i$ is Cohen-Macaulay on U'.*

(iv) *Let s be any integer and let $\sigma \in \mathrm{H}^0(U, \mathcal{M}^*(s))$ be a regular section on U. Let* $\mathbf{A} = \operatorname{coker}(\mathrm{H}^0_*(U, \mathcal{M}^{**}(-s)) \xrightarrow{\sigma^*} \mathbf{B})$. *Let $C = \operatorname{Proj}(\mathbf{A})$. Let $\overline{\mathbf{A}} := \mathbf{B}/\mathrm{H}^0_*(U', \mathcal{I}_{C/S})$.*

Under the above conditions, we can conclude the following

1. $\dim C = \dim S - r$ *and $\overline{\mathbf{A}}$ is the top dimensional component of $\mathbf{A}$.*
2. *If* $\mathrm{H}^{\dim C + i}_*(S, \mathcal{M}_i) = 0$ for $1 \leq i \leq r-1$ *then $\overline{\mathbf{A}}$ is quasi-Gorenstein.*
3. *If* $\mathrm{H}^{\dim C+1}_*(S, \mathcal{M}_1) = 0$ *and if* $\mathrm{H}^{\dim C+i}_*(S, \mathcal{M}_i) = \mathrm{H}^{\dim C+i}_*(S, \mathcal{M}_{i-1}) = 0$ for $2 \leq i \leq r-1$ *then $\mathbf{A}$ is quasi-Gorenstein and $\mathbf{A} \simeq \overline{\mathbf{A}}$.*

Remark 10. *(a) Note that* $\operatorname{depth}_{\mathbf{J}} \mathbf{B} = \dim \mathbf{B} - \dim \mathbf{B}/\mathbf{J}$ *for any homogeneous ideal $\mathbf{J}$ of $\mathbf{B}$ since $\mathbf{B}$ is Cohen-Macaulay, cf. [5]. Now if we suppose*

$$\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} \geq r+1,$$

in addition to the assumptions (i) to (iv) above, it follows from Theorem 9 that we may define $\overline{\mathbf{A}}$ using U instead of U', i.e. we have $\overline{\mathbf{A}} \simeq \mathbf{B}/\mathrm{H}^0_*(U, \mathcal{I}_{C/S})$. Indeed, by conclusion 1 of Theorem 9, $\overline{\mathbf{A}}$ is independent of any U' as long as U' satisfies (iii). With the depth condition of this remark, it is often natural to use Theorem 9 with $U' = U$. On the other hand, taking U' to be as large as possible may give additional information. For instance, if $\mathcal{M}_i$ (with $1 \leq i \leq \frac{r}{2}$) are Cohen-Macaulay on S (e.g. $\mathcal{M}$ is locally free on S), we may take $U' = S$ and we see that $\overline{\mathbf{A}}$ is the homogeneous coordinate ring of C in P (or S) because $\mathrm{H}^0_*(S, \mathcal{I}_{C/S})$ is the largest homogeneous ideal defining C in S.

(b) The assumptions appearing in conclusions 2 and 3 of Theorem 9, are equivalent to certain requirements on the modules $\mathcal{M}_i$ for $1 \leq i \leq \frac{r}{2}$. In view of Lemma 11, we will have that the condition $\mathrm{H}^{\dim C+i}_*(S, \mathcal{M}_i) = 0$ for $1 \leq i \leq r-1$ *of conclusion 2 is equivalent to*

$$\mathrm{H}^i_*(U', \mathcal{M}_i) = 0 \ \text{ for } \ 1 \leq i \leq \frac{r}{2} \quad \text{and} \quad \mathrm{H}^{\dim C+i}_*(S, \mathcal{M}_i) = 0 \ \text{ for } \ 1 \leq i \leq \frac{r-1}{2}$$

or to $\mathrm{H}^i_*(U', \mathcal{M}_i) = 0$ for $1 \leq i \leq r-1$ *(or to* $\mathrm{H}^i_*(U, \mathcal{M}_i) = 0$ for $1 \leq i \leq r-1$ provided $\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} \geq r+1$.*) Similarly, the condition* $\mathrm{H}^{\dim C+i}_*(S, \mathcal{M}_{i-1}) = 0$ *for $2 \leq i \leq r-1$ appearing in conclusion 3 of the theorem is equivalent to*

$$\mathrm{H}^{i-1}_*(U', \mathcal{M}_i) = 0 \ \text{ for } \ 2 \leq i \leq \frac{r}{2} \quad \text{and} \quad \mathrm{H}^{\dim C+i+1}_*(S, \mathcal{M}_i) = 0 \ \text{ for } \ 1 \leq i \leq \frac{r-1}{2}.$$

or to $\mathrm{H}^{i-1}_*(U', \mathcal{M}_i) = 0$ for $2 \leq i \leq r-1$.

(c) If there exists a coherent sheaf $\mathcal{M}'_i$ such that $\mathcal{M}'_i|_U \simeq \wedge^i\mathcal{M}|_U$ and such that $\operatorname{depth}_Z \mathcal{M}'_i \geq 2$*, then the module $\mathcal{M}_i$ of Theorem 9 satisfies $\mathcal{M}_i \simeq \mathcal{M}'_i$ by [8], Theorem 3.8 and Proposition 1.11. Since* $\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} \geq 2$*, such an $\mathcal{M}'_i$ exists, namely $\mathcal{M}'_i = (\wedge^i\mathcal{M})^{**}$, i.e. we have $\mathcal{M}_i = (\wedge^i\mathcal{M})^{**}$. If the sheaf $\wedge^i\mathcal{M}$ satisfies* $\operatorname{depth}_Z(\wedge^i\mathcal{M}) \geq 2$ *(e.g.* $\operatorname{depth}_{I(Z)}(\wedge^i\mathbf{M}) \geq 2$ *where $\mathbf{M} := \mathrm{H}^0_*(S,\mathcal{M})$), we can take $\mathcal{M}'_i = \wedge^i\mathcal{M}$. This allows us to verify condition (iii) and the conditions required for conclusions 2 and 3 of Theorem 9 using $\wedge^i\mathcal{M}$ instead of $\mathcal{M}_i$ if such a depth condition is present.*

With notation and conditions as in Theorem 9 $(i)-(iv)$ we prove the following lemmas:

Lemma 11. *Let $U_j = S - Z_j$ be an open subset of S where $\mathcal{E}xt^k_{\mathcal{O}_S}(\mathcal{M}_j,\mathcal{K}_S)|_{U_j} = 0$ for every $k > 0$ and suppose* $\operatorname{depth}_{\mathbf{I}(Z_j)} \mathbf{B} > r-i+1$*. Then*

$$\mathrm{H}^{\dim C+i}(S,\mathcal{M}_j(\nu))^\vee \simeq \mathrm{H}^{r-i}(U_j,\mathcal{M}_{r-j}(-t-\nu)).$$

In particular for any $i \geq 1$ and any $0 \leq j \leq r$ we have

$$\mathrm{H}^{\dim C+i}(S,\mathcal{M}_j(\nu))^\vee \simeq \mathrm{H}^{r-i}(U',\mathcal{M}_{r-j}(-t-\nu)).$$

Proof. Since $r = \dim S - \dim C$, by Serre duality

$$\mathrm{H}^{\dim C+i}(S,\mathcal{M}_j(\nu))^\vee \simeq \operatorname{Ext}^{r-i}_{\mathcal{O}_S}(\mathcal{M}_j(\nu),\mathcal{K}_S).$$

We conclude easily because the assumptions of the lemma lead to

$$\operatorname{Ext}^{r-i}_{\mathcal{O}_S}(\mathcal{M}_j(\nu),\mathcal{K}_S) \simeq \operatorname{Ext}^{r-i}_{\mathcal{O}_{U_j}}(\mathcal{M}_j(\nu)|_{U_j},\mathcal{K}_S|_{U_j}) \simeq \mathrm{H}^{r-i}(U_j,\mathcal{M}_{r-j}(-t-\nu)).$$

Indeed, since $\mathrm{H}^k_{\mathbf{I}(Z_j)}(\mathbf{B}) = \mathrm{H}^k_{\mathbf{I}(Z_j)}(\mathbf{K_B}) = 0$ for $k \leq r-i+1$, the left isomorphism follows from [7], exp. VI. Moreover the right isomorphism follows from the spectral sequence relating local and global Ext groups, from $\mathcal{E}xt^k_{\mathcal{O}_S}(\mathcal{M}_j,\mathcal{K}_S)|_{U_j} = 0$ and from $\mathcal{H}om(\mathcal{M}_j,\mathcal{K}_S)(t)|_{U_j} \simeq \mathcal{M}_{r-j}|_{U_j}$ cf. Theorem 7. $\square$

Lemma 12. *Let α be an integer such that $0 \leq \alpha \leq \dim C$, and suppose* $\operatorname{depth}_{\mathbf{I}(Z')} \mathbf{B} \geq r+1+\alpha$*. Then the following conditions are equivalent*

1. $\mathrm{H}^{\dim C+i-\alpha}_*(S,\mathcal{M}_i) = 0$ for $1 \leq i \leq r-1$
2. $\mathrm{H}^{i+\alpha}_*(U',\mathcal{M}_i) = 0$ for $1 \leq i \leq r-1$

Moreover, if one of them holds, then there is a natural injection

$$\mathrm{H}^{\alpha+1}_{\mathbf{I}(Z')}(\overline{\mathbf{A}}) \simeq \mathrm{H}^{\alpha+1}_{\mathbf{I}(Z')}(\mathbf{A}) \hookrightarrow \mathrm{H}^{r+1+\alpha}_{\mathbf{I}(Z')}(\mathbf{K_B}(t-rs)).$$

Proof. The equivalence of 1) and 2) follows immediately from Lemma 11. Now assume conditions 1) and 2) hold. We want to show the natural injection of Lemma 12. Consider the complex from Theorem 7:

$$0 \to \mathcal{K}_S(t-rs) \to \widetilde{\mathbf{M}}_{r-1}((1-r)s) \to \cdots \to \widetilde{\mathbf{M}}_2(-2s) \to \widetilde{\mathbf{M}}_1(-s) \xrightarrow{\sigma^*} \mathcal{O}_S \to \mathcal{O}_C \to 0.$$

Let

$$\mathcal{Z}_i = \ker[\mathcal{M}_i(-is) \to \mathcal{M}_{i-1}(-(i-1)s)],\ 1 \le i \le r-1 \tag{2}$$

and let $\mathcal{Z}_0 = \mathcal{I}_{C/S} = \widetilde{\mathbf{I}}_{\mathbf{B}/\mathbf{A}}$. Let $\mathbf{J} = \mathbf{I}(Z')$. Using the exact sequence

$$0 \to \mathbf{I}_{\mathbf{B}/\mathbf{A}} \to \mathbf{B} \to \mathbf{A} \to 0$$

we get $\mathrm{H}^{\alpha+1}_{\mathbf{J}}(\mathbf{A}) \simeq \mathrm{H}^{\alpha+1}_{*}(U', \mathcal{I}_{C/S})$. Sheafifying this exact sequence and applying the functor $\mathrm{H}^0_*(U', -)$, we have

$$\overline{\mathbf{A}} = \mathbf{B}/\,\mathrm{H}^0_*(U', \mathcal{I}_{C/S}) \simeq \mathbf{A}/\,\mathrm{H}^0_{\mathbf{J}}(\mathbf{A}) \quad \text{and} \quad \widetilde{\overline{\mathbf{A}}}|_{U'} \simeq \widetilde{\mathbf{A}}|_{U'}.$$

Hence $\mathrm{H}^{\alpha+1}_{\mathbf{J}}(\overline{\mathbf{A}}) \simeq \mathrm{H}^{\alpha+1}_{\mathbf{J}}(\mathbf{A})$. Now, by the definition of $\mathcal{Z}_i$, the following sequence is exact for $k > 0$

$$\to \mathrm{H}^{\alpha+k}_*(U', \mathcal{M}_k(-ks)) \to \mathrm{H}^{\alpha+k}_*(U', \mathcal{Z}_{k-1}) \to \mathrm{H}^{\alpha+k+1}_*(U', \mathcal{Z}_k) \to$$

Since we are working under the assumption $\mathrm{H}^{\alpha+k}_*(U', \mathcal{M}_k) = 0$, we get

$$\mathrm{H}^{\alpha+1}_*(U', \mathcal{Z}_0) \hookrightarrow \ldots \hookrightarrow \mathrm{H}^{\alpha+k+1}_*(U', \mathcal{Z}_k) \hookrightarrow \ldots \hookrightarrow \mathrm{H}^{\alpha+r}_*(U', \mathcal{Z}_{r-1})$$

i.e.

$$\mathrm{H}^{\alpha+1}_{\mathbf{J}}(\overline{\mathbf{A}}) \simeq \mathrm{H}^{\alpha+1}_*(U', \mathcal{Z}_0) \hookrightarrow \mathrm{H}^{\alpha+r}_*(U', \mathcal{K}_S(t-rs)) \simeq \mathrm{H}^{\alpha+r+1}_{\mathbf{J}}(\mathbf{K}_{\mathbf{B}}(t-rs))$$

and we are done. $\square$

Remark 13. *Lemma 12 holds with $\alpha = -1$ in the sense that the condition*

$$\mathrm{H}^{i-1}_*(U', \mathcal{M}_i) = 0 \ \text{ for } \ 2 \le i \le r-1$$

implies the injectivity of the natural map $\mathrm{H}^0_{\mathbf{J}}(\mathbf{A}) \to \mathrm{H}^r_{\mathbf{J}}(\mathbf{K}_{\mathbf{B}}(t-rs))$.

