Components of the Hilbert scheme of space curves on low-degree smooth surfaces

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Abstract

We study maximal families W of the Hilbert scheme, $\mathrm{H}(d,g)_{sc}$, of smooth connected space curves whose general curve C lies on a smooth surface S of degree s. We give conditions on C under which W is a generically smooth component of $\mathrm{H}(d,g)_{sc}$ and we determine $\dim W$. If $s = 4$ and W is an irreducible component of $\mathrm{H}(d,g)_{sc}$, then the Picard number of S is at most 2 and we explicitly describe, also for $s \geq 5$, non-reduced and generically smooth components in the case $\mathrm{Pic}(S)$ is generated by the classes of a line and a smooth plane curve of degree $s-1$. For curves on smooth cubic surfaces the first author finds new classes of non-reduced components of $\mathrm{H}(d,g)_{sc}$, thus making progress in proving a conjecture for such families.

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1 Introduction and Main Results

In this paper we study the Hilbert scheme of smooth connected space curves, $\mathrm{H}(d,g)_{sc}$, with regard to dimension and smoothness, with a special emphasis on existence of non-reduced components. The first example of a non-reduced component was found by Mumford [34]. There are several papers that consider such problems, see e.g. [2], [4,5,7], [15,16], [20–23,25,26], [29–31,36] and the book [17]. Here we generalize the approach that was used in [20] for curves on cubic surfaces to study families of curves on smooth surfaces of degree $s \geq 4$. In particular we investigate when maximal irreducible closed subsets W of the Hilbert scheme $\mathrm{H}(d,g)_{sc}$ whose general curve C lies on a *quartic* surface, form non-reduced, or generically smooth, irreducible components of $\mathrm{H}(d,g)_{sc}$. We find a pattern similar to what is known for maximal irreducible families of curves on smooth cubic surfaces; if $H^1(\mathcal{I}_C(s)) = 0$, $\mathcal{I}_C$ the sheaf ideal of C, then W turns out to be a generically smooth component of $\mathrm{H}(d,g)_{sc}$. If, however, $H^1(\mathcal{I}_C(s)) \neq 0$ *and* the genus is sufficiently large, then W is still an irreducible component, but it is now non-reduced. For $s = 4$ it suffices to take "g large" as $g > G(d,5)$, the maximum genus of curves of degree d not contained in a degree-4 surface (see (13)), or as the better bound

$$g \; > \; \min\{\, G(d,5) - 1, \, \frac{d^2}{10} + 21\} \quad \text{and} \quad d \geq 21\,, \tag{1}$$

see Theorem 4.1 and Corollary 4.5 of Section 4.

Let $s(C)$ denote the minimal degree of a surface containing a curve C. If W is an irreducible closed subset of $\mathrm{H}(d,g)_{sc}$, we define $s(W) := s(C)$ where C is a general curve of W. As in [20] we say W is $s(W)$-*maximal* if it is maximal with respect to $s(W)$, i.e. $s(V) > s(W)$ for any closed irreducible subset V properly containing W. We say W is an $s(W)$-maximal family or subset of $\mathrm{H}(d,g)_{sc}$ in this case. By Remark 2.3 below, if a very general curve of a 4-maximal family W sits on a smooth quartic surface S and $d > 16$, then the Picard number of S is *at most* 2.

Note that an s-maximal family W needs not be an irreducible component of $\mathrm{H}(d,g)_{sc}$, but the converse holds (with $s = s(W)$). For instance if W is a 2-maximal family whose general curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ has bidegree (p,q) , $p \leq q$, and degree $d = p+q \geq 6$, then W is not an irreducible component precisely when $p \leq 2$, e.g. if $p = 1$ then $g = (p-1)(q-1) = 0$ and the codimension of W in $\mathrm{H}(d,g)_{sc}$ is $4d - (2d + g + 8) = 2d - 8$, (cf. [40]). Indeed, by Remark 2.1, $g \geq 2d - 8$ is a necessary condition for W to be an irreducible component. This condition is sufficient by (6) below.

For a 3-maximal family W, $g \geq g_1 := 3d - 18$ is necessary (for $d > 9$) while [20, Cor. 17] shows that $g > g_2 := \lfloor (d^2 - 4)/8 \rfloor$ is sufficient for W to be an irreducible component. Indeed $g > g_2$ implies (6) by [20, Cor. 17]. Since the sufficient condition also implies that W is generically smooth, $W \subset \mathrm{H}(d,g)_{sc}$ may be a non-reduced component only when $g_1 \leq g \leq g_2$. For $g_1 = g_2$ and $d > 10$ we have $g = 24, d = 14$ and the existence of a non-reduced component W with $s(W) = 3$ as shown by Mumford in [34]. The first author generalized this result in [21] and showed the existence of 3-maximal families of non-reduced components of $\mathrm{H}(d, \lfloor (d^2-4)/8 \rfloor)_{sc}$ for every $d \geq 14$, where $d = 14$ corresponds to Mumford's example, see [4], [20] and the appendix for further generalizations and a conjecture.

In this paper we consider closely 4-maximal families W of curves on smooth quartic surfaces $S \subset \mathbb{P}^3$. To get interesting classes, we study surfaces S where the Picard group $\mathrm{Pic}(S)$ is freely generated over $\mathbb{Z}$ by the classes of two smooth connected curves Γ_1 and Γ_2 satisfying $\Gamma_1^2 = -2$, $\Gamma_2^2 = 0$, $\Gamma_1 \cdot \Gamma_2 = 3$, i.e. with intersection matrix $\left(\begin{smallmatrix} -2 & 3 \\ 3 & 0 \end{smallmatrix}\right)$, and such that $H = \Gamma_1 + \Gamma_2$ is a hyperplane section. If $C \equiv a\Gamma_1 + b\Gamma_2$ are linearly equivalent divisors, we show that $a, b \geq 0$ if C is a curve (i.e. effective divisor). The necessary condition $g \geq 4d - 33$ of Remark 2.1 for W to be an irreducible component implies $a > 4$ for $d > 16$ because $g = ad - 2a^2 + 1$. In Section 5 we prove:

Theorem 1.1. *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with Γ_1, Γ_2 and H as above, let $C \equiv a\Gamma_1 + b\Gamma_2$ be a smooth connected curve of degree $d > 16$ and suppose $a \neq b$. Then C belongs to a unique 4-maximal family $W \subseteq \mathrm{H}(d,g)_{sc}$. Moreover if $\tilde{S}$ is a quartic surface containing a very general member of W, then $\mathrm{Pic}(\tilde{S})$ is freely generated by the classes of a line and a smooth plane cubic curve, and every $C \equiv a\Gamma_1 + b\Gamma_2$ contained in some surface S as above belongs to W. Furthermore* $\dim W = g + 33$,

$$d = a + 3b\ , \quad g = 3ab - a^2 + 1 \quad \textit{and}$$

I) *W is a generically smooth, irreducible component of* $\mathrm{H}(d,g)_{sc}$ *provided*

$$4 < a < \frac{3b}{2} - 1 \qquad \text{or} \qquad (a,b) = (5,4).$$

II) *W is a non-reduced irreducible component of* $\mathrm{H}(d,g)_{sc}$ *provided*

$$\frac{3b}{2} - 1 \leq a \leq \frac{3b}{2} \quad , \qquad (a,b) \neq (5,4) \tag{2}$$

and (1) *holds. Explicitly, this region is given by the three families*

a) $(8+3k, 6+2k)$ *b*) $(10+3k, 7+2k)$ *c*) $(15+3k, 10+2k)$ *for* $k \geq 0$,

and the dimension of their tangent spaces of $\mathrm{H}(d,g)_{sc}$ *at* (C) *is*

$$\dim W + h^1(\mathcal{I}_C(4))$$

where $h^1(\mathcal{I}_C(4)) = 1$ *(resp.* $h^1(\mathcal{I}_C(4)) = 2$, $h^1(\mathcal{I}_C(4)) = 4$ *) for the family a) (resp. b), c)).*

One may show that W is non-empty, i.e. that there exist smooth connected curves $C \equiv a\Gamma_1 + b\Gamma_2$ if and only if $0 < a \leq \frac{3b}{2}$, or $(a,b) \in \{(1,0),(0,1)\}$. The case $a = b$ corresponds to C being a complete intersection of S with some other surface (a c.i. in S).

We also consider curves sitting on smooth surfaces containing a line and corresponding s-maximal families W for every $s \geq 5$, and we get similar results as in I) above while we in II) only prove that W is a component (i.e the non-reducedness is open), cf. Theorem 7.3. For $s = 5$ we get a little more:

Theorem 1.2. *Let $S \subset \mathbb{P}^3$ be a smooth quintic surface containing a line Γ_1, let $\Gamma_2 \equiv H - \Gamma_1$, H a hyperplane section, be a smooth quartic curve and suppose* $\mathrm{Pic}(S) \simeq \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$. *Let $C \equiv a\Gamma_1 + b\Gamma_2$ be a smooth connected curve of degree $d > 25$ with $a \neq b$ and $a, b > 1$. Then C belongs to a unique 5-maximal family $W \subseteq \mathrm{H}(d,g)_{sc}$. Moreover if $\tilde{S}$ is a quintic surface containing a very general member of W, then $\mathrm{Pic}(\tilde{S})$ is freely generated by the classes of a line and a smooth plane quartic curve, and every $C \equiv a\Gamma_1 + b\Gamma_2$ contained in some surface S as above belongs to W. Furthermore* $\dim W = -d + g + 56$, *where*

$$d = a + 4b\ , \quad g = 4ab + \frac{1}{2}(a + 4b - 3a^2) + 1 \quad \textit{and}$$

I) *W is a generically smooth, irreducible component of $\mathrm{H}(d,g)_{sc}$ provided $5 < a < \frac{4b}{3} - 1$.*

II) *W is a non-reduced irreducible component of $\mathrm{H}(d,g)_{sc}$ for $(a,b) = (4n, 3n)$, $n \geq 3$.*

To get II) we need a natural map $H^0(\mathcal{N}_C) \to H^1(\mathcal{I}_C(s))$ to be non-zero, cf. (4) below. Since this map is surjective for S smooth of degree $s = 4$, $H^1(\mathcal{I}_C(s)) \neq 0$ suffices to get II) in Theorem 1.1.

Note that we have $0 < a \leq \frac{4b}{3}$ for C as above and that quintic surfaces as in Theorem 1.2 exist.

Another main result (Theorem 3.1), related to I) above, is obtained by studying certain maps that involve the relative Picard scheme. Specializing to s-maximal families satisfying $d > s^2$, we get:

Theorem 1.3. *Let $s \geq 1$ be an integer and let $W \subseteq \mathrm{H}(d,g)_{sc}$ be an s-maximal family such that $d > s^2$. Let C be a member of W sitting on a smooth surface S of degree s satisfying*

$$H^1(\mathcal{I}_C(s)) = H^1(\mathcal{I}_C(s-4)) = 0.$$

Let E be a curve on S, H a hyperplane section and suppose $C \equiv eE + fH$ for some $e \neq 0, f \in \mathbb{Z}$. Let t be the non-negative integer $t := h^1(\mathcal{N}_E) - h^1(\mathcal{O}_E(s))$ where $\mathcal{N}_E$ is the normal sheaf of $E \subset \mathbb{P}^3$. If E is either arithmetically Cohen-Macaulay, or *$t = 0$ and $H^1(\mathcal{I}_E(s)) = H^1(\mathcal{I}_E(s-4)) = 0$, then W is a generically smooth irreducible component of $\mathrm{H}(d,g)_{sc}$ (indeed C and $eE + fH$ are unobstructed), and*

$$\dim W = (4 - s)d + g + \binom{s+3}{3} - 2 + h^0(\mathcal{I}_E(s-4)) + t\,.$$

As a consequence we get a non-trivial formula for $h^1(\mathcal{N}_C)$. Note that this theorem shows the unobstructedness of a curve with a multiplicity-e structure on S under some assumptions. The components in Theorems 1.1, 1.2 and 7.3 correspond to $h^0(\mathcal{I}_E(s-4))$ large in Theorem 1.3, cf. Remark 3.4.

Finally the first author consider in the appendix a conjecture about non-reduced components for maximal families $W \subseteq \mathrm{H}(d,g)_{sc}$ of linearly normal curves on a smooth cubic surface S [20, Conj. 4]. In Theorem 8.3 he extends the known range where the conjecture holds. We thank O. A. Laudal for interesting discussions on that subject. We also thank D. Eklund for a discussion of K3 surfaces and R. Hartshorne for his comments. Also thanks to the careful referee for very helpful comments.

