DIMENSION OF FAMILIES OF DETERMINANTAL SCHEMES.

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Abstract. A scheme $X \subset \mathbb{P}^{n+c}$ of codimension c is called *standard determinantal* if its homogeneous saturated ideal can be generated by the maximal minors of a homogeneous $t \times (t+c-1)$ matrix and X is said to be *good determinantal* if it is standard determinantal and a generic complete intersection. Given integers $a_0, a_1, ..., a_{t+c-2}$ and $b_1, ..., b_t$ we denote by $W(\underline{b};\underline{a}) \subset \mathrm{Hilb}^p(\mathbb{P}^{n+c})$ (resp. $W_s(\underline{b};\underline{a})$) the locus of good (resp. standard) determinantal schemes $X \subset \mathbb{P}^{n+c}$ of codimension c defined by the maximal minors of a $t \times (t+c-1)$ matrix $(f_{ij})^{i=1,...,t}_{j=0,...,t+c-2}$ where $f_{ij} \in k[x_0, x_1, ..., x_{n+c}]$ is a homogeneous polynomial of degree $a_j - b_i$.

In this paper we address the following three fundamental problems : To determine (1) the dimension of $W(\underline{b};\underline{a})$ (resp. $W_s(\underline{b};\underline{a})$) in terms of a_j and b_i, (2) whether the closure of $W(\underline{b};\underline{a})$ is an irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^{n+c})$, and (3) when $\mathrm{Hilb}^p(\mathbb{P}^{n+c})$ is generically smooth along $W(\underline{b};\underline{a})$. Concerning question (1) we give an upper bound for the dimension of $W(\underline{b};\underline{a})$ (resp. $W_s(\underline{b};\underline{a})$) which works for all integers $a_0, a_1, ..., a_{t+c-2}$ and $b_1, ..., b_t$, and we conjecture that this bound is sharp. The conjecture is proved for $2 \le c \le 5$, and for $c \ge 6$ under some restriction on $a_0, a_1, ..., a_{t+c-2}$ and $b_1, ..., b_t$. For questions (2) and (3) we have an affirmative answer for $2 \le c \le 4$ and $n \ge 2$, and for $c \ge 5$ under certain numerical assumptions.

Contents

1. Introduction

In this paper, we will deal with *determinantal* schemes, i.e. schemes defined by the vanishing locus of the minors of a homogeneous polynomial matrix. Some classical schemes that can be constructed in this way are the Segre varieties, the rational normal scrolls and the Veronese varieties. Determinantal schemes have been a central topic in both commutative algebra and algebraic geometry and, due to their important role, their study

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has attracted many researchers and has received considerable attention in the literature. Some of the most remarkable results about determinantal schemes are due to J.A. Eagon and M. Hochster in [8], and to J.A. Eagon and D.G. Northcott in [9]. Eagon and Hochster proved that generic determinantal schemes are arithmetically Cohen-Macaulay. Eagon and Northcott constructed a finite free resolution for any standard determinantal scheme and as a corollary they got that standard determinantal schemes are arithmetically Cohen-Macaulay. Since then many authors have made important contributions to the study of determinantal schemes and the reader can look at [5], [23], [4] and [10] for background, history and a list of important papers.

A scheme $X \subset \mathbb{P}^{n+c}$ of codimension c is called *standard determinantal* if its homogeneous saturated ideal can be generated by the maximal minors of a homogeneous $t \times (t+c-1)$ matrix and X is said to be *good determinantal* if it is standard determinantal and a generic complete intersection (see Remark 2.2). In this paper, we address the problem of determining the dimension of the family of standard (resp. good) determinantal schemes. The first important contribution to this problem is due to G. Ellingsrud [12]; in 1975, he proved that every arithmetically Cohen-Macaulay, codimension 2 closed subscheme X of $\mathbb{P}^{n+2}$ is unobstructed (i.e. the corresponding point in the Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}^{n+2})$ is smooth) provided $n \geq 1$ and he also computed the dimension of the Hilbert scheme at X. Recall also that the homogeneous ideal of an arithmetically Cohen-Macaulay, codimension 2 closed subscheme X of $\mathbb{P}^{n+2}$ is given by the maximal minors of a $t \times (t+1)$ homogeneous matrix, the Hilbert-Burch matrix. That is, such a scheme is standard determinantal. The purpose of this work is to extend Ellingsrud's Theorem, viewed as a statement on standard determinantal schemes of codimension 2, to arbitrary codimension. The case of codimension 3, is solved in [19]; Proposition 1.12. In the present work, using essentially the methods developed in [19]; §10, we succeed in generalizing to arbitrary codimension the formula for the dimension of families of determinantal schemes provided certain weak numerical conditions are satisfied (see Theorem 3.5, Proposition 3.13 and Corollaries 4.7, 4.10, 4.14, 4.15 and 4.18). We also address the problem whether the closure of the locus W of determinantal schemes in $\mathbb{P}^{n+c}$ is an irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^{n+c})$ and when $\mathrm{Hilb}^p(\mathbb{P}^{n+c})$ is generically smooth along W (see Corollaries 5.3, 5.7, 5.9 and 5.10).

Next we outline the structure of the paper. In section 2, we recall the basic facts on standard and good determinantal schemes $X \subset \mathbb{P}^{n+c}$ of codimension c defined by the maximal minors of a $t \times (t+c-1)$ homogeneous matrix and the associated complexes needed later on. Sections 3-5 are the heart of the paper. Given integers $b_1, ..., b_t$ and $a_0, a_1, ..., a_{t+c-2}$, we denote by $W(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^p(\mathbb{P}^{n+c})$ the locus of good determinantal schemes $X \subset \mathbb{P}^{n+c}$ of codimension $c \geq 2$ defined by the maximal minors of a homogeneous matrix $\mathcal{A} = (f_{ji})^{i=1,...,t}_{j=0,...,t+c-2}$ where $f_{ji} \in k[x_0, ..., x_{n+c}]$ is a homogeneous polynomial of degree $a_j - b_i$. The goal of section 3 is to give an upper bound for the dimension of $W(\underline{b}; \underline{a})$ in terms of $b_1, ..., b_t$ and $a_0, a_1, ..., a_{t+c-2}$ (cf. Theorem 3.5 and Proposition 3.13). To this end we proceed by induction on c by successively deleting columns of the largest possible degree and we repeatedly use the Eagon-Northcott complexes and the Buchsbaum-Rim complexes associated to a standard determinantal scheme. In section 4, using again induction on the codimension and the theory of Hilbert flag schemes, we analyze when the upper bound of $\dim W(\underline{b};\underline{a})$ given in section 3 is indeed the dimension of the determinantal locus. In turns out that the upper bound of $\dim W(\underline{b};\underline{a})$ given in Theorem 3.5 is sharp in a number of instances. More precisely, if $2 \le c \le 3$, this is known ([19], [12]), for $4 \le c \le 5$ it is a consequence of the main theorem of this section (see Corollaries 4.10 and 4.14), while for $c \ge 6$ we get the expected dimension formula for $W(\underline{b};\underline{a})$ under more restrictive assumptions (see Corollary 4.15). In section 5, we study when the closure of $W(\underline{b};\underline{a})$ is an irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^{n+c})$ and when $\mathrm{Hilb}^p(\mathbb{P}^{n+c})$ is generically smooth along $W(\underline{b};\underline{a})$, and other cases of unobstructedness. The main result of this section (Theorem 5.1) shows that the closure of $W(\underline{b};\underline{a})$ is a generically smooth irreducible component provided the zero degree pieces of certain Ext^1-groups vanish. The conditions of the Theorem can be shown to be satisfied in a wide number of cases which we make explicit through this section. In particular we show that the mentioned Ext^1-groups vanish if $3 \le c \le 4$ (Corollary 5.3). Similarly, in Corollaries 5.7, 5.9 and 5.10 and as a consequence of Theorem 5.1, we prove that under certain numerical assumptions the closure of $W(\underline{b};\underline{a})$ is indeed a generically smooth, irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^{n+c})$ of the expected dimension. In Examples 5.6 and 5.8, we show that this is not always the case, although the examples created are somewhat special because all the entries of the associated matrix are linear entries.

We end the paper with a Conjecture raised by this paper and proved in many cases (cf. Conjectures 6.1 and 6.2), and we correct an inaccuracy in [19].

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Notation: Throughout this paper $\mathbb{P}^N$ will be the N-dimensional projective space over an algebraically closed field k, $R = k[x_0, x_1, \ldots, x_N]$ and $\mathfrak{m} = (x_0, \ldots, x_N)$. The sheafification of a graded R-module M will be denoted by $\tilde{M}$ and the support of M by $Supp(M)$.

For any closed subscheme X of $\mathbb{P}^N$ of codimension c, we denote by $\mathcal{I}_X$ its ideal sheaf, $\mathcal{N}_X$ its normal sheaf, $I(X) = H^0_*(\mathcal{I}_X)$ its saturated homogeneous ideal and $\omega_X = \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^N})(-N-1)$ its canonical sheaf. If $\mathcal{F}$ and $\mathcal{G}$ are two coherent $\mathcal{O}_X$-modules, we denote the group of morphisms from $\mathcal{F}$ to $\mathcal{G}$ by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ while $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ denotes the sheaf of local morphisms of $\mathcal{F}$ into $\mathcal{G}$. We often omit $\mathcal{O}_X$ in $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ (resp. $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$) if the underlying scheme X is evident. Moreover we set $\hom(\mathcal{F}, \mathcal{G}) = \dim_k \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ and $\mathrm{aut}(\mathcal{F}) = \hom(\mathcal{F}, \mathcal{F})$ where $\dim_k$ denotes the dimension as k-vector space. These dimensions coincide with the dimensions of $\mathbb{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and $\mathbb{A}ut_{\mathcal{O}_X}(\mathcal{F})$ as schemes.

For any quotient A of R of codimension c, we let $I_A = \ker(R \twoheadrightarrow A)$, $N_A = \mathrm{Hom}_R(I_A, A)$ be the normal module and $K_A = \mathrm{Ext}^c_R(A, R)(-N-1)$ be its canonical module. When we write $X = Proj(A)$, we let $A = R/I(X)$ and $K_X = K_A$.

We denote the Hilbert scheme by $\mathrm{Hilb}^p(\mathbb{P}^N)$ (cf. [13]). Thus, any point $p_X \in \mathrm{Hilb}^p(\mathbb{P}^N)$ parameterizes a subscheme $X \subset \mathbb{P}^N$ with Hilbert polynomial $p \in \mathbb{Q}[s]$. By abuse of notation we will write $X \in \mathrm{Hilb}^p(\mathbb{P}^N)$. By definition $X \in \mathrm{Hilb}^p(\mathbb{P}^N)$ is unobstructed if $\mathrm{Hilb}^p(\mathbb{P}^N)$ is smooth at X.

The pullback of the universal family on $\mathrm{Hilb}^p(\mathbb{P}^N)$ via a morphism $\psi : W \longrightarrow \mathrm{Hilb}^p(\mathbb{P}^N)$ yields a flat family over W, and we will write $X \in W$ for a member of that family as well. By definition a general $X \in W$ has a certain property if there is a non-empty open dense subset U of W such that all members of U have this property.

2. Preliminaries

This section provides the background and basic results on standard (resp. good) determinantal schemes needed in the sequel, and we refer to [5] and [10] for more details.

Let $\mathcal{A}$ be a homogeneous matrix, i.e. a matrix representing a degree 0 morphism ϕ of free graded R-modules. In this case, we denote by $I(\mathcal{A})$ (or $I(\phi)$) the ideal of R generated by the maximal minors of $\mathcal{A}$.

Definition 2.1. A codimension c subscheme $X \subset \mathbb{P}^{n+c}$ is called a *standard determinantal* scheme if $I(X) = I(\mathcal{A})$ for some $t \times (t+c-1)$ homogeneous matrix $\mathcal{A}$. $X \subset \mathbb{P}^{n+c}$ is called a *good determinantal* scheme if additionally, $\mathcal{A}$ contains a $(t-1) \times (t+c-1)$ submatrix (allowing a change of basis if necessary) whose ideal of maximal minors defines a scheme of codimension $c+1$.

Remark 2.2. It is well known that a good determinantal scheme $X \subset \mathbb{P}^{n+c}$ is standard determinantal and the converse is true provided X is a generic complete intersection, cf. [22].

