Maximal families of Gorenstein algebras

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Abstract

The purpose of this paper is to study maximal irreducible families of (e.g. Artinian) Gorenstein quotients of a polynomial ring R. If $\mathrm{GradAlg}^H(R)$ is the scheme parametrizing graded quotients of R with Hilbert function H, we prove there is a close relationship between the generically smooth irreducible components of $\mathrm{GradAlg}^H(R)$ whose general member is a Gorenstein codimension $(c+1)$ quotient, and the generically smooth irreducible components of $\mathrm{GradAlg}^{H'}(R)$ whose general member B is a codimension c Cohen-Macaulay algebra. Moreover the dimension of the "Gorenstein" components are computed in terms of the dimension of the corresponding "Cohen-Macaulay" component and of other invariants of B. Furthermore we show that linkage is quite effective in computing all invariants of the dimension formula as well as verifying the assumptions which appears in a generalization of the main Theorem.

AMS Subject Classification. 14C05, 13D10, 13D03, 13D07, 13C40.
Keywords. Parametrization, Gorenstein algebra, Artinian algebra, liaison, licci, Cohen-Macaulay, canonical module, normal module, Hilbert scheme.

1 Introduction

The main goal of this paper is to contribute to the classification of (e.g. Artinian) Gorenstein quotients of a polynomial ring R. If $\mathrm{GradAlg}^H = \mathrm{GradAlg}^H(R)$ is the scheme parametrizing graded quotients A of R of depth $A \geq \min(1, \dim A)$ and with Hilbert function H, we let CM_c^H (resp. Gor_c^H) be the open subscheme parametrizing Cohen-Macaulay (resp. Gorenstein) quotients of codimension c in R. In the Artinian case Gor_c^H and the scheme $\mathrm{PGor}(H)$ defined by certain catalecticant minors ([13]) are essentially the same. We shortly call an irreducible component of $\mathrm{GradAlg}^H$ which properly intersects CM_c^H (resp. Gor_c^H) a "Cohen-Macaulay (resp. "Gorenstein) codimension c" component. The main Theorem of this paper establishes a strong connection between the generically smooth "Gorenstein codimension $(c+1)$" components of $\mathrm{GradAlg}^H(R)$ and the generically smooth "Cohen-Macaulay codimension c" components of $\mathrm{GradAlg}^{H'}(R)$ for some H'. Indeed the connection between these components allows us to deduce the generically smoothness and the dimension of the "Gorenstein codimension $(c+1)$" component from the corresponding property and number of the "Cohen-Macaulay codimension c" component. Heuristically speaking, there are at least as many generically smooth "Gorenstein codimension $(c+1)$" components of $\mathrm{GradAlg}^H$ as there are generically smooth "Cohen-Macaulay codimension c components of $\mathrm{GradAlg}^{H'}(R)$.

We all known the nice classification of CM_2^H, given by the Hilbert-Burch theorem, and of Gor_3^H, proved by Buchsbaum and Eisenbud, in terms of the maximal minors (resp. the Pfaffians) of a suitable matrix. In this case the schemes CM_2^H and Gor_3^H are smooth and irreducible. There is, in our opinion, little hope of finding one such nice classification of Gor_4^H

in codimension 4 because this scheme consists of several irreducible components ([3] and [14]). We think a more promising classification strategy should consist of classifying the irreducible components of Gor_4^H together with describing the members of each family concretely. In this context the main Theorem shows that the corresponding classification of $\mathrm{CM}_3^{H'}$ is sentral for the classification of Gor_4^H, and it provides a method for answering Iarrobino et al's question ([14]) about whether the non-irreducibility of Gor_4^H is a rare phenomenon, or not.

More precisely if we have A *well defined* (and hence B generically Gorenstein, see the beginning of section 2) by the sequence

$$0 \to K_B(-s) \to B \to A \to 0, \tag{1}$$

and K_B^* is the B-dual of the canonical module of $B \simeq R/I_B$, we show

Theorem 1. *Let $B \simeq R/I_B$ be licci (resp. $\mathrm{Proj}(B)$ licci and B Cohen-Macaulay and unobstructed) and let A be a codimension 1 quotient of B, well defined by a twist of the canonical module K_B. Then A is unobstructed (resp. unobstructed if $s >> 0$) as a graded R-algebra, A is Gorenstein and,*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + \dim(K_B^*)_s - 1 - \delta(B)_{-s}$$

$$(\textit{resp.}\ \dim_{(A)} \mathrm{GradAlg}^{H_A}(R) = \dim_{(B)} \mathrm{GradAlg}^{H_B}(R) + \dim(K_B^*)_s - 1)$$

where $\delta(B)_{-s} = \dim({}_{-s}\mathrm{Hom}_B(I_B/I_B^2, K_B)) - \dim({}_{-s}\mathrm{Ext}_B^1(I_B/I_B^2, K_B))$.

In the licci case of the Theorem, we have in fact the unobstructedness of A in the sense $\mathrm{H}^2(R, A, A) = 0$, which holds in the non-graded case as well (Theorem 21). Moreover we prove a more general result (Proposition 10) having Theorem 1 as a Corollary. That result requires some skill in verifying the vanishing of the homology groups involved. In the final section 3 we show how we can use linkage to compute these homology groups as well as finding the other dimension invariants of Theorem 1. Note that if $B = R/I$ and R/J are geometrically linked, then $A = R/(I + J)$ is given by (1), so Theorem 1 covers this important case. The unobstructedness of A in the last mentioned case is known ([28]).

To achieve our results we study more generally deformations of a quotient A of R which happens to be a quotient of some quotient B of R. Firstly we recall and further develop (Theorem 4) some partially known results on the unobstructedness and the "family-dimension" of such a quotient A of R with fixed Hilbert function H_A. Here we neither assume B to be Cohen-Macaulay, nor A to be Gorenstein. The Theorem is central for the main application of this paper because we manage to treat the two main cases of interest: when the *generic* member of an irreducible family $R \to A$ factorizes via some deformation of $B \to A$, and when it does not.

In this work the author is inspired by joint works with Chris Peterson where we in various ways construct Gorenstein quotients of R. We show for instance that if M is a maximal Cohen-Macaulay B-module and $\wedge^r \widetilde{M} \simeq \widetilde{K}_B(s)$ as sheaves (in a large enough open set) of $\mathrm{Proj}(B)$, then any regular section of $\widetilde{M}^*(s)$ defines a Gorenstein quotient $R \twoheadrightarrow A$ of codimension $r + c$ (cf. [20] for details). The idea of this paper is to look to the construction and vary (via deformation theory) every object and morphism in the construction as much as possible, to see how large the corresponding stratum in $\mathrm{GradAlg}^H$ will be. This paper looks at the $r = 1$ case because it is so far only in this case the author is able to prove his results also in the Artinian Gorenstein case. Note that C. Peterson and J. Migliore already have observed that not every codimension 3 quotent is obtained by (1) ([24]). On the other hand A. Geramita and J. Migliore ([7]) use geometric linkage to show that every non-empty Gor_3^H contains at least one point given by (1). All this fits nicely with Theorem 1 in which the number $\dim({}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, K_B))$ essentially measures the codimension of the strata. To apply Theorem 1 we need to have at least one point of Gor_c^H given by (1), to apply it. In a forthcoming paper the author generalizes the results of this paper considerable by considering deformations of regular sections of a maximal Cohen-Macaulay module M of any rank, but, unfortunately, where we must restrict to positively dimensional Gorenstein algebras. We thank Olav A. Laudal, Roy Skjelnes and Johannes Kleppe for clarifying discussions.

For the convinciency of the reader we include a section of preliminaries where we give some space to the theory of deformations. Indeed it is not always easy to find a nice reference for the results we use because the litterature is often to general (with general cohomology groups of algebras which are difficult to handle) or to special (to makes the mentioned groups vanish or managable for being computed).

1.1 Preliminaries

Let B be an n-dimensional graded quotient of a polynomial k-algebra R (k is algebraically closed) in $n+c$ variables (of degree 1), and let M and N be finitely generated graded B-modules. Let $\operatorname{depth}_J M$ denote the length of a maximal M-sequence in a homogeneous ideal J and let $\operatorname{depth} M = \operatorname{depth}_{\mathfrak{m}} M$ where $\mathfrak{m}$ the irrelevant maximal ideal. Let $\mathrm{H}^i_J(-)$ be the right derivative of the functor, $\Gamma_J(-)$, of sections with support in $\mathrm{Spec}(B/J)$. Recall that $\operatorname{depth}_J M = \inf\{\operatorname{depth} M_\wp \mid \wp \supseteq J,\ \wp \text{ graded}\}$ and that $\operatorname{depth}_J M \ge r$ iff $\mathrm{H}^i_J(M) = 0$ for $i < r$, cf. [8] or [10].

If $Y = \mathrm{Proj}(B)$ and Z is closed in Y and $U = Y - Z$, we let $\mathrm{H}^0_*(U, \widetilde{M}) = \oplus_v \mathrm{H}^0(U, \widetilde{M}(v))$. Then we have an exact sequence

$$0 \to \mathrm{H}^0_{I(Z)}(M) \to M \to \mathrm{H}^0_*(U, \widetilde{M}) \to \mathrm{H}^1_{I(Z)}(M) \to 0 \tag{2}$$

and isomorphisms $\mathrm{H}^i_{I(Z)}(M) \simeq \mathrm{H}^{i-1}_*(U, \widetilde{M})$ for $i \ge 2$. In the case $\operatorname{depth}_{I(Z)} N \ge 2$, the graded $\mathrm{Ext}^1_B(M, N)$ injects into the corresponding global $\mathrm{Ext}^1_{\mathcal{O}_U}$-group of sheaves. Indeed we have in this case an exact sequence ([8], exp. VI)

$${}_0\mathrm{Ext}^1_B(M,N) \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}_U}(\widetilde{M}|_U, \widetilde{N}|_U) \to {}_0\mathrm{Hom}_B(M, \mathrm{H}^2_{I(Z)}(N)) \to {}_0\mathrm{Ext}^2_B(M,N) \to \mathrm{Ext}^2_{\mathcal{O}_U}(\widetilde{M}|_U, \widetilde{N}|_U) \tag{3}$$

where the form of the middle term comes from a spectral sequence also treated in [8].

A Cohen-Macaulay (resp. maximal Cohen-Macaulay) B-module satisfies $\operatorname{depth} M = \dim M$ (resp. $\operatorname{depth} M = \dim B$) by definition, or equivalently, $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i < \dim M$ (resp. $i < \dim B$). If B is Cohen-Macaulay and and $K_B = \mathrm{Ext}^c_R(B, R(-n-c))$ is the canonical module of B, we know that v-graded piece of $\mathrm{H}^i_{\mathfrak{m}}(M)$ is by Gorenstein duality

$${}_v\mathrm{H}^i_{\mathfrak{m}}(M) \simeq {}_{-v}\mathrm{Ext}^{n-i}_B(M, K_B)^\vee .$$

Recall that two graded quotients, R/J and R/J', are said to be (complete intersection) linked if there exists a homogeneous complete intersection ideal L such that $J = L : J'$ and

$J' = L : J$ (with $L \subseteq J \cap J'$). The relationship of being linked generates the equivalence relation, "linkage". $B = R/I_B$ is said to be licci (and hence Cohen-Macaulay) if it is in the linkage class of a complete intersection (cf. [22] for a survey).

Even though we in this paper tries to avoid using algebra (co)homology groups, e.g. $\mathrm{H}_2(R,B,B)$ and $\mathrm{H}^2(R,B,B)$ (we will instead use the corresponding Tor- of Ext-groups), we occasionally need to consider them. For this we recall that the former group is given by an exact sequence

$$0 \to \mathrm{H}_2(R,B,B) \to \mathrm{H}_1 \to G_1 \otimes_R B \to I_B/I_B^2 \to 0. \tag{4}$$

in which G_1 is R-free, $G_1 \twoheadrightarrow I_B$ is surjective and minimal, and $\mathrm{H}_1 = \mathrm{H}_1(I_B)$ is the 1. Koszul homology of I_B [27]. An ideal I_B of R is called *syzygietic* if $\mathrm{H}_2(R, R/I_B, R/I_B) = 0$. If I_B is syzygietic, then $(I_B)_\wp$ is syzygietic for any prime ideal $\wp$ of R. Using (4) one shows that if $R \to B$ is generically a complete intersection and B is licci, then $\mathrm{H}_2(R,B,B) = 0$ because H_1 is a maximal Cohen-Macaulay module in the licci case [12]. Thus, generically complete intersection licci ideals are syzygietic. For the graded group $\mathrm{H}^2(R,B,B)$ we remark that there is an exact sequence

$$0 \to {}_0\mathrm{Ext}^1_B(I_B/I_B^2, N) \to {}_0\mathrm{H}^2(R,B,N) \to {}_0\mathrm{Hom}_B(\mathrm{H}_2(R,B,B), N) \to \tag{5}$$

(coming from a well known spectral sequence because $I_B/I_B^2 \simeq \mathrm{H}_1(R,B,B)$, cf. [1] or [21]).