1.1 Notations and terminology

In this paper the ground field k is *algebraically closed of characteristic zero* (and equal to the complex numbers in the statements where the concept "very general" is used). A surface S in $\mathbb{P}^3$ is a hypersurface, defined by a single equation. A curve C in $\mathbb{P}^3$ (resp. in S) is a *pure one-dimensional* subscheme of $\mathbb{P} := \mathbb{P}^3$ (resp. S) with ideal sheaf $\mathcal{I}_C$ (resp. $\mathcal{I}_{C/S}$) and normal sheaf $\mathcal{N}_C = \mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_C, \mathcal{O}_C)$ (resp. $\mathcal{N}_{C/S} = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{I}_{C/S}, \mathcal{O}_C)$). We denote by $d = d(C)$ (resp. $g = g(C)$) the degree (resp. arithmetic genus) of C. If $\mathcal{F}$ is a coherent $\mathcal{O}_\mathbb{P}$-Module, we let $H^i(\mathcal{F}) = H^i(\mathbb{P}, \mathcal{F})$,

$h^i(\mathcal{F}) = \dim H^i(\mathcal{F})$, $\chi(\mathcal{F}) = \Sigma(-1)^i h^i(\mathcal{F})$ and we often write $H^i(S, \mathcal{O}_S(C))$ as $H^i(\mathcal{O}_S(C))$ for a Cartier divisor C on S. Let

$$s(C) = \min\{\, n \mid h^0(\mathcal{I}_C(n)) \neq 0\}\,.$$

We denote by $\mathrm{H}(d,g)$ (resp. $\mathrm{H}(d,g)_{sc}$) the Hilbert scheme of (resp. smooth connected) space curves of Hilbert polynomial $\chi(\mathcal{O}_C(t)) = dt + 1 - g$ [9]. A curve C is called *unobstructed* if $\mathrm{H}(d,g)$ is smooth at the corresponding point (C). The curve in a small enough open irreducible subset U of $\mathrm{H}(d,g)$ is called a *general* curve of $\mathrm{H}(d,g)$. So any member of U has all the openness properties which we want to require. A *generization* $C' \subset \mathbb{P}^3$ of $C \subset \mathbb{P}^3$ in $\mathrm{H}(d,g)$ is the general curve of some irreducible subset of $\mathrm{H}(d,g)$ containing (C). By an irreducible component of $\mathrm{H}(d,g)$ we always mean a *non-embedded* irreducible component. We denote by $\mathrm{H}(s)$ the Hilbert scheme of surfaces of degree s in $\mathbb{P}^3$. A member of a closed irreducible subset V of $\mathrm{H}(s)$ or $\mathrm{H}(d,g)_{sc}$ is called *very general* in V if it is smooth and sits outside a countable union of proper closed subset of V.

2 Background

In this section we first recall some results from [20] needed in this paper. The proofs use the deformation theory developed by Laudal in [27]; in particular the results rely on [27, Thm. 4.1.14].

2.1 The Hilbert flag scheme

Let $\mathrm{D}(d,g;s)$ (resp. $\mathrm{D}(d,g;s)_{sc}$) be the Hilbert-flag scheme parameterizing pairs (C,S) of curves (resp. smooth connected curves) C contained in surfaces S in $\mathbb{P}^3$ with Hilbert polynomials $p(t) = dt + 1 - g$ and $q(t) = \binom{t+3}{3} - \binom{t-s+3}{3}$ respectively. Then the tangent space, $A^1 := A^1(C \subset S)$, of $\mathrm{D}(d,g;s)$ at (C,S) is given by the Cartesian diagram (i.e. pullback or fibered product);

$$\begin{array}{ccccccc} & & A^1 & \longrightarrow & H^0(\mathcal{N}_S) & \simeq & H^0(\mathcal{O}_S(s)) \\ & & \downarrow & \square & \downarrow m & & \\ 0 \to H^0(\mathcal{N}_{C/S}) & \to & H^0(\mathcal{N}_C) & \longrightarrow & H^0(\mathcal{N}_S|_C) & \simeq & H^0(\mathcal{O}_C(s)) \end{array} \tag{3}$$

where the morphisms are induced by natural (or restriction) maps to normal sheaves.

Suppose S is a smooth surface of degree s. If C is a curve on S, we have $\mathcal{N}_{C/S} \simeq \omega_C \otimes \omega_S^{-1}$ and a connecting homomorphism $\delta : H^0(\mathcal{N}_S|_C) \to H^1(\mathcal{N}_{C/S}) \simeq H^0(\mathcal{O}_C(s-4))^\vee$ continuing the lower horizontal sequence in (3). Let $\alpha = \alpha_C := \delta \circ m$ be the composed map and let $A^2 := \operatorname{coker} \alpha$. Using (3), cf. [20, (2.7) and Lem. 8] for details, we get $\dim A^1 - \dim A^2 = (4-s)d + g + \binom{s+3}{3} - 2$ and an exact sequence

$$0 \to H^0(\mathcal{I}_{C/S}(s)) \to A^1 \to H^0(\mathcal{N}_C) \to H^1(\mathcal{I}_C(s)) \to \operatorname{coker} \alpha_C \to H^1(\mathcal{N}_C) \to H^1(\mathcal{O}_C(s)) \to 0\,. \tag{4}$$

The map $A^1 \to H^0(\mathcal{N}_C)$ in (3) is the tangent map of the 1^{st} projection,

$$pr_1 : \mathrm{D}(d,g;s) \longrightarrow \mathrm{H}(d,g)\ , \quad \text{induced by} \quad pr_1((C_1,S_1)) = (C_1)\,, \tag{5}$$

at (C,S). Since we may view $\mathrm{D}(d,g;s)$ as a relative Hilbert scheme over $\mathrm{H}(d,g)$ (cf. [17, Thm. 24.7]), it follows that pr_1 is a projective morphism by [9]. By [20, Lem. A10] pr_1 is smooth at (C,S) under the assumption

$$H^1(\mathcal{I}_C(s)) = 0\,. \tag{6}$$

Moreover by [20, (2.6)] $A^2 = \operatorname{coker} \alpha_C$ contains the obstructions of deforming the pair (C,S), cf. [21, Thm. 1.2.7] for a detailed version where also the meaning of obstructions is explained.

Let C be a smooth connected curve. If we suppose $d > s^2$ and $s = s(C)$, then it is easy to see $H^0(\mathcal{I}_{C/S}(s)) = 0$ for some hypersurface $S \supset C$ of degree s, and hence, by the semi-continuity of $h^0(\mathcal{I}_C(v))$ for $v \in \{s-1, s\}$, that *the restricted* projection, $pr_1 : \mathrm{D}(d,g;s)_{sc} \to \mathrm{H}(d,g)_{sc}$, is injective in $pr_1^{-1}(U)$ for some neighborhood $U \subset \mathrm{H}(d,g)_{sc}$ of (C). An s-maximal (or just maximal) family W of $\mathrm{H}(d,g)_{sc}$ containing (C) is therefore nothing but the image under pr_1 of an irreducible component of $\mathrm{D}(d,g;s)_{sc}$ containing (C,S) ([22, Def. 1.24 and Cor. 1.26]). If we in addition suppose that α is surjective, then (C,S) belongs to a unique generically smooth component of $\mathrm{D}(d,g;s)_{sc}$ and

$$\dim W = h^0(\mathcal{N}_C) - h^1(\mathcal{I}_C(s)) = (4-s)d + g + \binom{s+3}{3} - 2\,. \tag{7}$$

Assuming also (6) it follows that W is a generically smooth irreducible component of $\mathrm{H}(d,g)_{sc}$ ([20, Thm. 10]).

Using the infinitesimal Noether-Lefschetz theorem for $s = 4$ ([13, p. 253]) as explained in the proof of [20, Lem. 13], we immediately get that α is surjective provided S is smooth of degree $s \le 4$, $d > s^2$ and $C \subset S$ is smooth and connected, but (for $s = 4$ only) not a complete intersection of S with some other surface. Hence $\mathrm{D}(d,g;s)$ is smooth at (C,S) and we get all conclusions above, assuming (6) for the final one.

Remark 2.1. *If W is an* irreducible component *of* $\mathrm{H}(d,g)_{sc}$ *containing a curve C sitting on a smooth surface S of degree $s := s(W)$ with α_C surjective and $d > s^2$, then* $\dim W = (4-s)d+g+\binom{s+3}{3}-2 \ge \chi(\mathcal{N}_C) = 4d$, *i.e.*

$$g \ge sd - \binom{s+3}{3} + 2\,, \tag{8}$$

or equivalently, $h^1(\mathcal{I}_C(s)) \le h^1(\mathcal{O}_C(s))$. Moreover if the general curve of W does not satisfy (6)*, we get by* (7) *that the component W is non-reduced (i.e. not generically smooth) and that* (8) *holds.*

2.2 The relative Picard scheme

We also need to consider the Hilbert scheme, $\mathrm{H}(s) \simeq \mathbb{P}^{\binom{s+3}{3}-1}$, of surfaces of degree s in $\mathbb{P}^3$ and the second projection;

$$pr_2 : \mathrm{D}(d,g;s) \longrightarrow \mathrm{H}(s), \quad \text{induced by} \quad pr_2((C_1,S_1)) = (S_1)\,.$$

Moreover let Pic be the relative Picard scheme over the open set in $\mathrm{H}(s)$ of smooth surfaces of degree s, (see [10]). Then there is a projection $p_2 : \mathrm{Pic} \to \mathrm{H}(s)$, forgetting the invertible sheaf, and a rational map,

$$\pi : \mathrm{D}(d,g;s) \dashrightarrow \mathrm{Pic}, \quad \text{induced by} \quad \pi((C_1,S_1)) = (\mathcal{O}_{S_1}(C_1), S_1) \tag{9}$$

which is defined (and denoted π_U) on the *open subscheme* $U \subset \mathrm{D}(d,g;s)$ given by pairs (C_1,S_1) where C_1 is Cartier on a smooth S_1. Obviously, if we restrict to U we have $p_2 \circ \pi = pr_2$. If $H^1(S, \mathcal{O}_S(C)) \simeq H^1(\mathbb{P}^3, \mathcal{I}_C(s-4))^\vee = 0$ then π is smooth at (C,S) by [10, Rem. 4.5]. Indeed, $H^1(S,\mathcal{L}) = 0$, $\mathcal{L} := \mathcal{O}_S(C)$, implies a surjective map $A^1 \to T_{\mathrm{Pic},\mathcal{L}}$ between the tangent spaces of $\mathrm{D}(d,g;s)$ at (C,S) and Pic at $(\mathcal{L})$ and an injection $\operatorname{coker}\alpha_C \to \operatorname{coker}\alpha_{\mathcal{L}}$ on their obstruction spaces (mapping obstructions onto obstructions), fitting into the following commutative diagram of exact horizontal sequences

$$\begin{array}{ccccccc} 0 \to H^0(\mathcal{N}_{C/S}) & \longrightarrow & A^1 & \longrightarrow & H^0(\mathcal{N}_S) & \xrightarrow{\alpha_C} & H^1(\mathcal{N}_{C/S}) \\ \downarrow & & \downarrow & & \| & & \downarrow \\ 0 \to H^1(\mathcal{O}_S) = 0 & \longrightarrow & T_{\mathrm{Pic},\mathcal{L}} & \longrightarrow & H^0(\mathcal{N}_S) & \xrightarrow{\alpha_{\mathcal{L}}} & H^2(\mathcal{O}_S)\,. \end{array} \tag{10}$$

Here $\alpha_{\mathcal{L}}$ is the composition of α_C with the connecting homomorphism $H^1(\mathcal{N}_{C/S}) \to H^2(\mathcal{O}_S)$ induced from the exact sequence $0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{N}_{C/S} \to 0$, cf. [6, Thm. 1], [25, Sect. 4] and [24] for some details and compare with [17, Ex. 10.6] and [8]. Indeed, using [8, Thm. 1] and its proof we get the following version of the infinitesimal Noether-Lefschetz theorem (due to Green and Voisin);

$$\dim \operatorname{im} \alpha_{\mathcal{L}} \geq s - 3\,, \tag{11}$$

making the surjectivity of α_C for $s = 4$ mentioned above a special case. In our applications, however, we consider divisors where a basis for $\operatorname{Pic}(S)$ is given, allowing us to compute $\dim \operatorname{coker} \alpha_C$ explicitly.

Lemma 2.2. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree s, H a hyperplane section, and let E and C be curves on S satisfying $C \equiv eE + fH$ for some $e \neq 0, f \in \mathbb{Z}$. Let $d = d(C), g = g(C)$ and suppose*

$$H^1(\mathcal{I}_E(s-4)) = H^1(\mathcal{I}_C(s-4)) = 0\,.$$

(i) *Then $\mathrm{D}(d, g; s)$ is smooth at (C, S) if and only if $\mathrm{D}(d(E), g(E); s)$ is smooth at (E, S), and*

$$\dim \mathrm{D}(d(C), g(C); s) - h^1(\mathcal{O}_C(s-4)) = \dim \mathrm{D}(d(E), g(E); s) - h^1(\mathcal{O}_E(s-4))\,,$$

noting that $\dim |C| = h^0(\mathcal{O}_S(C)) - 1 = h^1(\mathcal{O}_C(s-4))$ *and* $\dim |E| = h^1(\mathcal{O}_E(s-4))$. *Moreover*

$$\dim \operatorname{coker} \alpha_C + h^0(\mathcal{I}_{C/S}(s-4)) = \dim \operatorname{coker} \alpha_E + h^0(\mathcal{I}_{E/S}(s-4))$$

(ii) *If $H^1(\mathcal{I}_E(s)) = 0$ and $\mathrm{H}(d(E), g(E))_{sc} \ni (E)$ is a smooth irreducible scheme, then $\mathrm{D}(d, g; s)$ is smooth at (C, S) and every $(C', S') \in \mathrm{D}(d, g; s)$ satisfying $C' \equiv eE' + fH'$ for some $(E', S') \in \mathrm{D}(d(E), g(E); s)_{sc}$, H' a hyperplane section of a* smooth *surface $S' \subset \mathbb{P}^3$, belongs to the unique irreducible component of $\mathrm{D}(d, g; s)$ containing (C, S).*

Proof. (i) If $\mathrm{D}(d(E), g(E); s)$ is smooth at (E, S), it follows that Pic is smooth at $(\mathcal{O}_S(E), S)$ by [10, Rem. 4.5] and $H^1(\mathcal{I}_E(s-4)) = 0$. Then Pic is smooth at $(\mathcal{O}_S(C), S)$ because the local rings $\mathcal{O}_{\mathrm{Pic},(\mathcal{O}_S(C),S)}$ and $\mathcal{O}_{\mathrm{Pic},(\mathcal{O}_S(E),S)}$ are isomorphic, at least up to completion. Indeed, by [6, Prop. 2 and Constr. 2], $\alpha_{\mathcal{O}_S(C)} = e \cdot \alpha_{\mathcal{O}_S(E)}$ and then (10) shows that the morphism between the local deformation functors of Pic at $(\mathcal{O}_S(E), S)$ and Pic at $(\mathcal{O}_S(C), S)$ (induced by $\mathcal{O}_S(E) \mapsto \mathcal{O}_S(E)^{\otimes e}(f)$) is an isomorphism. Hence $\mathrm{D}(d, g; s)$ is smooth at (C, S) by $H^1(\mathcal{I}_C(s-4)) = 0$, and conversely if $\mathrm{D}(d, g; s)$ is smooth at (C, S) we get that $\mathrm{D}(d(E), g(E); s)$ is smooth at (E, S) by the same argument.