Now we are going to describe the generalized Koszul complexes associated to a codimension c standard determinantal scheme X. To this end, we denote by $\varphi : F \longrightarrow G$ the morphism of free graded R-modules of rank t and $t+c-1$, defined by the homogeneous matrix $\mathcal{A}$ of X. We denote by $\mathcal{C}_i(\varphi^*)$ the generalized Koszul complex:

$$\mathcal{C}_i(\varphi^*):\ 0 \longrightarrow \wedge^i G^* \otimes S_0(F^*) \longrightarrow \wedge^{i-1} G^* \otimes S_1(F^*) \longrightarrow \dots \longrightarrow \wedge^0 G^* \otimes S_i(F^*) \longrightarrow 0.$$

Let $\mathcal{C}_i(\varphi^*)^*$ be the R-dual of $\mathcal{C}_i(\varphi^*)$. The map φ induces graded morphisms

$$\mu_i : \wedge^{t+i} G^* \otimes \wedge^t F \longrightarrow \wedge^i G^*.$$

They can be used to splice the complexes $\mathcal{C}_{c-i-1}(\varphi^*)^* \otimes \wedge^{t+c-1} G^* \otimes \wedge^t F$ and $\mathcal{C}_i(\varphi^*)$ to a complex $\mathcal{D}_i(\varphi^*)$:

$$0 \longrightarrow \wedge^{t+c-1} G^* \otimes S_{c-i-1}(F) \otimes \wedge^t F \longrightarrow \wedge^{t+c-2} G^* \otimes S_{c-i-2}(F) \otimes \wedge^t F \longrightarrow \dots \longrightarrow$$
$$\wedge^{t+i} G^* \otimes S_0(F) \otimes \wedge^t F \longrightarrow \wedge^i G^* \otimes S_0(F^*) \longrightarrow \wedge^{i-1} G^* \otimes S_1(F^*) \longrightarrow \dots \longrightarrow \wedge^0 G^* \otimes S_i(F^*) \longrightarrow 0.$$

The complex $\mathcal{D}_0(\varphi^*)$ is called the *Eagon-Northcott complex* and the complex $\mathcal{D}_1(\varphi^*)$ is called the *Buchsbaum-Rim complex*. Let us rename the complex $\mathcal{C}_c(\varphi^*)$ as $\mathcal{D}_c(\varphi^*)$. Then we have the following well known result:

Proposition 2.3. *Let $X \subset \mathbb{P}^{n+c}$ be a standard determinantal subscheme of codimension c associated to a graded minimal (i.e. $\mathrm{im}(\varphi) \subset \mathfrak{m}G$) morphism $\varphi : F \longrightarrow G$ of free R-modules of rank t and $t+c-1$, respectively. Set $M = \mathrm{coker}(\varphi^*)$. Then*

(i) $\mathcal{D}_i(\varphi^*)$ is acyclic for $-1 \le i \le c$.

(ii) $\mathcal{D}_0(\varphi^*)$ is a minimal free graded R-resolution of $R/I(X)$ and $\mathcal{D}_i(\varphi^*)$ is a minimal free graded R-resolution of length c of $S_i(M)$, $1 \le i \le c$.

(iii) $K_X \cong S_{c-1}(M)$ up to degree shift . So, up to degree shift, $\mathcal{D}_{c-1}(\varphi^*)$ is a minimal free graded R-module resolution of K_X.

Proof. See, for instance [5]; Theorem 2.20 and [10]; Corollary A2.12 and Corollary A2.13. □

Remark 2.4. By Proposition 2.3(ii), any standard determinantal scheme $X \subset \mathbb{P}^{n+c}$ is arithmetically Cohen-Macaulay (briefly, ACM). Moreover, in codimension 2, the converse is true: If $X \subset \mathbb{P}^{n+2}$ is an ACM, closed subscheme of codimension 2 then it is standard determinantal (Hilbert-Burch Theorem).

The homogeneous matrix $\mathcal{A}$ associated to a standard determinantal scheme $X \subset \mathbb{P}^{n+c}$ of codimension c also defines an injective morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ of locally free $\mathcal{O}_{\mathbb{P}^{n+c}}$-modules of rank t and $t+c-1$. Since the construction of the generalized Koszul complexes globalizes, we can also associated to φ^* the *Eagon-Northcott complex* of $\mathcal{O}_{\mathbb{P}^{n+c}}$-modules

$$0 \longrightarrow \wedge^{t+c-1}\mathcal{G}^* \otimes S_{c-1}(\mathcal{F}) \otimes \wedge^t\mathcal{F} \longrightarrow \wedge^{t+c-2}\mathcal{G}^* \otimes S_{c-2}(\mathcal{F}) \otimes \wedge^t\mathcal{F} \longrightarrow \dots \longrightarrow$$

$$\wedge^{t}\mathcal{G}^* \otimes S_{0}(\mathcal{F}) \otimes \wedge^t\mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^{n+c}} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

and the *Buchsbaum-Rim complex* of locally free $\mathcal{O}_{\mathbb{P}^{n+c}}$-modules

$$0 \longrightarrow \wedge^{t+c-1}\mathcal{G}^* \otimes S_{c-2}(\mathcal{F}) \otimes \wedge^t\mathcal{F} \longrightarrow \wedge^{t+c-2}\mathcal{G}^* \otimes S_{c-3}(\mathcal{F}) \otimes \wedge^t\mathcal{F} \longrightarrow \dots \longrightarrow$$

$$\wedge^{t+1}\mathcal{G}^* \otimes S_{0}(\mathcal{F}) \otimes \wedge^t\mathcal{F} \longrightarrow \mathcal{G}^* \xrightarrow{\varphi^*} \mathcal{F}^* \longrightarrow \tilde{M} \longrightarrow 0.$$

Since the degeneracy locus of φ^* has codimension c, these two complexes are acyclic. Moreover, the kernel of φ^* is called the *1st Buchsbaum-Rim sheaf* associated to φ^*.

Let $X \subset \mathbb{P}^{n+c}$ be a standard (resp. good) determinantal scheme of codimension $c \ge 2$ defined by the vanishing of the maximal minors of a $t\times(t+c-1)$ matrix $\mathcal{A} = (f_{ji})_{i=1,...t}^{j=0,...,t+c-2}$ where $f_{ji} \in k[x_0, ..., x_{n+c}]$ are homogeneous polynomials of degree $a_j - b_i$ with $b_1 \le ... \le b_t$ and $a_0 \le a_1 \le ... \le a_{t+c-2}$. We assume without loss of generality that $\mathcal{A}$ is minimal; i.e., $f_{ji} = 0$ for all i, j with $b_i = a_j$. If we let $u_{ji} = a_j - b_i$ for all $j = 0, \dots, t+c-2$ and $i = 1, \dots, t$, the matrix $\mathcal{U} = (u_{ji})_{i=1,...t}^{j=0,...,t+c-2}$ is called the *degree matrix* associated to X. We have:

Lemma 2.5. *The matrix $\mathcal{U}$ has the following properties:*

(i) *For every j and i, $u_{j,i} \le u_{j+1,i}$ and $u_{j,i} \ge u_{j,i+1}$.*
(ii) *For every $i = 1, \dots, t$, $u_{i-1,i} = a_{i-1} - b_i > 0$.*

And vice versa, given a degree matrix $\mathcal{U}$ of integers verifying (i) and (ii) there exists a codimension c standard (resp. good) determinantal scheme $X \subset \mathbb{P}^{n+c}$ with associated degree matrix $\mathcal{U}$.

Proof. The first condition is obvious. For the second one we only need to observe that if for some $i = 1, \dots, t$, we have $u_{i-1,i} \leq 0$ then in the matrix $\mathcal{A}$ we have $f_{j,k} = 0$ for $j \leq i-1$ and $k \geq i$. But this would imply that the minor which is obtained by deleting the last $c-1$ columns has to be zero contradicting the minimality of $\mathcal{A}$.

The converse is trivial. Indeed, given a matrix of integers, $\mathcal{U}$, satisfying (i) and (ii), we can consider the standard (resp. good) determinantal scheme $X \subset \mathbb{P}^{n+c}$ of codimension c associated to the homogeneous matrix

$$\mathcal{A} = \begin{pmatrix} x_0^{a_{t+c-2}-b_t} & x_1^{a_{t+c-3}-b_t} & \cdots & \cdots & x_{c-1}^{a_{t-1}-b_t} & 0 & 0 & \cdots & \cdots \\ 0 & x_0^{a_{t+c-3}-b_{t-1}} & x_1^{a_{t+c-4}-b_{t-1}} & \cdots & \cdots & x_{c-1}^{a_{t-2}-b_{t-1}} & 0 & 0 & \cdots \\ 0 & 0 & x_0^{a_{t+c-4}-b_{t-2}} & x_1^{a_{t+c-5}-b_{t-2}} & \cdots & \cdots & x_{c-1}^{a_{t-3}-b_{t-2}} & 0 & \cdots \\ \cdots\cdot & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

(resp. $\mathcal{A} =$

$$\begin{pmatrix} x_0^{a_{t+c-2}-b_t} & x_1^{a_{t+c-3}-b_t} & \cdots & \cdots & x_{c-1}^{a_{t-1}-b_t} & 0 & 0 & \cdots & \cdots \\ x_c^{a_{t+c-2}-b_{t-1}} & x_0^{a_{t+c-3}-b_{t-1}} & x_1^{a_{t+c-4}-b_{t-1}} & \cdots & \cdots & x_{c-1}^{a_{t-2}-b_{t-1}} & 0 & 0 & \cdots \\ 0 & x_c^{a_{t+c-3}-b_{t-2}} & x_0^{a_2-b_{t-2}} & x_1^{a_{t+c-5}-b_{t-2}} & \cdots & \cdots & x_{c-1}^{a_{t-3}-b_{t-2}} & 0 & \cdots \\ \cdots\cdot & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}).$$

Up to re-ordering, we easily check that the degree matrix associated to X is $\mathcal{U}$. $\square$

Given integers $b_1, ..., b_t$ and $a_0, a_1, ..., a_{t+c-2}$, we denote by $W(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^p(\mathbb{P}^{n+c})$ (resp. $W_s(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^p(\mathbb{P}^{n+c})$) the locus of good (resp. standard) determinantal schemes $X \subset \mathbb{P}^{n+c}$ of codimension $c \geq 2$ defined by the maximal minors of a homogeneous matrix $\mathcal{A} = (f_{ji})^{i=1,\dots,t}_{j=0,\dots,t+c-2}$ where $f_{ji} \in k[x_0, ..., x_{n+c}]$ is a homogeneous polynomial of degree $a_j - b_i$. Clearly, $W(\underline{b}; \underline{a}) \subset W_s(\underline{b}; \underline{a})$. Moreover we have:

Corollary 2.6. *Assume $b_1 \leq ... \leq b_t$ and $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$. We have that $W(\underline{b}; \underline{a}) \neq \emptyset$ if and only if $W_s(\underline{b}; \underline{a}) \neq \emptyset$ if and only if $u_{i-1,i} = a_{i-1} - b_i > 0$ for $i = 1, ..., t$.*

Proof. It easily follows from Lemma 2.5. $\square$

Let $X \subset \mathbb{P}^{n+c}$ be a good determinantal scheme of codimension $c \geq 2$ defined by the homogeneous matrix $\mathcal{A} = (f_{ji})^{j=0,\dots,t+c-2}_{i=1,\dots t}$. It is well known that by successively deleting columns from the right hand side of $\mathcal{A}$, and taking maximal minors, one gets a flag of determinantal subschemes

$$(2.1) \qquad (\mathbf{X}.) : X = X_c \subset X_{c-1} \subset ... \subset X_2 \subset \mathbb{P}^{n+c}$$

where each $X_{i+1} \subset X_i$ (with ideal sheaf $\mathcal{I}_{X_{i+1}|X_i} = \mathcal{I}_i$) is of codimension 1, $X_i \subset \mathbb{P}^{n+c}$ is of codimension i $(i = 2, \dots, c)$ and where there exist $\mathcal{O}_{X_i}$-modules $\mathcal{M}_i$ fitting into short exact sequences

$$(2.2) \qquad 0 \to \mathcal{O}_{X_i}(-a_{t+i-1}) \to \mathcal{M}_i \to \mathcal{M}_{i+1} \to 0 \ \text{ for } \ 2 \leq i \leq c-1,$$

such that $\mathcal{I}_i(a_{t+i-1})$ is the $\mathcal{O}_{X_i}$-dual of $\mathcal{M}_i$, for $2 \leq i \leq c$, and $\mathcal{M}_2$ is a twist of the canonical module of X_2, cf. (3.4)-(3.7) for details.