A Cohen-Macaulay quotient $B = R/I_B$ of codimension $c = \dim R - \dim B$ in R has a minimal R-free resolution of the following form [6]

$$0 \to G_c \to \ldots \to G_1 \to R \to B \to 0 \qquad , \qquad G_j = \oplus_{i=1}^{r_j} R(-n_{j,i}) \tag{6}$$

Let G_i^* be the R-dual of G_i. Since $K_B = \mathrm{Ext}^c_R(B, R(-n-c))$, the R-dual complex of (6) yields a minimal R-free resolution of the twisted canonical module $K_B(n+c)$. In particular if $c = 3$, it induces a complex

$$0 \to N_B \to G_1^* \otimes_R B \xrightarrow{\psi_1} G_2^* \otimes_R B \xrightarrow{\psi_2} G_3^* \otimes_R B \to K_B(n+c) \to 0 \tag{7}$$

which is exact except in the middle, cf. [20], p.795. Indeed $\ker \psi_2 / \operatorname{im} \psi_1 \simeq \mathrm{Tor}_1^R(B, K_B)(n + c) \simeq I/I^2 \otimes K_B(n+c)$, and this module is a maximal Cohen-Macaulay module if B is licci [4]. If $c = 2$, then (7) makes an *exact* sequence provided we have removed $G_3^* \otimes_R B$.

Recall that a closed subscheme Y of $\mathbb{P} = \operatorname{Proj} R$ is called *unobstructed* if the Hilbert scheme, $\mathrm{Hilb}^p := \mathrm{Hilb}^p(\mathbb{P})$, is smooth at $(Y \subset \mathbb{P})$. Let $\mathrm{GradAlg}^H = \mathrm{GradAlg}^H(\mathbb{P})$ be the strata of Hilb^p given by deforming $(Y \subset \mathbb{P})$ with constant Hilbert function $H_Y = H$ (i.e. its functor deforms both $Y \subset \mathbb{P}$ and its *homogeneous coordinate ring*, B, flatly), cf. [17]. $\mathrm{GradAlg}^H$ allows a natural scheme structure whose tangent (resp. "obstruction") space at $(Y \subset \mathbb{P})$ is ${}_0\mathrm{Hom}_B(I_B/I_B^2, B) \simeq {}_0\mathrm{Hom}_R(I_B, B)$ (resp. ${}_0\mathrm{H}^2(R,B,B)$), i.e. it is given by deforming B as a graded R-algebra [15]. In the case $H(v)$ does not vanish for large v (i.e. B is non-Artinian), we may look upon $\mathrm{GradAlg}^H$ as parametrizing graded R-quotients, $R \to B$, satisfying $\operatorname{depth}_{\mathfrak{m}} B \geq 1$ and with Hilbert function $H_B = H$. If B is Artinian, $\mathrm{GradAlg}^H$ still represents a functor parametrizing graded R-quotients with Hilbert function $H_B = H$ (see Proposition 6), and its open subscheme of Gorenstein quotients coincides, at least topologically and infinitesimally, with $\mathrm{PGor}(H)$ (the corresponding scheme of forms with "catalectical structure"). B is called unobstructed as a graded R-algebra iff $\mathrm{GradAlg}^H$

is smooth at $(R \to B)$. This definition of unobstructedness coincides with the one given in Remark 2 by [21] or [25], p.151. By [15], Remark 3.7,

$$\mathrm{GradAlg}^H \simeq \mathrm{Hilb}^p \quad \text{at} \quad (Y \subset \mathbb{P}) \tag{8}$$

provided ${}_0\mathrm{Hom}_R(I_B, \mathrm{H}^1_{\mathfrak{m}}(B)) = 0$. This happenes for quotients B of $\mathrm{depth}_{\mathfrak{m}} B \geq 2$.

Similarly we let $\mathrm{GradAlg}(H_B, H_A)$ be the representing object of the functor deforming flags (surjections) $B \to A$ of graded quotients of R of positive depth (for non-Artinian quotients) and with Hilbert functions H_B and H_A of B and A respectively. The tangent space $T_{B\to A}$ of $\mathrm{GradAlg}(H_B, H_A)$ at $(B \to A)$ and the tangent maps T_{pr_i} of the projection morphisms $pr_1 : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}^{H_A}$ and $pr_2 : \mathrm{GradAlg}(H_B, H_A) \to \mathrm{GradAlg}^{H_B}$ are given by the *cartesian* square in the following diagram of exact sequences

$$\begin{array}{ccccccc}
 & & & & & & {}_0\mathrm{Hom}_R(I_B, I_{A/B}) \\
 & & & & & & \downarrow \\
 & & & & T_{B\to A} & \xrightarrow{T_{pr_2}} & {}_0\mathrm{Hom}_R(I_B, B) \\
 & & & & \downarrow & \square & \downarrow \\
0 & \to & {}_0\mathrm{Hom}_R(I_{A/B}, A) & \to & {}_0\mathrm{Hom}_R(I_A, A) & \to & {}_0\mathrm{Hom}_R(I_B, A)
\end{array} \tag{9}$$

Here $B \to A \simeq R/I_A$ is a surjection of graded R-algebras with kernel $I_{A/B}$. Since the lower sequence in the diagram above may continue by ${}_0\mathrm{H}^2(B, A, A)$ to make a long exact sequence of algebra cohomology groups, and since ${}_0\mathrm{H}^2(B, A, A)$ injects in general into ${}_0\mathrm{Ext}^1_B(I_{A/B}, A)$ ([9], exp. VI), it follows that ${}_0\mathrm{Hom}_R(I_A, A) \to {}_0\mathrm{Hom}_R(I_B, A)$ is surjective provided ${}_0\mathrm{Ext}^1_B(I_{A/B}, A) = 0$. (One may also prove this surjectivity more elementary by dualizing the exact sequence $0 \to I_B \otimes_R A \simeq I_B/I_A I_B \to I_A \otimes_R B \to I_{A/B} \to 0$.)

Remark 2. *Let $R \to B$ be a graded quotient. Let $(T, \mathfrak{m}_T) \to (S, \mathfrak{m}_S)$ be a small artin surjection (i.e. of local Artinian k-algebras with residue fields k whose kernel $\mathfrak{a}$ satisfies $\mathfrak{a}\cdot\mathfrak{m}_T = 0$). A graded deformation B_S of B to S is a graded S-flat quotient of $R \otimes_k S$ satisfying $B_S \otimes_S k \simeq B$. A graded deformation B_T of B_S to T is T-flat and satisfies $B_T \otimes_T S \simeq B_S$.*

A flag (i.e. a surjection) of graded quotients of R, $B \to A$, is said to be unobstructed if for every small artin $T \twoheadrightarrow S$ and for every graded deformation $B_S \to A_S$ of $B \to A$ to S, there exists a graded deformation $B_T \to A_T$ (of T-flat quotients) reducing to $B_S \to A_S$ via $(-) \otimes_T S$. Similarly a quotient A of R is unobstructed if every graded deformation A_S deforms further to T

The obstruction of deforming A_S (resp. $B_S \to A_S$) to T (resp. to a given graded deformation B_T of B_S) sits in ${}_0\mathrm{H}^2(R, A, A) \otimes_k \mathfrak{a}$ (resp. in ${}_0\mathrm{H}^2(B, A, A) \otimes_k \mathfrak{a}$) and ${}_0\mathrm{H}^1(R, A, A) \otimes_k \mathfrak{a}$ (resp. ${}_0\mathrm{H}^1(B, A, A) \otimes_k \mathfrak{a}$) corresponds to the set of deformations (e.g. the tangent space). Note that, by definition, the vanishing of the obstruction is equivalent to the existence of corresponding desired deformation. Hence the vanishing of the obstruction groups ("spaces") lead to the unobstructedness of the corresponding objects.

One may consider non-graded deformations of e.g. a quotient B of R in which case the whole algebra cohomology group $\mathrm{Hom}_R(I_B, B) \simeq \mathrm{H}^1(R, B, B)$ (resp. $\mathrm{H}^2(R, B, B)$) serves as tangent (resp. "obstruction") space of the deformations of B as an R-algebra. For graded deformations it is the degree zero piece of the cohomology groups which counts. For a generically a complete intersection, $R \twoheadrightarrow B$, (or weaker, for a B which satisfies ${}_0\mathrm{Hom}_B(\mathrm{H}_2(R,B,B),B) = 0$, its obstructions sit in ${}_0\mathrm{Ext}^1_B(I_B/I_B^2,B) \otimes \mathfrak{a}$ by (5).

It is rather straightforward to get (i) in the Proposition below from Remark 2 and the injection ${}_0\mathrm{H}^2(B,A,A) \hookrightarrow {}_0\mathrm{Ext}^1_B(I_{A/B},A)$. The latter statement is deeper and relies on a closer study of the obstructions involved, see [16], p.135 for a related proof.

Proposition 3. *Let B be unobstructed as a graded R-algebra.*

(i) If ${}_0\mathrm{Ext}^1_B(I_{A/B},A) = 0$ (or if ${}_0\mathrm{H}^2(B,A,A)$), then $B \to A$ is unobstructed, i.e. $\mathrm{GradAlg}(H_B,H_A)$ *is smooth at $(B \to A)$.*

(ii) If ${}_0\mathrm{Hom}_R(I_B,B) \to {}_0\mathrm{Hom}_R(I_B,A)$ is surjective, then the first projection pr_1 : $\mathrm{GradAlg}(H_B,H_A) \to \mathrm{GradAlg}^{H_A}$ *is smooth at $(B \to A)$.*

Using Proposition 3, we get easily the "resp." part of the following Theorem which is not new ([16], sect.1 or [19], ch.9), nor is special cases of the "non-resp" version ([19], ch.7). Note that the "resp." part and in particular the parenthetical condition in Theorem 4(iii) has as a consequence that the generic member of a family of $R \twoheadrightarrow A$ comes from a member of the family of $R \twoheadrightarrow B \twoheadrightarrow A$. Since it is in general unlikely that a general member factorizes via a deformation of B even though a special member does, we need a result to take care of this possibility. The "non-resp." part of the following theorem does that (in which we use small letters for the k-dimension), and we only include a proof of this variation.

Theorem 4. *Let R be a finitely generated graded k-algebra and let $B \simeq R/I_B \twoheadrightarrow A \simeq B/I_{A/B}$ be a graded morphism of quotients of R. If*

(i) ${}_0\mathrm{Ext}^1_B(I_B/I_B^2,B) = 0$ and $(I_B)_\wp$ is syzygietic for any graded $\wp$ of $Ass(A) \cup Ass(B)$ (resp. B is unobstructed as a graded R-algebra),

(ii) ${}_0\mathrm{Ext}^1_B(I_{A/B},A) = 0$, and

(iii) ${}_0\mathrm{Ext}^2_B(I_B/I_B^2,I_{A/B}) \hookrightarrow {}_0\mathrm{Ext}^2_B(I_B/I_B^2,B)$ is injective (resp. ${}_0\mathrm{Ext}^1_B(I_B/I_B^2,I_{A/B}) \hookrightarrow {}_0\mathrm{Ext}^1_B(I_B/I_B^2,B)$) is injective),

then A is unobstructed as a graded R-algebra. Moreover, if $\mathrm{depth}_{\mathfrak{m}} A \geq \min(1,\dim A)$, *then*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A} = {}_0\hom_R(I_B,B) + {}_0\hom_B(I_{A/B},A) - {}_0\hom_R(I_B,I_{A/B}) + {}_0ext^1_B(I_B/I_B^2,I_{A/B}).$$

$$(resp.\ \dim_{(A)} \mathrm{GradAlg}^{H_A} = {}_0\hom_R(I_B,B) + {}_0\hom_B(I_{A/B},A) - {}_0\hom_R(I_B,I_{A/B})\ .)$$

Furthermore if k is of characteristic zero and if $W(B \to A)$ and $W(A)$ are the irreducible components of $\mathrm{GradAlg}(H_A,H_B)$ *and* $\mathrm{GradAlg}^{H_A}$ *respectively to which $(B \to A)$ and (A) belong, then*

$$\dim W(A) - \dim pr_1(W(B \to A)) = {}_0ext^1_B(I_B/I_B^2,I_{A/B}).$$

Proof. Using the assumptions of (i) and (iii) and the long exact sequence of ${}_0\mathrm{Ext}^i_B(I_B/I_B^2,-)$, we get ${}_0\mathrm{Ext}^1_B(I_B/I_B^2,A) = 0$. By the definition of a syzygetic ideal we have $\mathrm{H}_2(R,B,B)_\wp = 0$ for any $\wp$ of $Ass(A)$. It follows that ${}_0\mathrm{Hom}_B(\mathrm{H}_2(R,B,B),A) = 0$ and hence that ${}_0\mathrm{H}^2(R,B,A) = 0$ by (5). Since we have in general an injection ${}_0\mathrm{H}^2(B,A,A) \hookrightarrow {}_0\mathrm{Ext}^1_B(I_{A/B},A)$ ([9], exp.

VI), we get ${}_0\mathrm{H}^2(B,A,A)=0$ by (ii) above. It follows that ${}_0\mathrm{H}^2(R,A,A)=0$ by the long exact sequence of algebra cohomology [21],

$$\to {}_0\mathrm{H}^2(B,A,A) \to {}_0\mathrm{H}^2(R,A,A) \to {}_0\mathrm{H}^2(R,B,A) \to \tag{10}$$

This proves the unobstructedness of A and $\dim_{(A)} \mathrm{GradAlg}^{H_A} = {}_0\hom_R(I_A,A)$. Now, looking to (9), noting that vanishing of ${}_0\mathrm{Ext}^1_B(I_{A/B},A)$ translates to a certain surjectivity in the cartesian square of (9) and that the assumption (i) above implies that the cokernel of the vertical Hom-map in the cartesian square of (9) is precisely ${}_0\mathrm{Ext}^1_B(I_B/I_B^2, I_{A/B})$, it is straightforward to get the dimension formula.