Moreover since the fiber of π in (9) over $(\mathcal{O}_S(C), S)$ is the complete linear system $|C|$ on S and since smooth morphisms have surjective tangent maps, we also get the dimension formulas, using duality.

Finally to determine $\dim \operatorname{coker} \alpha_C$, we use (10). Since $H^1(\mathcal{I}_C(s-4)) = 0$ the map $H^1(\mathcal{N}_{C/S}) \simeq H^0(\mathcal{O}_C(s-4))^\vee \to H^2(\mathcal{O}_S) \simeq H^0(\mathcal{O}_S(s-4))^\vee$ is injective with cokernel $H^0(\mathcal{I}_{C/S}(s-4))^\vee$, whence

$$0 \longrightarrow \operatorname{coker} \alpha_C \longrightarrow \operatorname{coker} \alpha_{\mathcal{O}_S(C)} \longrightarrow H^0(\mathcal{I}_{C/S}(s-4))^\vee \longrightarrow 0$$

is exact. Since $H^1(\mathcal{I}_E(s-4)) = 0$ there is a corresponding exact sequence replacing C by E, and the middle term in these sequences are isomorphic because $\alpha_{\mathcal{O}_S(C)} = e \cdot \alpha_{\mathcal{O}_S(E)}$. This implies the final dimension formula of (i).

(ii) By the assumption $H^1(\mathcal{I}_E(s)) = 0$, $\mathrm{D}(d(E), g(E); s)$ is smooth at (E, S), cf. (6), whence $\mathrm{D}(d, g; s)$ is smooth at (C, S) by (i). Moreover $\mathrm{D}(d(E), g(E); s)_{sc}$ is also irreducible since one knows that $pr_1 : \mathrm{D}(d(E), g(E); s)_{sc} \to \mathrm{H}(d(E), g(E))_{sc}$ is irreducible by [22, Thm. 1.16]). It follows that the image U' of $\pi' : \mathrm{D}(d(E), g(E); s)_{sc} \dashrightarrow \mathrm{Pic}$ defined as in (9) is irreducible and since the morphism

$$\eta : U' \to \mathrm{Pic} \quad \text{induced by} \quad (\mathcal{O}_{S_1}(E_1), S_1) \mapsto (\mathcal{O}_{S_1}(E_1)^{\otimes (e)}(f), S_1)$$

is smooth (in fact an isomorphism) onto its image $U'' \subset \mathrm{Pic}$ by the argument for their local deformation functors used in the proof of (i), we get that U'' is irreducible. Finally using that the fiber $\pi_U^{-1}((\mathcal{O}_{S_1}(C_1), S_1))$ of the morphism in (9) is given by the complete linear system $|C_1|$ which is irreducible, we get that $\pi_U^{-1}(U'')$ is irreducible (cf. [18, Prop. 1.8]). Since $(C', S') \in \pi_U^{-1}(U'')$ we are done. $\square$

Remark 2.3. *Using [19, Cor. 4, p. 222], or [3, Prop. 3.4], and, say, the smoothness of π restricted to the set $U \cap \mathrm{D}(d,g;4)_{sc}$ accompanying* (9)*, we get that a closed irreducible subset W of $\mathrm{H}(d,g)_{sc}$, $d > 16$, whose very general member C sits on a smooth quartic surface S with Picard number ρ, will satisfy $\dim W \le g + 35 - \rho$. Hence if $W \subset \mathrm{H}(d,g)_{sc}$ is 4-maximal (e.g. an irreducible component), then $\rho = 2$ (or $\rho = 1$ in the c.i. case).*

One should compare Remark 2.3 with the following result which is a special case [28, Cor. II 3.8] of a theorem of A. Lopez, see [28, Thm. II 3.1] for a proof.

Lemma 2.4. *Let $E \subset \mathbb{P}^3$ be a smooth irreducible curve, let $n \ge 4$ be an integer and suppose the degree of every minimal generator of the homogeneous ideal of E is at most $n-1$. Let S be a very general smooth surface of degree n containing E and let H be a hyperplane section. Then $\mathrm{Pic}(S) \simeq \mathbb{Z} \oplus \mathbb{Z}$ and we may take $\{\mathcal{O}_S(H), \mathcal{O}_S(E)\}$ as a $\mathbb{Z}$-basis for $\mathrm{Pic}(S)$.*

Finally we will need the following lemma to prove our theorems.

Lemma 2.5. *Let S be a smooth projective surface containing a smooth rational curve Γ and let D be a divisor such that $c = -D \cdot \Gamma > 0$ and $D - \Gamma - K \neq 0$ is effective, K the canonical divisor.*

- *If $H^1(S, \mathcal{O}_S(D - \Gamma)) \neq 0$, then $H^1(S, \mathcal{O}_S(D)) \neq 0$.*
- *If $c > 1$, then $H^1(S, \mathcal{O}_S(D)) \neq 0$. In fact, $\dim H^0(S, \mathcal{O}_S(D)) \ge c - 1$.*
- *If $c = 1$ and $H^1(S, \mathcal{O}_S(D - \Gamma)) = 0$, then $H^1(S, \mathcal{O}_S(D)) = 0$.*

Proof. Taking cohomology of the exact sequence

$$0 \to \mathcal{O}_S(D - \Gamma) \to \mathcal{O}_S(D) \to \mathcal{O}_\Gamma(-c) \to 0,$$

and using duality and the fact that $\Gamma \simeq \mathbb{P}^1$, we get

$$h^1(\mathcal{O}_S(D)) = h^1(\mathcal{O}_S(D - \Gamma)) + h^1(\mathcal{O}_{\mathbb{P}^1}(-c)) = h^1(\mathcal{O}_S(D - \Gamma)) + c - 1$$

and the result follows. $\square$

2.3 On the maximum genus of space curves

Finally we recall the definition of $G(d,s)$; the maximum genus of smooth connected space curves of degree d not contained in a surface of degree $s-1$, cf. [14]. By definition,

$$G(d,s) = \max\{\, g(C) \,|\, (C) \in \mathrm{H}(d,g)_{sc} \ \text{ and } \ H^0(\mathcal{I}_C(s-1)) = 0 \,\} \tag{12}$$

In the case where $d > s(s-1)$, Gruson and Peskine showed in [14] that

$$G(d,s) = 1 + \frac{d}{2}\left(\frac{d}{s} + s - 4\right) - \frac{r(s-r)(s-1)}{2s} \quad \text{where } d + r \equiv 0 \bmod s \ \text{ for } \ 0 \le r < s, \tag{13}$$

and that $g(C) = G(d,s)$ if and only if C is directly linked to a plane curve of degree r by a c.i. of type (s,f), $f := (d+r)/s$. Note that this description of a curve C of $\mathrm{H}(d,G(d,s))_{sc}$ makes it possible to use Theorem 3.1 below to find $\dim V$ where $V \subset \mathrm{H}(d,G(d,s))_{sc}$ is the irreducible component containing (C). Indeed, we may assume a general member of V is contained in a smooth surface of degree s because the inequality $r < s$ allows us to start with a smooth plane curve E of degree r contained in a smooth surface of degree s and then make a linkage via a c.i. of type (s,f) to get a curve C' which, by [22, Ex. 3.13], belongs to V. Since $C' \equiv fH - E$, H a hyperplane section, Theorem 3.1 applies and we get $\dim V$ from (14).

3 A criterion of unobstructedness

We will now prove a theorem which via Remark 3.2 implies Theorem 1.3. Note that Theorem 3.1 immediately gives us a formula for $h^1(\mathcal{N}_C)$ because C is unobstructed, whence $h^1(\mathcal{N}_C) = \dim W - 4d$.

Theorem 3.1. *Let W' be an irreducible component of $\mathrm{D}(d,g;s)_{sc}$ and let $W := pr_1(W') \subseteq \mathrm{H}(d,g)_{sc}$. Let (C,S) be a member of W' such that S is smooth of degree s and suppose that*

$$H^1(\mathcal{I}_C(s)) = H^1(\mathcal{I}_C(s-4)) = 0\,.$$

Let E be a curve on S, H a hyperplane section and suppose $C \equiv eE + fH$ for some $e, f \in \mathbb{Z}$. Let $u := h^0(\mathcal{I}_{C/S}(s)) + h^0(\mathcal{I}_{C/S}(s-4))$ and let t be the non-negative number $t := h^1(\mathcal{N}_E) - h^1(\mathcal{O}_E(s))$. If E is arithmetically Cohen-Macaulay (ACM), or more generally if E is unobstructed and satisfies $H^1(\mathcal{I}_E(s)) = H^1(\mathcal{I}_E(s-4)) = 0$, then W is a generically smooth irreducible component of $\mathrm{H}(d,g)_{sc}$ (indeed C and $eE + fH$ are unobstructed) of dimension

$$(4-s)d + g + \binom{s+3}{3} - 2 - u + h^0(\mathcal{I}_{E/S}(s-4)) + t \tag{14}$$

if $e \neq 0$; if $e = 0$ then replace $h^0(\mathcal{I}_{E/S}(s-4)) + t$ by $\binom{s-1}{3}$ in (14).

Remark 3.2. *Let (C,S) and W be as in Theorem 3.1 and suppose $d > s^2$. Then $s = s(C)$ and W is an s-maximal family of $\mathrm{H}(d,g)_{sc}$ (see paragraph after (6)), in which case $h^0(\mathcal{I}_{C/S}(s))$ and $h^0(\mathcal{I}_{C/S}(s-4))$ vanish. Moreover if $E \subset S$ is a curve satisfying $h^1(\mathcal{N}_E) = h^1(\mathcal{O}_E(s))$ and $H^1(\mathcal{I}_E(s)) = 0$, then the obstruction group $\operatorname{coker}\alpha_E = 0$ by (4), whence E is unobstructed by (6). Noting that $h^0(\mathcal{I}_{E/S}(s-4)) = h^0(\mathcal{I}_E(s-4))$ we get Theorem 1.3 from Theorem 3.1.*

Remark 3.3. *If E is ACM then $H^1(\mathcal{I}_E(v)) = 0$ for any v and E is unobstructed by [5].*

Proof. Firstly we suppose $e \neq 0$. Note that E is unobstructed by Remark 3.3 and assumption. Hence using $H^1(\mathcal{I}_E(s)) = 0$ and the text accompanying (6) it follows that $\mathrm{D}(d(E),g(E);s)$ is smooth at (E,S). By Lemma 2.2 (i), $\mathrm{D}(d,g;s)$ is smooth at (C,S) and at (C',S), $C' := eE + fH$. Then the smoothness of pr_1, due to $H^1(\mathcal{I}_C(s)) \simeq H^1(\mathcal{I}_{C'}(s)) = 0$ shows that $\mathrm{H}(d,g)_{sc}$ is smooth at (C) and (C') and that W is a generically smooth irreducible component of $\mathrm{H}(d,g)_{sc}$.

To find $\dim W' = \dim A^1$ and hence $\dim W = \dim W' - h^0(\mathcal{I}_{C/S}(s))$, we need to determine $\dim\operatorname{coker}\alpha_C$. For this we use Lemma 2.2 (i) to relate $\dim\operatorname{coker}\alpha_C$ in terms of $\dim\operatorname{coker}\alpha_E$. Then using (4), we get $\dim\operatorname{coker}\alpha_E = t \geq 0$ and hence we obtain the dimension of W' from the dimension formula accompanying (4).

Finally we suppose $e = 0$. Then C is a c.i. and it is well known that $\mathrm{H}(d,g)_{sc}$ is smooth at (C) and (C'). By the smoothness of pr_1, we get that W is a generically smooth irreducible component of $\mathrm{H}(d,g)_{sc}$. Moreover $H^1(\mathcal{N}_C) \simeq H^1(\mathcal{O}_C(s)) \oplus H^1(\mathcal{O}_C(f))$ and using (4), we see that $\operatorname{coker}\alpha_C \simeq H^1(\mathcal{O}_C(f))$ and we conclude the proof by $H^1(\mathcal{O}_C(f)) \simeq H^0(\mathcal{O}_C(s-4))^\vee$. $\square$

Remark 3.4. (i) *If $H^1(\mathcal{N}_E) = 0$, e.g. $H^1(\mathcal{O}_E(1)) = 0$ and E reduced, then E is unobstructed and $t = 0$ in Theorem 3.1. Observe, however, that in many cases where the unobstructedness of E is known, there is also a dimension formula of $h^1(\mathcal{N}_E)$, making the number t of Theorem 3.1 explicit, see e.g. [26, Thm. 1.1] for a formula covering both the ACM and Buchsbaum diameter-1 case.*

(ii) *Assuming $h^0(\mathcal{I}_{E/S}(s-4)) = 0$ and $t = 0$ in Theorem 3.1 we get generically smooth irreducible components of $\mathrm{H}(d,g)_{sc}$ equipped with a dimension ("expected dimension" if $h^0(\mathcal{I}_{C/S}(s)) = 0$, i.e. $u = 0$ according to [25]) which in [24] was considered to be "the good general components" in $\mathrm{H}(d,g)_{sc}$ in a certain range of the d,g-plane. These components correspond, at least infinitesimally, to general components in the Noether-Lefschetz locus, see [1] and its references for a discussion of this locus, while the components in our main Theorems 1.1, 1.2 and 7.3 have large $h^0(\mathcal{I}_{E/S}(s-4))$ and correspond to components in the Noether-Lefschetz locus of the smallest codimension, see [6, 8, 41].*

4 Irreducible components of $\mathrm{H}(d,g)_{sc}$

In the background section we noticed that the assumption $H^1(\mathcal{I}_C(s)) = 0$ for $s = 4$ implies that 4-maximal subsets form generically smooth irreducible components of $\mathrm{H}(d,g)_{sc}$. We are now looking for a converse, i.e. that $H^1(\mathcal{I}_C(s)) \neq 0$ for $s \leq 4$ implies that s-maximal subsets form non-reduced components of $\mathrm{H}(d,g)_{sc}$. If $s = 3$ this is essentially a conjecture that the first author partially prove in the appendix. In this section we will see that some ideas of [20] generalize to cover cases where $s > 3$ as well. Indeed, we will show the following result which, together with (13), will be used for proving some results of this paper.