We now prove Theorem 9.

Proof. By the proof of Lemma 12 we have

$$\overline{\mathbf{A}} := \mathbf{B}/\,\mathrm{H}^0_*(U', \mathcal{I}_{C/S}) \simeq \mathbf{A}/\,\mathrm{H}^0_{\mathbf{J}}(\mathbf{A}). \tag{3}$$

Hence we get conclusion 1 of Theorem 9 from Theorem 7 and Proposition 6.

For the proof of conclusion 2 of Theorem 9, we have the assumption $\operatorname{depth}_{\mathbf{I}(Z')} \mathbf{B} \ge r+1$. Suppose, momentarily, that we have the slightly stronger assumption $\operatorname{depth}_{\mathbf{I}(Z')} \mathbf{B} \ge r+2$. In this case, we can obtain Conclusion 2 from previous results. Indeed applying Lemma 12 with $\alpha = 0$, we get

$$\mathrm{H}^1_{\mathbf{J}}(\overline{\mathbf{A}}) \simeq \mathrm{H}^1_{\mathbf{J}}(\mathbf{A}) \hookrightarrow \mathrm{H}^{r+1}_{\mathbf{J}}(\mathbf{K}_{\mathbf{B}}(t-rs)) = 0$$

where $\mathbf{J} = \mathbf{I}(Z')$. To see that $\overline{\mathbf{A}}$ is quasi-Gorenstein, we use Theorem 7 which tells that $\mathcal{K}_C|_{U'} \simeq \mathcal{O}_C(l)|_{U'}$ for some integer l. Recalling $\operatorname{depth}_{\mathbf{J}} \mathbf{K}_{\mathbf{B}} = \operatorname{depth}_{\mathbf{J}} \mathbf{B} \ge r+2$, we get

$$\mathbf{K}_{\overline{\mathbf{A}}} = \operatorname{Ext}^r_{\mathbf{B}}(\overline{\mathbf{A}}, \mathbf{K}_{\mathbf{B}}) \simeq \bigoplus_{\nu} \operatorname{Ext}^r_{\mathcal{O}_{U'}}(\widetilde{\overline{\mathbf{A}}}|_{U'}, \mathcal{K}_S(\nu)|_{U'})$$

where the right isomorphism is due to [7], exp. VI. By Theorem 7, $C \cap U'$ is equidimensional and since S is arithmetically Cohen-Macaulay, we get $\mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{K}_S)|_{U'} = 0$ for $i < r$. Hence

$$\mathbf{K}_{\overline{\mathbf{A}}} \simeq \bigoplus_{\nu} \mathrm{H}^0(U', \mathcal{E}xt^r_{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{K}_S(\nu))) \simeq \mathrm{H}^0_*(U', \mathcal{K}_C) \simeq \mathrm{H}^0_*(U', \mathcal{O}_C(l)). \tag{4}$$

Since $\overline{\mathbf{A}} = \mathbf{B}/\,\mathrm{H}^0_*(U', \mathcal{I}_{C/S})$ and $\mathrm{H}^1_{\mathbf{J}}(\overline{\mathbf{A}}) = 0$, we have $\mathbf{K}_{\overline{\mathbf{A}}} \simeq \overline{\mathbf{A}}(l)$.

Now suppose $\mathrm{depth}_{\mathbf{J}}\,\mathbf{B} \geq r+1$. We still have (3), but not necessarily $\mathrm{H}^1_{\mathbf{J}}(\overline{\mathbf{A}}) = 0$. Indeed, there is an exact sequence

$$0 \to \overline{\mathbf{A}} \to \mathrm{H}^0_*(U', \mathcal{O}_C) \to \mathrm{H}^1_{\mathbf{J}}(\overline{\mathbf{A}}) \to 0. \tag{5}$$

Moreover, the isomorphism of (4) is now only an injection since [7], exp. VI demonstrates the existence of the exact sequence

$$0 \to \mathbf{K}_{\overline{\mathbf{A}}} \to \mathrm{H}^0_*(U', \mathcal{E}xt^r(\mathcal{O}_C, \mathcal{K}_S)) \to \mathrm{Hom}_{\mathbf{B}}(\overline{\mathbf{A}}, \mathrm{H}^{r+1}_{\mathbf{J}}(\mathbf{K}_{\mathbf{B}})) \to \tag{6}$$

By the long exact sequence of Theorem 7, the exact sequences (5) and (6) fit into a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \overline{\mathbf{A}}(l) & \to & \mathrm{H}^0_*(U', \mathcal{O}_C(l)) & \to & \mathrm{H}^1_{\mathbf{J}}(\overline{\mathbf{A}})(l) & \to & 0 \\
 & & & & \downarrow \simeq & \circ & \downarrow & & \\
0 & \to & \mathbf{K}_{\overline{\mathbf{A}}} & \to & \mathrm{H}^0_*(U', \mathcal{K}_C) & \to & \mathrm{Hom}_{\mathbf{B}}(\overline{\mathbf{A}}, \mathrm{H}^{r+1}_{\mathbf{J}}(\mathbf{K}_{\mathbf{B}})) & &
\end{array}$$

The right vertical map is injective by Lemma 12. Hence we get $\mathbf{K}_{\overline{\mathbf{A}}} \simeq \overline{\mathbf{A}}(l)$ in this case as well.

For conclusion 3 of Theorem 9, we have $\mathrm{H}^0_{\mathbf{J}}(\mathbf{A}) = 0$ by Remark 10 (b) and Remark 13. Hence $\overline{\mathbf{A}} = \mathbf{A}/\,\mathrm{H}^0_{\mathbf{J}}(\mathbf{A}) \simeq \mathbf{A}$ and we are done. □

Quasi-Gorenstein quotients of $\mathbf{R}$ satisfy S_2. Hence if the quasi-Gorenstein quotients $\mathbf{A}$ and $\overline{\mathbf{A}}$ of Theorem 9 are 1 dimensional (resp. 2 dimensional), they are Gorenstein, and since condition (iii) of Theorem 9 shows $\dim \mathbf{B}/\mathbf{I}(Z') = 0$ (resp. ≤ 1), we get $U' = S$ (resp. $S - U'$ a finite or empty set of points). In the case $U' = S$, $\overline{\mathbf{A}} = \mathbf{B}/\,\mathrm{H}^0_*(S, \mathcal{I}_{C/S})$ is the homogeneous coordinate ring of C in P. Combining with Remark 10 (b), we get the following corollaries:

Corollary 14. *In addition to the notation and conditions of Theorem 9 $(i) - (iv)$, suppose $r = n$. Then $U' = S$ and $C = \mathrm{Proj}(\mathbf{A})$ is zero-dimensional. In this case, we have the following:*

1. *If $\mathrm{H}^i_*(S, \mathcal{M}_i) = 0$ for $1 \leq i \leq \frac{r}{2}$, then $\overline{\mathbf{A}}$ is Gorenstein, i.e. C is arithmetically Gorenstein.*

2. *If $\mathrm{H}^i_*(S, \mathcal{M}_i) = \mathrm{H}^i_*(S, \mathcal{M}_{i-1}) = \mathrm{H}^{i-1}_*(S, \mathcal{M}_i) = 0$ for $2 \leq i \leq \frac{r}{2}$ and if $\mathrm{H}^1_*(S, \mathcal{M}_1) = 0$ and $\mathrm{H}^i(S, \mathcal{M}_{i-1}) = 0$ for $i = \frac{r+1}{2}$, then $\mathbf{A}$ is Gorenstein and $\mathbf{A} \simeq \overline{\mathbf{A}}$.*

Corollary 15. *In addition to the assumptions in Theorem 9 (i) to (iv), suppose $r = n - 1$. Then $U' = S$ or $\dim(S - U') = 0$ and $C = \mathrm{Proj}(\mathbf{A})$ is one-dimensional. In this case, we have the following:*

1. *If $\mathrm{H}_*^{i+1}(S,\mathcal{M}_i)=0$ for $1\le i\le r-1$, or equivalently,*
   if $\mathrm{H}_*^{i+1}(S,\mathcal{M}_i)=0$ for $1\le i\le \frac{r-1}{2}$ and $\mathrm{H}_*^{i}(U',\mathcal{M}_i)=0$ for $1\le i\le \frac{r}{2}$, then $\overline{\mathbf{A}}$ is Gorenstein.

2. *If $\mathrm{H}_*^{i+1}(S,\mathcal{M}_i)=\mathrm{H}_*^{i+2}(S,\mathcal{M}_i)=0$ for $1\le i\le r-2$ and $\mathrm{H}_*^{r}(S,\mathcal{M}_{r-1})=0$, or equivalently,*
   if $\mathrm{H}_*^{i+1}(S,\mathcal{M}_i)=\mathrm{H}_*^{i+2}(S,\mathcal{M}_i)=0$ for $1\le i\le \frac{r-1}{2}$, $\mathrm{H}_*^{1}(U',\mathcal{M}_1)=0$ and $\mathrm{H}_*^{i-1}(U',\mathcal{M}_i)=\mathrm{H}_*^{i}(U',\mathcal{M}_i)=0$ for $2\le i\le \frac{r}{2}$, then $\mathbf{A}$ is Gorenstein and $\mathbf{A}\simeq\overline{\mathbf{A}}$.

In the corollaries above, Conclusion 1 seems to be the most interesting because the property of $\overline{\mathbf{A}}$ being Gorenstein is equivalent to C being arithmetically Gorenstein (provided $U'=S$). Now we shall see that Theorem 9 also extends to the case where $\mathbf{A}$ is Artinian. Here it is the analogue of Conclusion 2 which is the most interesting.

Proposition 16. *Let $r=n+1\ge 2$, let $S\subseteq P$ be as in Theorem 9 (i) and let $\mathcal{M}$ be a coherent locally free $\mathcal{O}_S$-module satisfying $\wedge^r\mathcal{M}\simeq\mathcal{K}_S(t)$ for some t. Let $\sigma\in\mathrm{H}^0(S,\mathcal{M}^*(s))$ be a regular section and let $\mathbf{A}=\mathrm{coker}(\mathrm{H}_*^0(S,\mathcal{M}(-s))\xrightarrow{\sigma^*}\mathbf{B})$. If one of the following equivalent cohomological assumptions holds*

(i) $\mathrm{H}_*^{i}(S,\wedge^{i+1}\mathcal{M})=0\quad\text{for}\quad 1\le i\le r-2$

(ii) $\mathrm{H}_*^{i}(S,\wedge^{i}\mathcal{M})=0\quad\text{for}\quad 1\le i\le r-2$

(iii) $\mathrm{H}_*^{i}(S,\wedge^{i}\mathcal{M})=0\quad\text{for}\quad 1\le i\le \frac{r-1}{2}$ *and* $\mathrm{H}_*^{i}(S,\wedge^{i+1}\mathcal{M})=0\quad\text{for}\quad 1\le i\le \frac{r-2}{2}$

then $\mathbf{A}$ is an Artin Gorenstein algebra.

Proof. We easily get the equivalence of the cohomological assumptions by duality on S (e.g. see the proof of Lemma 11). Now exactly as in the very last part of the proof of Lemma 12 (cf. Remark 13) we get

$$\mathrm{H}_*^1(S,\mathcal{Z}_1)\hookrightarrow\mathrm{H}_*^2(S,\mathcal{Z}_2)\hookrightarrow\ldots\hookrightarrow\mathrm{H}_*^{r-1}(S,\mathcal{Z}_{r-1})=\mathrm{H}_*^{r-1}(S,\widetilde{\mathbf{K}}_{\mathbf{B}}(t-rs))$$

where now $r-1=n=\dim(S)$ and $\mathrm{H}_*^{r-1}(S,\widetilde{\mathbf{K}}_{\mathbf{B}}(t-rs))$ is nonzero. Moreover,

$$\mathrm{Hom}_{\mathbf{B}}(\mathbf{A},\mathrm{H}_*^n(S,\widetilde{\mathbf{K}}_{\mathbf{B}}))\simeq\mathrm{Ext}_{\mathbf{B}}^{n+1}(\mathbf{A},\mathbf{K}_{\mathbf{B}})\simeq\mathbf{K}_{\mathbf{A}}.$$

We claim $\mathbf{A}\hookrightarrow\mathrm{H}_*^1(S,\mathcal{Z}_1)$ is injective. Indeed, the sequence

$$\mathrm{H}_*^0(S,\mathcal{M})(-s)\to\mathbf{B}\to\mathbf{A}\to 0$$

is exact. Since $0\to\mathcal{Z}_1\to\mathcal{M}(-s)\to\widetilde{\mathbf{I}}_{\mathbf{B}/\mathbf{A}}\to 0$ implies the exactness of

$$\mathrm{H}_*^0(S,\mathcal{M})(-s)\to\mathrm{H}_*^0(S,\widetilde{\mathbf{I}}_{\mathbf{B}/\mathbf{A}})\to\mathrm{H}_*^1(S,\mathcal{Z}_1).$$

we get the claim because $\mathrm{Proj}(\mathbf{A})$ is empty, i.e. $\mathrm{H}_*^0(S,\widetilde{\mathbf{I}}_{\mathbf{B}/\mathbf{A}})=\mathrm{H}_*^0(S,\widetilde{\mathbf{B}})=\mathbf{B}$.
Now the composition $\mathbf{A}\hookrightarrow\mathrm{H}_*^1(S,\mathcal{Z}_1)\hookrightarrow\mathrm{H}_*^n(S,\widetilde{\mathbf{K}}_{\mathbf{B}})(t-rs)$ shows that we have an $\mathbf{A}$-linear injection

$$\mathbf{A}=\mathrm{Hom}_{\mathbf{B}}(\mathbf{A},\mathbf{A})\hookrightarrow\mathrm{Hom}_{\mathbf{B}}(\mathbf{A},\mathrm{H}_*^n(S,\widetilde{\mathbf{K}}_{\mathbf{B}}))(t-rs)\simeq\mathbf{K}_{\mathbf{A}}(t-rs).$$

Since $\mathbf{A}$ and $\mathbf{K}_{\mathbf{A}}$ have the same length, they are isomorphic and we are done. $\square$

If $\dim C \leq 1$, Theorem 9 gives conditions which imply that $\overline{\mathbf{A}}$ (resp. $\mathbf{A}$) is Gorenstein. If $\dim C > 1$, we need some additional assumptions in order to see that $\mathrm{Proj}(\overline{\mathbf{A}})$ is arithmetically Cohen-Macaulay and hence deduce that $\overline{\mathbf{A}}$ (resp. $\mathbf{A}$) is Gorenstein. Our first result plays on the proof of Lemma 12 by increasing the depth assumption on $\operatorname{depth}_{\mathbf{I}(Z')} \mathbf{B}$ to say $\operatorname{depth}_{\mathbf{I}(Z')} \mathbf{B} > r - 1 + \frac{\dim C}{2}$ while assuming the first condition of Lemma 12 holds for any α such that $0 \leq \alpha \leq \frac{\dim C}{2}$. Our second result uses the ideas of [20] and [21] in which they get the arithmetically Cohen-Macaulayness of $\overline{C} := \mathrm{Proj}(\overline{\mathbf{A}})$ from the Cohen-Macaulayness of $\overline{C} \cap H$, H a hyperplane section.