Remark 2.7. Assume $b_1 \le ... \le b_t$ and $a_0 \le a_1 \le ... \le a_{t+c-2}$. If X is general in $W(\underline{b};\underline{a})$ and $u_{i-min(\alpha,t),i} = a_{i-min(\alpha,t)} - b_i \ge 0$ for $min(\alpha,t) \le i \le t$, then $X_j = \operatorname{Proj}(D_j)$, for all $j = 2, \cdots, c$, is non-singular except for a subset of codimension at least $min\{2\alpha-1, j+2\}$, i.e.

$$\operatorname{codim}_{X_j} Sing(X_j) \ge min\{2\alpha - 1, j+2\}. \tag{2.3}$$

This follows from [6]; Theorem arguing as in [6]; Example 2.1. In particular, if $\alpha \ge 3$, we get that for each $i > 0$, the closed embeddings $X_i \subset \mathbb{P}^{n+c}$ and $X_{i+1} \subset X_i$ are local complete intersections outside some set Z_i of codimension at least $min(4, i+1)$ in X_{i+1} ($\operatorname{depth}_{Z_i} \mathcal{O}_{X_{i+1}} \ge \min(4, i+1)$).

Moreover, taking $\alpha = 1$, we deduce from (2.3) that a general X in $W(\underline{b};\underline{a})$ is reduced provided $a_{i-1} > b_i$ for all $1 \le i \le t$. This means (see Corollary 2.6) that a non-empty $W(\underline{b};\underline{a})$ always contains a reduced determinantal scheme. This Remark improves [14]; Proposition 2.7.

3. Upper bound for the dimension of the determinantal locus

The goal of this section is to write down an upper bound for the dimension of the locus $W(\underline{b};\underline{a})$ (resp. $W_s(\underline{b};\underline{a})$) of good (resp. standard) determinantal subschemes $X \subset \mathbb{P}^{n+c}$ of codimension c inside the Hilbert scheme $\operatorname{Hilb}^{p(s)}(\mathbb{P}^{n+c})$, where $p(s) \in \mathbb{Q}[s]$ is the Hilbert polynomial of X which can be computed explicitly using the minimal free R-resolution of $R/I(X)$ given by the Eagon-Northcott complex (see, Proposition 2.3 (ii) and [14], Proposition 2.4). In section 4, we will analyze when the mentioned upper bound is sharp and in section 5, we will discuss under which conditions the closure of $W(\underline{b};\underline{a})$ in $\operatorname{Hilb}^{p(s)}(\mathbb{P}^{n+c})$ is a generically smooth, irreducible component of $\operatorname{Hilb}^{p(s)}(\mathbb{P}^{n+c})$.

Let $X \subset \mathbb{P}^{n+c}$ be a good determinantal scheme of codimension $c \ge 2$ defined by the vanishing of the maximal minors of a $t \times (t+c-1)$ matrix $\mathcal{A} = (f_{ji})_{i=1,...t}^{j=0,...,t+c-2}$ where $f_{ji} \in k[x_0, ..., x_{n+c}]$ are homogeneous polynomials of degree $a_j - b_i$ and let $A = R/I(X)$ be the homogeneous coordinate ring of X. The matrix $\mathcal{A}$ defines a morphism of locally free sheaves

$$\varphi : \mathcal{F} := \bigoplus_{i=1}^{t} \mathcal{O}_{\mathbb{P}^{n+c}}(b_i) \longrightarrow \mathcal{G} := \bigoplus_{j=0}^{t+c-2} \mathcal{O}_{\mathbb{P}^{n+c}}(a_j)$$

and we may assume without loss of generality that φ is minimal; i.e., $f_{ji} = 0$ for all i, j with $b_i = a_j$.

Our aim is to determine an upper bound for $\dim W(\underline{b};\underline{a})$ in terms of $b_1, ..., b_t$ and $a_0, a_1, ..., a_{t+c-2}$. To this end, we consider the affine scheme $\mathbb{V} = \mathbb{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{F}, \mathcal{G})$ whose rational points are the morphisms from $\mathcal{F}$ to $\mathcal{G}$. Let $\mathbb{Y}$ be the non-empty, open, irreducible subscheme of $\mathbb{V}$ whose rational points are the morphisms $\varphi_\lambda : \mathcal{F} \longrightarrow \mathcal{G}$ such that their associated homogeneous matrix $\mathcal{A}_\lambda$ defines a good determinantal subscheme $X_\lambda \subset \mathbb{P}^{n+c}$.

The Eagon-Northcott complex of the universal morphism

$$\Psi : pr_2^* \mathcal{F} \longrightarrow pr_2^* \mathcal{G}$$

on $\mathbb{Y} \times \mathbb{P}^{n+c}$ (where $pr_2 : \mathbb{Y} \times \mathbb{P}^{n+c} \longrightarrow \mathbb{P}^{n+c}$ is the natural projection) induces a morphism

$$f : \mathbb{Y} \longrightarrow W(\underline{b}; \underline{a})$$

which is defined by $f(\varphi_\lambda) := X_\lambda$ on closed points. We consider the affine group scheme $G := \mathbb{A}ut(\mathcal{F}) \times \mathbb{A}ut(\mathcal{G})$ which is an irreducible open dense subset of

$$G := \mathbb{A}ut(\mathcal{F}) \times \mathbb{A}ut(\mathcal{G}) \subset \mathbb{H}\mathrm{om}(\mathcal{F}, \mathcal{F}) \times \mathbb{H}\mathrm{om}(\mathcal{G}, \mathcal{G}) \cong k^\Upsilon$$

where $\Upsilon = \sum_{i,j} \binom{b_i - b_j + n + c}{n+c} + \sum_{i,j} \binom{a_i - a_j + n + c}{n+c}$. $G := \mathbb{A}ut(\mathcal{F}) \times \mathbb{A}ut(\mathcal{G})$ operates on $\mathbb{Y}$:

$$\sigma : G \times \mathbb{Y} \longrightarrow \mathbb{Y}; \quad ((\alpha, \beta), \varphi_\lambda) \mapsto \beta \varphi_\lambda \alpha^{-1}.$$

The action σ is compatible with the morphism f. Thus, at least set-theoretically $f : \mathbb{Y} \longrightarrow W(\underline{b}; \underline{a})$ induces a surjective map from the orbit set $\mathbb{Y}//G$ to $W(\underline{b}; \underline{a})$. Moreover, since the map from $\mathbb{Y}$ to the closure $\overline{W(\underline{b}; \underline{a})}$ in $\mathrm{Hilb}^{p(s)}(\mathbb{P}^{n+c})$ is dominant, we get that $W(\underline{b}; \underline{a})$ is irreducible and we have (small letters denote dimension, cf. Notation):

$$\dim W(\underline{b}; \underline{a}) \le \hom_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{F}, \mathcal{G}) - \mathrm{aut}(\mathcal{G}) - \mathrm{aut}(\mathcal{F}) + \dim(G_\lambda) \tag{3.1}$$

where

$$G_\lambda = \{(\delta, \tau) \in \mathbb{A}ut(\mathcal{F}) \times \mathbb{A}ut(\mathcal{G}) \mid \tau \varphi_\lambda \delta^{-1} = \varphi_\lambda\}$$

is the isotropy group of any closed point $\varphi_\lambda \in \mathbb{Y}$. By [19]; Proposition 10.2, for all $\varphi_\lambda \in \mathbb{Y}$, we have (we let $\binom{n+a}{a} = 0$ for $a < 0$, as usual):

$$\dim(G_\lambda) = \mathrm{aut}(\mathcal{B}_\lambda) + \sum_{j,i} \binom{b_i - a_j + n + c}{n+c} \tag{3.2}$$

where $\mathcal{B}_\lambda = \mathrm{coker}(\varphi_\lambda)$. Therefore, we have

$$\begin{aligned} \dim W(\underline{b}; \underline{a}) \le \sum_{i,j} \binom{a_i - b_j + n + c}{n+c} - \sum_{i,j} \binom{a_i - a_j + n + c}{n+c} - \\ \sum_{i,j} \binom{b_i - b_j + n + c}{n+c} + \sum_{j,i} \binom{b_j - a_i + n + c}{n+c} + \mathrm{aut}(\mathcal{B}_\lambda). \end{aligned} \tag{3.3}$$

Our next goal is to bound the dimension $\mathrm{aut}(\mathcal{B})$ in terms of a_j and b_i, where $\mathcal{B} = \mathrm{coker}(\varphi)$ and φ is a closed point of $\mathbb{Y}$ (see, Proposition 3.3). To this end we need to fix some more notation.

Let $\mathcal{A}_i$ be the matrix obtained deleting the last $c - i$ columns. The matrix $\mathcal{A}_i$ defines a morphism

$$\varphi_i : F = \bigoplus_{i=1}^{t} R(b_i) \longrightarrow G_i := \bigoplus_{j=0}^{t+i-2} R(a_j) \tag{3.4}$$

of R-free modules and let B_i be the cokernel of φ_i. Put $\varphi = \varphi_0$, $G = G_c$ and $B = B_c$. Let M_i be the cokernel of $\varphi_i^* = \mathrm{Hom}_R(\varphi_i, R)$, i.e. let the sequence

$$G_i^* \xrightarrow{\varphi_i^*} F^* \longrightarrow M_i \cong \mathrm{Ext}^1_R(B_i, R) \longrightarrow 0 \tag{3.5}$$

be exact. If $D_i \cong R/I_{D_i}$ is the k-algebra given by the maximal minors of $\mathcal{A}_i$ and $X_i = \operatorname{Proj}(D_i)$ (i.e. $R \twoheadrightarrow D_2 \twoheadrightarrow D_3 \twoheadrightarrow \ldots \twoheadrightarrow D_c = A$), then M_i is a D_i-module and there is an exact sequence

$$(3.6) \qquad 0 \longrightarrow D_i \longrightarrow M_i(a_{t+i-1}) \longrightarrow M_{i+1}(a_{t+i-1}) \longrightarrow 0$$

in which $D_i \longrightarrow M_i(a_{t+i-1})$ is the regular section which defines D_{i+1} [22]. Indeed,

$$(3.7) \qquad 0 \longrightarrow M_i(a_{t+i-1})^* = \operatorname{Hom}_{D_i}(M_i(a_{t+i-1}), D_i) \longrightarrow D_i \longrightarrow D_{i+1} \longrightarrow 0$$

and we may put $I_i := I_{D_{i+1}/D_i} = M_i(a_{t+i-1})^*$. An R-free resolution of M_i is given by Proposition 2.2, and we get in particular that M_i is a maximal Cohen-Macaulay D_i-module. Using (3.7) we see that I_i is also a maximal Cohen-Macaulay D_i-module. Proposition 2.2 (iii) also gives us $K_{D_i}(n+c+1) \cong S_{i-1}M_i(\ell_i)$ where $\ell_i := \sum_{j=0}^{t+i-2} a_j - \sum_{q=1}^{t} b_q$.

In what follows we always let $Z_i \subset X_i$ be some closed subset such that $U_i = X_i - Z_i \hookrightarrow \mathbb{P}^{n+c}$ is a local complete intersection. By the well known fact that the 1. Fitting ideal of M_i is equal to $I_{t-1}(\varphi_i)$, we get that $\tilde{M}_i$ is locally free of rank 1 precisely on $X_i - V(I_{t-1}(\varphi_i))$ [4], Lemma 1.4.8. Since the set of non locally complete intersection points of $X_i \hookrightarrow \mathbb{P}^{n+c}$ is precisely $V(I_{t-1}(\varphi_i))$ by e.g. [26], Lemma 1.8, we get that $U_i \subset X_i - V(I_{t-1}(\varphi_i))$ and that $\tilde{M}_i$ and $\mathcal{I}_{X_i}/\mathcal{I}_{X_i}^2$ are locally free on U_i.