Theorem 4.1. *Let $W \subseteq \mathrm{H}(d,g)_{sc}$ be a 4-maximal family whose general member C is contained in a smooth surface $S \subset \mathbb{P}^3$ of degree 4, and suppose that C is not a complete intersection of S and some other surface. If $h^1(\mathcal{I}_C(1)) \leq d - 25$ and*

$$d \geq 31 \quad \text{and} \quad g \;>\; 21 + \frac{d^2}{10}\ ,$$

then W is an irreducible component of $\mathrm{H}(d,g)_{sc}$. Moreover W is non-reduced if and only if

$$H^1(\mathcal{I}_C(4)) \neq 0\,.$$

Remark 4.2. *Let C be a curve contained in a smooth quartic surface S. Using $\mathcal{I}_C = \ker(\mathcal{O}_{\mathbb{P}} \to \mathcal{O}_C)$ and the exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{I}_C \to \mathcal{I}_{C/S} \to 0\ ,$$

we get

$$H^1(\mathcal{I}_C(4)) \simeq H^1(\mathcal{I}_{C/S}(4)) \quad \text{and} \quad H^1(\mathcal{O}_C(4)) \simeq H^2(\mathcal{I}_C(4)) \simeq H^2(\mathcal{I}_{C/S}(4))\,.$$

Moreover since $\mathcal{I}_{C/S} \simeq \mathcal{O}_S(-C)$, we have by duality $H^i(\mathcal{I}_{C/S}(4))^\vee \simeq H^{2-i}(\mathcal{O}_S(C-4H))$, H a hyperplane section. So to explicitly find non-reduced components given by Theorem 4.1, one should look for curves C on S such that $H^i(\mathcal{O}_S(C-4H)) \neq 0$, for $i = 0, 1$ i.e. such that the linear system $|C - 4H|$ contains fixed components, or $C - 4H$ is composed with a pencil (cf. [39]).

To prove the theorem we will need

Proposition 4.3. *Let F be an integral surface in $\mathbb{P}^3$ of degree $s \geq 4$, let $S \to F$ be a desingularization and let $C \subset F$ be a smooth connected curve of degree d and genus g such that $Sing(F) \cap C$ is a finite set. If $\mathcal{N}_{C/S}$ is the normal sheaf of $C \hookrightarrow S$ (i.e. of the proper transform of $C \hookrightarrow F$) and $\mathrm{Hilb}(F)_{sc}$ is the Hilbert scheme of smooth connected curves on F, then*

$$\dim_{(C)} \mathrm{Hilb}(F)_{sc} \;\leq\; \dim H^0(\mathcal{N}_{C/S}) \;\leq\; \max\{\, \frac{d^2}{s} - g + 1, \frac{d^2}{2s} + 1\,\}\,.$$

Proof. The first inequality is [20, Lemma 22]. In [20, Lemma 23] we use Hodge's index theorem to show that $\deg \mathcal{N}_{C/S} = C^2 \leq d^2/s$. Therefore $H^1(\mathcal{N}_{C/S}) = 0$ implies $h^0(\mathcal{N}_{C/S}) = \chi(\mathcal{N}_{C/S}) = d^2/s + 1 - g$ by Riemann-Roch while if $H^1(\mathcal{N}_{C/S}) \neq 0$ then Clifford's theorem gives $\dim |\mathcal{N}_{C/S}| \leq \frac{1}{2} \deg \mathcal{N}_{C/S} \leq d^2/2s$, i.e. $h^0(\mathcal{N}_{C/S}) \leq d^2/2s + 1$. $\square$

In the proposition below we extend [20, Prop. 20], which should have assumed "$Sing(F) \cap C$ finite" or $d > (s(C)-1)^2$, from $s = 4$ to $s \geq 4$.

Proposition 4.4. *Let V be an irreducible component of $\mathrm{H}(d,g)_{sc}$ whose general curve C sits on some integral surface F of degree $s \geq 4$. If $d > s^2$, then*

$$\dim V \leq \binom{s+3}{3} - 1 + \max\left\{ \frac{d^2}{s} - g\,,\ \frac{d^2}{2s}\,,\ (4-s)d + g - 1 + h^0(\mathcal{O}_C(s-4)) \right\}.$$

Proof. Let W be any irreducible component of $\mathrm{D}(d,g;s)_{sc}$ containing (C,F). Since the 2^{nd} projection, $pr_2 : \mathrm{D}(d,g;s)_{sc} \to \mathrm{H}(s)$ has the Hilbert scheme $\mathrm{Hilb}(F)_{sc}$ as its fiber over (F), it follows that

$$\dim W \ \leq \ \dim pr_2(W) + \dim_{(C)} \mathrm{Hilb}(F)_{sc} \tag{15}$$

where $pr_2(W)$ is the scheme theoretic image of pr_2 restricted to W. Indeed endowing W with its reduced induced scheme structure, we may look upon the induced map $pr_2' : W \to pr_2(W)$ as a morphism between integral schemes whose fiber over (F) is at least contained in $\mathrm{Hilb}(F)_{sc}$. We get (15) by the fact that the dimension of every component in a fiber of pr_2' over (F) is not smaller than the relative dimension of pr_2', cf. [16, Ch. II, Ex. 3.22].

Suppose F is *smooth*. Then $\dim pr_2(W) \leq \dim_{(F)} \mathrm{H}(s) = \binom{s+3}{3} - 1$. Moreover, $\mathcal{N}_{C/F} \simeq \omega_C \otimes \omega_F^{-1}$ leads to $\chi(\mathcal{N}_{C/F}) = \chi(\omega_C(4-s)) = (4-s)d + g - 1$ and

$$\dim_{(C)} \mathrm{Hilb}(F)_{sc} \leq h^0(\mathcal{N}_{C/F}) = (4-s)d + g - 1 + h^0(\mathcal{O}_C(s-4)).$$

Suppose F is *not smooth, but integral*, then pr_2 is at least non-dominating, whence $\dim pr_2(W) \leq \binom{s+3}{3} - 2$. To use Proposition 4.3 to bound $\dim_{(C)} \mathrm{Hilb}(F)_{sc}$, we must show that $Sing(F) \cap C$ is a finite set. Indeed if this set is not finite, then the smooth connected curve C is contained in $Sing(F)$ which implies $d \leq (s-1)^2$ because there is a c.i. of type $(s-1, s-1)$ containing C (chosen among the partial derivatives of the form defining F). This contradicts an assumption of Proposition 4.4 while the other assumptions imply the existence of an irreducible component $W \ni (C,F)$ of $\mathrm{D}(d,g;s)_{sc}$ which dominates V under the first projection pr_1 given in (5). Since $d > s^2$ then $\dim V = \dim W$ and we can use (15) and the upper bounds of $\dim_{(C)} \mathrm{Hilb}(F)_{sc}$ to get Proposition 4.4. $\square$

Proof (of Theorem 4.1). To see that W is an irreducible component, we suppose there exists a component V of $\mathrm{H}(d,g)_{sc}$ satisfying $W \subset V$ and $\dim W < \dim V$. Then $s := s(V) \geq 5$ by the definition of a 4-maximal family. Moreover $s = 5$ since the case $s \geq 6$ can be excluded. Indeed using (12) and (13) we get $g > G(d,6)$ from $21 + d^2/10 > 1 + d^2/12 + d$ and the assumptions of the theorem. To get a contradiction we will use Proposition 4.4 for $s = 5$, and (7) that implies $\dim W = g + 33$. Indeed α_C is surjective by the infinitesimal Noether-Lefschetz theorem and by assumption, see the paragraph before Remark 2.1, and we get (7). Let C' be the general curve of V. Then $s(C') = 5$ and C' is a smooth connected curve. It follows that a surface F' containing C' of the least possible degree, namely 5, is integral. We get

$$g + 33 < 55 + \max\left\{ \lfloor \frac{d^2}{5} \rfloor - g\,,\ \lfloor \frac{d^2}{10} \rfloor\,,\ -d + g - 1 + 4 + h^1(\mathcal{I}_{C'}(1)) \right\}.$$

Suppose the maximum to the right is obtained by $\lfloor d^2/5 \rfloor - g$. Then since $g + 33 < 55 + d^2/5 - g$ is equivalent to $g < 11 + d^2/10$, we get a contradiction to the displayed assumption of the theorem. Similarly, $g + 33 < 55 + \lfloor d^2/10 \rfloor$ will lead to a contradiction. Finally if we suppose

$$g + 33 < 55 - d + g - 1 + 4 + h^1(\mathcal{I}_{C'}(1)),$$

i.e. $h^1(\mathcal{I}_{C'}(1)) > d - 25$ and we use that $h^1(\mathcal{I}_{C'}(1)) \le h^1(\mathcal{I}_C(1))$ by semi-continuity, we get $h^1(\mathcal{I}_C(1)) > d - 25$ which again is a contradiction to the assumptions. Thus we have proved that W is an irreducible component of $\mathrm{H}(d,g)_{sc}$.

Then using (7), i.e. $\dim W + h^1(\mathcal{I}_C(4)) = h^0(\mathcal{N}_C)$, it is straightforward to get the final statement of the theorem, and we are done. □

Corollary 4.5. *With notations and assumptions as in the first sentence in Theorem 4.1, suppose in addition*

$$g \ > \ \min\{\, G(d,5) - 1, \frac{d^2}{10} + 21\} \quad \text{and} \quad d \ge 21\,.$$

Then W is an irreducible component of $\mathrm{H}(d,g)_{sc}$, and W is non-reduced if and only if $H^1(\mathcal{I}_C(4)) \neq 0$.

Proof. One checks that the minimum value in the corollary is equal to $G(d,5) - 1$ (resp. $21 + d^2/10$) for $21 \le d \le 44$ (resp. $d \ge 45$). Moreover, $h^1(\mathcal{I}_C(1)) = h^0(\mathcal{O}_C(1)) - 4 \le \max\{d - g, \frac{d}{2}\} - 3$ for a non-plane curve by Clifford's theorem and Riemann-Roch. Hence if $d \ge 45$, we get $h^1(\mathcal{I}_C(1)) \le d - 25$ and we conclude by Theorem 4.1.

If $21 \le d \le 44$ we suppose there is an irreducible component V of $\mathrm{H}(d,g)_{sc}$ satisfying $W \subset V$ and $\dim W < \dim V$. We may suppose either $g > G(d,5)$ or $g = G(d,5)$. In the first case we get $s(V) = 4$ which contradicts the 4-maximality of W. In the remaining case, using (13) and the arguments after (13), we may assume the general curve C' of V is linked to a smooth plane curve E of degree $r < 5$ by a c.i. of type $(5, (d+r)/5)$ where the quintic surface S in the c.i. is smooth. Hence we can apply Theorem 3.1 (replacing W in Theorem 3.1 by V) with $s = 5$ and $e = -1$ (and $e = 0$ when $r = 0$) to compute $\dim V$. Since $\mathcal{N}_E \simeq \mathcal{O}_E(1) \oplus \mathcal{O}_E(r)$, whence $H^1(\mathcal{N}_E) \simeq H^1(\mathcal{O}_E(1))$ we easily compute $h^0(\mathcal{I}_{E/S}(1)) + t$ in Theorem 3.1 to be $4 - \chi(\mathcal{O}_E(1)) \le 2$. We get $\dim V \le 56 - d + g - h^0(\mathcal{I}_{C'/S}(5))$ (resp. $\dim V \le 58 - d + g$ for $r = 0$) by Theorem 3.1. In particular we have $\dim V \le 58 - d + g$ for $d \ge 25$ which contradicts $g + 33 = \dim W < \dim V$. For $21 \le d \le 24$ we have by (13) at least two quintic surfaces in the c.i. Y linking C' to E. Thus $h^0(\mathcal{I}_{C'/S}(5)) \ge 1$. In the case $d = 21$ we get $h^0(\mathcal{I}_{C'/S}(5)) = 2$ because $\omega_E \simeq \mathcal{I}_{C'/Y}(6)$ imply $h^0(\mathcal{I}_{C'/Y}(5)) = h^0(\mathcal{O}_E) = 1$. Hence $\dim V \le 56 - d + g - h^0(\mathcal{I}_{C/S}(5))$ implies $\dim V \le g + 33$, and we have a contradiction. Finally using the left equality of (7) we easily get the statement on non-reducedness of the corollary, and we are done. □

Remark 4.6. *Let C be a general curve of a 3-maximal family W. The analogue of Theorem 4.1 for $s(C) = 3$ states that W is an irreducible component of $\mathrm{H}(d,g)_{sc}$ provided*

$$g > 7 + (d-2)^2/8 \text{ and } d \ge 27$$

as one may easily deduce from the proof of [20, Theorem 5], paying a little extra attention to the case $(d,g) = (30,106)$. To show that W is a non-reduced irreducible component, the above result turned out to be quite useful in [20]. This result, together with Theorem 4.1 for $s(C) = 4$, improve upon what we may show by only using (13) by $k + d/2$, k a constant, cf. Corollary 4.5. This improvement is not necessary for Theorem 1.1 of this paper because the curves in II) satisfy $g > G(d,5) - 1$. We need, however, Theorem 4.1 in the appendix, and we hope it, or a refined version, applies to other classes of components where $s(C) = 4$, as it did for $s(C) = 3$.