Proposition 17. *Let $S \subseteq P$, $\mathcal{M}_i$, $\mathbf{A}$ and $\overline{\mathbf{A}}$ satisfy the assumptions of Theorem 9 (i) to (iv). Moreover*

1. *Suppose the following conditions are satisfied:*
   i) $\operatorname{depth}_{\mathbf{I}(Z')} \mathbf{B} > r - 1 + \frac{\dim C}{2}$
   ii) $\mathrm{H}_*^{j+i}(S, \mathcal{M}_i) = 0$ *for* $1 \leq i \leq r-1$, $\frac{1}{2}\dim C \leq j \leq \dim C$
   then $\overline{\mathbf{A}}$ is Gorenstein.

2. *Suppose the following conditions are satisfied:*
   i) $\operatorname{depth}_{\mathbf{I}(Z')} \mathbf{B} > r - 2 + \frac{\dim C}{2}$
   ii) $\mathrm{H}_*^{j+i}(S, \mathcal{M}_i) = 0$ *for* $1 \leq i \leq r-1$, $\frac{1}{2}\dim C \leq j \leq \dim C$
   iii) $\mathrm{H}_*^{\dim C+i}(S, \mathcal{M}_{i-1}) = 0$ *for* $2 \leq i \leq r-1$
   then $\mathbf{A}$ is Gorenstein and $\mathbf{A} = \overline{\mathbf{A}}$.

Proof. Proof of part 1: Let $\mathbf{J} = \mathbf{I}(Z')$. In the case $\dim C = 2g-1$ (resp. $\dim C = 2g$), then $\operatorname{depth}_{\mathbf{J}} \mathbf{B} > r - 1 + \frac{\dim C}{2}$ is equivalent to $\operatorname{depth}_{\mathbf{J}} \mathbf{B} \geq r + g - 1$ (resp. $\operatorname{depth}_{\mathbf{J}} \mathbf{B} \geq r + g$), i.e. to $\dim \mathbf{B}/\mathbf{J} \leq g+1$. In both cases we claim

$$\mathrm{H}_{\mathfrak{m}}^{j}(\overline{\mathbf{A}}) = 0 \quad \text{for} \quad g+1 \leq j < \dim \overline{\mathbf{A}}.$$

We can suppose $g \geq 1$ since Corollary 14 takes care of the case $\dim \overline{\mathbf{A}} = 1$. Hence

$$\mathrm{H}_{\mathfrak{m}}^{j}(\overline{\mathbf{A}}) \simeq \mathrm{H}_{\mathfrak{m}}^{j+1}(\mathbf{I}_{\mathbf{B}/\overline{\mathbf{A}}}) \simeq \mathrm{H}_*^{j}(S, \mathcal{Z}_0)$$

letting $\mathcal{Z}_0 = \widetilde{\mathbf{I}}_{\mathbf{B}/\overline{\mathbf{A}}}$ and elsewhere with notations as in (2). The cokernel of the right map in

$$0 \to \mathcal{Z}_i \to \mathcal{M}_i(-is) \to \mathcal{Z}_{i-1}$$

is supported at $Z' \subseteq S$ because the complex of Theorem 7 is exact on U'. If $\mathcal{Z}'_{i-1} = \mathrm{im}(\mathcal{M}_i(-is) \to \mathcal{Z}_{i-1})$ for $i > 0$, then $\mathcal{Z}'_0 = \widetilde{\mathbf{I}}_{\mathbf{B}/\mathbf{A}}$ and we have $\mathrm{H}_*^{j}(S, \mathcal{Z}_i/\mathcal{Z}'_i) = 0$ for $j \geq g+1$ because $\dim \mathbf{B}/\mathbf{J} \leq g+1$. Hence using $\mathcal{Z}_{r-1} \simeq \mathcal{K}_S(t-rs)$ and the assumptions of (ii), we get

$$\begin{aligned} \mathrm{H}_*^{j}(S, \mathcal{Z}_0) \twoheadleftarrow \mathrm{H}_*^{j}(S, \mathcal{Z}'_0) \hookrightarrow \mathrm{H}_*^{j+1}(S, \mathcal{Z}_1) \simeq \mathrm{H}_*^{j+1}(S, \mathcal{Z}'_1) \hookrightarrow \\ \mathrm{H}_*^{j+2}(S, \mathcal{Z}_2) \simeq \mathrm{H}_*^{j+2}(S, \mathcal{Z}'_2) \hookrightarrow \ldots \hookrightarrow \mathrm{H}_{\mathfrak{m}}^{j+r}(\mathbf{K}_{\mathbf{B}}(t-rs)) = 0 \end{aligned} \tag{7}$$

since $r + j < r + \dim \mathbf{A} = \dim \mathbf{B}$, and the claim is proved.

Suppose $\dim C = 2g-1$. Since $\mathrm{H}^i_{\mathfrak{m}}(\overline{\mathbf{A}}) = 0$ for $g+1 \le i < 2g$, then $\mathbf{K}_{\overline{\mathbf{A}}}$ satisfies the Serre condition S_{g+1} by Proposition 2. Since $\mathbf{K}_{\overline{\mathbf{A}}} \simeq \overline{\mathbf{A}}(l)$ by Theorem 9, $\overline{\mathbf{A}}$ satisfies S_{g+1}, i.e.

$$\operatorname{depth} \overline{\mathbf{A}}_{\mathfrak{p}} \ge \min\{g+1, \dim \overline{\mathbf{A}}_{\mathfrak{p}}\} \text{ for all } \mathfrak{p} \in \operatorname{Supp}(\overline{\mathbf{A}}).$$

Hence $\operatorname{depth} \overline{\mathbf{A}} \ge g+1$ which implies $\mathrm{H}^i_{\mathfrak{m}}(\overline{\mathbf{A}}) = 0$ for $i < g+1$. Combining with the proven claim, we see that $\overline{\mathbf{A}}$ is Cohen-Macaulay and hence Gorenstein.

If $\dim C = 2g$, then $\mathbf{K}_{\overline{\mathbf{A}}}$ satisfies S_{g+2} by Proposition 2. Hence $\overline{\mathbf{A}} \simeq \mathbf{K}_{\overline{\mathbf{A}}}(-l)$ satisfies $\operatorname{depth} \overline{\mathbf{A}} \ge g+2$, and we get $\mathrm{H}^i_{\mathfrak{m}}(\overline{\mathbf{A}}) = 0$ for $i < \dim \overline{\mathbf{A}}$ in this case as well.

Proof of part 2: By Theorem 9 and the proof above (cf. in particular (7)) we are done once we have proved that the assumption $\operatorname{depth}_{\mathbf{J}}(\mathbf{B}) > r - 2 + \frac{\dim C}{2}$ (which is equivalent to $\dim \mathbf{B}/\mathbf{J} \le g+2$) still implies $\mathrm{H}^j_*(S, \mathcal{Z}_i/\mathcal{Z}'_i) = 0$ for $j \ge g+1$. Looking to (7), it suffices to prove the slightly weaker:

$$\mathrm{H}^j_*(S, \mathcal{Z}_0/\mathcal{Z}'_0) = 0 \text{ for } j \ge g+1 \text{ and } \mathrm{H}^j_*(S, \mathcal{Z}_i/\mathcal{Z}'_i) = 0 \text{ for } j \ge g+2, i > 0 \tag{8}$$

By Theorem 9 we have $\mathbf{A} \simeq \overline{\mathbf{A}}$ and hence $\mathcal{Z}'_0 \simeq \widetilde{\mathbf{I}}_{\mathbf{B}/\mathbf{A}} \simeq \widetilde{\mathbf{I}}_{\mathbf{B}/\overline{\mathbf{A}}} = \mathcal{Z}_0$. Since, for each i, $\mathcal{Z}_i/\mathcal{Z}'_i$ is supported at $V(\mathbf{J}) \subseteq S$ and $\dim V(\mathbf{J}) \le g+1$, we get (8) and we can conclude the proof. □

If $\mathcal{M}$ is locally free on S, it is natural to take $U = U' = S$ in Theorem 9. In this case Proposition 17 allows the following important corollary.

Corollary 18. *In addition to the notation and conditions of Theorem 9 $(i)-(iv)$, we suppose $\mathcal{M}$ is locally free on S (and we may take $U = U' = S$). Then we have the following:*

1. *If $\mathrm{H}^{j+i}_*(S, \wedge^i \mathcal{M}) = 0$ for any i,j such that $1 \le i \le \frac{r}{2}$, $0 \le j \le \dim C$, then $\overline{\mathbf{A}}$ is Gorenstein, i.e. C is arithmetically Gorenstein.*

2. *If $\mathrm{H}^{j+i}_*(S, \wedge^i \mathcal{M}) = 0$ for any $1 \le i \le \frac{r}{2}$, $-1 \le j \le \dim C + 1$ and $j + i > 0$, then $\mathbf{A}$ is Gorenstein and $\mathbf{A} = \overline{\mathbf{A}}$.*

Proof. This is straightforward from Proposition 17 and Remark 10 (b). □

Remark 19. *If r is even we can "weaken" the assumptions in Corollary 18, assuming only half of the cohomology groups for $i = \frac{r}{2}$ to vanish. By duality, such a weakening is equivalent to the corresponding assumptions stated in Corollary 18.*

Remark 20. *We can skip assumption 2 (i) of Proposition 17 on $\operatorname{depth}_{\mathbf{I}(Z')} \mathbf{B}$ provided $2 \le r \le 3$ (or provided $r = 4$ and $\mathrm{H}^{\dim C+3}_*(S, \mathcal{M}_1) = 0$). More generally, letting $2 \le N \le r-2$ and assuming*

$$\mathrm{H}^{\dim C+i}_*(S, \mathcal{M}_{i-k}) = 0 \text{ for } 2 \le k \le N \text{ and } k+1 \le i \le r-1 \tag{9}$$

we get the conclusion of the second part of Proposition 17 under the assumption

$$\operatorname{depth}_{\mathbf{I}(Z')} \mathbf{B} > r - 1 - N + \frac{\dim C}{2}.$$

Indeed, slightly modifying the proof of Proposition 17 (see in particular (7) and (8)) we get the conclusion in Proposition 17.2 because the cohomological assumption in (9) implies $\mathcal{Z}'_i \simeq \mathcal{Z}_i$

for $1 \leq i \leq N-1$ *(use the arguments of the proof of Lemma 12). Note that we have* $\mathcal{Z}_{r-1} \simeq \mathcal{K}_S(t-rs)$ *which leads to* $\mathrm{H}^1_*(U', \mathcal{Z}_{r-1}) = 0$ *and* $\mathcal{Z}'_{r-2} \simeq \mathcal{Z}_{r-2}$*. Hence if (9) holds for* $N = r-2$*, we get* $\mathcal{Z}^!_i \simeq \mathcal{Z}_i$ *for all* i *and the complex of Conclusion 4 of Theorem 7 is exact (on S!). In this case we can drop the assumption on* $\mathrm{depth}_{\mathbf{I}(Z')} \mathbf{B}$ *above and hence, completely drop the assumption (2.i) of Proposition 17. This happens for* $2 \leq r \leq 3$ *because the condition (9) with* $N = r-2$ *is empty.*

The following theorem essentially generalizes the main theorem of [17] by replacing the maximal Cohen-Macaulay module condition on the modules $\mathbf{M}_i$, $1 \leq i \leq \frac{r}{2}$, with a condition of "sufficiently high depth". If we, in addition, suppose $\mathrm{codim}_S Z' > r + \frac{\dim C}{2}$, one may get Theorem 21 as a corollary of Proposition 17. In this generality, however, we prove it by induction on $\dim \mathbf{A}$ and by using Theorem 9.