To see that ${}_0ext^1_B(I_B/I_B^2,I_{A/B})$ measures the codimension, we claim that $B \twoheadrightarrow A$ is unobstructed and hence that $\dim_{(B\to A)} \mathrm{GradAlg}(H_A,H_B) = \dim T_{B\to A}$. Indeed the assumption (i) and (5) show that B is unobstructed as a graded R-algebra because we have $\mathrm{H}_2(R,B,B)_\wp = 0$ for any $\wp$ of $Ass(B)$. Using (ii) above and Proposition 3(i), we get the claim. Now we remark that $pr_1 : \mathrm{GradAlg}(H_B,H_A) \to \mathrm{GradAlg}^{H_A}$ is a morphism between smooth schemes (i.e. smooth at $(B\to A)$ and A respectively) and with fiber dimension ${}_0\hom_B(I_B/I_B^2,I_{A/B})$, from which we may conclude by using the Theorem of generic smoothness to pr_1 at $(B\to A)$, cf. [19], first part of the proof of Proposition 9.14 for more details (from a similar proof). $\square$

Remark 5. *a) Note that the "non-resp. part" of Theorem 4 leads to* ${}_0\mathrm{H}^2(R,A,A)=0$ *(and in fact to* ${}_v\mathrm{H}^2(R,A,A)=0$ *by twisting all three* Ext_B*-assumptions by* v*), cf. proof.*

b) It is possible to weaken assumption (ii) *in Theorem 4 to "* ${}_0\mathrm{H}^2(B,A,A)=0$*" and still conclude as in Theorem 4. Similarly one may also weaken "*$(I_B)_\wp$ *is syzygietic for any* $\wp$ *of* $Ass(A)\cup Ass(B)$*" to "*$(I_B)_\wp$ *is syzygietic for any* $\wp$ *of* $Ass(A)$ *and* ${}_0\mathrm{Hom}_B(\mathrm{H}_2(R,B,B),A) = 0$*" and still get the same conclusions. Moreover by possibly forgetting the conclusion that the codimension is equal to* ${}_0ext^1_B(I_B/I_B^2,I_{A/B})$ *in Theorem 4, we may replace the "non-resp. assumptions" of both* (i) *and* (iii) *by "* ${}_0\mathrm{Ext}^1_B(I_B/I_B^2,A)=0$ *and* ${}_0\mathrm{Hom}_B(\mathrm{H}_2(R,B,B),A)=0$*" and still conclude that* A *is unobstructed, but now that* $\dim_{(A)}\mathrm{GradAlg}^{H_A}$ *is given by*

$${}_0\hom_R(I_B,B) + {}_0\hom_B(I_{A/B},A) - {}_0\hom_R(I_B,I_{A/B}) + {}_0ext^1_B(I_B/I_B^2,I_{A/B}) - {}_0ext^1_B(I_B/I_B^2,B).$$

There is no real change in the proof. This variations turn out to have an application notably in the case where A *is Artinian.*

We finish this section by looking more closely to deformations of an Artinian algebra A. Firstly we recall the representability of the functor (the sheaf) $\mathrm{GRADALG}^H(-)$, locally given by letting $\mathrm{GRADALG}^H(\mathrm{Spec}(S))$ (S a noetherian k-algebra) be the set of graded S-flat quotients $R_S \to A_S$ such that $H_{A_s} = H$ and $\mathrm{depth}_\mathfrak{m} A_s \ge \min(1, \dim A_s)$ for any $s \in \mathrm{Spec}(S)$. Here $R_S = R\otimes_k S$ and A_s is the "fiber" of $\mathrm{Spec}(A_S) \to \mathrm{Spec}(S)$ at $s\in\mathrm{Spec}(S)$. To simplify and clarify [17] (being more careful with the topologies involved), we partially include proofs.

Proposition 6. *The functor* $\mathrm{GRADALG}^H(-)$ *is representable. Its representing object,* $\mathrm{GradAlg}^H$*, is a* k*-scheme of finite type.*

Proof. Here we sketch a proof which is slightly different from that in [17] in the Artinian case (i.e. when $H(v)=0$ for $v >> 0$). Indeed the flattening stratification described in [25] and Grothendieck's representability Theorems of the Hilbert functor are the results we need to conclude. In the non-Artinian case, $H(v)$ is a polynomial $p(v)$ for $v >> 0$ and $\mathrm{GradAlg}^H$ is the stratum of $\mathrm{Hilb}^p(\mathbb{P}^{n+c-1})$ of constant postulation (cf. [17], Theorem 1.1 for details). If $H(v) = 0$ for $v >> 0$ and $t = \sum_v H(v)$, then there exists a scheme $\mathrm{Hilb} := \mathrm{Hilb}^t(\mathbb{A}^{n+c})$ of finite type which parametrizes quotients $R \to A$ of length t. The restriction (pullback) $R_S \to A_S$ of the universal quotient to $\mathrm{Spec}(S) \to \mathrm{Hilb}$ defines the subscheme $\mathrm{GradAlg}^H$ of Hilb by requiring $\ker((R_S)_v \to A_S)$ to be flat over S, of rank $\dim R_v - H(v)$ for every $0 \le v \le t$ at each fiber of $\mathrm{Spec}(S)$. Since such a subscheme $\mathrm{GradAlg}^H$ of finite type exists by [25], Lect. 8 or [23], VI, Proposition 1.1, we are done. □

Remark 7. *We may in the case $H(v) = 0$ for $v >> 0$ use flattening stratification to show the existence of the scheme $Z(H)$ of* not neccessarily graded *Artinian quotients A of R with Hilbert function H (by requiring $gr(A_S)$ to be S-flat, of Hilbert function H at each fiber of* $\mathrm{Spec}(S)$*), to get $Z(H)$ as a locally closed subscheme (stratum) of* $\mathrm{Hilb}^t(\mathbb{A}^{n+c})$*. Its local deformation functor at a quotient $A \simeq R/I_A$ (with $\sqrt{I_A} = \mathfrak{m}$) of R, defined on the category of local Artinian k-algebras with residue field k, consists of quotients A_S of $R_S = S[X_1 ...., X_{n+c}]$ (or equivalently, of $S[[X_1 ...., X_{n+c}]]$) such that A_S and $gr(A_S)$ are deformations of A and $gr(A)$ respectively. If $R \to A$ is graded and satisfies ${}_v\mathrm{H}^2(R, A, A) = 0$ for $v \ge 0$, then [17], Theorem 1.10 and Remark 1.11 shows that $Z(H)$ is smooth at $(R \to A)$ and $\dim_{(A)} Z(H) = \sum_{v \ge 0} {}_v\mathrm{hom}_R(I_A, A)$.*

Inside $\mathrm{GradAlg}^H$ there is an open set (and hence a scheme Gor_c^H with induced scheme structure) consisting of Gorenstein quotients $R \to A$ with Hilbert function H. In the Artinian case there is another known scheme parametrizing graded Gorenstein quotients with Hilbert function H, namely the determinantal loci $\mathrm{PGor}(H)$ of forms of degree s in $\mathbb{P}^N$, $N = \binom{s+n+c-1}{n+c-1} - 1$, given by requiring the "catalecticant matrices" to have ranks given by the Hilbert function, see Iarrobino-Kanev's book [13], Sect.1.1 for an introduction, and denote the restriction of the generic s-form of $\mathbb{P}^N$ to $\mathrm{PGor}(H)$ by $F_{\mathrm{PGor}(H)}$ ([13], Definition 1.3). The underlying set of closed poins of Gor_c^H and $\mathrm{PGor}(H)$ are the same by apolarity (the Macaulay correspondence), and one may see from the proof of our next result that they are in fact isomorphic as topological spaces. Moreover since Iarrobino and Kanev have proved that the tangent space of $\mathrm{PGor}(H)$ at A is isomorphic to ${}_0\mathrm{Hom}_A(I_A/I_A^2, A)$ ([13], Theorem 3.9) and it is well known that the tangent space of $\mathrm{GradAlg}^H$ is the same group [15], we get

Theorem 8. *Let $R \to A$ be a graded Gorenstein Artinian quotient with Hilbert function H. Then* $\dim_{(A)} \mathrm{GradAlg}^H = \dim_{(A)} \mathrm{PGor}(H)$*. Hence* $\mathrm{PGor}(H)$ *is smooth at $(R \to A)$ if and only if* $\mathrm{GradAlg}^H$ *is smooth at $(R \to A)$. In particular* $\mathrm{PGor}(H)$ *is smooth at $(R \to A)$ provided ${}_0\mathrm{H}^2(R, A, A) = 0$, i.e. provided ${}_s\mathrm{H}_2(R, A, A) = 0$ where s is the socle degree of A.*

Proof. The vanishing of the two last mentioned groups is equivalent by duality. Hence we only need to prove $\dim_{(A)} \mathrm{GradAlg}^H = \dim_{(A)} \mathrm{PGor}(H)$ because we know their tangent spaces are isomorphic. Let $V \subset \mathrm{PGor}(H)$ be a closed irreducible subset (e.g. of the form $V = \mathrm{Spec}(T)$), and let V have the reduced scheme structure. By the definition of $\mathrm{PGor}(H)$, the restriction F_V of the "universal" s-form $F_{\mathrm{PGor}(H)}$ to V defines via apolarity a family of graded Artinian Gorenstein quotients over V (e.g. the family $A_T \simeq R_T \circ F_V$ over $V = \mathrm{Spec}(T)$ where R_T is interpreted as the T-algebra of contractions "partial derivations without coefficients") with constant Hilbert function H. Since V is integral, it follows that the family (e.g. the morphism

$\mathrm{Spec}(A_T) \to \mathrm{Spec}(T))$ is flat. Hence we have a morphism $\pi : V \to \mathrm{GradAlg}^H$ by the universal property of $\mathrm{GradAlg}^H$. $\pi(V)$ is irreducible and closed in Gor_c^H (it is closed because an "invers" $(\mathrm{Gor}_c^H)_{red} \to \mathrm{PGor}(H)$ on closed points exists by [13], Ch. 9). So chains of closed irreducible subsets in $\mathrm{PGor}(H)$ and Gor_c^H correspond, and we are done. $\square$

Problem 9. *It will be interesting to prove that* $\mathrm{PGor}(H)$ *and* Gor_c^H *are* isomorpic as schemes, *cf. [17], Remark 1.9. The arguments in the proof above, which leads to the isomorphism of* $\mathrm{PGor}(H)$ *and* Gor_c^H *as topological spaces, are implicitely contained in [17], Remark 1.9. In that Remark we also use the mentioned arguments to see the existence of a natural* $\pi : \mathrm{PGor}(H) \to \mathrm{Gor}_c^H$. *Here an index "red" is missing, i.e. we can from the arguments only be sure of the existence of a natural morphism* $\pi : (\mathrm{PGor}(H))_{red} \to \mathrm{Gor}_c^H$.

2 Families of Gorenstein quotients of codimension 1 on Cohen-Macaulay algebras

The main purpose of this paper is to study families of graded Gorenstein quotients A obtained by taking a regular section of the anticanonical sheaf $\widetilde{K}_B^*(s)|_U$, i.e. quotients A given by

$$0 \to K_B(-s) \to B \to A \to 0 \qquad (11)$$

for some integer s. Our main results apply to Artinian Gorenstein algebras as well as to Gorenstein quotients of higher dimension. In what follows B is Cohen-Macaulay. In order to make the description of A in (11) meaningfull, we must suppose $B \simeq R/I_B$ is generically Gorenstein (cf. [19], Lemma 5.2 and 5.4). Then we shortly say *"A is a codimension 1 quotient of B, well defined by a twist of the canonical module"*. In this case there is an open set $U = Y - Z$ of $Y = \mathrm{Proj}\, B$ such that $\dim B - \dim B/I(Z) \geq 1$ with $\widetilde{K_B}|_U$ locally free (let $I(Z) = \mathfrak{m}$ if $U = Y$), and a regular section $\sigma \in \mathrm{H}^0(U, \widetilde{K_B}^*(s))$ whose dual section $\sigma^* \in \mathrm{H}^0_*(Y, \widetilde{K_B}^{**}(-s)) \to \mathrm{H}^0_*(Y, \widetilde{B})$ factorizes via $K_B(-s) \to B$, thus defining A by $A := \mathrm{coker}(K_B(-s) \to B)$. Note that if U above is a little larger, namely such that $\dim B - \dim B/I(Z) \geq 2$ (i.e. B is Gorenstein in codimension ≤ 1), then

$$A := \mathrm{coker}(K_B(-s) = \mathrm{H}^0_*(U, \widetilde{K_B}^{**}(-s)) \xrightarrow{\sigma^*} B = \mathrm{H}^0_*(U, \widetilde{B})).$$

The following result determines $\dim_{(A)} \mathrm{GradAlg}^{H_A}$ in terms of invariants of B, such as $N_B := \mathrm{Hom}_B(I_B/I_B^2, B)$ and $K_B^* := \mathrm{Hom}_B(K_B, B) \simeq \mathrm{Hom}_B(S_2(K_B), K_B)$ ($S_2(K_B)$ is the second symmetric power of K_B), and leads to the main theorem of this section.