5 Components of $\mathrm{H}(d,g)_{sc}$ for $s=4$

In this section we prove Theorem 1.1 stated in the introduction. Let us start by considering the existence of the quartic surfaces that we need in the sequel, together with determining the smooth connected curves contained in the surfaces.

5.1 Quartic surfaces containing a line

Our main example comes from studying curves on smooth quartic surfaces containing a line. Such quartic surfaces appeared in the work of Mori [32], who showed the following result: If there exists a smooth quartic surface S_0 containing a nonsingular curve Γ_0 of degree d and genus g, then there also exists a smooth quartic surface S containing a smooth curve Γ of the same degree and genus, such that $\mathrm{Pic}(S) \simeq \mathbb{Z}\Gamma \oplus \mathbb{Z}H$, where H is the hyperplane section. (See also [17, p. 138]).

The following result describes the main properties of curves on such quartics (see also [38, Section 3.1]).

Proposition 5.1. *There exists a smooth quartic surface $S \subset \mathbb{P}^3$ with $H \equiv \Gamma_1 + \Gamma_2$ and $\mathrm{Pic}(S) \simeq \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$ where Γ_1, Γ_2 are smooth curves of genus 0 and 1 respectively, and intersection matrix given by*

$$\begin{pmatrix} \Gamma_1^2 & \Gamma_1 \cdot \Gamma_2 \\ \Gamma_1 \cdot \Gamma_2 & \Gamma_2^2 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 3 & 0 \end{pmatrix} .$$

Furthermore, for any such surface S with H, Γ_1, Γ_2 as above the following hold:

i) Any effective divisor class can be written as $a\Gamma_1 + b\Gamma_2$ for non-negative integers $a, b \geq 0$.

ii) A divisor class $a\Gamma_1 + b\Gamma_2$ is nef if and only if $3b \geq 2a \geq 0$.

iii) If $D \equiv a\Gamma_1 + b\Gamma_2$ is a divisor with $3b \geq 2a > 0$, then the general element in $|D|$ is a smooth irreducible curve. Conversely, the classes of the irreducible divisors correspond to classes $a\Gamma_1 + b\Gamma_2$ with $3b \geq 2a > 0$ or $(a,b) = (1,0), (0,1)$.

Proof. Smooth quartic surfaces S_0 containing a line $\{x_0 = x_1 = 0\}$ are defined by a homogeneous polynomial of the form $F = x_0 p + x_1 q = 0$ where $p, q \in k[x_0, x_1, x_2, x_3]$ are cubic forms. By Mori's result above there exists a smooth quartic surface S such that $\mathrm{Pic}(S)$ is generated by the classes of a smooth rational curve Γ_1 and the hyperplane section H. By the adjunction formula, we have $\Gamma_1^2 = -2$. In fact the diophantine equation $(xH + y\Gamma_1)^2 = 4x^2 + 2xy - 2y^2 = 2(2x-y)(x+y) = -2$ has the only solutions $(0, \pm 1)$, showing that Γ_1 is the unique (-2)-curve on S. The class $H - \Gamma_1$ has self-intersection 0 and is thus effective. It is in fact represented by the smooth elliptic curve given by $\{x_0 = q = 0\}$.

To prove *i)* we claim that every effective divisor is linearly equivalent to a non-negative integral linear combination of Γ_1 and Γ_2. Indeed, let D be any effective divisor class and write $D = a\Gamma_1 + b\Gamma_2$ for integers a, b. We may assume that $D \cdot \Gamma_1 \geq 0$ (otherwise Γ_1 is a fixed component of the linear system $|D|$ and we can instead consider $D - \Gamma_1$). Then we have $0 \leq D \cdot \Gamma_1 = 3b - 2a$ and $0 \leq D \cdot \Gamma_2 = 3a$ implying that $a, b \geq 0$. Dually we have also shown that the nef cone is determined by the inequalities $a \geq 0$ and $3b \geq 2a$, giving *i)* and *ii)*.

iii): If C is an irreducible curve with $C \neq \Gamma_1, \Gamma_2$, then C is nef and $C \cdot \Gamma_2 > 0$ (by the Hodge index theorem). So $C \equiv a\Gamma_1 + b\Gamma_2$ with $3b - 2a \geq 0$ and $a > 0$. Conversely, if these conditions are satisfied, the divisor $D = a\Gamma_1 + b\Gamma_2$ is base-point free [39, Corollary 3.2] and hence by Bertini's theorem the general element in $|D|$ is smooth and irreducible. □

For the existence of such a K3 surface one could also use a result of Nikulin [37] which states that for any even lattice of signature $(1, \rho-1)$ with $\rho \leq 10$, there exists a smooth projective K3 surface with this intersection form. Using this, and the embedding criteria of Saint-Donat [39], one can show that any surface with intersection matrix as above embeds as a smooth quartic surface.

To prove Theorem 1.1 it is necessary to determine when $H^1(S, \mathcal{O}_S(D)) \neq 0$. If $D = a\Gamma_1 + b\Gamma_2$ is effective, then by the proposition above, we must have $a, b \geq 0$. If $a = 0$ then $h^1(S, \mathcal{O}_S(D)) = h^1(S, \mathcal{O}_S(b\Gamma_2)) = \max\{b-1, 0\}$. We will assume $a > 0$. If now $c := -D \cdot \Gamma_1 = 2a - 3b \leq 0$, then D is nef and $D^2 = a(3b-2a) + 3ab > 0$ and so $h^1(S, \mathcal{O}_S(D)) = 0$. If $c = 1$, then $a > 1$ (due to $1 = 2a - 3b$) and $D - \Gamma_1$ is nef with $(D - \Gamma_1)^2 > 0$ and so $H^1(\mathcal{O}_S(D - \Gamma_1)) = 0$ and consequently $H^1(S, \mathcal{O}_S(D)) = 0$ by Lemma 2.5. The same lemma implies $H^1(S, \mathcal{O}_S(D)) \neq 0$ for $D \neq \Gamma_1$ and $c > 1$. Hence we obtain

Proposition 5.2. *Let S be a smooth quartic surface with* $\mathrm{Pic}(S) \simeq \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$ *and Γ_1, Γ_2 and H as above and suppose $D = a\Gamma_1 + b\Gamma_2$ is an effective divisor class with $a > 0$. Then for $D \neq \Gamma_1$,*

$$H^1(S, \mathcal{O}_S(D)) \neq 0 \qquad \text{if and only if} \qquad 2a > 3b + 1. \tag{16}$$

Moreover $H^1(S, \mathcal{O}_S(\Gamma_1)) = 0$ and if $a = 0$, then $h^1(S, \mathcal{O}_S(D)) = \max\{b-1, 0\}$.

Proof of Theorem 1.1. We get $d = a + 3b$, $g = 3ab - a^2 + 1$ from from $d = C \cdot H$, $g = 1 + C^2/2$ and since $C \notin |nH|$ for every $n \in \mathbb{Z}$ by assumption, it follows that C is not a c.i. in S. By [20, Thm. 10 and Lem. 13], see Section 2.1 of this paper, the Hilbert-flag scheme $\mathrm{D}(d, g; s)$ for $s = 4$ is smooth at (C, S) of dimension $\dim A^1 = g + 33$. Hence (C, S) belongs to a unique irreducible component of $\mathrm{D}(d, g; 4)_{sc}$ whose image under the 1^{st} projection, $pr_1 : \mathrm{D}(d, g; 4)_{sc} \to \mathrm{H}(d, g)_{sc}$, is the 4-maximal subset W of the theorem because the assumption $d > 16$ implies $s(W) = 4$.

By Lemma 2.2 we have the properties of W stated in Theorem 1.1 provided we can take a $\mathbb{Z}$-basis of $\mathrm{Pic}(\tilde{S})$ as in the theorem. Since we may assume that $\tilde{S}$ is very general by using the projections pr_i, $i = 1, 2$ and the very general assumption concerning W, this is straightforward by Lemma 2.4. Indeed there is a hyperplane section $\tilde{H}$ of $\tilde{S}$ containing $\tilde{E}$ such that $\Gamma := \tilde{H} - \tilde{E}$ is a smooth curve of degree 3 ([39]) and instead of the basis $\{\mathcal{O}_S(\tilde{E}), \mathcal{O}_S(\tilde{H})\}$ given by Lemma 2.4, we may take the classes of $\{\tilde{E}, \Gamma\}$ as a $\mathbb{Z}$-basis of $\mathrm{Pic}(\tilde{S})$.

For the rest of the proof we use (16), (1) as in Corollary 4.5, and Theorem 1.3.

I) By Theorem 1.3 it suffices to show that $H^1(S, \mathcal{O}_S(C - 4H)) = 0$ because $h^1(I_C(4)) = h^1(\mathcal{O}_S(C - 4H))$. But this is immediate from (16) since the inequality $4 < a < \frac{3b}{2} - 1$ implies $a - 4 > 0$ and $2(a-4) \leq 3(b-4) + 1$, and the case $(a, b) = (5, 4)$ corresponds to $C - 4H = \Gamma_1$.

II) By Corollary 4.5 it suffices to show that $H^1(S, \mathcal{O}_S(C - 4H)) \neq 0$ for the classes in (2) of Theorem 1.1. Also this is immediate from (16), since (2) and $d > 16$ imply $2(a-4) > 3(b-4) + 1$, $a - 4 > 0$ and $(a - 4, b - 4) \neq (1, 0)$. The lattice points in this region satisfying (1) are then found by inspection.

For the statement on the dimension of the tangent space of $\mathrm{H}(d, g)_{sc}$ at (C), we know that this dimension is equal to $g + 33 + h^1(I_C(4))$ by (7). We claim that $h^1(I_C(4)) = 1$ (resp. $h^1(I_C(4)) = 2$, $h^1(I_C(4)) = 4$) for the family $a)$ (resp. $b), c)$). To see it we use the short exact sequence in the proof of Lemma 2.5 for $\Gamma := \Gamma_1$ and $D := C - 4H$ (and then to $D := C - 4H - \Gamma_1$ for the class b)). Taking cohomology and counting dimensions, we get the claim. □

Remark 5.3. *Note that if $D := C - 4H = a\Gamma_1 + b\Gamma_2$ is effective and $h^1(S, \mathcal{O}_S(D)) > 0$, then either Γ_1 is a fixed component of $|D|$ or D is composed with a pencil, in which case $C = 4\Gamma_1 + r\Gamma_2$ for $r \geq 6$. In the latter case, it is easy to verify that C does not satisfy the constraints* (1)*, hence does not lead to non-reduced components by the theory we have so far.*

5.2 Other quartic surfaces

The surface appearing in Theorem 1.1 is an example of a quartic surface for which we can use the theory of this paper to describe the smoothness properties of the components of the Hilbert scheme. In fact, by the result of Mori quoted above, it is clear that there should exist many such examples, but finding ones with irreducible curves satisfying the bound (1) seems more difficult. Nevertheless, let us finish our study of curves on a smooth quartic by giving the main details of one more class.

Consider a homogeneous quartic form of the form $F = x_0p+q_1q_2$ where q_1, q_2 are quadrics defining the plane conics and p is a cubic. For q_1, q_2 general, F defines a smooth quartic surface $S_0 \subset \mathbb{P}^3$, where the hyperplane section splits into two plane conics ($\{x_0 = q_1 = 0\}$ and ($\{x_0 = q_2 = 0\}$). Then by Mori's theorem, for q_1, q_2 very general, one obtains a smooth quartic surface with the intersection matrix $(\Gamma_i \cdot \Gamma_j) = \left(\begin{smallmatrix} -2 & 4 \\ 4 & -2 \end{smallmatrix}\right)$. (Again we choose the basis $\{\mathcal{O}_S(\Gamma_1), \mathcal{O}_S(\Gamma_2)\}$ for $\operatorname{Pic}(S)$ rather than $\{\mathcal{O}_S(H), \mathcal{O}_S(\Gamma_1)\}$.) As before, one can show that Γ_1, Γ_2 define smooth irreducible (-2)-curves which generate the semigroup of effective divisors.

Proposition 5.4. *There exists a smooth quartic surface S with $\operatorname{Pic}(S) \simeq \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$ and Γ_1, Γ_2, $H \equiv \Gamma_1 + \Gamma_2$ as above.*

i) Any effective divisor class can be written as $a\Gamma_1 + b\Gamma_2$ for non-negative integers $a, b \geq 0$.

ii) A divisor class $a\Gamma_1 + b\Gamma_2$ is nef if and only if $\frac{b}{2} \leq a \leq 2b$.

iii) If $D \equiv a\Gamma_1 + b\Gamma_2$ is a divisor with $a, b > 0$, then $H^1(S, \mathcal{O}_S(D)) = 0$ if and only if D is nef.

iv) If $D \equiv a\Gamma_1 + b\Gamma_2$ is a divisor with $0 < \frac{b}{2} \leq a \leq 2b$, then the general element in $|D|$ is a smooth irreducible curve. Conversely, the classes of the irreducible curves correspond to classes $a\Gamma_1 + b\Gamma_2$ satisfying $a, b > 0$ and $\frac{b}{2} \leq a \leq 2b$ or $(a, b) = (1, 0), (0, 1)$.