Finally note that there is a close relation between $M_{i+1}(a_{t+i-1})$ and the normal module $N_{D_{i+1}/D_i} := \operatorname{Hom}_{D_i}(I_i, D_{i+1})$ of the quotient $D_i \to D_{i+1}$. If we suppose $\operatorname{depth}_{I(Z_i)} D_i \geq 2$, we get, by applying $\operatorname{Hom}_{D_i}(I_i, .)$ to (3.7), that

$$(3.8) \qquad 0 \longrightarrow D_i \longrightarrow M_i(a_{t+i-1}) \longrightarrow N_{D_{i+1}/D_i}$$

is exact. Hence we have an injection $M_{i+1}(a_{t+i-1}) \hookrightarrow N_{D_{i+1}/D_i}$, which in the case $\operatorname{depth}_{I(Z_i)} D_i \geq 3$ leads to an isomorphism $M_{i+1}(a_{t+i-1}) \cong N_{D_{i+1}/D_i}$. Indeed, this follows from the more general fact (by letting $M = N = I_i$) that if M and N are finitely generated D-modules such that $\operatorname{depth}_{I(Z)} M \geq r+1$ and $\tilde{N}$ is locally free on $U := X - Z$ ($X = \operatorname{Proj}(D)$), then the natural map

$$(3.9) \qquad \operatorname{Ext}^i_D(N, M) \longrightarrow H^i_*(U, \mathcal{H}om_{\mathcal{O}_X}(\tilde{N}, \tilde{M})) \simeq H^{i+1}_{I(Z)}(\operatorname{Hom}_D(N, M))$$

is an isomorphism, (resp. an injection) for $i < r$ (resp. $i = r$), Cf. [15], exp. VI.

Lemma 3.1. *Let M be an R-module. With the above notation, the sequence*

$$0 \to \operatorname{Hom}_R(M_i, M) \to F \otimes_R M \to G_i \otimes_R M \to B_i \otimes_R M \to 0$$

is exact and $\operatorname{Hom}_R(M_i, M) = \operatorname{Tor}_1^R(B_i, M)$.

Proof. We apply $\operatorname{Hom}(., R)$ to

$$(3.10) \qquad 0 \longrightarrow F \longrightarrow G_i \longrightarrow B_i \longrightarrow 0$$

and we get

$$0 \to \operatorname{Hom}(B_i, R) \to G_i^* \to F^* \to \operatorname{Ext}^1_R(B_i, R) = M_i \to 0.$$

Hence

$$0 \to \operatorname{Hom}(M_i, M) \to \operatorname{Hom}(F^*, M) \cong F \otimes M \to \operatorname{Hom}(G_i^*, M) \cong G_i \otimes M$$

and we get the first exact sequence and $\operatorname{Hom}(M_i, M) = \operatorname{Tor}_1^R(B_i, M)$ by applying $(.)\otimes_R M$ to (3.10). $\square$

Lemma 3.2. *With the notations above, if C is good determinantal, then* $\operatorname{depth}_{I(Z_i)} D_i \geq 1$ *for $2 \leq i \leq c$ and* $\operatorname{Hom}_{D_i}(M_i, M_i) = D_i$.

Proof. If X is a standard determinantal scheme, defined by some $t \times (t+i-1)$ matrix, and if we delete a column and let Y be the corresponding determinantal scheme, then Y is also standard determinantal [3]. Hence if X is good determinantal, it follows that Y is also good determinantal by the definition of a good determinantal scheme. In particular all X_i, $2 \leq i \leq c$, are good determinantal schemes and hence generic complete intersections. By the definition of Z_i, we get $\operatorname{depth}_{I(Z_i)} D_i \geq 1$.

Let $U_i = \operatorname{Proj}(D_i) - Z_i$ and note that $\tilde{M_i}|_{U_i}$ is an invertible sheaf. Let $S_r(M_i)$ be the r-th symmetric power of the D_i-module M_i. For $1 \leq r \leq i-1$, $S_r(M_i)$ are maximal Cohen-Macaulay modules and $S_{i-1}(M_i)(\ell_i) \cong K_{D_i}(n+c+1)$ (cf. Proposition 2.2 (iii)). By (3.9) we have injections

$$\operatorname{Hom}_{D_i}(S_r M_i, S_r M_i) \hookrightarrow H^0_*(U_i, \mathcal{H}om(S_r \tilde{M_i}, S_r \tilde{M_i})) \cong H^0_*(U_i, \tilde{D_i}).$$

Since $S_{i-1}(M_i)$ is a twist of the canonical module on D_i and since $\operatorname{Hom}(K_{D_i}, K_{D_i}) \cong D_i$, we get the lemma from the commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{D_i}(M_i, M_i) & \hookrightarrow & H^0_*(U_i, \tilde{D_i}) \\ \psi \downarrow & & \| \\ \operatorname{Hom}_{D_i}(S_{i-1}M_i, S_{i-1}M_i) & \hookrightarrow & H^0_*(U_i, \tilde{D_i}). \end{array}$$

Indeed, ψ is injective and we conclude by

$$D_i \to \operatorname{Hom}_{D_i}(M_i, M_i) \hookrightarrow \operatorname{Hom}_{D_i}(S_{i-1}M_i, S_{i-1}M_i) \cong D_i.$$

$\square$

Proposition 3.3. *Assume $b_1 \leq ... \leq b_t$, $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$ and $c \geq 2$. Set $\ell := \sum_{j=0}^{t+c-2} a_j - \sum_{i=1}^{t} b_i$. If $(c-1)a_{t+c-2} < \ell$ then,* $\operatorname{aut}(\mathcal{B}) = 1$. *Otherwise we have*

$$\operatorname{aut}(\mathcal{B}) \leq \sum_{i=1}^{c-3} \left(\sum_{r+s=i} \sum_{\substack{0 \leq i_i < \ldots < i_r \leq t+i \\ 1 \leq j_1 \leq \ldots \leq j_s \leq t}} (-1)^{i-r} \binom{h_i + a_{i_1} + \cdots + a_{i_r} + b_{j_1} \cdots + b_{j_s}}{n+c} \right)$$

$$+ \binom{h_0}{n+c} + 1$$

where we set $h_i := 2a_{t+1+i} + a_{t+2+i} + \cdots a_{c+t-3} + a_{c+t-2} - \ell + n + c$, for $i = 0, 1, \cdots, c-3$.

Proof. Set $S(\mathcal{B}) = Supp(\mathcal{E}xt^1(\mathcal{B}, \mathcal{O}_{\mathbb{P}^{n+c}}))$. Since $\operatorname{pd}(\mathcal{B}) = 1$ and $\operatorname{codim}(S(\mathcal{B}), \mathbb{P}^{n+c}) \geq 3$, $\mathcal{B}$ is a rank $c-1$ reflexive sheaf on $\mathbb{P}^{n+c}$ ([24]; Proposition 1.2). Moreover, if $(c-1)a_{t+c-2} < \ell$ then, by [19]; Lemma 10.1(ii), $\mathcal{B}$ is stable and $\operatorname{aut}(\mathcal{B}) = 1$ because stable reflexive sheaves are simple. So, from now on, we assume $(c-1)a_{t+c-2} \geq \ell$ and we will proceed by induction on c by successively deleting columns from the right side, i.e. of the largest degree. For $c=2$ the result was proved in [12] if $n \geq 1$ and in [19] for any $n \geq 0$. So, we will assume $c \geq 3$.

Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2}) & = & \mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2}) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_{i=1}^{t}\mathcal{O}_{\mathbb{P}^{n+c}}(b_i) & \xrightarrow{\varphi_c} & \oplus_{j=0}^{t+c-2}\mathcal{O}_{\mathbb{P}^{n+c}}(a_j) & \longrightarrow & \mathcal{B}_c & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_{i=1}^{t}\mathcal{O}_{\mathbb{P}^{n+c}}(b_i) & \xrightarrow{\varphi_{c-1}} & \oplus_{j=0}^{t+c-3}\mathcal{O}_{\mathbb{P}^{n+c}}(a_j) & \longrightarrow & \mathcal{B}_{c-1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and the exact sequence

$$0 \longrightarrow \operatorname{Hom}(\mathcal{B}_{c-1},\mathcal{B}_c) \longrightarrow \operatorname{Hom}(\mathcal{B}_c,\mathcal{B}_c) \xrightarrow{\alpha} \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2}),\mathcal{B}_c) = H^0(\mathcal{B}_c(-a_{t+c-2})) \longrightarrow$$
$$\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{B}_{c-1},\mathcal{B}_c) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{B}_c,\mathcal{B}_c) \longrightarrow 0.$$

Moreover, if we tensor with $. \otimes_R B_c$ the exact sequence

$$0 \to D_{c-1}(-a_{t+c-2}) \longrightarrow M_{c-1} \longrightarrow M_c \longrightarrow 0$$

we get

$$\operatorname{Tor}_1^R(B_c, M_{c-1}) \longrightarrow \operatorname{Tor}_1^R(B_c, M_c) \longrightarrow D_{c-1}(-a_{t+c-2}) \otimes B_c \longrightarrow$$
$$\longrightarrow M_{c-1} \otimes B_c \cong \operatorname{Ext}^1_R(B_{c-1}, B_c) \longrightarrow M_c \otimes B_c \cong \operatorname{Ext}^1_R(B_c, B_c) \longrightarrow 0.$$

Applying Lemmas 3.1 and 3.2 we get $\operatorname{Tor}_1^R(B_c, M_{c-1}) = \operatorname{Hom}(M_c, M_{c-1}) = 0$ (since M_c is supported in X_c which has codimension 1 in $X_{c-1} = Supp(M_{c-1})$) and $\operatorname{Tor}_1^R(B_c, M_c) = \operatorname{Hom}(M_c, M_c) = D_c = A$. Hence,

$$H^0(\mathcal{B}_c(-a_{t+c-2})) \to \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{B}_{c-1},\mathcal{B}_c)$$

coincides with $(D_{c-1}(-a_{t+c-2}) \otimes B_c)_0 \to (M_{c-1} \otimes B_c)_0$ whose kernel is $A_0 \cong k$, i.e. 1-dimensional. Therefore, $\dim(\operatorname{im}(\alpha)) = 1$ which gives us

$$\operatorname{aut}(\mathcal{B}_c) = \hom(\mathcal{B}_{c-1},\mathcal{B}_c) + 1. \tag{3.11}$$

We call $e \in \operatorname{Ext}^1(\mathcal{B}_{c-1}, \mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2}))$ the non-trivial extension ($e \neq 0$) satisfying

$$e: \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2}) \longrightarrow \mathcal{B}_c \longrightarrow \mathcal{B}_{c-1} \longrightarrow 0,$$

we have

$$0 \longrightarrow \operatorname{Hom}(\mathcal{B}_{c-1}, \mathcal{O}(a_{t+c-2})) \longrightarrow \operatorname{Hom}(\mathcal{B}_{c-1},\mathcal{B}_c) \xrightarrow{\eta} \operatorname{Hom}(\mathcal{B}_{c-1},\mathcal{B}_{c-1}) \xrightarrow{\delta} \operatorname{Ext}^1(\mathcal{B}_{c-1}, \mathcal{O}(a_{t+c-2})).$$

Since $\delta(1) = e \neq 0$, we have $\dim(\ker(\delta)) \leq \operatorname{aut}(\mathcal{B}_{c-1}) - 1$. On the other hand, using the hypothesis of induction to bound $\operatorname{aut}(\mathcal{B}_{c-1})$, we obtain

$$\begin{aligned} \hom(\mathcal{B}_{c-1},\mathcal{B}_c) &= \hom(\mathcal{B}_{c-1}, \mathcal{O}(a_{t+c-2})) + \dim(\operatorname{im}(\eta)) = \\ &\hom(\mathcal{B}_{c-1}, \mathcal{O}(a_{t+c-2})) + \dim(\ker(\delta)) \leq \end{aligned} \tag{3.12}$$

$$\hom(\mathcal{B}_{c-1},\mathcal{O}(a_{t+c-2}))+\operatorname{aut}(\mathcal{B}_{c-1})-1\leq$$

$$\hom(\mathcal{B}_{c-1},\mathcal{O}(a_{t+c-2}))+1-1+\binom{h_0}{n+c}+$$

$$\sum_{i=1}^{c-4}\left(\sum_{r+s=i}\sum_{\substack{0\leq i_1<\ldots<i_r\leq t+i\\ 1\leq j_1\leq\ldots\leq j_s\leq t}}(-1)^{i-r}\binom{h_i+a_{i_1}+\cdots+a_{i_r}+b_{j_1}\cdots+b_{j_s}}{n+c}\right)$$

where we set $h_i:=2a_{t+1+i}+a_{t+2+i}+\cdots+a_{t+c-2}-\ell+n+c$, for all $i=1,\cdots,c-3$. Now, we will compute $\hom(\mathcal{B}_{c-1},\mathcal{O}(a_{t+c-2}))$. To this end, we first observe that $\operatorname{Hom}(\mathcal{B}_{c-1},\mathcal{O})$ is the first Buchsbaum-Rim module associated to

$$\varphi^*_{c-1}:G^*_{c-1}=\bigoplus_{j=0}^{t+c-3}R(-a_j)\longrightarrow F^*:=\bigoplus_{i=1}^{t}R(-b_i).$$