Theorem 21. *Let* $S \subseteq P$*,* $\mathbf{M}_i$*,* $\mathbf{A}$ *and* $\overline{\mathbf{A}}$ *satisfy the conditions of Theorem 9* $(i)-(iv)$*. Then we have the following*

1. *If* $\mathrm{depth}_{\mathfrak{m}} \mathbf{M}_i \geq \dim C + 2 + i$ for $1 \leq i \leq \frac{r}{2}$, *then* $\overline{\mathbf{A}}$ *is Gorenstein.*
2. *If* $\mathrm{depth}_{\mathfrak{m}} \mathbf{M}_i \geq \min(\dim C + 3 + i, \dim S + 1)$ for $1 \leq i \leq \frac{r}{2}$, *then* $\mathbf{A}$ *is Gorenstein and* $\mathbf{A} \simeq \overline{\mathbf{A}}$.

Remark 22. *If* r *is even, the depth conditions on* $\mathbf{M}_{r/2}$ *of Theorem 21 turn out to be equivalent to* $\mathrm{depth}_{\mathfrak{m}} \mathbf{M}_{r/2} \geq \dim C + 2 + \frac{r}{2} - \frac{1}{2}(\mathrm{depth}_{\mathbf{I}(Z')} \mathbf{B} - r)$*. Indeed, using the same proof as in [17], Remark 9 (i.e. use lemma 5 and the proof of lemma 11), one shows that the last mentioned depth condition implies the Cohen-Macaulayness of* $\mathbf{M}_{r/2}$*. We get in particular the depth condition on* $\mathbf{M}_{r/2}$ *of Theorem 21.*

Proof. First, we prove that $\overline{\mathbf{A}}$ (resp. $\mathbf{A}$) is quasi-Gorenstein (and hence is S_2) by showing that the corresponding assumptions of Theorem 9 are satisfied. The depth conditions on $\mathrm{depth}_{\mathfrak{m}} \mathbf{M}_i$ in 1. (resp. 2.) imply at once

$$\mathrm{H}^j_*(S, \widetilde{\mathbf{M}}_i) = 0 \ \text{ for } \ j = \dim C + i \ \ (\text{resp.} \dim C + i \leq j \leq \min(\dim C + i + 1, \dim S - 1)) \quad (10)$$

provided $1 \leq i \leq \frac{r}{2}$. Moreover by Lemma 11,

$$\mathrm{H}^{\dim C + i}_*(S, \widetilde{\mathbf{M}}_i)^\vee \simeq \mathrm{H}^{r-i}_*(U', \widetilde{\mathbf{M}}_{r-i}) \ \ (\text{up to shift})$$

and the vanishing of $\mathrm{H}^{\dim C + i}_*(S, \widetilde{\mathbf{M}}_i)$ for $\frac{r}{2} < i \leq r-1$ follows if we can show the vanishing of $\mathrm{H}^i_*(U', \widetilde{\mathbf{M}}_i) \simeq \mathrm{H}^{i+1}_{\mathbf{J}}(\mathbf{M}_i)$ ($\mathbf{J} = \mathbf{I}(Z')$) for $1 \leq i \leq \frac{r}{2}$. Correspondingly for the case $\mathrm{H}^{\dim C + i + 1}_*(S, \mathcal{M}_i)$, we need to show that $\mathrm{H}^i_{\mathbf{J}}(\mathbf{M}_i)$ for $2 \leq i \leq \frac{r}{2}$ vanishes. Using Lemma 5 and the assumptions $\mathrm{depth}_{\mathbf{J}} \mathbf{B} \geq r + 1 = n + 1 - \dim C$, we get $\mathrm{depth}_{\mathbf{J}} \mathbf{M}_i \geq 2 + i$ for both 1. and 2. and in particular $\mathrm{H}^j_{\mathbf{J}}(\mathbf{M}_i) = 0$ for $j \leq i + 1$ and we are done.

To prove that $\overline{\mathbf{A}}$ (resp. $\mathbf{A}$) is Gorenstein, we use induction on $\dim \overline{\mathbf{A}}$. If $\dim \overline{\mathbf{A}} \leq 2$, then $\overline{\mathbf{A}}$ (resp. $\mathbf{A}$) is S_2 and hence Cohen-Macaulay and Gorenstein by the first part of the proof. Suppose $\dim \overline{\mathbf{A}} > 2$, the depth conditions on $\mathbf{M}_i$ ($0 \leq i \leq \frac{r}{2}$ where $\mathbf{M}_0 = \mathbf{B}$) imply that the irrelevant maximal ideal $\mathfrak{m} \notin \mathrm{Ass}(\mathbf{M}_i)$. Hence, we can choose a homogeneous F of degree f which is outside all associated primes of $\mathbf{M}_i$ ($0 \leq i \leq \frac{r}{2}$) and ouside all minimal primes of

$\mathbf{B}/\mathbf{I}(Z)$ and $\mathbf{B}/\mathbf{I}(Z')$ and furthermore outside all associated primes in $\mathrm{Ass}(\mathbf{A}) - \{\mathfrak{m}\}$ (and hence of $\overline{\mathbf{A}}$).

Hence, if $S_F = S \cap V(F), Z_F = Z \cap V(F), Z'_F = Z' \cap V(F)$ and $U_F = S_F - Z_F$, then

$$0 \to \mathbf{M}_i(-f) \xrightarrow{F} \mathbf{M}_i \to \mathbf{M}_i/F \cdot \mathbf{M}_i \to 0, \quad 0 \le i \le \frac{r}{2} \tag{11}$$

is exact, $\mathrm{codim}_{S_F} Z_F \ge r$, $\mathrm{codim}_{S_F} Z'_F \ge r+1$ and F is $\overline{\mathbf{A}}$-regular.

Let $\mathbf{B}_F = \mathbf{B}/(F), \mathbf{A}_F = \mathbf{A}/(F)\mathbf{A}, (\overline{\mathbf{A}})_F = \overline{\mathbf{A}}/(F)\overline{\mathbf{A}}$ and $C_F = \mathrm{Proj}(\mathbf{A}_F)$. We claim that

$$\sigma_F := \sigma \otimes id_{\widetilde{\mathbf{B}}_F|_{U_F}} \quad \text{where} \quad \sigma : \widetilde{\mathbf{B}}|_U \to \widetilde{\mathbf{M}^*(s)}|_U \simeq \mathcal{M}^*(s)|_U$$

(id, the identity and $\mathbf{M} := \mathbf{M}_1$) is a regular section satisfying

$$\mathrm{coker}(\mathrm{H}^0_*(U_F, \widetilde{\mathbf{M}}(-s) \otimes \widetilde{\mathbf{B}}_F) \xrightarrow{\sigma_F^*} \mathbf{B}_F) \simeq \mathbf{A}_F.$$

Indeed, by the definition of $\mathbf{A}_F$, the sequence

$$\mathrm{H}^0_*(U, \widetilde{\mathbf{M}}(-s)) \otimes_{\mathbf{B}} \mathbf{B}_F \to \mathbf{B} \otimes_{\mathbf{B}} \mathbf{B}_F \to \mathbf{A}_F \to 0$$

is exact. It suffices, therefore, to show that

$$\mathrm{H}^0_*(U, \widetilde{\mathbf{M}}(-s)) \otimes_{\mathbf{B}} \mathbf{B}_F \to \mathrm{H}^0_*(U_F, \widetilde{\mathbf{M}}(-s) \otimes \widetilde{\mathbf{B}}_F)$$

is an isomorphism. Now, from the first part of the proof, we have $\mathrm{depth}_{\mathbf{J}} \mathbf{M} \ge 3$ (hence $\mathrm{H}^1_*(U', \widetilde{\mathbf{M}}) = 0$), and we have $\mathrm{depth}_{\mathbf{I}(Z)} \mathbf{M} \ge 2$ (hence $\mathrm{H}^0_*(U', \widetilde{\mathbf{M}}) \simeq \mathrm{H}^0_*(U, \widetilde{\mathbf{M}})$) by similar arguments. We therefore get the desired isomorphism (and hence the claim) by applying $\mathrm{H}^0_*(U', -)$ to the sheafification of the exact sequence (11) with $i = 1$.

We can apply Theorem 9 to the schemes P, S_F, U_F, U'_F and the modules $\widetilde{\mathbf{M}_i/F \cdot \mathbf{M}_i}$. Indeed, Theorem 9 (i) holds (with c increased by 1), and we have (ii) since $\mathcal{K}_{S_F} \simeq \mathcal{K}_S \otimes \mathcal{O}_{S_F}$. Moreover, the Cohen-Macaulayness of $\widetilde{\mathbf{M}_i/F \cdot \mathbf{M}_i}|_{U'_F}$ for $0 \le i \le \frac{r}{2}$ follows from (11). Using the proven claim, the zero locus of the section is precisely $\mathbf{A}_F$. Since

$$\mathrm{depth}_{\mathfrak{m}} \mathbf{M}_i/F \cdot \mathbf{M}_i \ge \dim C_F + 2 + i \quad \text{for} \quad 1 \le i \le \frac{r}{2}$$

$$(\text{resp.} \quad \mathrm{depth}_{\mathfrak{m}} \mathbf{M}_i/F \cdot \mathbf{M}_i \ge \min(\dim C_F + 3 + i, \dim S_F + 1)) \quad \text{for} \quad 1 \le i \le \frac{r}{2}$$

holds by (11), all assumptions of Theorem 21 holds as well. By induction, the top dimensional component, $\overline{(\mathbf{A}_F)}$ of $\mathbf{A}_F$ (resp. $\mathbf{A}_F$) is Gorenstein. Note that $\overline{\mathbf{A}}$ is S_2 and equidimensional and

$$0 \to \overline{\mathbf{A}} \xrightarrow{F} \overline{\mathbf{A}} \to \overline{\mathbf{A}}/(F)\overline{\mathbf{A}} \to 0 \tag{12}$$

is exact by the first part of the proof and the choice of F. It follows that all associated primes of $(\overline{\mathbf{A}})_F := \overline{\mathbf{A}}/(F)\overline{\mathbf{A}}$ are minimal and that $(\overline{\mathbf{A}})_F$ is equidimensional. Since, by reasons of dimension, no minimal prime of $(\overline{\mathbf{A}})_F$ can contain $\mathbf{J}(\overline{\mathbf{A}})_F$, $(\overline{\mathbf{A}})_F$ is the top dimensional component $\mathbf{A}_F$, i.e. $(\overline{\mathbf{A}})_F \simeq \overline{(\mathbf{A}_F)}$ is Gorenstein. Hence, $\overline{\mathbf{A}}$ (resp. $\mathbf{A}_F$) is Gorenstein by (12) and we are done. $\square$

Theorem 21 allows us to prove a variation of Corollary 18 in which $\mathcal{M}$ is not necessarily locally free on S. With the much weaker assumption $\operatorname{depth}_Z(\wedge^i \mathcal{M}) \geq 2$ (i.e. $\operatorname{depth}(\wedge^i \mathcal{M})_x \geq 2$ for any $x \in Z$ which means that $\wedge^i \mathcal{M}$ is reflexive) and, unfortunately, a certain strengthening (if $r \geq 4$) of the vanishing of cohomology groups, we get

Corollary 23. *In addition to the assumptions in Theorem 9 (i) to (iv), we suppose $\wedge^i \mathcal{M}$ is reflexive for $1 \leq i \leq \frac{r}{2}$. Then $\Gamma_*(S, \wedge^i \mathcal{M}) = \Gamma_*(U, \wedge^i \mathcal{M})$ for $1 \leq i \leq \frac{r}{2}$ and we have the following:*

1. *If $\mathrm{H}^j_*(S, \wedge^i \mathcal{M}) = 0$ for any $1 \leq i \leq \frac{r}{2}$, $1 \leq j \leq \dim C + i$, then $\overline{\mathbf{A}}$ is Gorenstein.*

2. *If $\mathrm{H}^j_*(S, \wedge^i \mathcal{M}) = 0$ for any $1 \leq i \leq \frac{r}{2}$, $1 \leq j \leq \min(\dim C + i + 1, \dim S - 1)$, then $\mathbf{A}$ is Gorenstein and $\mathbf{A} = \overline{\mathbf{A}}$.*

Proof. Since the dual of an $\mathcal{O}_S$-module has depth at least 2 if $\mathcal{O}_S$ has, it follows that $\operatorname{depth}_Z(\wedge^i \mathcal{M}) \geq 2$, i.e. we get the left isomorphism in $\Gamma_*(S, \wedge^i \mathcal{M}) = \Gamma_*(U, \wedge^i \mathcal{M}) = \mathbf{M}_i$ by [8]. In particular $\wedge^i \mathcal{M} = \widetilde{\mathbf{M}_i}$, and now the vanishing of the groups in Conclusion 1. and 2. immediately transfers to the depth conditions of Theorem 21, cf. (1). $\square$

3 Applications and Examples

3.1 Rank One Sheaves

An immediate application of Theorem 7 (and Remark 8) is that if S is locally Gorenstein outside a closed set, Z, of codimension at least 2 in S then the required conditions of the theorem are satisfied. We can conclude that sections of $\mathcal{K}_S^*(s)$ degenerate on arithmetically Gorenstein subschemes of S of codimension one in S. This result is well known. A discussion of this result and some applications can be found in [16] and [17].

3.2 Rank Two Sheaves

If $\mathcal{M}$ is a sheaf of rank 2 on an arithmetically Cohen-Macaulay scheme, S, then the cohomological conditions of Theorem 9 often force $\mathcal{M}$ to have no middle cohomology, i.e. that $\mathbf{M}_1 = \mathrm{H}^0_*(U, \mathcal{M})$ is a maximal Cohen-Macaulay module. This situation was treated in [17] and generalized (in rank ≥ 3 case) in Theorem 21 of this paper. We would, however, like to include some further examples that illustrate this principle.