Proof. The first part of the proposition and $i)$ follow as in the proof of Proposition 5.1. If $D = a\Gamma_1 + b\Gamma_2$ is nef and non-zero, then intersecting with Γ_1 and Γ_2 gives the above inequality for $ii)$. In $iii)$, if D is nef and $\not\equiv 0$, we have $D^2 = 8ab - 2a^2 - 2b^2 = 2a(2b - a) + 2b(2a - b) > 0$ and so by the Kawamata-Viehweg vanishing theorem, $H^1(S, \mathcal{O}_S(D)) = 0$. Conversely, if D is not nef, then we can without loss of generality assume $d = -D.\Gamma_1 > 0$. But d must be an even number, hence $d > 1$ and so $H^1(S, \mathcal{O}_S(D)) \neq 0$ by Lemma 2.5. $\square$

Note that Proposition 5.4 (iii) allows us to see exactly when $h^1(\mathcal{I}_C(4)) = h^1(S, \mathcal{O}_S(C-4H)) = 0$, and we get at least:

Proposition 5.5. *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface with Γ_1, Γ_2 as above, let $C \equiv a\Gamma_1 + b\Gamma_2$ be a smooth connected curve and suppose $a \neq b$ and $d > 16$. Then C belongs to a unique 4-maximal family $W \subseteq \mathrm{H}(d, g)_{sc}$. Moreover if $\tilde{S}$ is a quartic surface containing a very general member of W, then $\operatorname{Pic}(\tilde{S})$ is freely generated by the classes of two rational conics, and every $C \equiv a\Gamma_1 + b\Gamma_2$ contained in some surface S as above belongs to W. Furthermore $\dim W = g + 33$,*

$$d = 2a + 2b\ , \quad g = 4ab - a^2 - b^2 + 1 \quad \textit{and}$$

W is a generically smooth, irreducible component of $\mathrm{H}(d, g)_{sc}$ provided $\frac{b}{2} + 2 \leq a \leq 2b - 4$.

Proof. The proof of the properties of the maximal family W follows as in the first part of Theorem 1.1. The remaining part follows from Theorem 1.3 and Proposition 5.4 since $H^1(S, \mathcal{O}_S(C - 4H)) = 0$ if and only if $C - 4H$ is nef if and only if $\frac{b+4}{2} \leq a \leq 2(b - 2)$. $\square$

Remark 5.6. (i) *We may by symmetry restrict the range of Proposition 5.5 to $a > b$. Then there are 4 families in the range $2b - 4 < a \leq 2b$ which satisfy $H^1(\mathcal{I}_C(4)) \neq 0$. They are of the form $(5+2k, 4+k)$ $(6+2k, 4+k)$ $(7+2k, 4+k)$ $(8+2k, 4+k)$, $k \geq 1$. Unfortunately,* (1) *does not hold for any of these classes, so we can not conclude that they correspond to non-reduced components by the results we have so far. We* expect, *however, that they are non-reduced components.*

(ii) *We are informed by H. Nasu that he, using methods appearing in [33, 35, 36] is able to show that the family $(5 + 2k, 4 + k)$ corresponds to non-reduced components of $\mathrm{H}(d, g)_{sc}$, and that these methods also apply to show the non-reducedness of family a) of Theorem 1.1, cf. Remark 7.4.*

6 Components of $\mathrm{H}(d, g)_{sc}$ for $s = 5$

Let S be a very general smooth surface of degree 5 in $\mathbb{P}^3$ defined by an equation $x_0P + x_1Q$, where P, Q are very general homogeneous degree-4 polynomials. Let $\Gamma_1 = \{x_0 = x_1 = 0\}$ (a line) and $\Gamma_2 = \{x_0 = Q = 0\}$ (a plane quartic). The hyperplane section $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|_S|$ satisfies $H \equiv K \equiv \Gamma_1 + \Gamma_2$ and $H^2 = 5$ where K is the canonical divisor, and we may suppose $\mathrm{Pic}(S) \simeq \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$ by Lemma 2.4. Then $\Gamma_1 \cdot H = 1$, $H^2 = 5$ and the adjunction formula imply that the intersection matrix is $(\Gamma_i \cdot \Gamma_j) = \left(\begin{smallmatrix} -3 & 4 \\ 4 & 0 \end{smallmatrix}\right)$.

Let $C \subset S$ be a smooth, connected curve of degree d and genus g with $C \equiv a\Gamma_1 + b\Gamma_2$. We have $d = C \cdot H$, $g = 1 + (C^2 + C \cdot K)/2$, and we deduce

$$d = a + 4b \quad \text{and} \quad g = 1 + 4ab + \frac{1}{2}(a + 4b - 3a^2)\,.$$

As in the case of quartic surfaces, we easily deduce the following result:

Lemma 6.1. *Any effective divisor on S is linearly equivalent to $a\Gamma_1 + b\Gamma_2$ where $a, b \geq 0$. Every nef divisor is linearly equivalent to $a\Gamma_1 + b\Gamma_2$ where $4b \geq 3a \geq 0$.*

It will be of interest to study the divisor $4\Gamma_1 + 3\Gamma_2$, which is on the boundary of the nef cone.

Lemma 6.2. *Let S be a quintic surface with Γ_1, Γ_2 as above. Then the divisor $D = 4\Gamma_1 + 3\Gamma_2$ is base-point free. Moreover, for each $m \geq 1$, $|mD|$ contains a smooth irreducible curve.*

Proof. Choose global sections x and y_1, y_2 as bases of $H^0(S, \mathcal{O}_S(\Gamma_1))$ and $H^0(S, \mathcal{O}_S(\Gamma_2))$ respectively. Note that as Γ_2 is base-point free, so is the linear system $V = \langle y_1^3, y_1^2y_2, y_1y_2^2, y_2^3\rangle \subseteq H^0(S, \mathcal{O}_S(3\Gamma_2))$. Note that $x^4 \cdot V = \langle x^4y_1^3, x^4y_1^2y_2, x^4y_1y_2^2, x^4y_2^3\rangle \subseteq H^0(S, \mathcal{O}_S(D))$, so if D has a base-point, it is necessarily contained in Γ_1. But a general divisor $M \in |D|$ does not intersect Γ_1: In particular this is true for the curve $M = \{P = Q = 0\}$, for P, Q general. The last part now follows from Bertini's theorem since mD is not composed with a pencil. $\square$

Lemma 6.3. *Let C be a general element of the linear system $|a\Gamma_1 + b\Gamma_2|$, where $4b \geq 3a$ and $a > 1$. Then C is a smooth irreducible curve with $H^i(S, \mathcal{O}_S(C)) = 0$ for $i > 0$.*

Proof. Since the inequality $4b \geq 3a \geq 0$ describes exactly the nef cone of S, i.e., $C \cdot \Gamma_i \geq 0$ for $i = 1, 2$, it follows that C is a nef divisor. Assume first that C is ample, i.e., that $4b > 3a > 0$, then $C - K = C - \Gamma_1 - \Gamma_2$ is nef, and big by $a > 1$ which implies $b > 1$, and so by Kawamata-Viehweg, $H^i(S, \mathcal{O}_S(C)) = H^i(S, \mathcal{O}_S(K + (C - K))) = 0$ for $i > 0$. Moreover, in this case, Bertini's theorem gives that the general element is smooth and irreducible, since $|C|$ is base-point free.

It remains to consider the case $C \equiv 4m\Gamma_1 + 3m\Gamma_2$. It was shown above that C is smooth and irreducible. Let $D \in |4\Gamma_1 + 3\Gamma_2|$ be a general smooth element. Then we get $H^1(S, \mathcal{O}_S(D)) = 0$ by the exact sequence

$$0 \to \mathcal{O}_S(3\Gamma_1 + 3\Gamma_2) \to \mathcal{O}_S(D) \to \mathcal{O}_{\Gamma_1} \to 0\,.$$

Moreover, there is also an exact sequence

$$0 \to \mathcal{O}_S((m-1)D) \to \mathcal{O}_S(mD) \to \mathcal{O}_S(mD)|_D \to 0\,.$$

By induction on m, $H^1(S, \mathcal{O}_S((m-1)D)) = 0$. Also, a computation gives that $\mathcal{O}_S(mD)|_D$ has degree $> 2g(D) - 2$ on D for $m \geq 2$, so $H^1(D, \mathcal{O}_S(mD)|_D) = 0$. Hence $H^i(S, \mathcal{O}_S(mD)) = 0$ for $i > 0$. □

Proof of Theorem 1.2. If $E := \Gamma_1$, we get $H^1(\mathcal{I}_E(v)) = 0$ for any v, whence $\mathrm{D}(1, 0; 5)$ is smooth at (E, S) by (6). Then Lemma 2.2 and (4) imply that the Hilbert-flag scheme $\mathrm{D}(d, g; 5)_{sc}$ is smooth of dimension $\dim A^1 = \dim \operatorname{coker} \alpha_C - d + g + 54$ at (C, S) because $H^1(\mathcal{I}_C(1)) \simeq H^1(\mathcal{O}_S(C))^\vee = 0$ by Lemma 6.3, since C is smooth and irreducible. Hence (C, S) belongs to a unique irreducible component of $\mathrm{D}(d, g; 5)_{sc}$ whose image under the 1^{st} projection, $pr_1 : \mathrm{D}(d, g; 5)_{sc} \to \mathrm{H}(d, g)_{sc}$, is a 5-maximal subset W because the assumption $d > 25$ implies $s(W) = 5$. This W is the one given in the theorem. Thanks to Lemma 2.2 we get the properties of W stated in Theorem 1.2. Also $\dim \operatorname{coker} \alpha_C = 2$ is easily found using Lemma 2.2, and we get $\dim W = -d + g + 56$.

Now to get I) it suffices by (6) to show $H^1(\mathcal{I}_C(5)) = 0$. Since $H^1(\mathcal{I}_C(5))^\vee \simeq H^1(\mathcal{O}_S(C + K - 5H)) = H^1(\mathcal{O}_S(C - 4H))$ this group vanishes by Lemma 6.3: Indeed $C - 4H \equiv (a-4)\Gamma_1 + (b-4)\Gamma_2$ satisfies $4(b-4) \geq 3(a-4)$ and $a - 4 > 1$ by the assumption $5 < a < \frac{4b}{3} - 1$ of I).

To get II) we will show $g - G(d, 6) > 0$ where $G(d, 6) = 1 + d + d^2/12 - 5r(6-r)/12$ and $0 \leq r < 6$ are given by (13). Since $g = 1 + 8n + 24n^2$ and $d = 16n$ it is straightforward to get $g - (1 + d + d^2/12) = 8n(n-3)/3$, whence $g - G(d, 6) > 0$ for $n > 3$ and $g = G(d, 6)$ for $n = 3$. Then we can use exactly the arguments in the 2^{nd} paragraph of the proof of Corollary 4.5 to see that W is an irreducible component of $\mathrm{H}(d, g)_{sc}$, i.e. we only need to show that $\dim W \geq \dim V$ for the irreducible component V of $\mathrm{H}(d, g)_{sc}$ containing a curve C' of maximum genus: $g = G(d, 6) = 241$ where $d = 48$. By (13) the curve C' is a c.i. of type $(6, 8)$, whence with dualizing sheaf $\omega_{C'} \simeq \mathcal{O}_{C'}(10)$. Then we conclude by $\dim V = 4d + h^1(\mathcal{O}_{C'}(6)) + h^1(\mathcal{O}_{C'}(8)) = 237$ and $\dim W = -d + g + 56 = 249$.

This component is non-reduced if we can show $\dim W < h^0(\mathcal{N}_C)$ for C general. Since $\dim W = \dim A^1$, $H^0(\mathcal{I}_{C/S}(5)) = 0$ and $\dim \operatorname{coker} \alpha_C = 2$ it suffices by (4) to prove $h^1(\mathcal{I}_C(5)) \geq 3$. This follows from the exact sequence in the proof of Lemma 2.5, because $\Gamma_1 \simeq \mathbb{P}^1$. □

7 Components of $\mathrm{H}(d, g)_{sc}$ for $s \geq 5$

Let, as in the case of quintic surfaces, S be a very general smooth surface of degree $s \geq 5$ in $\mathbb{P}^3$ defined by an equation $x_0 P + x_1 Q$, where P, Q are very general homogeneous polynomials of degree $s - 1$. Let $\Gamma_1 = \{x_0 = x_1 = 0\}$ and $\Gamma_2 = \{x_0 = Q = 0\}$. The hyperplane section satisfies $H \equiv \Gamma_1 + \Gamma_2$, $H^2 = s$ and we may suppose $\mathrm{Pic}(S) \simeq \mathbb{Z}\Gamma \oplus \mathbb{Z}H$ by Lemma 2.4. If $C \equiv a\Gamma_1 + b\Gamma_2$ then $d = C \cdot H$, $K = (s-4)H$ and the adjunction formula imply that the intersection matrix is $(\Gamma_i \cdot \Gamma_j) = \left(\begin{smallmatrix} 2-s & s-1 \\ s-1 & 0 \end{smallmatrix}\right)$, and that

$$d = a + (s-1)b \quad \text{and} \quad g = 1 + (s-1)ab + \frac{1}{2}((s-4)a + (s-4)(s-1)b - (s-2)a^2)\,. \tag{17}$$

The first two lemmas of Section 6 generalize easily and we get

Lemma 7.1. *Any effective divisor on S is linearly equivalent to $a\Gamma_1 + b\Gamma_2$ where $a, b \geq 0$. Every nef divisor is linearly equivalent to $a\Gamma_1 + b\Gamma_2$ where $(s-1)b \geq (s-2)a \geq 0$.*

Lemma 7.2. *Let S be a smooth surface of degree s with Γ_1, Γ_2 as above. Then the divisor $D = (s-1)\Gamma_1 + (s-2)\Gamma_2$ is base-point free and $|mD|$ contains a smooth irreducible curve for each $m \geq 1$. Moreover if $C \equiv a\Gamma_1 + b\Gamma_2$ is any divisor satisfying $C \cdot \Gamma_1 > 0$ and $a > 1$ then $|C|$ contains a smooth irreducible curve.*

Indeed, for the final sentence we remark that $C-H$ is nef, hence base-point free since it is a linear combination of base-point free divisors, $\Gamma_2, H, (s-1)\Gamma_1+(s-2)\Gamma_2$, with non-negative coefficients.