Therefore, we have the following free graded R-resolution

$$0\longrightarrow\wedge^{t+c-2}G^*_{c-1}\otimes S_{c-3}(F)\otimes\wedge^tF\longrightarrow\cdots\longrightarrow\wedge^{t+i+1}G^*_{c-1}\otimes S_i(F)\otimes\wedge^tF$$
$$\longrightarrow\ldots\longrightarrow\wedge^{t+1}G^*_{c-1}\otimes S_0(F)\otimes\wedge^tF\longrightarrow\operatorname{Hom}(\mathcal{B}_{c-1},\mathcal{O})\longrightarrow 0.$$

Since,

$$\wedge^tF=R(\sum_{i=1}^{t}b_i),$$

$$S_m(F)=\bigoplus_{1\leq j_1\leq\ldots\leq j_m\leq t}R(b_{j_1}+\ldots+b_{j_m}),\ \text{and}$$

$$\wedge^r(G^*_{c-1})=\bigoplus_{0\leq i_1<\ldots<i_r\leq t+c-3}R(-a_{i_1}-\ldots-a_{i_r})$$
$$=\bigoplus_{0\leq i_1<\ldots<i_{t+c-2-r}\leq t+c-3}R(-\sum_{j=0}^{t+c-3}a_j+a_{i_1}+\ldots+a_{i_{t+c-2-r}})$$
$$=\bigoplus_{0\leq i_1<\ldots<i_{t+c-2-r}\leq t+c-3}R(-\sum_{j=0}^{t+c-2}a_j+a_{t+c+2}+a_{i_1}+\ldots+a_{i_{t+c-2-r}})$$

we have

$$\wedge^{t+i+1}(G^*_{c-1})\otimes S_i(F)\otimes\wedge^tF=$$
$$\bigoplus_{\substack{0\leq i_1<\ldots<i_{c-3-i}\leq t+c-3\\ 1\leq j_1\leq\ldots\leq j_i\leq t}}R(-\ell+a_{t+c-2}+a_{i_1}+\ldots+a_{i_{c-3-i}}+b_{j_1}+\ldots+b_{j_i}).$$

So, $\dim_k(\wedge^{t+i+1}(G^*_{c-1})\otimes S_i(F)\otimes\wedge^t F)=$

$$\sum_{\substack{0\le i_1<\ldots<i_{c-3-i}\le t+c-3\\ 1\le j_1\le\ldots\le j_i\le t}}\binom{-\ell+a_{t+c-2}+a_{i_1}+\ldots+a_{i_{c-3-i}}+b_{j_1}+\ldots+b_{j_i}+n+c}{n+c}$$

and, we conclude that

$$hom(\mathcal{B}_{c-1},\mathcal{O}(a_{t+c-2}))= \tag{3.13}$$

$$\sum_{r+s=c-3}\left(\sum_{\substack{0\le i_1<\ldots<i_r\le t+c-3\\ 1\le j_1\le\ldots\le j_s\le t}}(-1)^{c-3-r}\binom{h_{c-3}+a_{i_1}+\ldots+a_{i_r}+b_{j_1}+\ldots+b_{j_s}}{n+c}\right)$$

being $h_{c-3}=2a_{t+c-2}-\ell+n+c$. Putting altogether (3.11), (3.12) and (3.13), we obtain

$$\operatorname{aut}(\mathcal{B}_c)\le\binom{h_0}{n+c}+1+$$

$$\sum_{i=1}^{c-3}\left(\sum_{r+s=i}\sum_{\substack{0\le i_1<\ldots<i_r\le t+i\\ 1\le j_1\le\ldots\le j_s\le t}}(-1)^{i-r}\binom{h_i+a_{i_1}+\cdots+a_{i_r}+b_{j_1}\cdots+b_{j_s}}{n+c}\right)$$

where we set $h_i:=2a_{t+1+i}+a_{t+2+i}+\cdots+a_{t+c-2}-\ell+n+c$, for $i=0,1,\ldots,c-3$, which proves Proposition 3.3. $\square$

Remark 3.4. Note that $\operatorname{aut}(\mathcal{B})=1$ provided $\ell>2a_{t+c-2}+a_{t+c-3}+\ldots+a_{t+1}$ and $c>3$. (Indeed all binomials in the expression in Proposition 3.3 vanish.)

We are now ready to state the main result of this section.

Theorem 3.5. *Assume* $a_0\le a_1\le\ldots\le a_{t+c-2}$, $b_1\le\ldots\le b_t$ *and* $c\ge 2$. *Set* $\ell:=\sum_{j=0}^{t+c-2}a_j-\sum_{i=1}^{t}b_i$ *and* $h_i:=2a_{t+1+i}+a_{t+2+i}+\cdots+a_{t+c-2}-\ell+n+c$, *for* $i=0,1,\ldots,c-3$. *Then*
(i) If $(c-1)a_{t+c-2}<\ell$, *then*

$$\dim W(\underline{b};\underline{a})\le\sum_{i,j}\binom{a_i-b_j+n+c}{n+c}-\sum_{i,j}\binom{a_i-a_j+n+c}{n+c}-$$

$$\sum_{i,j}\binom{b_i-b_j+n+c}{n+c}+\sum_{j,i}\binom{b_j-a_i+n+c}{n+c}+1.$$

(ii) If $(c-1)a_{t+c-2}\ge\ell$, *then*

$$\dim W(\underline{b};\underline{a})\le\sum_{i,j}\binom{a_i-b_j+n+c}{n+c}-\sum_{i,j}\binom{a_i-a_j+n+c}{n+c}-$$

$$\sum_{i,j}\binom{b_i-b_j+n+c}{n+c}+\sum_{j,i}\binom{b_j-a_i+n+c}{n+c}+\binom{h_0}{n+c}+1+$$

$$\sum_{i=1}^{c-3}\left(\sum_{r+s=i}\sum_{\substack{0\le i_1<\ldots<i_r\le t+i\\ 1\le j_1\le\ldots\le j_s\le t}}(-1)^{i-r}\binom{h_i+a_{i_1}+\cdots+a_{i_r}+b_{j_1}\cdots+b_{j_s}}{n+c}\right).$$

Proof. These follow from the inequality (3.3) and Proposition 3.3. $\square$

Remark 3.6. Note that if $c>3$ and $\ell>2a_{t+c-2}+a_{t+c-3}+...+a_{t+1}$ then

$$\dim W(\underline{b};\underline{a})\le\sum_{i,j}\binom{a_i-b_j+n+c}{n+c}-\sum_{i,j}\binom{a_i-a_j+n+c}{n+c}-$$
$$\sum_{i,j}\binom{b_i-b_j+n+c}{n+c}+\sum_{j,i}\binom{b_j-a_i+n+c}{n+c}+1.$$

Indeed, this follows from Theorem 3.5 and Remark 3.4.

Remark 3.7. Given integers $a_0,a_1,\cdots,a_{t+c-2}$ and $b_1,\cdots,b_t$, we always have $\dim W_s(\underline{b};\underline{a})=\dim W(\underline{b};\underline{a})$. In fact, it is an easy consequence of Corollary 2.6 and the fact that a standard determinantal scheme is good determinantal if it is a generic complete intersection, and being a generic complete intersection is an open condition.

Example 3.8. (i) According to Ellingsrud's Theorem ([12]; Théorème 2), in codimension 2 case, the bound given in Theorem 3.5 is sharp provided $n\ge1$.

(ii) According to [19]; Proposition 1.12, in codimension 3 case, the bound given in Theorem 3.5 is sharp, provided $n\ge1$ and $\operatorname{depth}_{I(Z_2)}D_2\ge4$.

(iii) A *Rational normal scroll* $X\subset\mathbb{P}^N$ is a non-degenerate variety of minimal degree (i.e. $deg(X)=\operatorname{codim}(X)+1$) defined by the maximal minors of a $2\times(c+1)$ matrix with linear entries ($c=\operatorname{codim}(X)$). As example of rational normal scrolls we have the smooth, rational normal curves of degree d in $\mathbb{P}^d$. It is well known that the family of rational normal scrolls of degree d and codimension $d-1$ in $\mathbb{P}^N$ is irreducible of dimension $d(2N+2-d)-3$. So, again in this case the bound given in Theorem 3.5 is sharp.

(iv) Every closed subscheme $X\subset\mathbb{P}^{n+c}$ with Hilbert polynomial $p(t)=\binom{t+n}{n}$ is a linear space of dimension n and it is defined by c linear forms. Hence, $W(0;1,...,1)=Gr(n+1,n+c+1)=\operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+c})$. It is well known that the *Grassmannian*, $Gr(n+1,n+c+1)$, is a smooth, irreducible variety of dimension $c(n+1)$. So, again the bound given in our Theorem 3.5 is sharp.

We are led to pose the following questions.

Question 3.9. (i) When is the closure of $W(\underline{b};\underline{a})$ an irreducible component of $\operatorname{Hilb}^p(\mathbb{P}^{n+c})$?

(ii) Is $W(\underline{b};\underline{a})$ smooth or, at least, generically smooth?

(iii) Under which extra assumptions are the bounds given in Theorem 3.5 sharp?

We will address questions (i) and (ii) in next section 5 and question (iii) in section 4.

Finally we will show that the inequality for $\operatorname{aut}(\mathcal{B})$ in Proposition 3.3 is indeed an equality. One may show this by construing the proof of Proposition 3.3 more carefully. We will, however, take the opportunity to compute $\operatorname{aut}(\mathcal{B})$ by a different method, leading to an apparently new formula, and then prove that they coincide provided we assume $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$ and $b_1 \leq ... \leq b_t$ (see Proposition 3.12). This new formula will be used in the next section. We assume the notation of (3.4)-(3.5)

Lemma 3.10. *There is an exact sequence*

$$0 \to \operatorname{Hom}_R(B_c, F) \to \operatorname{Hom}_R(B_c, G_c) \to \operatorname{Hom}_R(B_c, B_c) \to \operatorname{Hom}_R(M_c, M_c) \to 0.$$

Proof. We apply $\operatorname{Hom}_R(B_c, .)$ to

$$0 \longrightarrow F \longrightarrow G_c \longrightarrow B_c \longrightarrow 0$$

and we get

$$0 \to \operatorname{Hom}(B_c, F) \to \operatorname{Hom}_R(B_c, G_c) \to \operatorname{Hom}_R(B_c, B_c) \to$$
$$\to \operatorname{Ext}^1_R(B_c, F) = M_c \otimes_R F \to \operatorname{Ext}^1(B_c, G_c) = M_c \otimes_R G_c.$$

By Lemma 3.1, we have the exact sequence

$$0 \to \operatorname{Hom}_R(M_c, M_c) \to M_c \otimes_R F \to M_c \otimes_R G_c$$

and we are done. □

Using the Buchsbaum-Rim resolution

$$0 \longrightarrow \wedge^{t+c-1} G_c^* \otimes S_{c-2}(F) \otimes \wedge^t F \longrightarrow \cdots \longrightarrow \wedge^{t+i+1} G_c^* \otimes S_i(F) \otimes \wedge^t F$$
$$\longrightarrow \ldots \longrightarrow \wedge^{t+1} G_c^* \otimes S_0(F) \otimes \wedge^t F \longrightarrow \operatorname{Hom}(B_c, R) \longrightarrow 0,$$

we immediately get the following Corollary:

Corollary 3.11. *Set* $\tau_\nu := \hom_R(B_c, R)_\nu$. *Then,*

$$\operatorname{aut}(B_c) = 1 + \sum_{j=0}^{t+c-2} \tau_{a_j} - \sum_{i=1}^{t} \tau_{b_i}.$$