Let S be an arithmetically Cohen-Macaulay curve and t any integer. Let $\mathcal{M}$ be any locally free rank 2 sheaf on S whose determinant is $\mathcal{K}_S(t)$. Then $\mathcal{M}$ satisfies the conditions of Proposition 16. Thus, a regular section of $\mathcal{M}^*$ will determine an Artinian Gorenstein algebra. It is well known that any rank two bundle on a smooth curve can be written as the extension of one line bundle by another line bundle. Conversely, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two line bundles on S satisfying $\mathcal{L}_1 \otimes \mathcal{L}_2 = \mathcal{K}_S(t)$. If $\mathcal{M}$ is any extension of $\mathcal{L}_1$ by $\mathcal{L}_2$ i.e. if we have $0 \to \mathcal{L}_2 \to \mathcal{M} \to \mathcal{L}_1 \to 0$ then $\mathcal{M}$ is a locally free rank 2 sheaf on S whose determinant is $\mathcal{K}_S(t)$. In general, if we can construct rank two bundles with canonical determinant on arithmetically Cohen-Macaulay curves, then we can use these bundles to construct Artinian Gorenstein algebras. One method of construction of such bundles is as extensions of suitable line bundles. Another general method is illustrated in the following example.

Example 24. *Let $S = \mathrm{Proj}(\mathbf{B})$ be the twisted cubic curve in $\mathbb{P}^3 = \mathrm{Proj}(\mathbf{R})$. The kernel of a general surjective morphism $\mathcal{O}_S^3 \twoheadrightarrow \mathcal{K}_S(2)$, is a rank two bundle, $\mathcal{E}^*$, on S. The determinant of $\mathcal{E} = (\mathcal{E}^*)^*$ is $\mathcal{K}_S(2)$. A general regular section, $\sigma \in \mathrm{H}^0_*(S, \mathcal{E}^*(3))$, determines an Artinian Gorenstein algebra, $\mathbf{A} = \mathrm{coker}(\mathrm{H}^0_*(S, \mathcal{E}(-3)) \xrightarrow{\sigma^*} \mathbf{B})$. The h-vector of $\mathbf{A}$ is $(1,\ 4,\ 7,\ 4,\ 1)$. The $\mathbf{R}$-free resolution of $\mathbf{A}$ is*

$$0 \to \mathbf{R}(-8) \to \mathbf{R}(-5)^6 \oplus \mathbf{R}(-6)^3 \to \mathbf{R}(-3)^2 \oplus \mathbf{R}(-4)^{12} \oplus \mathbf{R}(-5)^2$$
$$\to \mathbf{R}(-2)^3 \oplus \mathbf{R}(-3)^6 \to \mathbf{R} \to \mathbf{A} \to 0$$

Remark 25. *It is an interesting problem to study when an arithmetically Gorenstein subscheme, C, of codimension 3 in $\mathbb{P}^n$ is an intersection of two Gorenstein linked subschemes of codimension 2. If the Gorenstein linked schemes happen to be Gorenstein linked divisors on some hypersurface S, the problem is closely connected to whether the rank 2 sheaf, which defines C on S by some regular section, is indecomposable or not. To see it, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be line bundles on an arithmetically Cohen-Macaulay scheme S, and let S be a curve (the cohomological assumptions of Proposition 16 are empty in this case). The zero locus of a regular section of $\mathcal{L}_1^*$ and the zero locus of a regular section of $\mathcal{L}_2^*$ are both zeroschemes. Suppose there exists an integer t such that $\mathcal{L}_1 \otimes \mathcal{L}_2 = \mathcal{K}_S(t)$, then the union of these two zeroschemes corresponds to a section of $\mathcal{L}_1^* \otimes \mathcal{L}_2^* = \mathcal{K}_S^*(-t)$. Thus the union of the two zero loci is an arithmetically Gorenstein zero-scheme (i.e. the two zeroschemes are Gorenstein linked). If the linkage is geometric, then the sum of the ideals of the zeroschemes is an Artinian Gorenstein algebra. It is the algebra determined by a regular section of the* decomposable *rank 2 sheaf $\mathcal{L}_1^* \oplus \mathcal{L}_2^*$. Now if $\mathcal{E}$ is any extension of $\mathcal{L}_1^*$ by $\mathcal{L}_2^*$ then Proposition 16 shows that regular sections of $\mathcal{E}$ also determine Artinian Gorenstein quotients. And if $\mathcal{E}$ is indecomposable, its ideal can not be the sum of two geometrically Gorenstein linked ideals as above.*

Identifying an Artinian Gorenstein algebra with a regular section of an extension, $\mathcal{E}$, of two line bundles on a curve, S, has also other advantages. Namely, one may identify the deformations of an Artinian Gorenstein algebra as deformations of S, $\mathcal{E}$ and $\mathrm{H}^0(\mathcal{E})$ in a precise way. This has some bearing on the study of the smoothness and the dimension of $\mathrm{PGor}(H)$ *(a recent treatment of this problem can be found in [2], [4], [13], [14]).*

Example 26. *The Grassmann variety, $GR(2,5)$, of lines in $\mathbb{P}^4$ can be embedded as a quintic, arithmetically Gorenstein 6-fold in $\mathbb{P}^9$. The universal rank 2 bundle, $\mathcal{E}$, on $GR(2,5)$, satisfies the conditions of Theorem 21.2 as does the universal quotient bundle, $\mathcal{Q}$. The two bundles are related by the exact sequence*

$$0 \to \mathcal{E} \to \mathcal{O}^5_{GR(2,5)} \to \mathcal{Q} \to 0.$$

Zero-loci of sections of $\mathcal{E}^*$ give Gorenstein 4-folds in $\mathbb{P}^9$ while zero-loci of sections of $\mathcal{Q}^*$ give Gorenstein 3-folds in $\mathbb{P}^9$. An interesting example comes from taking a section of $\mathcal{E}(2)$ to get an arithmetically Gorenstein scheme, C, and taking a section of $\mathcal{Q}^*(2)$ to get an arithmetically Gorenstein scheme, D. Both C and D are cut out by 10 quadrics. The h-vector of C is $(1,\ 5,\ 5,\ 1)$. The h-vector of D is $(1,\ 6,\ 11,\ 6,\ 1)$. The free resolutions of the ideal sheaf, $\mathcal{I}_C$, and of the ideal sheaf, $\mathcal{I}_D$, are given below.

$$0 \to \mathcal{O}_{\mathbb{P}^9}(-8) \to \mathcal{O}_{\mathbb{P}^9}(-6)^{10} \to \mathcal{O}_{\mathbb{P}^9}(-5)^{16} \to \mathcal{O}_{\mathbb{P}^9}(-3)^{16} \to \mathcal{O}_{\mathbb{P}^9}(-2)^{10} \to \mathcal{I}_C \to 0$$
$$0 \to \mathcal{O}_{\mathbb{P}^9}(-10) \to \mathcal{O}_{\mathbb{P}^9}(-8)^{10} \to \mathcal{O}_{\mathbb{P}^9}(-6)^{25} \oplus \mathcal{O}_{\mathbb{P}^9}(-7)^{10} \to \mathcal{O}_{\mathbb{P}^9}(-5)^{52}$$
$$\to \mathcal{O}_{\mathbb{P}^9}(-3)^{10} \oplus \mathcal{O}_{\mathbb{P}^9}(-4)^{25} \to \mathcal{O}_{\mathbb{P}^9}(-2)^{10} \to \mathcal{I}_D \to 0$$

3.3 Rank Three Sheaves

If $\mathcal{E}$ is a locally free sheaf of rank n then $\wedge^r\mathcal{E} \simeq (\wedge^{n-r}\mathcal{E})^* \otimes \wedge^n\mathcal{E}$. Thus, if the rank of $\mathcal{E}$ is three then $\wedge^2\mathcal{E} = \mathcal{E}^* \otimes \wedge^3\mathcal{E}$. If $\mathcal{E}$ is a rank three sheaf on a threefold, S, and if $\wedge^3\mathcal{E} \simeq \mathcal{K}_S(s)$ (for some integer s) then by Serre duality, $\mathrm{H}^1_*(S,\mathcal{E}) = 0$ if and only if $\mathrm{H}^2_*(S,\wedge^2\mathcal{E}) = 0$. In this setting, to satisfy the cohomological conditions of Theorem 9.2, we need only check that $\mathrm{H}^1_*(S,\mathcal{E}) = 0$ (cf. Corollary 14.1). We can produce large families of such sheaves as follows. Consider the exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{E} \to 0$. We have $\det(\mathcal{E}) = \det(\mathcal{B}) \otimes \det(\mathcal{A}^*)$. If $\mathrm{H}^1_*(S,\mathcal{B}) = \mathrm{H}^2_*(S,\mathcal{A}) = 0$ then $\mathrm{H}^1_*(S,\mathcal{E}) = 0$. Assume $\det(\mathcal{B}) \otimes \det(\mathcal{A}^*)$ is a twist of the canonical sheaf on S. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on S which split as the sum of hypersurface sections then $\det(\mathcal{B}\oplus\mathcal{F})\otimes\det(\mathcal{A}^*\oplus\mathcal{G}^*)$ is still a twist of the canonical sheaf on S. Furthermore, if $\mathcal{A}$ is any rank r bundle on S and if $\mathcal{B}$ is any rank $r+3$ bundle on S then, for $t \gg 0$, there exist injections $\mathcal{A} \hookrightarrow \mathcal{B}(t)$ whose cokernel is locally free on S. Thus, to produce a locally free rank three sheaf, $\mathcal{E}$, on a threefold, S, with $\mathrm{H}^1_*(S,\mathcal{E}) = 0$ and with determinant equal to a twist of the canonical sheaf, it is enough to produce any pair of locally free sheaves $\mathcal{A}$ and $\mathcal{B}$ with $\mathrm{H}^1_*(S,\mathcal{B}) = \mathrm{H}^2_*(S,\mathcal{A}) = 0$ and with $\det(\mathcal{B}) \otimes \det(\mathcal{A}^*)$ a twist of the canonical sheaf on S. Infinite families of such pairs can be produced on any arithmetically Cohen-Macaulay threefold.

Example 27. *Let S be a threefold in $\mathbb{P}^5$ determined by the two by two minors of a two by three matrix of general linear forms. S is arithmetically Cohen-Macaulay and is licci. As a consequence, $\mathrm{H}^2_*(S,\mathcal{K}_S^*) = 0$ (cf. [17], Proposition 23 and its proof). Let $\mathcal{E}$ be the cokernel of a general injection $\mathcal{K}_S^* \hookrightarrow \mathcal{O}_S^4(2)$. The degeneracy locus of a general such map has codimension four. Since S is a threefold, $\mathcal{E}$ is locally free. By construction, $\mathcal{E}$ has rank three and the determinant of $\mathcal{E}$ is a twist of $\mathcal{K}_S$. A general regular section of $\mathcal{E}$ determines a non-saturated ideal, $\mathbf{I_C}$. The saturation of $\mathbf{I_C}$ is the homogeneous ideal of an arithmetically Gorenstein zeroscheme, C. The h-vector of the corresponding Gorenstein algebra is* (1, 5, 5, 1). *The free resolution of the ideal sheaf, $\mathcal{I}_C$, is given below.*

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-8) \to \mathcal{O}_{\mathbb{P}^5}(-6)^{10} \to \mathcal{O}_{\mathbb{P}^5}(-5)^{16} \to \mathcal{O}_{\mathbb{P}^5}(-3)^{16} \to \mathcal{O}_{\mathbb{P}^5}(-2)^{10} \to \mathcal{I}_C \to 0$$

Note the similarity to the resolution given in Example 26.

Let $\mathbf{R}$ be a graded Gorenstein k-algebra and let $P = \mathrm{Proj}(\mathbf{R})$. Let $\mathcal{F}$ and $\mathcal{G}$ be locally free sheaves of ranks f and g. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a generically surjective morphism. The expected codimension of the degeneracy locus of ϕ is $f-g+1$. If this is the case and if $\mathcal{F}$ and $\mathcal{G}$ split as the sum of hypersurface sections then the kernel of ϕ is called a Buchsbaum-Rim sheaf. Thus, Buchsbaum-Rim sheaves on $\mathbf{R}$ are the reflexive sheaves which arise as the sheafification of kernels of sufficiently general maps between free $\mathbf{R}$-modules. The Buchsbaum-Rim complex and its generalizations (see [5]) are acyclic in this setting. From these complexes one can show that the cohomological conditions of Theorem 21.1 are satisfied for the duals of odd rank Buchsbaum-Rim sheaves [20], [21]. Indeed [21], Remark 3.7 shows that all assumptions of Corollary 23.1 are satisfied, and we get Gorenstein quotients of $\mathbf{R}$ (as the top dimensional component) in this manner.