Proposition 10. *Let B be a generically Gorenstein, graded Cohen-Macaulay quotient of a polynomial ring R and let A be a codimension 1 quotient of B, defined by* (11) *for some s. If*

(i) ${}_0\mathrm{Ext}^1_B(I_B/I_B^2, B) = 0$ *and* $(I_B)_\wp$ *is syzygietic for any graded prime* $\wp \supseteq I_B$ *of R satisfying* $\dim B_\wp \leq 1$ *(resp. B is unobstructed as a graded algebra),*

(ii) ${}_s\mathrm{Ext}^1_B(S_2(K_B), K_B) = 0$, *and*

(iii) ${}_{-s}\mathrm{Ext}^2_B(I_B/I_B^2, K_B) = 0$ *(resp.* ${}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, K_B) = 0$*),*

then A is unobstructed as a graded R-algebra, A is Gorenstein and

$$\dim_{(A)} \mathrm{GradAlg}^{H_A} = \dim(N_B)_0 + \dim(K_B^*)_s - 1 - \delta(B)_{-s}$$

$$(resp.\ \dim_{(A)} \mathrm{GradAlg}^{H_A} = \dim(N_B)_0 + \dim(K_B^*)_s - 1 - {}_{-s}\hom_R(I_B, K_B))$$

where $\delta(B)_{-s} = {}_{-s}\hom_R(I_B, K_B) - {}_{-s}ext^1_B(I_B/I_B^2, K_B)$. *Moreover if the characteristic of k is zero, then*

$${}_{-s}ext^1_B(I_B/I_B^2, K_B) = \dim W(A) - \dim pr_1(W(B \to A))$$

where $W(B \to A)$ and $W(A)$ are the irreducible components of $\mathrm{GradAlg}(H_A, H_B)$ *and* $\mathrm{GradAlg}^{H_A}$ *respectively to which $(B \to A)$ and (A) belong.*

Remark 11. *(a) With the notations,* $0 \to \oplus_i R(-n_{c,i}) \to ... \to \oplus_i R(-n_{1,i}) \to I_B \to 0$, *of* (6), *recalling* $\dim R = n + c$ *and* $\dim B = n$, *we claim that*

$$\delta(B)_{-s} = 0 \text{ provided } s > \max n_{c,i} + \max n_{2,i} - n - c$$

(in fact ${}_{-s}\mathrm{Ext}^i_B(I_B/I_B^2, K_B) = 0$ *for $i = 0$ and 1 for such s). Indeed dualizing* (6) *we get a surjection* $\oplus_i R(n_{c,i}) \twoheadrightarrow K_B(n+c)$, *and it follows that* $(K_B)_{-v} = 0$ *for* $v > \max n_{c,i} - n - c$. *We conclude by applying* ${}_0\mathrm{Hom}_B(-, K_B)$ *to the right-exact sequence* $\oplus_i B(-n_{2,i}) \to \oplus_i B(-n_{1,i}) \to I_B/I_B^2 \to 0$ *and the definiton* $\mathrm{Ext}^i_B(I_B/I_B^2, K_B)$.

(b) If $\dim B = 1$, *we have* $\mathrm{Ext}^i_B(-, K_B) = 0$ *for $i \geq 2$ by duality. In this case the assumption* ${}_{-s}\mathrm{Ext}^2_B(I_B/I_B^2, K_B) = 0$ *of Proposition 10 is always satisfied. Moreover looking to Remark 5 and using that* $A \simeq K_A(-s)$, *we may weaken "I_B is syzygietic in codimension ≤ 1" to "I_B is generically syzygietic and* ${}_s\mathrm{H}_2(R, B, B) = 0$*" and still conclude as in Proposition 10. By possibly forgetting the final conclusion on* ${}_{-s}ext^1_B(I_B/I_B^2, K_B)$ *we may replace the whole "non-resp. assumption" of (i) by the assumption "*$(\mathrm{H}_2(R, B, B) \otimes_B A)_s = 0$*" and conclude that A is unobstructed and that*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A} = \dim(N_B)_0 + \dim(K_B^*)_s - 1 - \delta(B)_{-s} - {}_0ext^1_B(I_B/I_B^2, B)$$

(c) Slightly reformulating Proposition 10 (by replacing the vanishing of the three ${}_0\mathrm{Ext}^i_B$ *group of the ordinary assumptions by the corresponding vanishing of* Ext^i_B*) one may use the proof below to show that A is unobstructed as an R-algebra in the sense* $\mathrm{H}^2(R, A, A) = 0$, *cf. Remark 5. A further slight reformulation (replacing* Proj *by* Spec *and skip the twists) allows us to prove a corresponding result for non-graded objects as well. Since all three* Ext_B *groups vanish if B is licci ([4], [12], [27] Thm. 4.2.6, or Corollary 27 of this paper), we get the unobstructedness of the Gorenstein quotient A if B is licci. In the interesting codimension $c = 3$ case, the Cohen-Macaulayness of $S_2(K_B)$ and the Koszul homology group,* $\mathrm{H}_1 = \mathrm{H}_1(I_B)$, *are equivalent [27], Thm. 4.2.9. This Cohen-Macaulayness holds in particular if the Cohen-Macaulay type of B is 2 [27], Cor. 4.2.11.*

(d) If I_B is generically syzygietic, then $\mathrm{Hom}_B(\mathrm{H}_2(R, B, B), K_B) = 0$. *Hence applying* $Hom_B(-, K_B)$ *to (4), we see that* ${}_{-s}\mathrm{Ext}^2_B(I_B/I_B^2, K_B) \hookrightarrow {}_{-s}\mathrm{Ext}^1_B(\mathrm{H}_1, K_B)$ *is injective and that*

$$\delta(B)_{-s} = \dim(G_1^* \otimes_R K_B)_{-s} - {}_{-s}\hom_B(\mathrm{H}_1, K_B)$$

allowing us to restate the dimension formula in terms of H_1 *and* G_1. *If B is licci, then* ${}_{-s}\mathrm{Hom}_B(\mathrm{H}_1, K_B) \simeq (\mathrm{H}_{r-1})_{-t-s}$ *by [20], Prop. 18, where r and t is given by* $\wedge^r \widetilde{\mathrm{H}_1}|_U \simeq \widetilde{K}_B(t)|_U$.

Proof. Note that $A \simeq K_A(-s)$ and hence that A is Gorenstein by applying the mapping cone construction to (11), (or see [19], Lemma 5.2 and 5.4). We *claim* that the Proposition above follows from Theorem 4 and Remark 5 by letting $I_{A/B} = K_B(-s)$. Indeed all assumptions of Theorem 4 are straightforward to get from the corresponding assumptions of the Proposition above, except possibly (ii). Suppose temporarily that B is Gorenstein in codimension ≤ 1. Then ${}_0\mathrm{Ext}^1_B(K_B(-s), B) \simeq {}_0\mathrm{Ext}^1_B(K_B \otimes K_B(-s), K_B) \simeq {}_0\mathrm{Ext}^1_B(S_2(K_B)(-s), K_B) = 0$ by some spectral sequence of [11]. Moreover, since K_B is a maximal Cohen-Macaulay B-module, one knows ${}_0\mathrm{Ext}^i_B(K_B, K_B) = 0$ for $i > 0$ and we get ${}_0\mathrm{Ext}^1_B(K_B(-s), A) = 0$ by applying ${}_0\mathrm{Hom}(K_B(-s), -)$ to (11), i.e. we get (ii) of Theorem 4. In the general case where B is generically Gorenstein we get the conclusion of Theorem 4 by combining Lemma 12 and Remark 5(b), recalling $K_B(-s) \simeq I_{A/B}$. Finally to get the dimension formula we note that ${}_0\mathrm{Hom}(K_B(-s), K_B(-s))$ is 1 dimensional. Hence by again applying ${}_0\mathrm{Hom}(K_B(-s), -)$ to (11), we get the dimension formula from the corresponding formula of Theorem 4. □

Lemma 12. *If B is Cohen-Macaulay and $A \simeq B/I_{A/B}$ is Gorenstein of socle degree s and if $\dim B - \dim A = c$, then ${}_0\mathrm{H}^2(B, A, A)$ and ${}_{-s}\mathrm{Ext}^c_B(S_2(I_{A/B}), K_B)$ vanish simultaneously.*

Proof. Using (5) we see the vanishing of ${}_0\mathrm{H}^2(B, A, A)$ is equivalent to ${}_0\mathrm{Ext}^1_A(I_{A/B}/I^2_{A/B}, A) = 0$ *and* the injectivity of the natural map ${}_0\mathrm{Hom}_A(\mathrm{H}_2(B, A, A), A) \hookrightarrow {}_0\mathrm{Ext}^2_A(I_{A/B}/I^2_{A/B}, A)$. Moreover since $A \simeq K_A(-s)$ we have by Gorenstein duality (applied both to A and B) that

$${}_0\mathrm{Ext}^i_A(I_{A/B}/I^2_{A/B}, A) \simeq {}_s\mathrm{H}^{\dim A - i}_{\mathfrak{m}}(I_{A/B}/I^2_{A/B})^\vee \simeq {}_{-s}\mathrm{Ext}^{i+c-1}_B(I^2_{A/B}, K_B).$$

for $i \geq 1$ because the middle term is isomorphic to ${}_s\mathrm{H}^{\dim A+1-i}_{\mathfrak{m}}(I^2_{A/B})$ by the long exact sequence of local cohomology applied to $0 \to I^2_{A/B} \to I_{A/B} \to I_{A/B}/I^2_{A/B} \to 0$. Correspondingly ${}_0\mathrm{Hom}_A(\mathrm{H}_2(B, A, A), A) \simeq {}_{-s}\mathrm{Ext}^c_B(\mathrm{H}_2(B, A, A), K_B)$. Hence ${}_0\mathrm{H}^2(B, A, A)$ is equivalent to ${}_{-s}\mathrm{Ext}^c_B(I^2_{A/B}, K_B) = 0$ *and* the injectivity of of the natural map ${}_{-s}\mathrm{Ext}^c_B(\mathrm{H}_2(B, A, A), K_B) \hookrightarrow {}_{-s}\mathrm{Ext}^{c+1}_B(I^2_{A/B}, K_B)$. Recalling the exact sequence

$$0 \to \mathrm{H}_2(B, A, A) \to S_2(I_{A/B}) \to I^2_{A/B} \to 0 \tag{12}$$

(cf. [1], p.106 or [27], Section 2.1) and that ${}_{-s}\mathrm{Ext}^{c-1}_B(\mathrm{H}_2(B, A, A), K_B) = 0$ since $\mathrm{H}_2(B, A, A)$ is supported at the codimension c quotient A of B, we get the conclusion by the long exact sequence of ${}_{-s}\mathrm{Ext}^i_B(-, K_B)$ applied to (12). □

Theorem 13. *Let $B \simeq R/I_B$ be licci (resp.* $\mathrm{Proj}(B)$ *licci and B Cohen-Macaulay and unobstructed) and let A be a codimension 1 quotient of B, well defined (hence B generically Gorenstein) by a twist of the canonical module K_B. Then A is unobstructed (resp. unobstructed if $s >> 0$) as a graded R-algebra, A is Gorenstein and,*

$$\dim_{(A)} \mathrm{GradAlg}^{H_A} = \dim(N_B)_0 + \dim(K^*_B)_s - 1 - \delta(B)_{-s}$$

$$(\textit{resp.}\ \dim_{(A)} \mathrm{GradAlg}^{H_A} = \dim(N_B)_0 + \dim(K^*_B)_s - 1)$$

where $\delta(B)_{-s} = {}_{-s}\mathrm{hom}_R(I_B, K_B) - {}_{-s}ext^1_B(I_B/I^2_B, K_B)$.

Remark 14. *We may think of Theorem 13 (e.g. for fixed large enough s) as an application from the set of generically smooth irreducible components of* $\mathrm{GradAlg}^{H_B}$ *containing locally licci, Cohen-Macaulay codimension c quotients B of R, to the set of genercally smooth irreducible components of* $\mathrm{GradAlg}^{H_A}$ *containing Gorenstein codimension* $c+1$ *quotients A of R. We expect this map to be* injective*, because the dimension formula seems to show that the "Cohen-Macaulay" components of* $\mathrm{GradAlg}^{H_B}$ *above which have different dimension, yields "Gorenstein" components of* $\mathrm{GradAlg}^{H_A}$ *of different dimension (their* $\dim(N_B)_0$ *are different while their* $\dim(K_B^*)_s$ *usually (always?) coincide, cf. preceding examples).*

Proof. Since B is licci and generically Gorenstein, it is generically syzygietic by e.g. [19], Proposition 6.17 which tells $\mathrm{H}^2(R,B,B)_\wp = 0$ which in turn is equivalent to $\mathrm{H}_2(R,B,B)_\wp = 0$ in the case $\dim B_\wp = 0$ by Gorenstein-duality and the spectral sequence relating algebra-homology to algebra-cohomology. Since all Ext^i_B groups of the ordinary assumptions of Proposition 10 vanish by Remark 11(c), we conclude by Proposition 10.