Proof. It follows from Lemmas 3.2 and 3.10 and the isomorphisms

$$\operatorname{Hom}_R(B_c, F) \cong \operatorname{Hom}(B_c, R) \otimes F \cong \oplus_{i=1}^t \operatorname{Hom}(B_c, R(b_i)).$$

□

Proposition 3.12. *Set* $K_i := \hom(B_{i-1}, R(a_{t+i-2}))_0$, *for* $3 \leq i \leq c$. *Suppose* $b_1 \leq ... \leq b_t$ *and* $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$. *Then we have*

$$\operatorname{aut}(\mathcal{B}) = 1 + K_3 + K_4 + \cdots + K_c,$$

and the inequality for $\operatorname{aut}(\mathcal{B})$ *in Proposition 3.3 is an equality.*

Proof. Dualizing the exact sequence $0 \to R(a_{t+c-2}) \to B_c \to B_{c-1} \to 0$, we get

$$0 \to \operatorname{Hom}(B_{c-1}, R) \to \operatorname{Hom}_R(B_c, R) \to R(-a_{t+c-2}) \to M_{c-1} \to M_c \to 0$$

which together with (3.6) gives us the exact sequence

$$0 \to \operatorname{Hom}(B_{c-1}, R) \to \operatorname{Hom}_R(B_c, R) \to I_{D_{c-1}}(-a_{t+c-2}) \to 0. \tag{3.14}$$

Look at the commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \operatorname{Hom}(B_{c-1}, F)_0 & \longrightarrow & \operatorname{Hom}(B_{c-1}, G_c)_0 \\ \downarrow & & \downarrow \\ \operatorname{Hom}(B_c, F)_0 & \longrightarrow & \operatorname{Hom}(B_c, G_c)_0 \\ \downarrow & & \downarrow \\ 0 = \operatorname{Hom}(R(a_{t+c-2}), F)_0 & \longrightarrow & \operatorname{Hom}(R(a_{t+c-2}), G_c)_0. \end{array}$$

By (3.14), $\operatorname{Hom}(B_c, G_c)_0 \to \operatorname{Hom}(R(a_{t+c-2}), G_c)_0$ is zero because its image is $(I_{D_{c-1}} \otimes G_c(-a_{t+c-2}))_0$. Hence, we get

$$\hom(B_c, G_c)_0 - \hom(B_c, F)_0 =$$

$$\hom(B_{c-1}, G_{c-1})_0 + \hom(B_{c-1}, R(a_{t+c-2}))_0 - \hom(B_{c-1}, F)_0.$$

Since, we have

$$\operatorname{aut}(\mathcal{B}_c) = 1 + \hom(B_c, G_c)_0 - \hom(B_c, F)_0$$

by Lemma 3.10 and we may suppose

$$\operatorname{aut}(\mathcal{B}_{c-1}) = 1 + \hom(B_{c-1}, G_{c-1})_0 - \hom(B_{c-1}, F)_0$$

we have proved

$$\operatorname{aut}(\mathcal{B}_c) = K_c + \operatorname{aut}(\mathcal{B}_{c-1}). \tag{3.15}$$

Now, we conclude by induction taking into account that $\operatorname{aut}(\mathcal{B}_2) = \hom(I_{D_2}, I_{D_2})_0 = 1$. Moreover combining (3.15) and the definition of K_c with (3.13) we see that the expression we have for $\operatorname{aut}(\mathcal{B}_c)$ coincides with the corresponding binomials in the expression of $\operatorname{aut}(\mathcal{B}_c)$ in Proposition 3.3, and it follows that the inequality must be an equality. □

So, we can rewrite Theorem 3.5 and we have

Proposition 3.13. *With the above notation*

$$\dim W(\underline{b};\underline{a}) \leq \sum_{i,j} \binom{a_i - b_j + n + c}{n+c} - \sum_{i,j} \binom{a_i - a_j + n + c}{n+c} -$$

$$\sum_{i,j} \binom{b_i - b_j + n + c}{n+c} + \sum_{j,i} \binom{b_j - a_i + n + c}{n+c} + 1 + K_3 + \cdots + K_c.$$

Proof. It follows from the inequality (3.3) and Proposition 3.12. □

Remark 3.14. One may show that the right hand side of the inequality for $\dim W(\underline{b};\underline{a})$ in Proposition 3.13 is equal to $\dim \operatorname{Ext}^1_R(B_c, B_c)_0$. This indicates an interesting connection to the deformations of the R-module B_c.

4. The dimension of the determinantal locus

The purpose of this section is to analyze when the bound given in Theorem 3.5 is sharp. We will see that under mild conditions the upper bound for $\dim W(\underline{b};\underline{a})$ given in the preceding section is indeed the dimension of the determinantal locus $W(\underline{b};\underline{a})$ provided the codimension c is small. Indeed, if $2 \le c \le 3$ and $n \ge 1$, this is known ([19], [12]) while for $4 \le c \le 5$ it is a consequence of the main theorem of this section. If $c \ge 6$ we also get the expected dimension formula for $W(\underline{b};\underline{a})$ under more restrictive assumptions. As in the preceding section the proofs use induction on c by successively deleting columns of the largest possible degree.

We keep the notation introduced in §2 and §3: see in particular (3.4)-(3.8). If we denote by $W(F,G) := W(\underline{b};\underline{a})$ and by $\mathbb{V}(F,G_i) := \mathbb{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\tilde{F},\tilde{G}_i)$ the affine scheme whose rational points are the morphisms from $\tilde{F}$ to $\tilde{G}_i$, we have by the definition of $W(F,G_c)$ and $W(F,G_{c-1})$ a diagram of rational maps

$$\begin{tikzcd}
\mathbb{V}(F,G_c) \arrow[r] \arrow[d, two heads] & \mathbb{V}(F,G_{c-1}) \arrow[d, two heads] \\
W(F,G_c) & W(F,G_{c-1})
\end{tikzcd}$$

where the vertical down arrows are dominating and rational and $\mathbb{V}(F,G_c) \longrightarrow \mathbb{V}(F,G_{c-1})$ is defined by deleting the last column.

To prove that the upper bound of $\dim W(F,G_c)$ of Proposition 3.13 is also a lower bound, we need a deformation-theoretic technical result which computes the dimension of $W(F,G_c)$ in terms of the dimension of $W(F,G_{c-1})$. To do so, we consider the Hilbert flag scheme $D(p,q)$ parameterizing "pairs" $X \subset Y$ of closed subschemes of $\mathbb{P}^{n+c}$ with Hilbert polynomial p and q respectively and the subset $D(F,G_i,G_{i-1})$ of "pairs" $X \subset Y$ where $X \in W(F,G_i)$ is a good determinantal scheme defined by a matrix $\mathcal{A}_i \in \mathbb{V}(F,G_i)$ and Y is a good determinantal scheme defined by the matrix $\mathcal{A}_{i-1} \in \mathbb{V}(F,G_{i-1})$ obtained by deleting the last column of $\mathcal{A}_i$. Then the diagram above fits into

$$\begin{tikzcd}
\mathbb{V}(F,G_c) \arrow[rr] \arrow[dr, two heads] \arrow[dd, two heads] & & \mathbb{V}(F,G_{c-1}) \arrow[d, two heads] \\
& D(F,G_c,G_{c-1}) \arrow[r, "p_2"'] \arrow[dl, "p_1"] & W(F,G_{c-1}) \\
W(\underline{b};\underline{a}) = W(F,G_c) & &
\end{tikzcd}$$

where p_1 and p_2 are the restriction of the natural projections $pr_1 : D(p,q) \longrightarrow \mathrm{Hilb}^p(\mathbb{P}^{n+c})$ and $pr_2 : D(p,q) \longrightarrow \mathrm{Hilb}^q(\mathbb{P}^{n+c})$ respectively, and where $\mathbb{V}(F,G_c) \twoheadrightarrow D(F,G_c,G_{c-1})$ is dominating and rational by definition. Denoting

$$m_i(\nu) = \dim_k M_i(a_{t+i-2})_\nu$$

we have

Proposition 4.1. *Let $c \geq 3$. Suppose that $W(\underline{b};\underline{a}) \neq \emptyset$ and that* $\operatorname{depth}_{I(Z_{c-1})} D_{c-1} \geq 2$ *for a general $D_{c-1} \in W(F;G_{c-1})$. Then*

(1) *p_2 is dominating and*

$$\dim D(F,G_c,G_{c-1}) \geq \dim W(F,G_{c-1}) + m_c(0);$$

(2) $\dim W(\underline{b};\underline{a}) \geq \dim W(F,G_{c-1}) + m_c(0) - \hom_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{I}_{X_{c-1}}, \mathcal{I}_{c-1})$.

Proof. Due to [22], Proposition 3.2, we see that for any $Y \in W(F,G_{c-1})$ there exists a regular section $R/I(Y) \hookrightarrow M_{c-1}(a_{t+c-2})$ whose cokernel is supported at some X with $\dim X < \dim Y$, and such that M_{c-1} is the cokernel of the morphism $\varphi^*_{c-1} : G^*_{c-1} \to F^*$ as in §3. Moreover, for a given Y, the mapping cone construction shows that for any regular section $R/I(Y) \hookrightarrow M_{c-1}(a_{t+c-2})$ there is a morphism $\varphi^*_c : G^*_c \to F^*$ which reduces to the given φ^*_{c-1} by deleting the extra (say the last) column of the corresponding matrix. This shows that p_2 is dominating and that the fiber $p_2^{-1}(Y)$ "contains" the space of regular sections of $M_{c-1}(a_{t+c-2})$ in a natural way.

More precisely note that every $Y \in W(F,G_{c-1})$ corresponds to some morphism φ_{c-1} between the same graded modules F and G_{c-1} . These modules determine all free graded modules in the Buchsbaum-Rim resolution $D_1(\varphi^*_{c-1})$ of $M_{c-1} = \operatorname{coker}(\varphi^*_{c-1})$ (cf. Proposition 2.3). Hence, for all $Y \in W(F,G_{c-1})$ the corresponding vector spaces $M_{c-1}(a_{t+c-2})_0$ have the same dimension. Since by (3.6) $M_c(a_{t+c-2})_0 = M_{c-1}(a_{t+c-2})_0/k$, it follows that $m_c(0)$ is the same number for all $Y \in W(F,G_{c-1})$. Now if $Y \in W(F,G_{c-1})$ is general, we have

$$\dim D(F,G_c,G_{c-1}) = \dim W(F;G_{c-1}) + \dim p_2^{-1}(Y)$$

by generic flatness. Hence it suffices to see that $\dim p_2^{-1}(Y) \geq m_c(0)$. Pick $(X \subset Y) \in p_2^{-1}(Y)$, look at (3.8) and consider the injection $M_c(a_{t+c-2})_0 \hookrightarrow (N_{X/Y})_0$. In the tangent space $(N_{X/Y})_0$ of $pr_2^{-1}(Y) \supseteq p_2^{-1}(Y)$ at $(X \subset Y)$ we therefore have a $m_c(0)$-dimensional family arising from deforming the matrix $\mathcal{A} = [\mathcal{A}_{c-1}, L]$ of φ^*_c leaving φ^*_{c-1} (i.e. $\mathcal{A}_{c-1}$) fixed (L is the last column of $\mathcal{A}$). We may think of the last column of such a deformation of φ^*_c as $L + \sum_{i=1}^{m_c(0)} t_i L^{(i)}$ mod. $(t_1,t_2,...,t_{m_c(0)})^2$ where the t_i's are indeterminates and where the degree matrix of the columns $L^{(i)}$ are exactly the same as that of L. Since the degeneracy locus of the $t \times (t+c-1)$ matrix $[\mathcal{A}_{c-1}, L + \sum_{i=1}^{m_c(0)} t_i L^{(i)}]$ defines a flat family over some open subset T of $Spec(k[t_1,...,t_{m_c(0)}])$ containing the origin (because the Eagon-Northcott complex over $Spec(k[\underline{t}])$ must be acyclic over some T provided the pullback to $(0) \in Spec(k[\underline{t}])$ is acyclic), we see that the fiber $p_2^{-1}(Y)$ contains a $m_c(0)$-dimensional (linear) family, as required. This proves (1).