Example 28. *Let $\mathcal{E}^*$ be the kernel of a general morphism $\phi : \mathcal{O}_{\mathbb{P}^4}^4 \to \mathcal{O}_{\mathbb{P}^4}(1)$ (thus $\mathcal{E}^*$ is a rank 3 Buchsbaum-Rim sheaf). Dualizing, we get the short exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-1) \to \mathcal{O}_{\mathbb{P}^4}^4 \to \mathcal{E} \to 0.$$

Note that $\mathcal{E}$ is not locally free on $\mathbb{P}^4$ but is locally free away from the degeneracy locus of ϕ (which consists of a single point). A general section of $\mathcal{E}^*(3)$ vanishes along a non-equidimensional scheme, D. The top dimensional component of D is a non-singular arithmetically Gorenstein curve, C, of degree 20 and genus 31. The h-vector of the corresponding Gorenstein algebra is $(1,\ 3,\ 6,\ 6,\ 3,\ 1)$. *The free resolution of $\mathcal{I}_C$ is*

$$0 \to O_{\mathbb{P}^4}(-8) \to O_{\mathbb{P}^4}(-4) \oplus O_{\mathbb{P}^4}(-5)^4 \to O_{\mathbb{P}^4}(-3)^4 \oplus O_{\mathbb{P}^4}(-4) \to \mathcal{I}_C \to 0.$$

3.4 Sheaves on Licci Schemes

Let $\mathbf{R}$ be a graded Gorenstein k-algebra of finite type. Recall that two graded quotients, $\mathbf{R}/\mathbf{J}$ and $\mathbf{R}/\mathbf{J}'$, are said to be directly linked if there exists a homogeneous complete intersection ideal $\mathbf{L}$ such that $\mathbf{J} = \mathbf{L} : \mathbf{J}'$ and $\mathbf{J}' = \mathbf{L} : \mathbf{J}$ (with $\mathbf{L} \subseteq \mathbf{J} \cap \mathbf{J}'$). The relationship of being directly linked generates the equivalence relation, "linkage". $\mathbf{B} = \mathbf{R}/\mathbf{I}$ is said to be licci (and hence Cohen-Macaulay) if it is in the linkage class of a complete intersection, cf. [19] for details. Throughout this section we suppose $\mathbf{B} = \mathbf{R}/\mathbf{I}$ is licci of codimension $c \geq 2$ and that $S = \mathrm{Proj}(\mathbf{B}) \hookrightarrow P = \mathrm{Proj}(\mathbf{R})$ is a local complete intersection in some open $U = S - Z$ satisfying $\dim S - \dim Z \geq 2$ (letting $\dim Z = -1$ if $Z = \emptyset$).

For an ideal $\mathbf{I} \subset \mathbf{R}$, the 1^{st} Koszul homology module of $\mathbf{I}$, $\mathbf{H}_1$, is given by an exact sequence

$$0 \to \mathbf{H}_1 \to \mathbf{F}_1 \otimes_{\mathbf{B}} \mathbf{B} \to \mathbf{I}/\mathbf{I}^2 \to 0. \tag{13}$$

in which $\mathbf{F}_1$ is $\mathbf{R}$-free and $\mathbf{F}_1 \twoheadrightarrow \mathbf{I}$ is surjective and minimal. The normal sheaf of $S = \mathrm{Proj}(\mathbf{R}/\mathbf{I})$ is given by $\mathcal{N}_S = \widetilde{\mathbf{I}/\mathbf{I}^2}^*$. In [17] we applied the form of Theorem 7 given by Remark 8(b) to obtain the following proposition.

Proposition 29. *Let $\mathbf{R}$ be a graded Gorenstein k-algebra of finite type. Let $\mathbf{B} = \mathbf{R}/\mathbf{I}$ be licci of codimension $c \geq 2$ and of finite projective dimension over $\mathbf{R}$. Let $S = \mathrm{Proj}(\mathbf{B})$ be a local complete intersection in an open set $U = S - Z$. Let μ denote the number of minimal generators of $\mathbf{I}$.*

(i) If $\dim S - \dim Z \geq c$ then

1. $\mathrm{H}^0_*(U, \wedge^i \mathcal{N}_S)$ *is a maximal Cohen-Macaulay $\mathbf{B}$-module for every $1 \leq i \leq c$*

2. *For any s, every regular section of $\mathrm{H}^0(U, \mathcal{N}^*_S(s))$ defines a Gorenstein quotient $\mathbf{R} \twoheadrightarrow \mathbf{A}$ of codimension $2c$.*

(ii) If $\dim S - \dim Z \geq \max(2, \mu - c)$ then

1. $\mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{H}}_1)$ *is a maximal Cohen-Macaulay $\mathbf{B}$-module for every $1 \leq i \leq \mu - c$*

2. *For any s, every regular section of $\mathrm{H}^0(U, \widetilde{\mathbf{H}}^*_1(s))$ defines a Gorenstein quotient $\mathbf{R} \twoheadrightarrow \mathbf{A}$ of codimension μ.*

The modules $\mathrm{H}^0_*(U, \wedge^i \mathcal{N}_S)$ in Proposition 29 turn out to be isomorphic to a twist of the modules $\mathrm{Tor}^{\mathbf{R}}_i(\mathbf{B}, \mathbf{K}_{\mathbf{B}})$ whose Cohen-Macaulayness is known by [3]. The modules $\mathrm{H}^0_*(U, \wedge^i \widetilde{\mathbf{H}}_1)$

turn out to be isomorphic to a twist of the i^{th} Koszul homology modules, $\mathbf{H}_i$, whose Cohen-Macaulayness is known by [10]. If $\mathbf{B}$ is Gorenstein, it is also a consequence of the conclusion 1 statements in (i) resp. (ii) of Proposition 29 and Remark 8(b) that regular sections of $\mathrm{H}^0(U, \mathcal{N}_S(t))$, resp. $\mathrm{H}^0(U, \widetilde{\mathbf{H}}_1(t))$, define Gorenstein quotients of $\mathbf{R}$.

For the rest of this subsection we suppose the codimension of $S = \mathrm{Proj}(\mathbf{B})$ in $P = \mathrm{Proj}(\mathbf{R})$ is $c = 3$. Let

$$0 \to \mathbf{F}_3 \to \mathbf{F}_2 \to \mathbf{F}_1 \to \mathbf{I} \to 0 \tag{14}$$

be a minimal $\mathbf{R}$-free resolution. Letting $\mathbf{F}_i^*$ be the $\mathbf{R}$-dual of $\mathbf{F}_i$, we get a complex

$$0 \to \mathbf{N_B} \to \mathbf{F}_1^* \otimes_{\mathbf{R}} \mathbf{B} \xrightarrow{\psi_1} \mathbf{F}_2^* \otimes_{\mathbf{R}} \mathbf{B} \xrightarrow{\psi_2} \mathbf{F}_3^* \otimes_{\mathbf{R}} \mathbf{B} \to \mathbf{K_B}(-e) \to 0. \tag{15}$$

This complex is exact except in the middle [17]. Indeed, $\ker \psi_2 / \operatorname{im} \psi_1 \simeq \mathrm{Tor}_1^{\mathbf{R}}(\mathbf{B}, \mathbf{K_B})(-e) \simeq \mathbf{I}/\mathbf{I}^2 \otimes \mathbf{K_B}(-e)$, where e is the twist such that $\mathbf{R}(e)$ is the canonical module of $\mathbf{R}$. For a $\mathbf{B}$-module $\mathbf{M}$, we let $\mathbf{M}^*$ denote its $\mathbf{B}$-dual. We showed in [17] that each of the modules, $(\operatorname{im} \psi_1)^*, \ker \psi_2, \operatorname{coker} \psi_1$ and $(\operatorname{im} \psi_2)^*$ are maximal Cohen-Macaulay whenever they are non-trivial, that each one has determinant equal to a twist of $\mathbf{K_B}$ and that $(\operatorname{im} \psi_1)^* \simeq \mathbf{H}_1$. This suggests the following:

Question 30. *Let $\mathbf{R}$ be a graded Gorenstein k-algebra of finite type. Let $\mathbf{B} = \mathbf{R}/\mathbf{I}$ be licci. Let $S = \mathrm{Proj}(\mathbf{B})$ be a local complete intersection in an open $U = S - Z$. If $\dim S - \dim Z \geq \max(2, r)$, do regular sections of twists of $(\widetilde{\ker \psi_2})^*$, $(\widetilde{\operatorname{coker} \psi_1})^*$ and $\widetilde{\operatorname{im} \psi_2}$ on U define Gorenstein quotients of $\mathbf{R}$ (where r denotes the rank of each of these modules)?*

To use Theorem 7 to answer this question, we would need to prove that $\mathrm{H}^0_*(U, \wedge^i(-))$ is a maximal Cohen-Macaulay module for $2 \leq i \leq r/2$. If we let $g = \operatorname{rank} \mathbf{F}_3$ denote the Cohen-Macaulay type of the ideal $\mathbf{I}$ and let $\mu = \operatorname{rank} \mathbf{F}_1$ denote the number of minimal generators of $\mathbf{I}$ then $\operatorname{rank}((\ker \psi_2)^*) = \mu$, $\operatorname{rank}((\operatorname{coker} \psi_1)^*) = g + 2$ and $\operatorname{rank}(\operatorname{im} \psi_2) = g - 1$. It is straightforward to answer Question 30 affirmatively when $r \leq 3$. For these values of r, the only interesting sheaf is $\widetilde{\operatorname{im} \psi_2}$ and we have the following:

Corollary 31. *Let $\mathbf{R}$ be a graded Gorenstein k-algebra of finite type. Let $\mathbf{B} = \mathbf{R}/\mathbf{I}$ be licci of codimension 3. Let g denote the Cohen-Macaulay type of $\mathbf{I}$. Let $S = \mathrm{Proj}(\mathbf{B})$ be a local complete intersection in an open $U = S - Z$. Suppose that $\dim S - \dim Z \geq \max(2, g-1)$. If $g \leq 4$, then, for any s, every regular section of $\mathrm{H}^0(U, \widetilde{\operatorname{im} \psi_2}(s))$ defines a Gorenstein quotient $\mathbf{R} \twoheadrightarrow \mathbf{A}$ of codimension $g + 2$.*

The application of Theorem 7 to Question 30 is limited beyond the case $r = 3$. However, the weaker cohomological assumptions of Proposition 17 and Theorem 21 allow one to proceed further, to at least hope for the top dimensional component of a regular section to be Gorenstein. To simplify, we will use the theorems of the preceding section with $U = U'$.

Example 32. *Let $\mathbf{R}$ be a graded Gorenstein k-algebra of finite type. Let $\mathbf{B} = \mathbf{R}/\mathbf{I}$ be licci of codimension 3. Let $g = \operatorname{rank} \mathbf{F}_3$ be the Cohen-Macaulay type of $\mathbf{I}$. Let $S = \mathrm{Proj}(\mathbf{B})$ be a local complete intersection in an open set $U = S - Z$ with $\dim S - \dim Z \geq g$. Assume $\operatorname{depth} S_3(\mathbf{K_B}) \geq \dim \mathbf{B} - 1$. If $g \leq 6$, we claim that any regular section of $\mathrm{H}^0(U, \widetilde{\operatorname{im} \psi_2}(s))$ defines a Gorenstein quotient $\mathbf{R} \twoheadrightarrow \mathbf{A}$ of codimension $g + 2$. To see this, note that*

$$0 \to \operatorname{im} \psi_2 \to \mathbf{F}_3^* \otimes_{\mathbf{R}} \mathbf{B} \to \mathbf{K_B}(-e) \to 0 \tag{16}$$

is exact and its $\mathbf{B}$*-dual sequence is also short exact (see [17], proof of Proposition 23). Using [9], II, exercise 5.16, (and taking the "global" sections over* U*), we get the exact sequence*

$$0 \to (\mathbf{K_B}(-e) \otimes \operatorname{im} \psi_2)^* \to \wedge^2 \mathbf{F}_3^* \otimes \mathbf{B} \to (\wedge^2 \operatorname{im} \psi_2)^* \to 0,$$

By the assumption $\operatorname{depth} S_3(\mathbf{K_B}) \geq \dim \mathbf{B} - 1$ *and the maximal Cohen-Macaulayness of* $S_2(\mathbf{K_B})$*, one proves that* $(\mathbf{K_B} \otimes \operatorname{im} \psi_2)^*$ *is maximal Cohen-Macaulay (cf. Remark 26 of [17] for details which treats the case* $g = 5$*). Hence* $\operatorname{depth}(\wedge^2 \operatorname{im} \psi_2)^* \geq \dim \mathbf{B} - 1$ *and conclusion 2 of Theorem 21, cf. Remark 22, applies to show that* $\mathbf{A}$ *is Gorenstein.*

Example 33. *i) It is not always true that the modules appearing in Question 30 satisfy the maximal Cohen-Macaulay requirements of Theorem 7. If* $\mathbf{B}$ *is an almost complete intersection then* $\mu(\mathbf{I_B}) = \operatorname{rank} \mathbf{F}_1 = 4$*. If, in addition,* $\dim S - \dim Z \geq 5$*, we claim that*

$$\mathrm{H}^1_*(U, \wedge^2 \widetilde{\ker \psi_2}) \neq 0 \quad \text{and} \quad \mathrm{H}^{\dim S - 1}_*(S, \wedge^2 \widetilde{\ker \psi_2}) \neq 0.$$

To prove the claim, note that $\operatorname{rank} \mathbf{H}_1 = 1$ *by* (13)*. Since the determinant of* $\mathbf{H}_1$ *is a twist of* $\mathbf{K_B}$*, we get* $\mathbf{H}_1 \simeq \mathbf{K_B}(t)$ *for some integer* t*. Since* $\operatorname{im} \psi_1 \simeq \mathbf{H}_1^* \simeq \mathbf{K}_\mathbf{B}^*(-t)$*, by* (15) *we have*

$$0 \to \mathbf{H}_1^* \to \ker \psi_2 \to \mathbf{I}/\mathbf{I}^2 \otimes \mathbf{K_B}(-e) \to 0.$$