If $Y = \mathrm{Proj}(B)$ is licci and B unobstructed, the sheaf $\widetilde{S_2(K_B)}$ is Cohen-Macaulay by Remark 11(c). Hence $\mathcal{E}xt^1_{\mathcal{O}_Y}(\widetilde{S_2(K_B)}, \widetilde{K_B}) = 0$ and it follows that ${}_s\mathrm{Ext}^1_B(S_2(K_B), K_B)$ vanishes for large s because of the exact sequence

$$\to\ {}_s\mathrm{Hom}_B(S_2(K_B), \mathrm{H}^1_\mathfrak{m}(K_B)) \to\ {}_s\mathrm{Ext}^1_B(S_2(K_B), K_B) \to \mathrm{H}^1(Y, \mathcal{H}om_B(\widetilde{S_2(K_B)}, \widetilde{K_B})(s))$$

(which is analogously to (3)). Since the condition (iii) of Proposition 10 is obviously true for $s >> 0$, we conclude by the resp. assumptions of the Proposition. $\square$

To apply Proposition 10 and Theorem 13, we need to compute $\dim(K_B^*)_s$ and verify its assumptions. In several cases it is easy to compute $\dim(K_B^*)_s$ at least for large s. For instance if B is 2-dimensional and $C = \mathrm{Proj}(B)$ is *smooth*, then $\widetilde{K}_B^*$ is just the tangent sheaf (sheaf of derivations), θ_C, of C, and we can get the information we need from vanishing of cohomology and Riemann-Roch's theorem. In this case the most difficult assumption to verify is perhaps the unobstructedness of B (or ${}_0\mathrm{Ext}^1_B(I_B/I_B^2, B) = 0$), for which we point out a couple of important cases where the unobstructedness is known;

1. ${}_0\mathrm{Ext}^1_B(I_B/I_B^2, B) = 0$ provided C is linked (in possibly several steps) to a generically Gorenstein curve $C' = \mathrm{Proj}(B')$ satisfying ${}_0\mathrm{Ext}^1_{B'}(I_{B'}/I_{B'}^2, B') = 0$ (cf. next section).

2. B is unobstructed if $C = \mathrm{Proj}(B)$ sits on a smooth rational surface in $\mathbb{P}^4$ and the degree of C is ≥ 24 ([19], Proposition 9.13, see Theorem 9.4 for a generalization).

3. C is determinantal curve with some conditions on the degree matrix ([19], Proposition 10.7 and Corollary 10.11).

By 1) we get ${}_0\mathrm{Ext}^1_B(I_B/I_B^2, B) = 0$ if B is licci (e.g. if C' is a complete intersection), or if C' is a local complete intersection curve and $\mathrm{H}^1(\mathcal{N}_{C'}) = 0$ (because ${}_0\mathrm{Ext}^1_{B'}(I_{B'}/I_{B'}^2, B') \hookrightarrow \mathrm{H}^1(\mathcal{N}_{C'})$ is injective by (3)). Indeed for a local complete intersection curve $C = \mathrm{Proj}(B)$ we have $\mathrm{H}^1(\mathcal{N}_C(v)) \simeq \mathrm{Ext}^1_{\mathcal{O}_C}(\widetilde{I_B/I_B^2}, \widetilde{B}(v))$ and by (3) an exact sequence

$${}_v\mathrm{Ext}^1_B(I_B/I_B^2, B) \hookrightarrow \mathrm{H}^1(\mathcal{N}_C(v)) \to\ {}_v\mathrm{Hom}_B(I_B/I_B^2, \mathrm{H}^2_\mathfrak{m}(B)) \to\ {}_v\mathrm{Ext}^2_B(I_B/I_B^2, B) \to 0\ . \quad (13)$$

We illustrate by taking a relatively simple example, remarking that far more complex examples can be given by the same methods (by increasing s).

Example 15. *Let B be the homogeneous coordinate ring of a smooth rational arithmetically Cohen-Macaulay curve of degree $d = 4$ in $\mathbb{P}^4 = \mathrm{Proj}(R)$, with minimal resolution*

$$0 \to R(-4)^3 \to R(-3)^8 \to R(-2)^6 \to R \to B \to 0,$$

and let A be defined by a regular section of $K_B^*(s)$ for $s \geq 0$. C is the determinantal curve given by the maximal minors of a 2 by 4 matrix with linear entries. Even though B is not licci, Theorem 13 applies for $s >> 0$. Indeed, looking to Proposition 10 and Remark 11(a), we claim that the conclusions of Theorem 13 holds for $s \geq 3$ (and in fact for $s \geq 0$, see the next section). Indeed we only need to verify its assumptions. Since C is smooth and $\theta_C := \widetilde{K}_B^*$ is of degree $\deg \theta_C = -\deg \widetilde{K}_B = 2 - 2g$ (g = the genus of C), we get $\mathrm{H}^1(\theta_C(v)) = 0$ and $h^0(\theta_C(v)) = dv + 3 - 3g = 4v + 3$ for $v \geq 0$ ([25], Lect.11). Since we have an exact sequence

$$0 \to \theta_C \to \theta_{\mathbb{P}}|_C \to \mathcal{N}_C \to 0$$

where $\theta_{\mathbb{P}}|_C = \mathrm{coker}(\mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus 5})$ is the tangent sheaf of $\mathbb{P} = \mathbb{P}^4$ restricted to C, we get

$${}_0\mathrm{Ext}^1_B(I_B/I_B^2, B) \hookrightarrow \mathrm{H}^1(\mathcal{N}_C) = 0.$$

Moreover ${}_v\mathrm{Ext}^1_B(S_2(K_B), K_B)^\vee \simeq {}_{-v}\mathrm{H}^1_{\mathfrak{m}}(K_B)$ and since $\mathrm{H}^0_*(Y, \widetilde{S_2(K_B)}) \to \mathrm{H}^1_{\mathfrak{m}}(K_B)$ is surjective and $\mathrm{H}^0(Y, \widetilde{S_2(K_B)}(-v))^\vee \simeq \mathrm{H}^1(\theta_C(v))$ we get

$${}_v\mathrm{Ext}^1_B(S_2(K_B), K_B) = 0 \qquad \text{for} \quad v \geq 0.$$

Finally by Remark 11(a), we know ${}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, K_B) = 0$ for $s > 2$ and the claim is proved for $s \geq 3$. In conclusion the quotient A is an unobstructed Gorenstein algebra for $s \geq 3$ and

$$\dim_{(A)} \mathrm{GradAlg}^H = h^0(\mathcal{N}_C) + h^0(\theta_C(s)) - 1 = 5d + 1 - g + ds + 3 - 3g - 1 = 4s + 23$$

Since it is straightforward to see that the Hilbert function H of A is given by $\dim(A)_v = 4v+1$ for $0 \leq v \leq s$ and $\dim(A)_v = 4s+2$ for $v > s$, it follows that $\dim_{(A)} \mathrm{GradAlg}^H = 21 + H(s+1)$.

Now we will look to examples of Artinian Gorenstein quotients, and in particular concentrate on Artinian reductions (by modding out with a general linear form L) of the class of examples which we can treat according to the methods illustrated in Example 15. More generally let $B \simeq R/I_B$ be an n-dimensional Cohen-Macaulay algebra, let $A \simeq B/K_B(-s)$, and let $R_L := R/(L) \to B_L = B/(L) \to A_L = A/(L)$ be their quotients modulo (L). Note that an R_L-free minimal resolution of B_L has the same graded Betti numbers as the R-free minimal resolution of B ([22], p.28), i.e. their numbers $n_{j,i}$ given in (6) coincide. Dualizing (6) we see the minimal resolutions of $K_B(n+c)$ and $K_{B_L}(n+c-1)$ also coincide, i.e. $K_B \otimes B_L \simeq K_{B_L}(-1)$. Looking to (11) it follows that A_L is given by

$$0 \to K_{B_L}(-s-1) \to B_L \to A_L \to 0, \tag{14}$$

(note $-s-1$ instead of $-s$!) and A_L is an Artinian Gorenstein quotient, well defined by a twist of the canonical module. Hence we may apply Proposition 10 and Theorem 13. To compute $\dim(K_{B_L}^*)_{s+1}$ and verify its assumptions, we see, thanks to the arguments of Remark 11(a), that $\delta(B_L)_{-s-1} = 0$ and ${}_{-s-1}\mathrm{Ext}^1_{B_L}(I_{B_L}/I_{B_L}^2, K_{B_L}) = 0$ (where $I_{B_L} = \ker(R_L \to B_L)$), provided $s+1 > \max n_{c,i} + \max n_{2,i} - (n+c-1)$ which is equivalent to the inequality of that Remark, used to show $\delta(B)_{-s} = 0$. To check the other assumptions, we claim that

Lemma 16. *(i) If* ${}_{v-1}\mathrm{Ext}^1_B(S_2(K_B),K_B)=0$*, then* $\dim(K^*_{B_L})_{v+1}=\dim(K^*_B)_v-\dim(K^*_B)_{v-1}$.

(ii) If ${}_v\mathrm{Ext}^1_B(S_2(K_B),K_B)={}_{v-1}\mathrm{Ext}^2_B(S_2(K_B),K_B)=0$*, then*

$$ {}_{v+1}\mathrm{Ext}^1_{B_L}(S_2(K_{B_L}),K_{B_L})=0. $$

(iii) If B is an unobstructed graded R-algebra and the map ${}_{-1}\mathrm{Ext}^1_B(I_B/I_B^2,B)\hookrightarrow {}_0\mathrm{Ext}^1_B(I_B/I_B^2,B)$ *induced by L is injective, then B_L is an unobstructed graded R_L-algebra and*

$$ \dim(N_{B_L})_0=\dim(N_B)_0-\dim(N_B)_{-1}. $$

(iv) If ${}_0\mathrm{Ext}^1_B(I_B/I_B^2,B)={}_{-1}\mathrm{Ext}^2_B(I_B/I_B^2,B)=0$*, then* ${}_0\mathrm{Ext}^1_{B_L}(I_{B_L}/I_{B_L}^2,B_L)=0$.

Proof. (i) and (ii). Apply $\mathrm{Hom}_B(S_2(K_B),-)$ to the exact sequence $0\to K_B\to K_B(1)\to K_{B_L}\to 0$ and use that $S_2(K_B)\otimes B_L\simeq S_2(K_{B_L})(-2)$ and that $K^*_B=\mathrm{Hom}_B(S_2(K_B),K_B)$.

(iii) The unobstructedness is just a very particular case of Theorem 4 letting $I_{A/B}\simeq B(-1)$. To compute N_{B_L}, just apply ${}_0\mathrm{Hom}_B(I_B/I_B^2,-)$ to $0\to B(-1)\to B\to B_L\to 0$. Continuing this sequence into a long exact sequence of Ext-groups, we prove (iv) as well because ${}_0\mathrm{Ext}^1_{B_L}(I_{B_L}/I_{B_L}^2,B_L)\simeq {}_0\mathrm{Ext}^1_B(I_B/I_B^2,B_L)$. □

Remark 17. *(a) In the case* $\mathrm{Proj}(B)$ *is a curve, we have by duality* ${}_v\mathrm{Ext}^2_B(S_2(K_B),K_B)^\vee\simeq {}_{-v}\mathrm{H}^0_{\mathfrak{m}}(S_2(K_B))\hookrightarrow S_2(K_B)_{-v}$*. Looking to Remark 11(a), we see there is a surjection* $S_2(\otimes R(n_{c,i}))\twoheadrightarrow S_2(K_B(n+c))$ *showing that*

$$ {}_v\mathrm{Ext}^2_B(S_2(K_B),K_B)=0 \textit{ provided } v>2\max n_{c,i}-2(n+c) $$

(b) If the curve $C=\mathrm{Proj}(B)$ is a local complete intersection, then (13) shows that ${}_{-1}\mathrm{Ext}^2_B(I_B/I_B^2,B)=0$ *provided* $\mathrm{H}^1(\mathcal{O}_C(n_{1,i}-1))\simeq \mathrm{H}^0(\widetilde{K}_B(-n_{1,i}+1))^\vee=0$*, i.e.*

$$ {}_{-1}\mathrm{Ext}^2_B(I_B/I_B^2,B)=0 \textit{ provided } \min n_{1,i}>\max n_{c,i}-n-c+1. $$

Example 18. *We consider the Artinian reduction A_L of the Gorenstein algebra of Example 15. Since B is 2-dimensional we get the vanishing of* ${}_0\mathrm{Ext}^1_{B_L}(I_{B_L}/I_{B_L}^2,B_L)$*, and hence the unobstructedness of B_L, from Lemma 16(iii) and Remark 17(b). Moreover since we have seen that Remark 11(a) also applies to the Artinian reduction, we get* ${}_{-s-1}\mathrm{Ext}^1_{B_L}(I_{B_L}/I_{B_L}^2,K_{B_L})=0$ *for $s>2$. Finally Lemma 16(ii) and Remark 17(a) shows that* ${}_{v+1}\mathrm{Ext}^1_{B_L}(S_2(K_{B_L}),K_{B_L})=0$ *for $v\geq 0$. Hence all "resp.-assumtions" of Proposition 10 concerning B_L are satisfied. It follows that A_L is an unobstructed Artinian Gorenstein algebra for $s\geq 3$, i.e.* $\mathrm{PGor}(H_L)$ *is smooth at A_L by Theorem 8 and*

$$ \dim_{(A_L)}\mathrm{PGor}(H_L)=\dim(N_{B_L})_0+\dim(K^*_{B_L})_{s+1}-1=3d+d-1=15 $$

Finally we claim that this conclusion (about A_L) holds for any Artinian Gorenstein quotient A (well defined by a twist of the canonical module of B) with numerically the same minimal resolution (i.e. the same graded Betti numbers) as A_L. Indeed to use Remark 11(a) we only need the numbers of the minimal resolution, and since we may prove Remark 17 for a 1-dimensional B by replacing $\mathrm{Ext}^2_B(-,-)$ *by the corresponding* $\mathrm{Ext}^1_B(-,-)$ *(in (b) we use* ${}_0\mathrm{Ext}^1_B(I_B/I_B^2,B)\simeq {}_0\mathrm{Ext}^1_B(I_B\otimes_R K_B,K_B)\simeq {}_0\mathrm{H}^0_{\mathfrak{m}}(I_B\otimes_R K_B)^\vee$ *to see that this group vanishes provided* $\min n_{1,i}>\max n_{c,i}-n-c+1$*), we get the claim.*

We end this section by using Theorem 13 to reprove the smoothness and give a dimension formula of $\mathrm{GradAlg}^H$ for (e.g. Artinian) Gorenstein codimension 3 quotients equivalent to those in [17], [18] or [5].