Even though it seems that we do not get a result similar to Lemma 6.3 in full generality, we can at least use the first paragraph of its proof to get

$$H^i(S, \mathcal{O}_S(C)) = 0 \text{ for } i > 0 \text{ provided } a > s-4 \text{ and } (s-1)b \geq (s-2)a + s - 4 \tag{18}$$

because the assumptions on a, b imply that $C - K$ is nef and big. From this, we are led to

Theorem 7.3. *Let $S \subset \mathbb{P}^3$ be a smooth degree-s surface containing a line Γ_1, let $\Gamma_2 \equiv H - \Gamma_1$ be a smooth curve and suppose* $\mathrm{Pic}(S) \simeq \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2$ *and $s \geq 5$. Let $C \equiv a\Gamma_1 + b\Gamma_2$ be a smooth connected curve of degree $d > s^2$ with $a \neq b$.*

(i) *Suppose $a > s-4$ and $(s-1)b \geq (s-2)a + s - 4$. Then C belongs to a unique s-maximal family $W \subseteq \mathrm{H}(d,g)_{sc}$. Moreover if $\tilde{S}$ is a degree-s surface containing a very general member of W, then* $\mathrm{Pic}(\tilde{S})$ *is freely generated by the classes of a line and a smooth plane degree-$(s-1)$ curve, and every $C \equiv a\Gamma_1 + b\Gamma_2$ contained in some surface S as above belongs to W. Furthermore*

$$\dim W = (4-s)d + g + \binom{s+3}{3} + \binom{s-1}{3} - s + 1 \quad \textit{with } d, g \textit{ as in } (17),$$

and if $(a,b) \neq (2s-2, 2s-4)$ for $s = 5, 6$, then W is an irreducible component of $\mathrm{H}(d,g)_{sc}$.

(ii) *Suppose $s < a < \frac{(s-1)(b-1)}{s-2}$. Then all conclusions of* (i) *hold and W is a generically smooth irreducible component of* $\mathrm{H}(d,g)_{sc}$.

Proof. The proof follows the proof of the first part and I) of Theorem 1.2 except for W being an irreducible component in (i). Let us go through the main points.

The assumptions on a, b in (i) imply that $H^1(\mathcal{O}_S(C)) = 0$ by the Kawamata-Viehweg vanishing theorem (18). Moreover, if we replace (a,b) in (i) by $(a-4, b-4)$, we get exactly the assumptions of (ii), leading also to $H^1(\mathcal{O}_S(C-4H)) = 0$. Given the first vanishing, we now use Lemma 2.2 to get the stated properties of W in (i). We also get

$$\dim \operatorname{coker} \alpha_C = h^0(\mathcal{I}_{E/S}(s-4) = h^0(\mathcal{O}_{\mathbb{P}^3}(s-4)) - h^0(\mathcal{O}_E(s-4))\,.$$

Indeed since E is a line we have $\operatorname{coker} \alpha_E = 0$ by (4). It follows that $\dim W = \dim A^1(C \subset S)$ is given by the formula accompanying (4) recalling $A^2 \simeq \operatorname{coker} \alpha_C$, i.e. $\dim W$ is as stated. Finally having both vanishings, we also get (ii) using the smoothness of pr_1 at (C,S), cf. (6).

It remains to prove that W is an irreducible component also when $H^1(\mathcal{O}_S(C-4H)) \neq 0$, e.g. to show that $g > G(d, s+1)$, and in the case $g = G(d, s+1)$ to show $\dim V \leq \dim W$ for the component $V \subset H(d, G(d, s+1))_{sc}$ of curves of maximum genus mentioned in subsection 2.3. Noticing that

$$g - G(d, s+1) = \frac{d}{2}\left(2a - 1 - \frac{d}{s+1}\right) - \frac{a^2 s}{2} + \epsilon \tag{19}$$

by (13) and (17) where $\epsilon = \frac{r(s+1-r)s}{2(s+1)}$ and r is given by $d + r \equiv 0 \bmod (s+1)$ for $0 \leq r \leq s$, we first consider curves on the boundary $(s-1)b = (s-2)a$, i.e. where $C \equiv n(s-1)\Gamma_1 + n(s-2)\Gamma_2$. Inserting $a = n(s-1)$, $d = n(s-1)^2$ and $a^2 = nd$ into (19) and denoting $\epsilon(C) := \epsilon$ we get

$$g - G(d, s+1) = \frac{d}{2(s+1)}\left(n(s-3) - (s+1)\right) + \epsilon(C)\,.$$

It follows easily that $g \geq G(d, s+1)$ except in the cases $s = 5, 6$ and $(a, b) = (2s-2, 2s-4)$, and moreover that $g = G(d, s+1)$ only for $(s, n) \in \{(7, 2), (5, 3)\}$. The case $(s, n) = (5, 3)$ yields $d = 48$ and $g = 241$, cf. proof of Theorem 1.2. We get $\dim V = 237 < \dim W$. The case $(s, n) = (7, 2)$ is similar. Indeed let V be the component of $\mathrm{H}(d, g)_{sc}$ containing a curve C' of maximum genus $G(d, 6) = 469$ where $d = 72$. By (13) C' is a c.i. of type $(8, 9)$ with dualizing sheaf $\omega_{C'} \simeq \mathcal{O}_{C'}(13)$. We get $\dim V = 4d + h^1(\mathcal{O}_{C'}(8)) + h^1(\mathcal{O}_{C'}(9)) = 379$, while $\dim W = -3d + g + 134 = 387$ by Theorem 7.3, i.e. W is an irreducible component.

Finally, it suffices to show $g(D) > G(d(D), s+1)$ for $D \in |C + kH|$, $k > 0$ and C on the mentioned boundary. Using $d(D) = d + ks$ and (19), or directly $g(D) = g + kd + sk(s-4+k)/2$, we prove that

$$g(D) - G(d(D), s+1) - [g - G(d, s+1)] = \frac{k(2d+sk)}{2(s+1)} - \frac{sk}{2} + \epsilon(D) - \epsilon(C) \ . \tag{20}$$

To compute $\epsilon(D) - \epsilon(C)$, we remark that $\epsilon(C)$ does not change using the number ρ satisfying $d \equiv \rho \mod (s+1)$ for $0 < \rho \leq s+1$ instead of r in $\epsilon(C)$ because $r = s+1-\rho$. Since $d(D) = d + ks \equiv \rho - k \mod (s+1)$, we have (where the inequality is an equality if $\rho - k > 0$)

$$\epsilon(D) - \epsilon(C) \geq [(\rho - k)(s+1-\rho+k)s - \rho(s+1-\rho)s]/(2s+2) = -k(s+1-2\rho+k)s/(2s+2).$$

Combining with (20) we get

$$\frac{k(2d+sk)}{2(s+1)} - \frac{sk}{2} + \epsilon(D) - \epsilon(C) = \frac{k}{s+1}(d - s(s+1) + \rho s) > 0$$

because $d > s^2$ and $\rho \geq 1$. This shows $g(D) > G(d(D), s+1)$ for $(s, n) \notin \{(5, 2), (6, 2)\}$. In the cases $(s, n) \in \{(5, 2), (6, 2)\}$, a direct computation show $(d - s(s+1) + \rho s)/(s+1) = 2$ and $g - G(d, s+1) = -2$ (resp. -1) for $s = 5$ (resp. 6). Hence $g(D) > G(d(D), s+1)$ except in the case $(s, n, k) = (5, 2, 1)$ where we have $g(D) = G(d(D), 6) = 150$ and $d(D) = 37$, $D = C + H$. Since $\dim W = -d(D) + g(D) + 56 = 169$ by Theorem 7.3 while Theorem 1.3 with $s = 6$ (replacing W in Theorem 1.3 by V) implies $\dim V = -2d(D) + g(D) + 82 + 4 + 3 = 165 < \dim W$, W is also now an irreducible component and we are done. $\square$

Remark 7.4. *The components W of Theorem 7.3 may be non-reduced components of $\mathrm{H}(d, g)_{sc}$ in a range close to the boundary of the nef cone even though we only succeed to prove it for $s = 5$ and $(a, b) = (4n, 3n)$, $n \geq 3$ (Theorem 1.2). If we had been able to compute $h^0(\mathcal{N}_C)$ for a general curve C of W, we could conclude that all components satisfying $\dim W < h^0(\mathcal{N}_C)$ were non-reduced. Another promising approach to deal with this problem is to compute the cup-product of an element of $H^0(\mathcal{N}_C)$ and show that it is non-zero, as Mukai and Nasu do in [33, 35, 36] by using the role of effective divisors with negative self-intersection in linear systems corresponding to $|C - 4H|$.*

8 Appendix on non-reduced components of $\mathrm{H}(d, g)_{sc}$ for $s = 3$

by Jan O. Kleppe

In this section we look at progress to the conjecture below. Note that a maximal family W is closed and irreducible by our definition, and that $\dim W = d + g + 18$ always holds provided $d > 9$.

Conjecture 8.1. *Let W be a 3-maximal family of smooth connected, linearly normal space curves of degree $d > 9$ and genus g, whose general member C is contained in a smooth cubic surface. Then W is a non-reduced irreducible component of $\mathrm{H}(d, g)_{sc}$ if and only if*

$$d \geq 14, \quad 3d - 18 \leq g \leq (d^2 - 4)/8 \quad \textit{and} \quad H^1(\mathcal{I}_C(3)) \neq 0\,.$$

This conjecture, originating in [21], is here presented by modifications proposed by Ellia [4] (see also [2] by Dolcetti, Pareshi), because they found counterexamples which heavily depended on the fact the general curves were *not* linearly normal (i.e. the curves satisfied $H^1(\mathcal{I}_C(1)) \neq 0$).

The conjecture is known to be true in many cases. Indeed Mumford's well known example ([34]) of a non-reduced component is in the range of Conjecture 8.1 (minimal with respect to both degree and genus). Also the main result by the author in [20] shows that the conjecture holds provided $g > 7 + (d-2)^2/8$, $d \geq 18$, and Ellia makes further progresses in [4] which we comment on later. Recently Nasu proves (and reproves) a part of the conjecture by showing that the cup-product

$$H^0(\mathcal{N}_C) \times H^0(\mathcal{N}_C) \to H^1(\mathcal{N}_C)$$

is nonzero if $h^1(\mathcal{I}_C(3)) = 1$ ([36]). In this section we will see that the methods of [20] and a nice result of Ellia in [4] imply that we can greatly enlarge the range where Conjecture 8.1 holds.

Now recall that a smooth cubic surface S is obtained by blowing up $\mathbb{P}^2$ in six general points (see [16] and [15]). Taking the linear equivalence classes of the inverse image of a line in $\mathbb{P}^2$ and $-E_i$ (minus the exceptional divisors), $i = 1,..,6$, as a basis for $\mathrm{Pic}(S)$, we can associate a curve C on S and its corresponding invertible sheaf $\mathcal{O}_S(C)$ with a 7-tuple of integers $(\delta, m_1, .., m_6)$ satisfying

$$\delta \geq m_1 \geq .. \geq m_6 \quad and \quad \delta \geq m_1 + m_2 + m_3 \,. \tag{21}$$

The degree and the (arithmetic) genus of the curve are given by

$$d = 3\,\delta - \sum_{i=1}^{6} m_i \;, \quad g = \binom{\delta - 1}{2} - \sum_{i=1}^{6} \binom{m_i}{2} .$$

In terms of a 7-tuple $(\delta, m_1, .., m_6)$ satisfying (21) one may use Kodaira vanishing theorem and a further analysis (see [20, Lem. 16 and Cor. 17]) to verify the following facts for a curve C;

(A) If $m_6 \geq 3$ and $(\delta, m_1, .., m_6) \neq (\lambda + 9, \lambda + 3, 3, ..3)$ for any $\lambda \geq 2$, then $\mathrm{H}^1(\mathcal{I}_C(3)) = 0$. In particular if a curve on a smooth cubic satisfies $g > (d^2 - 4)/8$, then

$$H^1(\mathcal{I}_C(3)) = 0 \,.$$

(B) If $m_6 \geq 1$ and $(\delta, m_1, .., m_6) \neq (\lambda + 3, \lambda + 1, 1, ..1)$ for any $\lambda \geq 2$, then $H^1(\mathcal{I}_C(1)) = 0$. Moreover, in the range $d \geq 14$ and $g \geq 3d - 18$, we have

$$H^1(\mathcal{I}_C(3)) \neq 0 \text{ and } H^1(\mathcal{I}_C(1)) = 0 \quad \text{if and only if} \quad 1 \leq m_6 \leq 2 \,.$$