Using again [8], II, exercise 5.16, we have an exact sequence

$$0 \to \widetilde{\mathbf{K}}_\mathbf{B}^*(-t) \otimes \widetilde{\mathbf{I}/\mathbf{I}^2 \otimes \mathbf{K_B}}(-e)|_U \to \wedge^2 \widetilde{\ker \psi_2}|_U \to \wedge^2 (\widetilde{\mathbf{I}/\mathbf{I}^2 \otimes \mathbf{K_B}}(-e))|_U \to 0.$$

Noting that $\widetilde{\mathbf{I}/\mathbf{I}^2}^* \simeq \widetilde{\mathbf{N}}_\mathbf{B}$*,* $\widetilde{\mathbf{I}/\mathbf{I}^2}$ *has rank 3 and* $\det(\widetilde{\mathbf{I}/\mathbf{I}^2}) \simeq \mathbf{K}_\mathbf{B}^*(e)$*, we observe that*

$$\wedge^2 \widetilde{\mathbf{I}/\mathbf{I}^2} \otimes \widetilde{\mathbf{K}}_\mathbf{B}(-e) \otimes \widetilde{\mathbf{K}}_\mathbf{B}(-e)|_U \simeq \widetilde{\mathbf{N}}_\mathbf{B} \otimes \widetilde{\mathbf{K}}_\mathbf{B}^*(e) \otimes \widetilde{\mathbf{K}}_\mathbf{B}(-e) \otimes \widetilde{\mathbf{K}}_\mathbf{B}(-e)|_U$$

and deduce the exact sequence

$$0 \to \widetilde{\mathbf{I}/\mathbf{I}^2}(-t-e)|_U \to \wedge^2 \widetilde{\ker \psi_2}|_U \to \widetilde{\mathbf{N}}_\mathbf{B} \otimes \widetilde{\mathbf{K}}_\mathbf{B}(-e)|_U \to 0. \tag{17}$$

Now, since $\mathbf{H}_1$ *is a maximal Cohen-Macaulay module, by* (13) *we have* $\mathrm{H}^i_\mathfrak{m}(\mathbf{I}/\mathbf{I}^2) = 0$ *for* $i \leq \dim \mathbf{B} - 2$*. By [11] we have* $\mathrm{H}^{\dim \mathbf{B} - 1}_\mathfrak{m}(\mathbf{I}/\mathbf{I}^2) \neq 0$*. By Lemma 5,* $\mathrm{H}^i_{\mathbf{I}(Z)}(\mathbf{I}/\mathbf{I}^2) = 0$ *for* $i \leq \operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} - 2$*. Since* $\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} \geq 5$ *we get* $\mathrm{H}^i_*(U, \widetilde{\mathbf{I}/\mathbf{I}^2}) = 0$ *for* $i = 1, 2$ *and we get* $\mathrm{H}^{\dim S - 1}_*(S, \widetilde{\mathbf{I}/\mathbf{I}^2}) \neq 0$*. Exact sequence* (17) *and Serre duality (cf. proof of Lemma 11) yield*

$$\mathrm{H}^1_*(U, \wedge^2 \widetilde{\ker \psi_2})^\vee \simeq \mathrm{H}^1_*(U, (\widetilde{\mathbf{I}/\mathbf{I}^2})^* \otimes \widetilde{\mathbf{K}}_\mathbf{B}(-e))^\vee \simeq \mathrm{H}^{\dim S - 1}_*(S, \widetilde{\mathbf{I}/\mathbf{I}^2}(e)) \neq 0$$

where $(-)^\vee$ *means graded dual. Finally, Lemma 11 shows that* $\mathrm{H}^{\dim S - 1}_*(S, \wedge^2 \widetilde{\ker \psi_2})$ *and* $\mathrm{H}^1_*(U, \wedge^2 \widetilde{\ker \psi_2})^\vee$ *are isomorphic (up to twist) and the claim is proved.*

ii) In the class of schemes considered above, the conditions of conclusion 1 of Theorem 21 are not fulfilled. However, using (17)*, we claim that the conditions of conclusion 1 of Proposition 17 are satisfied provided* $\dim S - \dim Z \geq 4 + \dim C/2$*, i.e.* $\dim Z \leq \min(\dim S - 5, (\dim S - 4)/2)$ *which means that the top dimensional component is Gorenstein. Indeed, by*

Serre duality (as in the proof of Lemma 11), $\mathrm{H}^i_*(U, (\widetilde{\mathbf{I}/\mathbf{I}^2})^* \otimes \widetilde{\mathbf{K}}_{\mathbf{B}}(-e))^\vee \simeq \mathrm{H}^{\dim S - i}_*(S, \widetilde{\mathbf{I}/\mathbf{I}^2}(e))$ *provided* $\operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} \geq i+2$. *Since we have seen that* $\mathrm{H}^i_{\mathbf{I}(Z)}(\mathbf{I}/\mathbf{I}^2) = 0$ *for* $i \leq \operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} - 2$, *we get*

$$\mathrm{H}^i_*(U, (\widetilde{\mathbf{I}/\mathbf{I}^2})^* \otimes \widetilde{\mathbf{K}}_{\mathbf{B}}(-e)) = \mathrm{H}^i_*(U, \widetilde{\mathbf{I}/\mathbf{I}^2}) = 0 \quad \text{for} \quad 2 \leq i \leq \operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} - 2. \tag{18}$$

Applying $\mathrm{H}^0_*(U, -)$ *to* (17)*, we obtain a short exact sequence and the corresponding vanishing of* $\mathrm{H}^i_*(U, \widetilde{\mathbf{M}}_2)$*, where* $\mathbf{M}_j = \mathrm{H}^0_*(U, \wedge^j \widetilde{\ker \psi_2})$*. Thanks to Lemma 11, the vanishing* $\mathrm{H}^i_{\mathbf{I}(Z)}(\mathbf{I}/\mathbf{I}^2) = 0$ *for* $i \leq \operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} - 2$ *is equivalent to*

$$\mathrm{H}^i_*(S, \widetilde{\mathbf{M}}_2) = 0 \quad \text{for} \quad \dim S + 2 - \operatorname{depth}_{\mathbf{I}(Z)} \mathbf{B} \leq i \leq \dim S - 2.$$

Hence we obtain the vanishing required of Proposition 17 for the sheaf $\widetilde{\mathbf{M}}_2$*. Since* $\mathbf{M}_1$ *and hence* $\mathbf{M}_3$ *are maximal Cohen-Macaulay modules, the claim is verified. Letting* $\mathbf{A}$ *be defined by some regular section of* $(\widetilde{\ker \psi_2})^*(s)|_U$*, it follows from Proposition 17 that the top dimensional component of* $\mathbf{A}$ *is Gorenstein.*

Example 34. *Suppose still that* $\mathbf{R}$ *is a graded Gorenstein k-algebra of finite type and that* $\mathbf{B} = \mathbf{R}/\mathbf{I}$ *is licci of codimension 3. Assume the Cohen-Macaulay type of* $\mathbf{I}$ *is 2, that* $S = \operatorname{Proj}(\mathbf{B})$ *is a local complete intersection in an open set* $U = S - Z$ *and that* $\dim Z \leq \min(\dim S - 5, (\dim S - 4)/2)$*. Let* $\mathbf{A}$ *be defined by a regular section of* $(\widetilde{\operatorname{coker} \psi_1})^*(s)$ *on* U*. Using almost exactly the same arguments as in Example 33, we claim that the top dimensional component of* $\mathbf{A}$ *is Gorenstein. Indeed, the first exact sequence of Example 32 shows that* $\operatorname{rank} \operatorname{im} \psi_2 = 1$ *and we get* $(\operatorname{im} \psi_2)^* \simeq \mathbf{K}_{\mathbf{B}}(t)$ *for some integer* t*. By* (15)*,*

$$0 \to \mathbf{I}/\mathbf{I}^2 \otimes \mathbf{K}_{\mathbf{B}}(-e) \to \operatorname{coker} \psi_1 \to \operatorname{im} \psi_2 \to 0$$

is exact. Taking a filtration of $\wedge^2 \widetilde{\operatorname{coker} \psi_1}|_U$ *([8], II, exercise 5.16) we deduce an exact sequence*

$$0 \to \wedge^2 (\widetilde{\mathbf{I}/\mathbf{I}^2 \otimes \mathbf{K}_{\mathbf{B}}}(-e))|_U \to \wedge^2 \widetilde{\operatorname{coker} \psi_1}|_U \to \widetilde{\mathbf{I}/\mathbf{I}^2 \otimes \mathbf{K}_{\mathbf{B}}}(-e) \otimes \widetilde{\mathbf{K}}^*_{\mathbf{B}}(-t)|_U \to 0.$$

Exactly as in Example 33, the sequence reduces to

$$0 \to \widetilde{\mathbf{N}}_{\mathbf{B}} \otimes \widetilde{\mathbf{K}}_{\mathbf{B}}(-e)|_U \to \wedge^2 \widetilde{\operatorname{coker} \psi_1}|_U \to \widetilde{\mathbf{I}/\mathbf{I}^2}(-t-e)|_U \to 0.$$

Thus $\wedge^2 \widetilde{\operatorname{coker} \psi_1}|_U$ *is determined by the same exact sequence as the sheaf* $\wedge^2 \widetilde{\ker \psi_2}|_U$ *of Example 33 with the "arrows reversed". Since we still have* (18)*, we get the same vanishing of* $\mathrm{H}^i_*(S, \wedge^2 \widetilde{\operatorname{coker} \psi_1})$ *as in Example 33, and since* $\operatorname{coker} \psi_1$ *is a maximal Cohen-Macaulay module, the claim follows from Proposition 17.*

Finally note that, because of the "reversed arrows" in this example compared to Example 33, it is not possible to prove the non-vanishing of $\mathrm{H}^{\dim S - 1}_*(S, \wedge^2 \widetilde{\operatorname{coker} \psi_1})$ *as we did for* $\mathrm{H}^{\dim S - 1}_*(S, \wedge^2 \widetilde{\ker \psi_2})$ *in Example 33(i). Indeed in the case* $\mu(\mathbf{I}) = 4$*, we see from [17], Remark 27, that the first mentioned group vanishes.*

3.5 Sheaves on Determinantal Schemes

Let $\mathbf{R} = k[x_0, x_1, \ldots, x_n]$ be a polynomial ring with the standard grading. Let $\mathbb{P}^n = \mathrm{Proj}(\mathbf{R})$. Let $\mathbf{M}$ be an $f \times g$ homogeneous matrix over $\mathbf{R}$. Let $\mathbf{I_A}$ denote the ideal of maximal minors of $\mathbf{M}$. If $C = \mathrm{Proj}(\mathbf{R}/\mathbf{I_A})$ has codimension $f - g + 1$ in $\mathbb{P}^n$ then C is said to be a standard determinantal scheme. The scheme C is said to be a good determinantal scheme if, after performing row operations on $\mathbf{M}$, the resulting matrix can be made to contain an $(f-1) \times g$ submatrix whose ideal of maximal minors defines a scheme of codimension $f - g + 2$. A scheme is a good determinantal scheme if and only if it is a standard determinantal scheme and is a generic complete intersection. See [18] for details.

With the above notation we have the following:

Lemma 35. *Let $C = \mathrm{Proj}(\mathbf{A}) \subseteq \mathbb{P}^{n+3}$ be a good determinantal scheme of codimension 3 given by the maximal minors of an $r \times (r+2)$ matrix $\mathbf{M}$. Let $S = \mathrm{Proj}(\mathbf{B})$ be a good determinantal scheme obtained by deleting some column of $\mathbf{M}$. Let $Z \subseteq S$ be a closed set such that $S \subseteq \mathbb{P}^{n+3}$ is a local complete intersection in $U = S - Z$. If $\mathrm{depth}_{I(Z)} \mathbf{B} \geq 4$ then $\mathrm{H}^i_*(C, \mathcal{N}_C) = 0$ for $1 \leq i \leq n-2$.*

Proof. Let $\mathbf{K} = \mathrm{Ext}^2_{\mathbf{R}}(\mathbf{B}, \mathbf{R}(-n-4))$ be the canonical module of $\mathbf{B}$. By [18] we have an exact sequence

$$0 \to \mathbf{K}(t)^* \to \mathbf{B} \to \mathbf{A} \to 0 \tag{19}$$

thus $\mathbf{I_{B/A}} = \mathbf{K}(t)^*$. Let $\mathbf{N_{B/A}} = \mathrm{Hom}_{\mathbf{B}}(\mathbf{I_{B/A}}, \mathbf{A})$ be the normal module of $\mathbf{B} \to \mathbf{A}$. We claim that $\mathbf{N_{B/A}}$ is maximal Cohen-Macaulay as an $\mathbf{A}$-module, i.e. that $\mathrm{H}^i_{\mathfrak{m}}(\mathbf{N_{B/A}}) = 0$ for $0 \leq i \leq n$. To see this, note that $\mathbf{K}(t)^*$ is a maximal Cohen-Macaulay $\mathbf{B}$-module by (19), hence $\mathrm{depth}_{I(Z)} \mathbf{K}(t)^* = \mathrm{depth}_{I(Z)} \mathbf{B} \geq 4$. Using the fact that $\widetilde{\mathbf{K}}(t)^*|_U$ is invertible, this implies

$$\mathrm{Hom}_{\mathbf{B}}(\mathbf{K}(t)^*, \mathbf{K}(t)^*) \simeq \mathrm{H}^0_*(U, \mathcal{H}om(\widetilde{\mathbf{K}}(t)^*, \widetilde{\mathbf{K}}(t)^*)) \simeq \mathrm{H}^0_*(U, \widetilde{\mathbf{B}}) = \mathbf{B} \qquad \text{and}$$

$$\mathrm{Hom}_{\mathbf{B}}(\mathbf{I_{B/A}}, \mathbf{B}) \simeq \mathrm{H}^0_*(U, \mathcal{H}om(\widetilde{\mathbf{K}}(t)^*, \widetilde{\mathbf{B}})) \simeq \mathrm{H}^0_*(U, \widetilde{\mathbf{K}}(t)) = \mathbf{K}(t)$$

where the isomorphisms to the left follow from [7], exp. VI. For the same reasons, we have

$$\mathrm{Ext}^1_{\mathbf{B}}(\mathbf{K}(t)^*, \mathbf{K}(t)^*) \simeq \oplus_\nu \mathrm{Ext}^1_{\mathcal{O}_U}(\widetilde{\mathbf{K}}(t)^*, \widetilde{\mathbf{K}}(t)^*(\nu)) \simeq \mathrm{H}^1_*(U, \mathcal{H}om(\widetilde{\mathbf{K}}(t)^*, \widetilde{\mathbf{K}}(t)^*)) = 0.$$

Applying the functor $\mathrm{Hom}(\mathbf{I_{B/A}}, -)$ to (19), we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{B}}(\mathbf{I_{B/A}}, \mathbf{K}(t)^*) \to \mathrm{Hom}_{\mathbf{B}}(\mathbf{I_{B/A}}, \mathbf{B}) \to \mathrm{Hom}_{\mathbf{B}}(\mathbf{I_{B/A}}, \mathbf{A})$$

which, by the above isomorphisms, yields

$$0 \to \mathbf{B} \to \mathbf{K}(t) \to \mathbf{N_{B/A}} \to 0. \tag{20}$$

The Cohen-Macaulayness of $\mathbf{N_{B/A}}$ follows immediately.