Example 19. *Recall that if $A \simeq R/I_A$ is Gorenstein of codimension 3 with minimal resolution*

$$0 \to R(-f) \simeq F_3 \to F_2 \to F_1 \to I_A \to 0. \tag{15}$$

one knows that A is unobstructed and that $N_A \simeq \wedge^2 I_A(f)$ (char $k \neq 2$). It follows that

$$\dim_{(A)} \mathrm{GradAlg}^{H_A} = \dim(\wedge^2 F_1)_f - \dim(F_1 \otimes F_2)_f + \dim(S_2(F_2))_f \tag{16}$$

cf. [18], Theorem 2.6 and its Remark. In (16) it is not really neccessary to suppose (15) to be minimal because we easily see that (16) is invariant under adding a common free factor to both F_1 and F_2.

If $B \simeq R/I_B$ is Cohen-Macaulay of codimension 2 (and hence licci), and A is given as in Theorem 13, then we claim *the dimension formula of $\dim_{(A)} \mathrm{GradAlg}^{H_A}$ in Theorem 13 coincides with (16). Note that since every irreducible component of the Hilbert scheme of Gorenstein codimension 3 quotients of R contains a quotient constructed via (11) by [7], this example reproves the smoothness of $\mathrm{GradAlg}^H$ and $\mathrm{PGor}(H)$, as well as (16). The argument holds in the case char $k = 2$ as well.*

Firstly we compute $\dim(K_B^*)_s$. Indeed the dual of (7) (for $c = 2$) yields the exact sequence

$$0 \to K_B^*(-n-c) \to G_2 \otimes B \to G_1 \otimes B \to I_B/I_B^2 \to 0 \tag{17}$$

in which H_1 is the kernel of $G_1 \otimes B \to I_B/I_B^2$ by (4). By [2] one knows that

$$0 \to \wedge^{i+1} G_2 \to \wedge^{i+1} G_1 \to \wedge^i G_2 \to \mathrm{H}_i \to 0 \tag{18}$$

is exact for any i. Let $f = s + n + c$. Using the last mentioned sequence for $i = 1$, (17) and the R-free resolution of $G_2 \otimes B$ which we easily deduce from (6), we get

$$\dim(K_B^*)_s = \dim(\wedge^2 G_1)_f - \dim(G_1 \otimes G_2)_f + \dim(S_2(G_2))_f. \tag{19}$$

Next we will compute $\delta = \delta(B)_{-s} = \dim(G_1^* \otimes_R K_B)_{-s} - {}_{-s}\mathrm{hom}_B(\mathrm{H}_1, K_B)$, cf. Remark 11(d). If $r = \operatorname{rank} \mathrm{H}_1$, we have by (18) a resolution of H_{r-1} leading to

$$0 \to G_2^* \to (\wedge^2 G_1)^* \to (\wedge^2 G_2)^* \to \mathrm{H}_{r-1} \otimes (\wedge^{r+1} G_2)^* \simeq \mathrm{Hom}(\mathrm{H}_1, K_B(n+c)) \to 0 \tag{20}$$

We have an R-free resolution of K_B by dualizing (6) and hence a resolution of $G_1^* \otimes K_B$. Combining with (20), we get

$$-\delta = \dim(G_1 \otimes G_2)^*_{-f} - \dim(S_2(G_1)^*)_{-f} - \dim(\wedge^2 G_2)^*_{-f} + \dim(G_1)^*_{-f} - \dim(G_2)^*_{-f}. \tag{21}$$

Note that $\dim(N_B)_0$ is well known, e.g. one may easily deduce it from the exact sequence

$$0 \to G_1^* \otimes G_2 \to ((G_1^* \otimes G_1) \oplus (G_2^* \otimes G_2))/k \to G_2^* \otimes G_1 \to N_B \to 0 \tag{22}$$

of [20], Remark 16 (k is a 1-dimensional subspace). Putting these formulas patiently together, using Theorem 13, we get exactly (16) because we find the following resolution of I_A,

$$0 \to R(-f) \to G_1^*(-f) \oplus G_2 \to G_2^*(-f) \oplus G_1 \to I_A \to 0. \tag{23}$$

by applying the mapping cone construction onto (11).

Inside the scheme $Z(H)$ of Remark 7 we have the open subscheme $ZGor(H)$ consisting of *not neccessarily graded* Gorenstein quotients $R \to A$ with Hilbert function H. $ZGor(H)$ has been studiet in different contexts (see [13], p.126 for references). Since we by Remark 11(c) easily see that ${}_v\mathrm{H}^2(R,A,A)=0$ for $v \geq 0$ provided B is licci, we can use Remark 7 to prove the smoothness and find the dimension of the scheme $ZGor(H)$ at A. Indeed ${}_v\hom_R(I_A,A)$ is found using (9) as in the proof of Theorem 4. And if we apply ${}_v\mathrm{Hom}(K_B(-s),-)$ to (11), we get ${}_v\hom(K_B(-s),A)=\dim(K_B^*)_{s+v}-\dim B_v$. Since $A_v=0$ for $v>s$, we have

Proposition 20. *Let $B \simeq R/I_B$ be licci and generically Gorenstein, and let A be a codimension 1 quotient of B, defined by* (11) *for some s. Then $ZGor(H)$ is smooth at $(R \to A)$ and*

$$\dim_{(A)} Z(H) = \sum_{v\geq 0}^{s} (\dim(N_B)_v - \delta(B)_{-s+v}) + \sum_{v\geq 0}^{s} (\dim(K_B^*)_{s+v} - \dim B_v).$$

Invoking Remark 11(c), we take the opportunity to state the main result of this section in the non graded case. The proof is omited because it is the same as in the graded case.

Theorem 21. *Let $B \simeq R/I_B$ be Cohen-Macaulay, generically Gorenstein and generically syzygietic of dimension ≥ 1 and suppose B satisfies*

$$\mathrm{Ext}^1_B(I_B/I_B^2,B)=0\ ,\quad \mathrm{Ext}^1_B(S_2(K_B),K_B)=0 \quad \textit{and} \quad \mathrm{Ext}^2_B(I_B/I_B^2,K_B)=0.$$

Consider K_B as an ideal of B and let $A := B/K_B$. Then A is Gorenstein and strongly unobstructed in the sense $\mathrm{H}^2(R,A,A)=0$.

In particular if $B \simeq R/I_B$ be licci and generically Gorenstein, then A is Gorenstein and strongly unobstructed in the sense $\mathrm{H}^2(R,A,A)=0$. Indeed licci and generically Gorenstein implies Cohen-Macaulay and generically syzygietic as well as the vanishing of the three Ext-groups above.

3 Computing the dimension via liaison

It is not really difficult to compute $\delta(B)_{-s} = {}_{-s}\hom_R(I_B,K_B) - {}_{-s}ext^1_B(I_B/I_B^2,K_B)$ and $\dim(K_B^*)_s$. For instance it turns out that the difference $\dim(K_B^*)_s-\delta(B)_{-s}$ (which appears in Theorem 13) is quite easy to compute because, after adding a simple term to it, it becomes a liaison invariant among syzygetic ideals I_B. Moreover $\delta(B)_{-s}$ is computable in terms of $\dim(K_{B'}^*)_s$ of the linked algebra B'. To allow any complete intersection linkage of Cohen-Macaulay quotients and still get the liaison invariance described above, we need to define $\delta(B)$ also for Cohen-Macaulay quotients $R \to B$ which satisfy $\mathrm{H}_2(R,B,B) \neq 0$ generically. To do this we consider the exact sequence [27]

$$0 \to \mathrm{H}_2(R,B,B) \to S_2I_B \to I_B \to I_B/{I_B}^2 \to 0. \tag{24}$$

Let $\psi : \mathrm{H}^n_{\mathfrak{m}}(\mathrm{H}_2(R,B,B)) \to \mathrm{H}^n_{\mathfrak{m}}(S_2I_B)$, $\dim B = n$, be the induced map.

Definition 22. *Let $B = R/I_B$ be Cohen-Macaulay of dimension n, not neccessarily (generically) syzygietic. Then*

$$\delta^*(B)_v = {}_v\hom_R(I_B,K_B) - {}_vext^1_B(I_B/I_B^2,K_B) - \dim(\operatorname{im}\psi)_{-v}$$

Remark 23. *If B is generically syzygietic, then ${}_v\mathrm{Hom}_B(\mathrm{H}_2(R,B,B),K_B)=0$, and we get ${}_{-v}\mathrm{H}^n_{\mathfrak{m}}(\mathrm{H}_2(R,B,B))=0$ by duality and hence $\delta^*(B)_v=\delta(B)_v$, i.e. the "old and the new" definition of δ coincides.*

Using (24) and ${}_v\mathrm{Ext}^i_B(I_B/I_B^2,K_B)\simeq {}_{-v}\mathrm{H}^{n-i}_{\mathfrak{m}}(I_B/I_B^2)^\vee$ and letting $h^j_{\mathfrak{m}}(-)_v=\dim({}_v\mathrm{H}^j_{\mathfrak{m}}(-))$, one shows

$$\delta^*(B)_v=h^{n+1}_{\mathfrak{m}}(S_2I_B)_{-v}-h^n_{\mathfrak{m}}(S_2I_B)_{-v}-h^{n+1}_{\mathfrak{m}}(I_B)_{-v} \tag{25}$$

Indeed splitting (24) into two short exact sequences we see that $h^{n-1}_{\mathfrak{m}}(I_B/I_B^2)_{-v}=h^n_{\mathfrak{m}}(I_B^2)_{-v}$ and $h^n_{\mathfrak{m}}(I_B/I_B^2)_{-v}=h^{n+1}_{\mathfrak{m}}(I_B^2)_{-v}-h^{n+1}_{\mathfrak{m}}(I_B)_{-v}$ and moreover $h^{n+1}_{\mathfrak{m}}(I_B^2)_{-v}=h^{n+1}_{\mathfrak{m}}(S_2I_B)_{-v}$ and $h^n_{\mathfrak{m}}(I_B^2)_{-v}=h^n_{\mathfrak{m}}(S_2I_B)_{-v}-\dim(\operatorname{im}\psi)_{-v}$ and we get (25) from Definition 22.

Proposition 24. *Let R be a finitely generated polynomial k-algebra with canonical module $R(e)$ (so $e=-n-c$), and let B and B' be graded Cohen-Macaulay quotients, algebraically linked by a complete intersection $R\twoheadrightarrow D$ of dimension $n\geq 1$, and with minimal resolution*

$$0\to R(-a)\to \ldots\to \oplus_{i=1}^c R(-c_i)\to I_D\to 0.$$

Let $\beta(B,D)_v:=\dim B_v+\sum_{i=1}^c\dim(I_{B/D})_{v+c_i}$. Then $I_{B/D}\simeq K_{B'}(-e-a)$ and

- *(i) $\dim(N_B)_0+\sum_{i=1}^c\dim(I_{B/D})_{c_i}=\dim(N_{B'})_0+\sum_{i=1}^c\dim(I_{B'/D})_{c_i}$*
- *(ii) If B and B' are generically Gorenstein (resp. Gorenstein in codimension ≤ 1), then ${}_v\mathrm{Ext}^i_B(I_B/I_B^2,B)$ and ${}_v\mathrm{Ext}^i_{B'}(I_{B'}/I_{B'}^2,B')$ are isomorphic for $i=1$ (resp. $i=2$).*
- *(iii) $\delta^*(B)_{v-2e-2a}=\beta(B',D)_{v-e-a}-\dim(K^*_{B'})_v$ for any v. Moreover if $\beta(B',D)_{v-e-a}=0$ and I_B is generically syzygietic, then ${}_{v-2e-2a}ext^1_B(I_B/I_B^2,K_B)=\dim(K^*_{B'})_v$.*
- *(iv) $\dim(K^*_B)_v-\delta^*(B)_{-v}-\beta(B,D)_{v-e-a}=$*
  $\dim(K^*_{B'})_{-v+2e+2a}-\delta^*(B')_{v-2e-2a}-\beta(B',D)_{-v+e+a}$.
- *(v) If I_B is generically syzygietic (resp. syzygietic in codimension ≤ 1), then*
  ${}_v\mathrm{Ext}^2_B(I_B/I_B^2,K_B)\hookrightarrow {}_{v+2e+2a}\mathrm{Ext}^1_{B'}(S_2(K_{B'}),K_{B'})$ is injective (resp. an isomorphism).
- *(vi) The conclusion of (ii) holds if B is linked to B' in possibly several steps. Here the algebras of the intermediate steps need not neccessarily be generically Gorenstein.*

Proof. (i) is proved in [19], Proposition 9.17. (ii) Also this part, i.e. the liaison invariance of $I_B/I_B^2\otimes K_B$, is known ([4], or [19], ch.6). Here the Gorenstein assumption is required for having ${}_v\mathrm{Ext}^i_B(I_B/I_B^2,B)\simeq {}_v\mathrm{Ext}^i_B(I_B/I_B^2\otimes K_B,K_B)$. Note that the liaison invariance of $I_B/I_B^2\otimes K_B$ takes care of (vi) as well.