(2) It is straightforward to get (2) from (1). Indeed,

$$\dim D(F,G_c,G_{c-1}) - \dim W(\underline{b};\underline{a}) \leq \dim p_1^{-1}(D_c)$$

and since $p_1^{-1}(D_c)$ is contained in the full fiber of the first projection $pr_1 : D(p,q) \to \operatorname{Hilb}^p(\mathbb{P}^{n+c})$ whose fiber dimension is known to have $\hom(\mathcal{I}_{X_{c-1}}, \mathcal{I}_{c-1})$ as an upper bound (e.g. [19], Chapter 9), we easily conclude. $\square$

Proposition 4.1 allows us, under some assumptions, to find a lower bound for $\dim W(\underline{b};\underline{a})$ provided we have a lower bound for $\dim W(F,G_{c-1})$. Indeed, since it is easy to find $m_i(0)$

using the Buchsbaum-Rim resolution of M_i or by using (3.6) recursively, it remains to find $\hom(\mathcal{I}_{X_i}, \mathcal{I}_i)$ in terms of $\hom(\mathcal{I}_{X_{i-1}}, \mathcal{I}_{i-1})$.

Lemma 4.2. *Set $a = a_{t+i-2} - a_{t+i-1}$.*

(a) *If $\operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1})_{\nu+a} = 0$ and $\operatorname{depth}_{I(Z_{i-1})} D_{i-1} \geq 3$, then*

$$\hom(I_{D_i}, I_i)_\nu \leq \dim(D_i)_{\nu+a} + \hom(I_{D_{i-1}}, I_{i-1})_{\nu+a}.$$

(b) *If $\operatorname{Ext}^2_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1})_{\nu+a} = 0$ and $\operatorname{depth}_{I(Z_{i-1})} D_{i-1} \geq 4$, then*

$$\operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}}/I^2_{D_{i-1}}, I_{i-1})_{\nu+a} = 0 \Rightarrow \operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i)_\nu = 0.$$

Remark 4.3. Since $I_{i-1} = M_{i-1}(a_{t+i-2})^*$, we have also

$$\operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1}(a)) \cong \operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}} \otimes M_{i-1}, M^*_{i-1}(-a_{t+i-2} - a_{t+i-1})).$$

Proof. (a) We consider the two exact sequences

$$0 \to \operatorname{Hom}_R(I_{i-1}, I_i) \to \operatorname{Hom}_R(I_{D_i}, I_i) \to \operatorname{Hom}_R(I_{D_{i-1}}, I_i) \tag{4.1}$$

$$0 \to \operatorname{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1}) \to \operatorname{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, D_{i-1}) \to \operatorname{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, D_i). \tag{4.2}$$

We have $\operatorname{depth}_{I(Z_{i-1})} D_{i-1} \geq 3$ and hence $\operatorname{depth}_{I(Z_{i-1})} D_i \geq 2$ and $\operatorname{depth}_{I(Z_{i-1})} I_i \geq 2$ and we get by (3.9)

$$\begin{aligned} \operatorname{Hom}(I_{i-1}, I_i) &\cong H^0_*(U_{i-1}, \mathcal{H}om(\mathcal{I}_{i-1}, \mathcal{I}_i)) \cong \\ &H^0_*(U_{i-1}, \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{I}_{i-1} \otimes_{\mathcal{O}_{X_{i-1}}} \mathcal{O}_{X_i} \otimes \mathcal{I}^*_i, \mathcal{O}_{X_i})) \cong D_i(a) \end{aligned} \tag{4.3}$$

because, by (3.6), $\tilde{M}_{i-1}(a_{t+i-2}) \otimes_{\mathcal{O}_{X_{i-1}}} \mathcal{O}_{X_i}|_{U_{i-1}} \cong \tilde{M}_i(a_{t+i-1})(a)|_{U_{i-1}}$ and hence

$$\mathcal{I}_{i-1} \otimes_{\mathcal{O}_{X_{i-1}}} \mathcal{O}_{X_i}|_{U_{i-1}} \cong \mathcal{I}_i(-a)|_{U_{i-1}}.$$

For similar reasons;

$$\begin{aligned} \operatorname{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, D_{i-1}) &\cong H^0_*(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_{i-1}}/\mathcal{I}^2_{X_{i-1}} \otimes \mathcal{I}^*_{i-1}, \mathcal{O}_{X_{i-1}})) \cong \\ &H^0_*(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_{i-1}}/\mathcal{I}^2_{X_{i-1}}, \mathcal{I}_{i-1})) \cong \operatorname{Hom}_R(I_{D_{i-1}}, I_{i-1}) \end{aligned}$$

and

$$\operatorname{Hom}_R(I_{D_{i-1}}, I_i) \cong H^0_*(U_{i-1}, \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{I}_{X_{i-1}}/\mathcal{I}^2_{X_{i-1}} \otimes \mathcal{I}^*_i, \mathcal{O}_{X_i}))$$

is further isomorphic to

$$\operatorname{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, D_i(a)) \cong H^0_*(U_{i-1}, \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{I}_{X_{i-1}}/\mathcal{I}^2_{X_{i-1}} \otimes \mathcal{I}^*_{i-1} \otimes_{\mathcal{O}_{X_{i-1}}} \mathcal{O}_{X_i}, \mathcal{O}_{X_i}(a))).$$

Putting all this together, we get that the exact sequences (4.1) and (4.2) reduce, in degree ν and $\nu + a$ resp., to

$$0 \to (D_i)_{\nu+a} \to \operatorname{Hom}_R(I_{D_i}, I_i)_\nu \to \operatorname{Hom}_R(I_{D_{i-1}}, I_i)_\nu \cong \operatorname{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, D_i)_{\nu+a} \to \tag{4.4}$$

$$0 \to \operatorname{Hom}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1})_{\nu+a} \to \operatorname{Hom}_R(I_{D_{i-1}}, I_{i-1})_{\nu+a} \to \operatorname{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, D_i)_{\nu+a} \to 0 \tag{4.5}$$

where (4.5) is short-exact by assumption. Taking dimensions, we immediately get (a).

(b) As in (4.3) we see that

$$\operatorname{Ext}^1_{D_{i-1}}(I_{i-1}, I_i) \cong H^1_*(U_{i-1}, \mathcal{O}_{X_i}(a)) = 0. \tag{4.6}$$

Sheafifying (4.4) and (4.5) and taking global sections, we get

$$\begin{array}{l} 0 \to H^1_*(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_i}/\mathcal{I}^2_{X_i}, \mathcal{I}_i(-a))) \to H^1_*(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_{i-1}}/\mathcal{I}^2_{X_{i-1}}, \mathcal{I}_i(-a))) \to \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \to H^1_*(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_{i-1}} \otimes \mathcal{I}^*_{i-1}, \mathcal{O}_{X_{i-1}})) \to H^1_*(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_{i-1}} \otimes \mathcal{I}^*_{i-1}, \mathcal{O}_{X_i})) \to H^2_*(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_{i-1}} \otimes \mathcal{I}^*_{i-1}, \mathcal{I}_{i-1})). \end{array} \tag{4.7}$$

Since $\mathcal{H}om(\mathcal{I}_{X_{i-1}} \otimes \mathcal{I}^*_{i-1}, \mathcal{O}_{X_{i-1}}) \cong \mathcal{H}om(\mathcal{I}_{X_{i-1}}/\mathcal{I}^2_{X_{i-1}}, \mathcal{I}_{i-1})$, then $\operatorname{depth}_{I(Z_{i-1})} D_{i-1} \geq 4$ and (3.9) show that the H^i_*-groups of (4.7) are isomorphic to the Ext^i-groups in the following diagram

$$\begin{array}{l} 0 \to \operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i) \to \operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}}/I^2_{D_{i-1}}, I_i) \to \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \to \operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}}/I^2_{D_{i-1}}, I_{i-1}(a)) \to \operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, D_i(a)) \to \operatorname{Ext}^2_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1}(a)) \end{array} \tag{4.8}$$

of exact horizontal sequences. Using (4.8) we easily get (b). □

Remark 4.4. By (4.4) the conclusion of Lemma 4.2 obviously holds provided we have $\operatorname{Hom}_R(I_{D_{i-1}}, I_i)_\nu = 0$. Using the Eagon-Northcott resolution of $I_{D_{i-1}}$ (i.e. of D_{i-1}), one may see that this Hom_ν-group vanishes if a_{t+i-2} is large enough.

Put

$$\lambda_c := \sum_{i,j} \binom{a_i - b_j + n + c}{n + c} + \sum_{i,j} \binom{b_j - a_i + n + c}{n + c} - \sum_{i,j} \binom{a_i - a_j + n + c}{n + c} - \sum_{i,j} \binom{b_i - b_j + n + c}{n + c} + 1$$

where the indices belonging to a_j (resp. b_i) ranges over $0 \leq j \leq t + c - 2$ (resp. $1 \leq i \leq t$). We define λ_{c-1} by the analogous expression where now a_j (resp b_i) ranges over $0 \leq j \leq t + c - 3$ (resp. $1 \leq i \leq t$). It follows after a straightforward computation that

$$\lambda_c = \lambda_{c-1} + \sum_{i=1}^{t} \binom{a_{t+c-2} - b_i + n + c}{n + c} - \sum_{j=0}^{t+c-3} \binom{a_{t+c-2} - a_j + n + c}{n + c} - \sum_{j=0}^{t+c-2} \binom{a_j - a_{t+c-2} + n + c}{n + c}. \tag{4.9}$$

We now come to the main theorem of this section which shows that the inequalities in Theorem 3.5 are equalities under certain assumptions. Recalling the equivalent expression of the upper bound of $\dim W(\underline{b}; \underline{a})$ given in Proposition 3.13, we have

Theorem 4.5. *Let $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$ and $b_1 \leq ... \leq b_t$ and assume $a_{i-min(2,t)} \geq b_i$ for $min(2, t) \leq i \leq t$. Let $c \geq 3$ and let $W(\underline{b}; \underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \geq 0$ if $c \geq 4$ and $n \geq 1$ if $c = 3$. For a general* $\operatorname{Proj}(A) \in W(\underline{b}; \underline{a})$, *let $R \twoheadrightarrow D_2 \twoheadrightarrow D_3 \twoheadrightarrow ... \twoheadrightarrow D_c = A$ be the flag obtained by successively deleting columns from the right hand side. If*

$$\operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1})_\nu = 0 \textit{ for } \nu \leq 0 \textit{ and } 3 \leq i \leq c - 1$$

then

$$\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + K_4 + ... + K_c$$

where $K_i = \hom(B_{i-1}, R(a_{t+i-2}))_0$ *for* $3 \le i \le c$.

Remark 4.6. If $c = 2$ and $n \ge 1$ one knows by [12] that

$$\dim W(\underline{b};\underline{a}) = \lambda_2.$$

The same formula holds if $c = 2$ and $n = 0$ as well. In this case one may get the formula by taking a general $\mathrm{Proj}(A) \in W(\underline{b};\underline{a})$ and show that

$$\hom_R(I_A, A)_0 = \mathrm{ext}^1_R(I_A, I_A)_0 = \lambda_2$$

by e.g. using [20], (26). We leave the details to the reader.

Proof. Due to Remark 2.7 and the assumption $a_{i-min(2,t)} \ge b_i$ for $min(2,t) \le i \le t$, the set $Z_i = Sing(X_i)$ satisfies $\mathrm{depth}_{I(Z_i)} D_i \ge 3$ for $2 \le i \le c-2$, $\mathrm{depth}_{I(Z_{c-1})} D_{c-1} \ge 2$ (and also $\mathrm{depth}_{I(Z_2)} D_2 \ge 3$ in case $c = 3$ since $n \ge 1$) by choosing $X = \mathrm{Proj}(A)$ general in $W(\underline{b};\underline{a})$.