Remark 8.2. *i) The explicit size of the interval where $H^1_*(\mathcal{I}_C) := \oplus_v H^1(\mathcal{I}_C(v))$ is non-vanishing (and a proof of it) was originally found by Peskine and Gruson (see [21, Prop. 3.1.3]).*

ii) The case $m_6 = 0$ is treated by Dolcetti and Pareshi in [2]. In this case they found a range in the (d, g)-plane where the maximal subsets W were contained *in a non-reduced component of dimension $> d + g + 18$, see also [4, Rem. VI.6].*

Using (A) and the fact that $H^1(\mathcal{I}_C(3)) = 0$ implies unobstructedness and $\dim W = d + g + 18$ (for $d > 9$), one may easily see that the conditions of Conjecture 8.1 are necessary for W to be a non-reduced component. The conjecture therefore really deals with the converse, and we may suppose $m_6 = 1$ or 2 by (B). For both values the main theorem of this section shows that the conjecture is true under weak assumptions, thus generalizing the main results of [4] and [20] to:

Theorem 8.3. *Let W be a 3-maximal family of smooth connected space curves, whose general member sits on a smooth cubic surface S and corresponds to the 7-tuple $(\delta, m_1, .., m_6)$, $\delta \geq m_1 \geq .. \geq m_6$ and $\delta \geq m_1 + m_2 + m_3$, of* $\mathrm{Pic}(S)$. *Then*

i) W is a generically smooth, irreducible component of $\mathrm{H}(d, g)_{sc}$ *provided*

$$m_6 \geq 3 \text{ and } (\delta, m_1, .., m_6) \neq (\lambda + 9, \lambda + 3, 3, ..3) \text{ for any } \lambda \geq 2,$$

ii) W is a non-reduced irreducible component of $\mathrm{H}(d, g)_{sc}$ *provided;*

a) $m_6 = 2$, $m_5 \geq 4$, $d \geq 21$ *and* $(\delta, m_1, .., m_6) \neq (\lambda + 12, \lambda + 4, 4, .., 4, 2)$ *for any* $\lambda \geq 2$, *or*

b) $m_6 = 1$, $m_5 \geq 6$, $d \geq 35$ *and* $(\delta, m_1, .., m_6) \neq (\lambda + 18, \lambda + 6, 6, .., 6, 1)$ *for any* $\lambda \geq 2$, *or*

c) $m_6 = 1, m_5 = 5, m_4 \geq 7$, $d \geq 35$ *and* $(\delta, m_1, .., m_6) \neq (\lambda + 21, \lambda + 7, 7, .., 7, 5, 1)$ *for* $\lambda \geq 2$.

In the exceptional case $(\lambda+9, \lambda+3, 3, .., 3)$ of i) we have $H^1(\mathcal{O}_C(3)) = 0$; whence W is contained in a unique generically smooth irreducible component V of $\mathrm{H}(d, g)_{sc}$ and $\dim V - \dim W = h^1(\mathcal{I}_C(3))$ (cf. [20, Thm. 1]). For $m_6 = 2$ in ii) Nasu's result in [36] gives a better range, see Remark 8.6 ii).

To prove Theorem 8.3, we will need the following two results:

Proposition 8.4. *(Ellia) Let d and g be integers such that $d \geq 21$ and $g \geq 3d - 18$, let W be as in Theorem 8.3 and suppose the general curve C of W satisfies $H^1(\mathcal{I}_C(1)) = 0$. If C' is a generization of C in* $\mathrm{H}(d, g)_{sc}$ *satisfying $H^0(\mathcal{I}_{C'}(3)) = 0$, then $H^0(\mathcal{I}_{C'}(4)) = 0$.*

Proof. See [4, Prop. VI.2]. □

We remark that Ellia uses this key proposition to prove the conjecture provided $d \geq 21$ and $g > G(d, 5)$, cf. (12). His result is in most cases clearly better than the result in [20] which requires $g > 7 + (d-2)^2/8$, $d \geq 18$, because $G(d, 5) = d^2/10 + d/2 + \epsilon$, ϵ a correction term. There are, however, many cases where Theorem 8.3 implies the conjecture while Ellia's result does not.

Lemma 8.5. *Let C be a curve sitting on a smooth cubic surface S, whose corresponding invertible sheaf is given by $(\delta, m_1, .., m_6)$, $\delta \geq m_1 \geq .. \geq m_6$ and $\delta \geq m_1 + m_2 + m_3$. If v is a non-negative integer such that $m_3 \geq v$, and $(\delta, m_1, .., m_6) \neq (\lambda + 3v, \lambda + v, v, v, m_4, m_5, m_6)$ for any $\lambda \geq 2$, then*

$$h^0(\mathcal{I}_C(v)) - h^1(\mathcal{I}_C(v)) \geq \binom{v}{3} - \sum_{m_i < v} \binom{v + 1 - m_i}{2}$$

where the sum is taken among those $i \in \{4, 5, 6\}$ satisfying $m_i < v$.

Proof. Let $b_i := \max\{0, m_i - v\}$ and notice that the invertible sheaf $\mathcal{L}$, given by $(\delta - 3v, b_1, .., b_6)$, is generated by global sections because $b_6 \geq 0$ and $\delta - 3v \geq b_1 + b_2 + b_3$ (cf. [15, Sect. 2]). Moreover $(\delta - 3v, b_1, .., b_6) \neq (\lambda, \lambda, 0, .., 0)$ for $\lambda \geq 2$ by assumption; whence $H^0(\mathcal{L})$ contains a smooth connected curve $\overline{D}$ (take $\overline{D} = 0$ in the special case $(\delta - 3v, b_1, .., b_6) = (\lambda, \lambda, 0, .., 0)$ with $\lambda = 0$).

Let $n_i := -\min\{0, m_i - v\}$ for $i \in \{4, 5, 6\}$, let $F := \sum n_i E_i$ and observe that $D := \overline{D} + F$ is an effective divisor (or zero) of the linear system $|C - vH|$ corresponding to $(\delta - 3v, m_1 - v, .., m_6 - v)$. By e.g. the algorithm of [11, Rem. 2.7], for finding the Zariski decomposition, it is clear that F is the fixed component of $|D|$. Now, as in Lem. 2.5 and Cor. 2.6 of [36], taking global sections of the sequence $0 \to \mathcal{I}_{C/S}(v) \simeq \mathcal{O}_S(-C + vH) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$, we get

$$h^1(\mathcal{I}_C(v)) = h^1(\mathcal{I}_{C/S}(v)) = h^0(\mathcal{O}_D) - 1 = h^0(\mathcal{O}_{\overline{D}}) + h^0(\mathcal{O}_F) - 1 = h^0(\mathcal{O}_F)$$

for $\overline{D} \neq 0$ because $\overline{D} \cdot F = 0$ ($h^1(\mathcal{I}_C(v)) = h^0(\mathcal{O}_F) - 1$ for $\overline{D} = 0$). The lines E_i are skew and we get $h^0(\mathcal{O}_F) = \sum h^0(\mathcal{O}_{n_i E_i}) = \sum \binom{n_i+1}{2}$. Finally $h^0(\mathcal{I}_C(v)) = h^0(\mathcal{I}_{C/S}(v)) + \binom{v}{3} \geq \binom{v}{3}$ (equality holds, but we don't need it) and we are done. □

Proof of Theorem 8.3. i) is a special case of [20, Thm. 1] since $m_3 \geq 3$ easily implies $d > 9$ or that the general curve C is a c.i. of type (or bidegree) $(3,3)$.

ii) By (7) we get

$$\dim W + h^1(\mathcal{I}_C(3)) = h^0(\mathcal{N}_C) \ . \tag{22}$$

Since $h^1(\mathcal{I}_C(3)) \neq 0$, it suffices to prove that W is an irreducible *component* of $\mathrm{H}(d,g)_{sc}$ because if it is, then $\dim W < h^0(\mathcal{N}_C)$ implies that the general curve C of W is obstructed, i.e. W is non-reduced.

a) To get a contradiction, suppose W is *not* a component. Since W is a maximal family of curves on a cubic surface, there exists a generization C' of C satisfying $h^0(\mathcal{I}_{C'}(3)) = 0$. By semi-continuity, $h^1(\mathcal{O}_{C'}(4)) \leq h^1(\mathcal{O}_C(4))$. Combining with $\chi(\mathcal{I}_{C'}(4)) = \chi(\mathcal{I}_C(4))$, it follows that $h^0(\mathcal{I}_{C'}(4)) - h^1(\mathcal{I}_{C'}(4)) \geq h^0(\mathcal{I}_C(4)) - h^1(\mathcal{I}_C(4))$. However, by Lemma 8.5, we have $h^0(\mathcal{I}_C(4)) - h^1(\mathcal{I}_C(4)) \geq 1$, hence $h^0(\mathcal{I}_{C'}(4)) \geq 1$. Since the curve is linearly normal by (B), this inequality contradicts the conclusion of Proposition 8.4.

b) Again it suffices to prove that W is an irreducible component of $\mathrm{H}(d,g)_{sc}$. To get a contradiction we suppose there is a generization C' of C satisfying $h^0(\mathcal{I}_{C'}(3)) = 0$. By semi-continuity of $h^1(\mathcal{O}_C(v))$ and Lemma 8.5, we get

$$h^0(\mathcal{I}_{C'}(v)) - h^1(\mathcal{I}_{C'}(v)) \geq h^0(\mathcal{I}_C(v)) - h^1(\mathcal{I}_C(v)) \geq \tbinom{v}{3} - \tbinom{v}{2} \text{ for } 1 \leq v \leq 6 \ .$$

Hence $h^0(\mathcal{I}_{C'}(6)) - h^1(\mathcal{I}_{C'}(6)) \geq 5$. Since $s(C') \geq 5$ by Proposition 8.4 and (B), C' is contained in a c.i. of bidegree (5,6) or (6,6). Hence $d \leq 36$ and we have a contradiction except when $d = 35$ or 36. In the case $d = 36$, C' is a c.i. satisfying $h^0(\mathcal{I}_{C'}(6)) \geq 5$, and if $d = 35$, we can link C' to a line D satisfying $h^1(\mathcal{O}_D(2)) \neq 0$ (because $h^0(\mathcal{I}_{C'}(6)) > 2$), i.e. we get a contradiction in both cases, and we are done.

c) The proof is similar to b), remarking only that we now have $h^0(\mathcal{I}_{C'}(6)) \geq 4$ and $h^0(\mathcal{I}_{C'}(7)) \geq 11$ by Lemma 8.5, i.e. C' is contained in a c.i. of bidegree (5,7) or (6,6), and since the case where C' is a c.i. of bidegree (5,7) can not occur (the dimension of an irreducible component of $\mathrm{H}(d,g)_{sc}$ whose general curve is a c.i. of type $(5,7)$ is much smaller than $d+g+18$) we conclude as in b). $\square$

Remark 8.6. *i) Theorem 8.3 (without c) of ii)) was lectured at a workshop organized by the "Space Curves group" of Europroj, at the Emile Borel Center, Paris in May 1995, and may be known to some experts in the field (cf. [17, p. 95]), but it has not been published. The appendix in the preprint [23] covers the important results of the talk, and much of the material is included here. Note that we in Lemma 19 of [23] should replace equality by inequality, exactly as we now do in the displayed formula of Lemma 8.5 (we see from its proof that equality almost always holds, except when $\overline{D} = 0$). This correction does no harm to the arguments of Theorem 8.3 since it is precisely the inequality we need in its proof. In the proof of Lemma 8.5 we follow closely corresponding results in [36] which is based on making the fixed component of $|C - vH|$ explicit. Lemma 8.5 for $v = 4$ imply Lemma 18 of [20].*

ii) The case a) of Theorem 8.3 ii) is fully generalized in [36]. Indeed Nasu shows that the cup-product (primary obstruction) of the general curve of any maximal family W satisfying $m_6 = 2$ and $m_5 \geq 3$ is non-vanishing. We think his approach may be adequate for proving the whole conjecture.

Finally using Proposition 4.4 for $s = 5$ and closely following the proof of Theorem 4.1 (replacing $\dim W = g + 33$ by $\dim W = d + g + 18$ in the argument and noticing that a generization C' satisfies $s(C') \geq 5$ by Ellia's Proposition 8.4), we immediately get the following result.

Proposition 8.7. *Let W be a 3-maximal family of smooth connected space curves, whose general member is linearly normal and sits on a smooth cubic surface. If*

$$g > \max\{\frac{d^2}{10} - \frac{d}{2} + 18,\ G(d,6)\}\,,\ d \geq 31\,, \tag{23}$$

then W is an irreducible component of $\mathrm{H}(d,g)_{sc}$. Moreover, W is non-reduced if and only if $H^1(\mathcal{I}_C(3)) \neq 0$. In particular Conjecture 8.1 holds in the range (23).

Note that we in (23) have $G(d,6) \geq \frac{d^2}{10} - \frac{d}{2} + 18$ if and only if $d \leq 74$. We can weaken the assumption $g > G(d,6)$ by using Proposition 4.4 also for $s = 6$ and 7. Indeed for any t such that $6 \leq t \leq 8$ we can conclude as in Proposition 8.7 provided $g > \max\{ \frac{d^2}{10} - \frac{d}{2} + 18,\ G(d,t) \}$, $d > t(t-1)$. Since $G(d,8) \leq \frac{d^2}{10} - \frac{d}{2} + 18$ for $d \geq 58$, we obtain all conclusions of Proposition 8.7 in the range

$$g > \frac{d^2}{10} - \frac{d}{2} + 18,\ d \geq 58\,. \tag{24}$$

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