To prove (iii), recall that $K_D\simeq D(e+a)$ and hence, $I_{B/D}\simeq \mathrm{Hom}_D(B,D)\simeq K_{B'}(-e-a)$, i.e. we get exact sequences

$$0\to I_D\cdot I_B\to S_2I_B\to S_2(K_{B'})(-2e-2a)\to 0 \tag{26}$$

$$0\to I_D\cdot I_B\to I_D\to I_D\otimes_R R/I_B\simeq I_D/I_D^2\otimes_D B\simeq \oplus_{i=1}^c B(-c_i)\to 0, \tag{27}$$

cf. [27], Lemma 4.2.7 to see (26). Since $\beta(B',D)_v:=\dim B'_v+\sum_{i=1}^c\dim(I_{B'/D})_{v+c_i}$ and $I_{B'/D}\simeq K_B(-e-a)$, then $\beta(B',D)_v=h^n_{\mathfrak{m}}(K_{B'})_{-v}+\sum_i h^n_{\mathfrak{m}}(B)_{e-v+a-c_i}$ by Gorenstein duality.

Using (25) and (26), we get $\dim(K_{B'}^*)_v = h_{\mathfrak{m}}^n(S_2K_{B'})_{-v} = -\delta^*(B)_{-2e+v-2a} - h_{\mathfrak{m}}^{n+1}(I_B)_{2e-v+2a} + h_{\mathfrak{m}}^{n+1}(I_DI_B)_{2e-v+2a}$ while (27) and $h_{\mathfrak{m}}^n(K_{B'})_{-v+e+a} = h_{\mathfrak{m}}^{n+1}(I_D)_{2e-v+2a} - h_{\mathfrak{m}}^{n+1}(I_B)_{2e-v+2a}$ shows $-h_{\mathfrak{m}}^{n+1}(I_B)_{2e-v+2a} + h_{\mathfrak{m}}^{n+1}(I_DI_B)_{2e-v+2a} = \beta(B',D)_{v-e-a}$. Hence we have proved

$$\dim(K_{B'}^*)_v = -\delta^*(B)_{-2e+v-2a} + \beta(B',D)_{v-e-a} \tag{28}$$

Moreover if $\beta(B',D)_{v-e-a} = 0$, then $\dim B'_{v-e-a} = 0$, i.e. $v-e-a < 0$ and hence $H_{\mathfrak{m}}^{n+1}(I_D)_{2e-v+2a} = 0$ by duality (e.g. if $\dim R > n+1$, then $h_{\mathfrak{m}}^{n+1}(I_D)_{2e-v+2a} = h_{\mathfrak{m}}^n(D)_{2e-v+2a} = h_{\mathfrak{m}}^n(K_D)_{-v+e+a} = 0$ by duality). Since $h_{\mathfrak{m}}^n(K_{B'})_{-v+e+a} = 0$, it follows that $h_{\mathfrak{m}}^{n+1}(I_DI_B)_{2e-v+2a} = \beta(B',D)_{v-e-a} = 0$ which easily leads to the final conclusion of (iii).

Note that (28) also shows $\dim(K_B^*)_v = -\delta^*(B')_{v-2e-2a} + \beta(B,D)_{v-e-a}$. Moreover if we replace v by $-v+2e+2a$ in (28), we get (iv) by combining these formulas.

To prove (v), we use Gorensteinduality. Indeed

$${}_v\mathrm{Ext}_B^2(I_B/I_B^2, K_B) \simeq {}_{-v}\mathrm{H}_{\mathfrak{m}}^{n-2}(I_B/I_B^2)^\vee \simeq {}_{-v}\mathrm{H}_{\mathfrak{m}}^{n-1}(I_B^2)^\vee \hookrightarrow {}_{-v}\mathrm{H}_{\mathfrak{m}}^{n-1}(S_2I_B)^\vee,$$

where the final injection follows from Gorenstein duality and ${}_v\mathrm{Hom}_B(\mathrm{H}_2(R,B,B),K_B) = 0$ because I_B is generically syzygietic. Note that the injection is an isomorphism if I_B is syzygietic in codimension ≤ 1. Now combining (26) and (27) we get

$${}_{-v}\mathrm{H}_{\mathfrak{m}}^{n-1}(S_2I_B)^\vee \simeq {}_{-v}\mathrm{H}_{\mathfrak{m}}^{n-1}(S_2(K_{B'})(-2e-2a))^\vee \simeq {}_{v+2e+2a}\mathrm{Ext}_{B'}^1(S_2(K_{B'}), K_{B'}) \tag{29}$$

and we conclude easily. $\square$

Example 25. *We now complete Example 15 for the remaining cases ($0 \le s \le 2$) by verifying the assumptions of Proposition 10 and computing its invariants, using liaison. Indeed if we link B to a B' via a complete intersection of three hypersurfaces, all of degree 2, the linked curve $C' = \mathrm{Proj}(B')$ has degree $d' = 8-4 = 4$ and genus $g' = 0$, i.e. we see that C and C' are essentially the same curve. Hence ${}_v\mathrm{Ext}_{B'}^1(S_2(K_{B'}), K_{B'}) = 0$ for $v \ge 0$ since B satisfies this vanishing. By Proposition 24 it follows that ${}_{-s}\mathrm{Ext}_B^2(I_B/I_B^2, K_B) \hookrightarrow {}_{-s+2}\mathrm{Ext}_{B'}^1(S_2(K_{B'}), K_{B'}) = 0$ for $s \le 2$, which precisely covers the cases of Example 15 which is not already analyzed. So, for $0 \le s \le 2$, the non-resp. version of Proposition 10 applies! In Example 15 we saw that $\dim(K_B^*)_s = 4s+3$ for any $s \ge 0$. It remains to compute $\delta(B)_{-s}$. Since $\delta(B)_{v-2} = \beta(B',D)_{v-1} - \dim(K_{B'}^*)_v$ and $\beta(B',D)_v = \dim B'_v + 3\dim(I_{B'/D})_{v+2}$ by Proposition 24, we get $(\delta(B)_0, \delta(B)_{-1}, \delta(B)_{-2})$ to be equal to $(15,3,-3)$, and as previously*

$$\dim_{(A)} \mathrm{GradAlg}^H = h^0(\mathcal{N}_C) + h^0(\theta_C(s)) - 1 - \delta(B)_{-s} = 4s + 23 - \delta(B)_{-s}$$

Note that it is easy, for $s = 2$, to see that ${}_{-s}\mathrm{Hom}(I_B/I_B^2, K_B) = 0$ and hence ${}_{-2}ext_B^1(I_B/I_B^2, K_B) = -\delta(B)_{-2} = 3$ by looking to the minimal resolution of I_B (or use Proposition 24(iii)). By the interpretation of ${}_{-s}ext_B^1(I_B/I_B^2, K_B)$ of Proposition 10 we get that the codimension of the strata of Gorenstein quotients given by (11) is 3 in this case.

Example 26. *Now we consider the Artinian reduction A_L of the Example 18 of socle degree $s+1$, to treat the remaining cases of interest ($0 \le s \le 2$). Since $s \le 2$ and I_B has no linear generator, ${}_{s+1}\mathrm{H}_2(R,B,B) \hookrightarrow (S_2I_B)_{s+1} = 0$. Hence in using Proposition 10, Remark 17(b) shows that no syzygietic assumptions on I_B is neccessary to assume and that Proposition 10(iii) always holds. Looking to Example 18 we see that all assumptions of the* *"non-resp."version of Proposition 10 are satisfied. Now examining* $\mathrm{H}^i(\theta_C(v))$ *of Example 15 for* $v=-1$ *a little further and invoking* $(S_2K_B)_1=0$ *we get* ${}_{-1}ext^1_B(S_2(K_B),K_B)=1$ *in addition to* ${}_v\mathrm{Ext}^1_B(S_2(K_B),K_B)=0$ *for* $v\geq 0$. *By Lemma 16(i) we get* $\dim(K_B^*)_v=4$ *for* $v\geq 1$. *Moreover if we apply* $\mathrm{Hom}_B(I_B/I_B^2,-)$ *to the exact sequence* $0\to K_B\to K_B(1)\to K_{B_L}\to 0$, *we get*

$$\delta(B_L)_{v-1}=\delta(B)_v-\delta(B)_{v-1}\ +\ {}_vext^2_B(I_B/I_B^2,K_B)\ -\ {}_{v-1}ext^2_B(I_B/I_B^2,K_B).$$

Since ${}_{-3}ext^2_B(I_B/I_B^2,K_B)=\ {}_{-1}ext^1_{B'}(S_2(K_{B'}),K_{B'})=1$ *and* ${}_{-s}\mathrm{Ext}^2_B(I_B/I_B^2,K_B)=0$ *for* $s\leq 2$, *we get that* $(\delta(B_L)_{-1},\delta(B_L)_{-2},\delta(B_L)_{-3})$ *is equal to* $(12,6,-4)$ *from the values of* $\delta(B)_v$ *in Example 25. It follows that* A_L *is an unobstructed Artinian Gorenstein algebra for* $0\leq s\leq 2$, *i.e.* $\mathrm{PGor}(H_L)$ *is smooth at* A_L *by Theorem 8 and*

$$\dim_{(A_L)}\mathrm{PGor}(H_L)=\dim(N_{B_L})_0+\dim(K^*_{B_L})_{s+1}-1-\delta(B_L)_{-s-1}=15-\delta(B_L)_{-s-1}\ .$$

Moreover, for $s=2$, *we see that* $-\delta(B_L)_{-3}=\ {}_{-3}ext^1_{B_L}(I_{B_L}/I^2_{B_L},K_{B_L})=4$ *is the codimension of the strata of Gorenstein quotients given by* (14).

Finally we remark that the conclusions (about A_L*) also hold for any Artinian Gorenstein quotient* A *(well defined by a twist of the canonical module of* B*) with the same graded Betti numbers as* A_L *(cf. Example 18). In this case we can compute the numbers* $\delta(B)_v$ *of a* 1*-dimensional* B *as we did in Example 25 by essentailly the same linkage argument.*

Mainly using Proposition 24 twice (for (i) below, we utilize (29)), we get

Corollary 27. *If* B *and* B' *are Cohen-Macaulay quotients of* R, *algebraically linked by a complete intersection* D *of type* $(c_1,c_2,..,c_c)$, *and* B' *and* B'' *are algebraically linked by a complete intersection* D' *of type* $(c'_1,c'_2,..,c'_c)$, *and if* $a=\sum c_i$, $a'=\sum c'_i$ *and* $\dim B>0$, *then*

- *(i)* ${}_{v+2a'}\mathrm{Ext}^1_{B''}(S_2(K_{B''}),K_{B''})\simeq\ {}_{v+2a}\mathrm{Ext}^1_B(S_2(K_B),K_B)$ *for any* v.
- *(ii)* ${}_{v-2a'}\mathrm{Ext}^2_{B''}(I_{B''}/I^2_{B''},K_{B''})\simeq\ {}_{v-2a}\mathrm{Ext}^2_B(I_B/I_B^2,K_B)$ *provided* I_B *and* $I_{B''}$ *are syzygietic in codimension* ≤ 1.
- *(iii)* $\delta^*(B'')_{v-2e-2a'}=\delta^*(B)_{v-2e-2a}-\beta(B',D)_{v-e-a}+\beta(B',D')_{v-e-a'}$.
- *(iv)* $\dim(K^*_{B''})_{v+2e+2a'}=\dim(K^*_B)_{v+2e+2a}-\beta(B',D)_{v+e+a}+\beta(B'',D')_{v+e+a'}$ *or equivalently,* $\dim(K^*_{B''})_{v+2e+2a'}=\dim(K^*_B)_{v+2e+2a}+\dim B''_{v+e+a'}-\dim B_{v+e+a}+\sum_{i=1}^c(\dim(K_{B'})_{v+c'_i}-\dim(K_{B'})_{v+c_i})$.

We include one example of biliaison starting with the smooth rational curve of Example 15. Of course starting with a licci curve, the verification of the assumptions of Proposition 10 is immediate (Theorem 13). Some part of the argument below is, however, still needed to find $\dim(K^*_B)_s-\delta^*(B)_{-s}$ and to see how small we can let the number s in Teorem 13 be.