Let $\mathbf{N_B}$ denote the normal module of $\mathbf{R} \to \mathbf{B} = \mathbf{R}/\mathbf{I}$. We wish to show the exactness of the following two sequences:

$$0 \to \mathbf{N_{B/A}} \to \mathbf{N_A} \to \mathrm{Hom}_{\mathbf{R}}(\mathbf{I}, \mathbf{A}) \to 0 \tag{21}$$

$$0 \to \mathbf{I}/\mathbf{I}^2_{\mathbf{B}}(n+4-t) \simeq \mathrm{Hom}_{\mathbf{R}}(\mathbf{I},\mathbf{K}(t)^*) \to \mathrm{Hom}_{\mathbf{R}}(\mathbf{I},\mathbf{B}) \to \mathrm{Hom}_{\mathbf{R}}(\mathbf{I},\mathbf{A}) \to 0. \tag{22}$$

If we apply the functor $\mathrm{Hom}_{\mathbf{R}}(-,\mathbf{A})$ to the exact sequence $0 \to \mathbf{I} \to \mathbf{I}_{\mathbf{A}} \to \mathbf{I}_{\mathbf{B}/\mathbf{A}} \to 0$ we obtain (21) provided we show that $\mathrm{Ext}^1_{\mathbf{R}}(\mathbf{I}_{\mathbf{B}/\mathbf{A}},\mathbf{A}) = 0$. The vanishing of $\mathrm{Ext}^1_{\mathbf{R}}(\mathbf{I}_{\mathbf{B}/\mathbf{A}},\mathbf{A})$ will follow if we can show that $\mathrm{Ext}^1_{\mathbf{B}}(\mathbf{I}_{\mathbf{B}/\mathbf{A}},\mathbf{A}) = 0$ (e.g. see [16], proof of Theorem 6.1). Since $\mathrm{depth}_{I(Z)}\,\mathbf{B} \geq 4$, we get

$$\mathrm{Ext}^2_{\mathbf{B}}(\mathbf{K}(t)^*,\mathbf{K}(t)^*) \simeq \mathrm{H}^2_*(U,\mathcal{H}om(\widetilde{\mathbf{K}}(t)^*,\widetilde{\mathbf{K}}(t)^*)) \simeq \mathrm{H}^3_{I(Z)}(\mathbf{B}) = 0 \qquad \text{and}$$

$$\mathrm{Ext}^1_{\mathbf{B}}(\mathbf{K}(t)^*,\mathbf{B}) \simeq \mathrm{H}^1_*(U,\mathcal{H}om(\widetilde{\mathbf{K}}(t)^*,\widetilde{\mathbf{B}})) \simeq \mathrm{H}^2_{I(Z)}(\mathbf{K}(t)) = 0.$$

From $\mathrm{Hom}(\mathbf{K}(t)^*,-)$ applied to (19) we have the exact sequence

$$\to \mathrm{Ext}^1_{\mathbf{B}}(\mathbf{K}(t)^*,\mathbf{B}) \to \mathrm{Ext}^1_{\mathbf{B}}(\mathbf{K}(t)^*,\mathbf{A}) \to \mathrm{Ext}^2_{\mathbf{B}}(\mathbf{K}(t)^*,\mathbf{K}(t)^*) \to \dots$$

Using the above isomorphisms, we see that $\mathrm{Ext}^1_{\mathbf{B}}(\mathbf{I}_{\mathbf{B}/\mathbf{A}},\mathbf{A}) = 0$ and we have established the exactness of (21). To establish the exactness of (22), we consider the exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{B}}(\mathbf{I}/\mathbf{I}^2,\mathbf{K}(t)^*) \to \mathrm{Hom}_{\mathbf{B}}(\mathbf{I}/\mathbf{I}^2,\mathbf{B}) \to \mathrm{Hom}_{\mathbf{B}}(\mathbf{I}/\mathbf{I}^2,\mathbf{A}) \to \dots$$

It suffices to show that $\mathrm{Hom}_{\mathbf{B}}(\mathbf{I}/\mathbf{I}^2,\mathbf{K}(t)^*) \simeq \mathbf{I}/\mathbf{I}^2(n+4-t)$ and $\mathrm{Ext}^1(\mathbf{I}/\mathbf{I}^2,\mathbf{K}(t)^*) = 0$. Since $\mathrm{pd}\,\mathbf{I} = 1$, we have

$$\mathbf{N}_{\mathbf{B}} = \mathrm{Hom}_{\mathbf{R}}(\mathbf{I},\mathbf{B}) \simeq \mathrm{Ext}^1_{\mathbf{R}}(\mathbf{I},\mathbf{I}) \simeq \mathrm{Ext}^1_{\mathbf{R}}(\mathbf{I},\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{I} \simeq \mathbf{K}(n+4) \otimes_{\mathbf{B}} \mathbf{I}/\mathbf{I}^2.$$

Since $\mathrm{depth}_{I(Z)}\,\mathbf{K}(t)^*$ is large enough and since $\mathcal{I}_S/\mathcal{I}_S^2|_U$ is locally free, we get

$$\mathrm{Hom}_{\mathbf{B}}(\mathbf{I}/\mathbf{I}^2,\mathbf{K}(t)^*) \simeq \mathrm{H}^0_*(U,\widetilde{\mathbf{N}}_{\mathbf{B}} \otimes \mathbf{K}(t)^*) \simeq \mathrm{H}^0_*(U,\widetilde{\mathbf{I}/\mathbf{I}^2}(n+4-t))$$

$$\mathrm{Ext}^1_{\mathbf{B}}(\mathbf{I}/\mathbf{I}^2,\mathbf{K}(t)^*) \simeq \mathrm{H}^1_*(U,\widetilde{\mathbf{N}}_{\mathbf{B}} \otimes \mathbf{K}(t)) \simeq \mathrm{H}^2_{I(Z)}(\mathbf{I}/\mathbf{I}^2(n+4-t)).$$

The first Koszul homology module $\mathbf{H}_1$ is a maximal Cohen-Macaulay module (cf. Proposition 29) and fits into an exact sequence (13). From this we deduce

$$\mathrm{H}^2_{I(Z)}(\mathbf{I}/\mathbf{I}^2(n+4-t)) \simeq \mathrm{H}^3_{I(Z)}(\mathbf{H}_1(n+4-t)) = 0 \qquad \text{and}$$

$$\mathrm{H}^0_*(U,\widetilde{\mathbf{I}/\mathbf{I}^2}(n+4-t)) \simeq \mathbf{I}/\mathbf{I}^2(n+4-t)$$

taking into account $\mathrm{depth}_{I(Z)}\,\mathbf{B} = \mathrm{depth}_{I(Z)}\,\mathbf{H}_1 \geq 4$. This establishes the exactness of (22).

Finally, to show that $\mathrm{H}^i_{\mathfrak{m}}(\mathbf{N}_{\mathbf{A}}) = 0$ for $0 \leq i < n$, we will use exact sequence (22) in the form

$$0 \to \mathbf{I}/\mathbf{I}^2(n+4-t) \to \mathbf{N}_{\mathbf{B}} \to \mathrm{Hom}_{\mathbf{R}}(\mathbf{I},\mathbf{A}) \to 0.$$

Noting that $\mathbf{N}_{\mathbf{B}}$ is maximal Cohen-Macaulay by Proposition 29 and that $\mathbf{I}/\mathbf{I}^2$ has codepth 1 by (13), the above exact sequence shows

$$\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Hom}_{\mathbf{R}}(\mathbf{I},\mathbf{A})) \simeq \mathrm{H}^{i+1}_{\mathfrak{m}}(\mathbf{I}/\mathbf{I}^2(n+4-t)) = 0 \quad \text{for} \quad i+1 < \dim \mathbf{B} - 1 \quad (\text{i.e. for} \quad i < n).$$

Since $\mathbf{N}_{\mathbf{B}/\mathbf{A}}$ is maximal Cohen-Macaulay as an $\mathbf{A}$-module, we conclude by using (21). $\square$

In light of Lemma 35 we conclude that sections of the conormal sheaf to a codimension 3 determinantal scheme can be used to construct codimension 6 arithmetically Gorenstein schemes. Indeed, the top dimensional component is Gorenstein by Theorem 21.1.

Example 36. *Let C be a fourfold in $\mathbb{P}^7$ determined by the two by two minors of a two by four matrix of general linear forms. C is arithmetically Cohen-Macaulay but is not licci. A general section of $\mathbf{N}_C^*(3)$ determines a non saturated ideal. Upon saturation, we obtain the homogeneous ideal of a smooth, degree 14, genus 8, arithmetically Gorenstein curve, Z. The h-vector of Z is (1,6,6,1). The free resolution of the ideal sheaf, $\mathcal{I}_Z$, is*

$$0 \to \mathcal{O}(-8) \to \mathcal{O}(-6)^{15} \to \mathcal{O}(-5)^{35} \to \mathcal{O}(-4)^{42} \to \mathcal{O}(-3)^{35} \to \mathcal{O}(-2)^{15} \to \mathcal{I}_Z \to 0.$$

Remark 37. *Let C be a codimension r, determinantal scheme in $\mathbb{P}^n$ defined by the maximal minors of a $t \times (t + r - 1)$ matrix, $\mathbf{M}$. $\mathbf{M}$ determines a map of free $\mathbf{R}$-modules. Let $\mathbf{L}$ be the cokernel of this map. Let $\mathcal{L}$ denote the sheafification of $\mathbf{L}$ restricted to C. Assume the ideal of $(t-1) \times (t-1)$ minors of $\mathbf{M}$ is $\mathfrak{m}$-primary. Then $\mathcal{L}$ is a line bundle on C. By the exactness of the generalized Buchsbaum-Rim complexes, $\mathcal{L}^i$ is maximal Cohen-Macaulay for $1 \leq i \leq r-1$. Additionally, $\mathcal{L}^{r-1} \simeq \mathcal{K}_C(t)$ (for some t) by the Eagon-Northcott complex. Let $a_1, a_2, \ldots, a_s$ be a sequence of positive integers such that $a_1 + a_2 + \ldots a_s = r - 1$. Then the sheaf $\mathcal{E} = \oplus_{i=1}^s \mathcal{L}^{a_i}$ satisfies the conditions of Theorem 9 and regular sections of $\mathcal{E}^*$ give arithmetically Gorenstein schemes (provided $2r \leq n$).*

A final example will close the paper. The example was carried out on a computer. We have not worked out a higher dimensional version of Lemma 35 to support this example. It includes an unexpected vanishing of the cohomology of the second exterior power of the normal bundle to a codimension four determinantal scheme. We did not expect such vanishing. Perhaps it hints at an interesting general phenomenon that should be further investigated.

Example 38. *Let $S \subseteq \mathbb{P}^9$ be defined by the maximal minors of a 2×5 matrix of indeterminants. The Normal bundle, $\mathcal{N}_S$, of S in $\mathbb{P}^9$ is a rank four bundle with codepth one. However, the second exterior power of $\mathcal{N}_S$ is Cohen-Macaulay! By Theorem 21.1, regular sections of $\mathcal{N}_S^*(n)$ vanish on codimension 8 arithmetically Gorenstein schemes.*

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Jan O. Kleppe
Oslo University College
Faculty of Engineering
Cort Adelers gt. 30
N-0254, Oslo, Norway
JanOddvar.Kleppe@iu.hio.no
http://www.iu.hio.no/∼jank

Chris Peterson
Department of Mathematics
Colorado State University
Fort Collins, CO 80523-1874, U.S.A.
peterson@math.colostate.edu