Example 28. *If we link the homogeneous coordinate ring* B *of the smooth rational quartic curve of Example 15 with minimal resolution*

$$0\to R(-4)^3\to R(-3)^8\to R(-2)^6\to R\to B\to 0,$$

to a B' using a complete intersection of type $(2,2,3)$ and further link B' to a B'' via a complete intersection of type $(2,3,4)$ (using the same hypersurface of degree 2 and 3 as previously) one may (notably due to $I_{B'/D} \simeq K_B(-2)$ and the mapping cone construction, used twice, or see [26]) see that the minimal resolution of B'' is

$$0 \to R(-7) \oplus R(-6)^3 \to R(-5)^{10} \to R(-4)^5 \oplus R(-3) \oplus R(-2) \to R \to B'' \to 0.$$

If A is defined by a regular section of $K^*_{B''}(s)$, we can easily find the minimal resolution of A and its Hilbert function by using (11). *For instance* $\dim(A)_v = 16v - 19$ *for* $3 \le v \le s-3$ *and* $\dim(A)_v = 16s - 38$ *for* $v > s$. *Let us suppose* $s > 3$ *since otherwise* $\operatorname{Proj}(A)$ *is degenerate. To use Proposition 10, we may argue as in Example 15 to see that* $\delta(B'')_{-s} = 0$ *for* $s > 7$ *by Remark 11(a), and that* $\mathrm{H}^1(\theta_{C''}(v)) = 0$ *and hence* $\dim(K^*_{B''})_v = h^0(\theta_{C''}(v)) = 16v - 57$ *for* $v \ge 5$ *because* $C'' := \operatorname{Proj}(B'')$ *is smooth curve of degree* $d'' = 16$ *and genus* $g'' = 20$. *Corollary 27 leads, however, to more precise information. Indeed it is rather straightforward to combine the Corollary and the computations of Example 15 and Example 25 to see that the conclusions of Theorem 13 holds for* $s \ge 4$, *that* $\dim(K^*_{B''})_4 = 8$, *and that* $(\delta(B'')_{-4}, \delta(B'')_{-5}, \delta(B'')_{-6})$ *is equal to* $(7, -6, -6)$ *and* $\delta(B'')_{-s} = 0$ *for* $s > 6$. *Finally patiently using Proposition 24(i) several times, one shows* $\dim(N_{B''})_0 = 62$ *(It is, however, faster to show* ${}_0\mathrm{Ext}^i_{B''}(I_{B''}/I^2_{B''}, B'') = 0$ *for* $i = 1$ *and* 2 *by Proposition 24(ii), using the computations of Example 15 and* (13), *and then use* (13) *once more to see* $h^1(\mathcal{N}_{C''}) = 1$ *and hence* $\dim(N_{B''})_0 = 5d'' + 1 - g'' + h^1(\mathcal{N}_{C''}) = 62$*). Putting this together we have by Proposition 10 that A is an unobstructed Gorenstein algebra for* $s \ge 4$ *and*

$$\dim_{(A)} \operatorname{GradAlg}^H = \dim(N_{B''})_0 + \dim(K^*_{B''})_s - 1 - \delta(B'')_{-s} = 16s + 4 \quad \text{for} \quad s \ge 7,$$

$\dim_{(A)} \operatorname{GradAlg}^H = 16s + 10$ *for* $5 \le s \le 6$ *and* $\dim_{(A)} \operatorname{GradAlg}^H = 62$ *for* $s = 4$. *By the interpretation of* ${}_{-s}ext^1_{B''}(I_{B''}/I^2_{B''}, K_{B''})$ *of Proposition 10, we get by e.g. Proposition 24(iii) that the codimension of the strata of Gorenstein quotients given by* (11) *is* 6 *for* $5 \le s \le 6$.

We may treat the Artinian reduction of $A \simeq B''/K_{B''}(-s)$ in Example 28 satisfactorily by remarking that ${}_{s+1}\mathrm{H}_2(R, B'', B'') = 0$ for $4 \le s \le 6$ (since one may show $(S_2 I_{B''})_{s+1} \simeq (I^2_{B''})_{s+1}$ by using that the generators of $I_{B''}$ of degree 2 and 3 form a regular sequence), and that B''_L is unobstructed by Lemma 16(*iii*) for any $s > 3$. For the latter, we use the liaison invariance of ${}_v\mathrm{Ext}^1_B(I_B/I^2_B, B)$ and the fact that $\mathrm{H}^1(\mathcal{N}_C(v)) = 0$ for $v \ge -1$ for a smooth rational curve C. Instead of working out these details, we will finish this subsection by considering two general classes of examples where B is licci. We concentrate on licci codimension 3 quotients, leading to unobstructed Gorenstein codimension 4 quotients by Theorem 13. In one of the cases the use of Proposition 24 is quite effective.

Example 29. *Let B be a Gorenstein quotient of R of codimension 3 given by*

$$0 \to R(-f) \simeq F_3 \to F_2 \to F_1 \to I_B \to 0. \tag{30}$$

(as A in Example 19). In this case $e = -n - 3$ *and* $K_B \simeq B(-f + n + 3)$ *and since B is licci ([27]) and therefore satisfies* ${}_v\mathrm{Ext}^1_B(I_B/I^2_B, B) = 0$ *for any v by Proposition 24, we get* $\delta(B)_{-s} = \dim(N_B)_{-(f+s-n-3)}$. *If A is defined by a section of $K^*_B(s)$ as in Theorem 13, then A is Gorenstein and unobstructed and*

$$\dim_{(A)} \operatorname{GradAlg}^{H_A} = \dim(N_B)_0 + \dim(B)_{f+s-n-3} - 1 - \dim(N_B)_{-f-s+n+3}$$

where $\dim(N_B)_v$ *is given by [18], Theorem 2.6, i.e. by modifying* (16) *to*

$$\dim(N_B)_v = \dim(\wedge^2 F_1)_{f+v} - \dim(F_1 \otimes F_2)_{f+v} + \dim(S_2(F_2))_{f+v} + \dim(F_1)_v - \dim(F_2)_v$$

Example 30. *Let* B *be an almost complete intersection of* R *of codimension 3 with a minimal resolution*

$$0 \to G_3 \to G_2 \to G_1 = \oplus_{i=1}^{3} R(-c_i) \oplus R(-h) \to I_B \to 0 \tag{31}$$

and let B' *be the Gorenstein quotient obtained by linking* B *to* B' *via a complete intersection whose minimal generators (of degree* c_i*) corresponds to three of those of* I_B*. A standard mapping cone construction and duality yields*

$$0 \to R(h-a) \to G_2^*(-a) \to G_3^*(-a) \oplus (\oplus_{i=1}^{3} R(-c_i)) \to I_{B'} \to 0 \tag{32}$$

and we see that $K_{B'} \simeq B'(a-h-n-3)$ *where* $a = \sum c_i$*. If* A *is defined by the zero locus of a section of* $K_B^*(s)$ *as in Theorem 13, then* A *is Gorenstein and unobstructed and invoking Proposition 24 and* $\dim(K_{B'}^*)_{-s+2a-2n-6} - \delta(B')_{s-2a+2n+6} = \dim(B')_{(-s+h+a-n-3)} - \dim(N_{B'})_{(s-h-a+n+3)}$ *we get*

$$\dim_{(A)} \operatorname{GradAlg}^{H_A} = \dim(N_{B'})_0 + \dim(B')_{(-s+h+a-n-3)} - \dim(N_{B'})_{(s-h-a+n+3)} - 1 + \gamma$$

where $\gamma := \sum_{i=1}^{3} (\dim(I_{B'/D})_{c_i} - \dim(I_{B/D})_{c_i} + \dim(I_{B/D})_{s-a+n+3+c_i} - \dim(I_{B'/D})_{-s+a-n-3+c_i}) + \dim B_{s-a+n+3} - \dim B'_{-s-n-3+a}$*. Using* $I_{B/D} \simeq K_{B'}(n+3-a) \simeq B'(-h)$*, we easily express* γ *in terms of the Hilbert function of* B' *(or of* B*) and* D*. Moreover* $\dim(N_{B'})_v$ *is given by the final formula of Example 29.*

References

[1] M. André. Homologie des algèbres commutatives. Die Grundlehren der mathematischen Wissenschaften, Band 206. Springer-Verlag, Berlin-New York, 1974.

[2] L. Avramov and J. Herzog. The Koszul algebra of a codimension 2 embedding. *Math. Z.* 175 (1980), 249–260.

[3] Boij, M. Components of the space parametrizing graded Gorenstein Artin algebras with a given Hilbert function. *Pacific J. of Math.* 187 no. 1 (1999), 1–11.

[4] R. Buchweitz and B. Ulrich. Homological properties which are invariant under linkage. *Preprint* 1983.

[5] A. Conca and G. Valla. Hilbert function of powers of ideals of low codimension. *Math. Z.* 230 no. 4 (1999) 753–784.

[6] D. Eisenbud. Commutative algebra. With a view toward algebraic geometry. *Graduate Texts in Math.*, volume 150, Springer–Verlag, New York, 1995.

[7] A.Geramita and J. Migliore A construction of codimension three arithmetically Gorenstein subschemes of projective space. *Trans. Amer. Math. Soc.* 349 no. 9 (1997), 3803–3821.

[8] A. Grothendieck. Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (*SGA* 2). Augmenté d'un exposé par Michèle Raynaud. Séminaire de Géométrie Algébrique du Bois-Marie, 1962. *Advanced Studies in Pure Mathematics* Vol. 2. North-Holland Publishing Co., Amsterdam; Masson & Cie, Éditeur, Paris, 1968. vii+287 pp.

[9] A. Grothendieck, M. Raynaud and D.S. Rim. Groupes de Monodromie en Géométrie Algébrique (*SGA* 7). *Lecture Notes in Math.* Vol. 288 (1972) and Vol. 340 (1973) Springer–Verlag.

[10] R. Hartshorne. Local Cohomology. *Lectures Notes in Math.* Vol. 41 Springer–Verlag, New York, 1967.

[11] J. Herzog. Deformationen von Cohen-Macaulay Algebren. *J. reine angew. Math.* 318 (1980), 83–105.

[12] C. Huneke. Linkage and the Koszul homology of ideals. *Amer. J. Math.* 104 no. 5 (1982), 1043-1062.

[13] A. Iarrobino, V. Kanev. Power sums, Gorenstein Algebras and Determinantal Loci. *Lectures Notes in Math.* Vol. 1721 Springer–Verlag, , New York, 1999.

[14] A. Iarrobino, H. Srinivasan. Some Gorenstein Atrin algebras of embedding dimension four,I: components of $\mathrm{PGor}(H)$ for $H = (1, 4, 7, ..., 1)$. Preprint 2002

[15] J. O. Kleppe. Deformations of graded algebras. *Math. Scand.* 45 (1979), 205–231.

[16] J. O. Kleppe. Liaison of families of subschemes in $\mathbb{P}^n$, in *"Algebraic Curves and Projective Geometry, Proceedings (Trento, 1988)," Lectures Notes in Math.* Vol. 1389 Springer–Verlag, New York, 1989.

[17] J. O. Kleppe. The smoothness and the dimension of $\mathrm{PGor}(H)$ and of other strata of the punctual Hilbert scheme. *J. Algebra*, 200 no. 2 (1998), 606–628.

[18] J. O. Kleppe and R. Miro-Roig. The dimension of the Hilbert scheme of Gorenstein codimension 3 subschemes. *J. Pure Appl. Algebra*, 127 no. 1 (1998), 73–82.

[19] J.O. Kleppe, J. Migliore, R. Miro-Roig, U. Nagel and C. Peterson. Gorenstein Liaison, Complete Intersection Liaison Invariants and Unobstructedness. *Mem. Amer. Math. Soc.* Vol. 154, no 732, (2001)

[20] J. O. Kleppe and C. Peterson. Gorenstein algebras and maximal Cohen-Macaulay modules. *J. Algebra*, 238, (2001), 776-800.

[21] A. Laudal. Formal Moduli of Algebraic Structures. *Lectures Notes in Math.*, Vol. 754, Springer–Verlag, New York, 1979.

[22] J. Migliore. Introduction to liaison theory and deficiency modules. *Progress in Math.*, Vol. 165, Birkhäuser Boston, Inc., Boston, MA, 1998.

[23] M. Martin-Deschamps and D. Perrin. Sur la classification des courbes gauches, I. *Astérisque*, 184–185 (1990).

[24] J. Migliore and C. Peterson. A construction of codimension three arithmetically Gorenstein subschemes of projective space. *Trans. Amer. Math. Soc* 349 no. 9 (1997), 3803–3821.

[25] D. Mumford. Lectures on Curves on an Algebraic Surface. *Annals of Math. Studies*, Vol. 59, Princeton Univ. Press, 1966.

[26] C. Peskine and L. Szpiro. Liaison des variétés algébriques. I. *Invent. Math.* 26 (1974), 271–302.

[27] W. Vasconcelos. Arithmetic of Blowing up Algebras. *London Mathematical Society. Lecture Note Series*, Vol. 195, Cambridge University Press, 1994.

[28] B. Ulrich. Sums of linked ideals. *Trans. Amer. Math. Soc.* 318 (1990), 1–42.

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