To use Proposition 4.1, we only need to compute $m_c(0)$ and $\hom(I_{D_{c-1}}, I_{c-1})_0$ because we may by induction suppose that $\dim W(F, G_{c-1}) = \lambda_{c-1} + K_3 + ... + K_{c-1}$ for $c \ge 3$ (interpreting the expression as λ_2 when $c - 1 = 2$). By (3.6) and (3.5) we get

$$\begin{aligned} m_0(c) &= \dim M_{c-1}(a_{t+c-2})_0 - 1 = \\ &\dim F^*(a_{t+c-2})_0 - \dim G^*_{c-1}(a_{t+c-2})_0 + \hom(B_{c-1}, R(a_{t+c-2}))_0 - 1 = \\ &= \sum_{i=1}^{t} \binom{a_{t+c-2} - b_i + n + c}{n + c} - \sum_{j=0}^{t+c-3} \binom{a_{t+c-2} - a_j + n + c}{n + c} + K_c - 1. \end{aligned} \tag{4.10}$$

Thanks to Lemma 4.2, we can find an upper bound of $\hom(I_{D_{c-1}}, I_{c-1})_0$. We have

$$\hom(I_{D_{c-1}}, I_{c-1})_0 \le \binom{a + n + c}{n + c} + \hom(I_{D_{c-2}}, I_{c-2})_a$$

because $a = a_{t+c-3} - a_{t+c-2} \le 0$ and $\dim(D_i)_a$, which is either 0 or 1, must be equal to the binomial coefficient above. Repeated use of Lemma 4.2 implies

$$\hom(I_{D_{c-1}}, I_{c-1})_0 \le \sum_{j=t+1}^{t+c-3} \binom{a_j - a_{t+c-2} + n + c}{n + c} + \hom(I_{D_2}, I_2)_{a_{t+1} - a_{t+c-2}}. \tag{4.11}$$

It remains to compute $\hom(I_{D_2}, I_2)_\alpha$ with $\alpha = a_{t+1} - a_{t+c-2}$. Using (3.9) (cf. the proof of Lemma 4.2), we get

$$\mathrm{Hom}(I_{D_2}, I_2) \cong \mathrm{Hom}_{D_2}(I_{D_2} \otimes I_2^*, D_2) \cong \mathrm{Hom}_{D_2}(I_{D_2} \otimes M_2(a_{t+1}), D_2).$$

Moreover, if $\ell_2 = \sum_{j=0}^t a_j - \sum_{i=1}^t b_i$, then $M_2 \cong K_{D_2}(-\ell_2 + n + c + 1)$ by Proposition 2.2. In codimension $c = 2$, one knows

$$(I_{D_2}/I_{D_2}^2)^* \cong \mathrm{Ext}^1_R(I_{D_2}, I_{D_2}) \cong \mathrm{Ext}^1_R(I_{D_2}, D_2) \otimes I_{D_2} \cong K_{D_2}(n + c + 1) \otimes I_{D_2}$$

and since $\mathrm{depth}_{I(Z_2)} D_2 \ge 3$ and hence $\mathrm{depth}_{I(Z_2)} I_{D_2}/I_{D_2}^2 \ge 2$ (because the codepth of $I_{D_2}/I_{D_2}^2$ is ≤ 1 by [1]), we get

$$\begin{aligned}(4.12)\qquad \operatorname{Hom}(I_{D_2}, I_2)_\alpha &\cong \operatorname{Hom}(I_{D_2}\otimes K_{D_2}(n+c+1), D_2)(\ell_2 - a_{t+1})_\alpha \cong \\ &\cong (I_{D_2}/I_{D_2}^2)^{**}(\ell_2 - a_{t+1})_\alpha \cong (I_{D_2}/I_{D_2}^2)_{\ell_2 - a_{t+c-2}}.\end{aligned}$$

Thus the inequality $a_j \le a_{t+c-2}$ and the exact sequences

$$0 \to F \to G_2 = \oplus_{j=0}^{t} R(a_j) \to I_{D_2}(\ell_2) \to 0$$

$$(4.13)\qquad 0 \to \wedge^2 F \to F \otimes G_2 \to S_2 G_2 \to I_{D_2}^2(2\ell_2) \to 0$$

show

$$\hom(I_{D_2}, I_2)_\alpha = \dim(G_2)_{-a_{t+c-2}} = \sum_{j=0}^{t} \binom{a_j - a_{t+c-2} + n + c}{n+c}.$$

Using this last inequality together with (4.10), (4.11) and Proposition 4.1, we get by induction

$$\begin{aligned}\dim W(\underline{b};\underline{a}) \ge \lambda_{c-1} + K_3 + \dots + K_{c-1} + \sum_{i=1}^{t} \binom{a_{t+c-2} - b_i + n + c}{n+c} \\ - \sum_{j=0}^{t+c-3} \binom{a_{t+c-2} - a_j + n + c}{n+c} + K_c - 1 - \sum_{j=0}^{t+c-3} \binom{a_j - a_{t+c-2} + n + c}{n+c} \\ = \lambda_c + K_3 + K_4 + \dots + K_c\end{aligned}$$

where the last equality is due to (4.9). Combining with Proposition 3.13, we get the Theorem. $\square$

Note that the vanishing assumption of Theorem 4.5 is empty if $c = 3$. Hence, we have

Corollary 4.7. *Let $W(\underline{b};\underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \ge 1$ and $c = 3$. If $a_{i-min(2,t)} \ge b_i$ for $min(2,t) \le i \le t$, then*

$$\dim W(\underline{b};\underline{a}) = \lambda_3 + K_3.$$

$\square$

Remark 4.8. The above Corollary essentially generalizes [19], Corollary 10.15(i) where the depth condition is slightly stronger than the one we use in the proof of Theorem 4.5. The only missing part is that the assumption $n \ge 1$ excludes the interesting case of 0-dimensional good determinantal schemes. See Corollary 4.18 for the 0-dimensional case.

To apply Theorem 4.5 in the codimension $c = 4$ case, it suffices to prove that

$$\operatorname{Ext}^1_{D_2}(I_{D_2} \otimes I_2^*, I_2) = 0.$$

Due to Remark 4.3 and Proposition 2.2, the Ext^1-group above is isomorphic to a twist of

$$(4.14)\qquad \operatorname{Ext}^1_{D_2}(I_{D_2} \otimes M_2, M_2^*) \cong \operatorname{Ext}^1_{D_2}(I_{D_2} \otimes K_{D_2}, K_{D_2}^*)(2\ell_2 - 2n - 2c - 2).$$

Hence, all we need follows from

Lemma 4.9. *Let $R \twoheadrightarrow D = R/I_D$ be a Cohen-Macaulay codimension 2 quotient and suppose* $\mathrm{Proj}(D) \hookrightarrow \mathbb{P}^{n+c}$ *is a local complete intersection outside a closed subset $Z \subset$* $\mathrm{Proj}(D)$ *which satisfies* $\mathrm{depth}_{I(Z)} D \geq 4$. *Then*

$$\mathrm{depth}_{\mathfrak{m}} \mathrm{Hom}_D(I_D \otimes K_D, K_D^*) \geq \mathrm{depth}_{\mathfrak{m}} D - 1.$$

In particular, $\mathrm{depth}_{I(Z)} \mathrm{Hom}_D(I_D \otimes K_D, K_D^*) \geq 3$ *and hence*

$$\mathrm{Ext}^1_D(I_D \otimes K_D, K_D^*) = 0.$$

Proof. D is determinantal, say $D = D_2$ and we have a minimal free R-resolution

$$0 \to F \to G_2 \to I_D(\ell_2) \to 0 \tag{4.15}$$

as previously. If H_i is the i-th Koszul homology built on some set of minimal generators of I_D, it suffices to show that there are two exact sequences

$$0 \to \mathrm{Hom}_D(K_D(n+c+1), H_1) \to \wedge^2(F(-\ell_2)) \otimes D \to H_2 \to 0 \tag{4.16}$$

$$0 \to \mathrm{Hom}_D(K_D, H_1) \to K_D^* \otimes G_2(-\ell_2) \to \mathrm{Hom}(I_D \otimes K_D(n+c+1), K_D^*) \to 0. \tag{4.17}$$

Indeed, H_i are maximal Cohen-Macaulay modules by [18]. Hence, the first sequence shows that $\mathrm{Hom}_D(K_D(n+c+1), H_1)$ is maximal Cohen-Macaulay while the second shows that the codepth of $\mathrm{Hom}(I_D \otimes K_D(n+c+1), K_D^*)$ is at most 1 and all conclusions of the lemma follow easily (cf. (3.9) for the last conclusion).

To see that (4.16) is exact we deduce, from (4.15), the exact sequence

$$0 \to K_D(n+c+1)^* \to F(-\ell_2) \otimes_R D \to G_2(-\ell_2) \otimes_R D \to I_D/I_D^2 \to 0.$$

Indeed, we only need to prove that $K_D(n+c+1)^* = \ker[F(-\ell_2) \otimes_R D \to G_2(-\ell_2) \otimes_R D]$ which follows by applying $\mathrm{Hom}_R(.,D)$ to

$$.. \to G_2(-\ell_2)^* \to F(-\ell_2)^* \to \mathrm{Ext}^1_R(I_D, R) \cong K_D(n+c+1) \to 0.$$

Since one moreover knows

$$0 \to H_1 \to G_2(-\ell_2) \otimes_R D \to I_D/I_D^2 \to 0 \tag{4.18}$$

we get the exact sequence

$$0 \to K_D(n+c+1)^* \to F(-\ell_2) \otimes_R D \to H_1 \to 0, \tag{4.19}$$

from which we see that the Cohen-Macaulayness of $K_D(n+c+1)^*$ follows from that of H_1. Sheafifying (4.19) and using [16], Ch II, exec. 5.16, we get an exact sequence

$$0 \to \tilde{K}_D(n+c+1)^* \otimes \tilde{H}_1|_U \to \wedge^2(\tilde{F}(-\ell_2)) \otimes \tilde{D}|_U \to \wedge^2 \tilde{H}_1|_U$$

where $U = \mathrm{Proj}(D) - Z$. Applying $H^0_*(U,.)$ and recalling that $H^0_*(U, \wedge^2 \tilde{H}_1) \cong H_2$ [20], Proposition 18, we get the exact sequence (4.16) because $\mathrm{depth}_{I(Z)} H_1 \geq 2$ implies $\mathrm{Hom}(K_D(n+c+1), H_1) \cong H^0_*(U, \tilde{K}_D(n+c+1)^* \otimes \tilde{H}_1)$ and the right most map in the exact sequence

$$0 \to \wedge^3(F(-\ell_2)) \to \wedge^3(G_2(-\ell_2)) \to \wedge^2(F(-\ell_2)) \to H_2 \to 0$$

(see [1]) must correspond to the map $\wedge^2(F(-\ell_2)) \otimes D \to H_2$ in (4.16) and the later is surjective (which one may prove directly as well, by applying $H^0_*(U, \tilde{K}_D^* \otimes (.))$ to (4.19), to see $H^0_*(U, \tilde{K}_D^* \otimes \tilde{H}_1) = 0$).

To see that (4.17) is exact we dualize (4.18) and we get

$$0 \to (I_D/I_D^2)^* \to G_2(-\ell_2)^* \otimes D \to H_1^* \to 0$$

because $\mathrm{Ext}^1_D(I_D/I_D^2, D) \cong \mathrm{Ext}^1_D((I_D/I_D^2) \otimes K_D, K_D) = 0$ by the Cohen-Macaulayness of $(I_D/I_D^2) \otimes K_D(n+c+1) \cong (I_D/I_D^2)^*$, cf. the proof of Theorem 4.5 for the last isomorphims and [19], Ch. 6, for the Cohen-Macaulayness. Applying $\mathrm{Hom}_D(.,K_D^*)$ to the last exact sequence we get (4.17) because $\mathrm{depth}_{I(Z)} D \geq 3$ implies $\mathrm{Hom}_D(H_1^*, K_D^*) \cong \mathrm{Hom}_D(K_D, H_1)$ and $\mathrm{Ext}^1_D(H_1^*, K_D^*) \cong H^1_*(U, \mathcal{H}om(\tilde{K_D}, \tilde{H}_1)) = 0$ where the vanishing is due to the Cohen-Macaulayness of $\mathrm{Hom}(K_D, H_1)$, which holds because we already have proved the exactness of (4.16). This concludes the proof. $\square$

Corollary 4.10. *Let $W(\underline{b};\underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \geq 1$ and $c = 4$. If $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$, then*

$$\dim W(\underline{b};\underline{a}) = \lambda_4 + K_3 + K_4.$$

Proof. Due to Remark 2.7 and the assumption $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$, when $X = \mathrm{Proj}(A)$ is chosen general in $W(\underline{b};\underline{a})$, then the set $Z_2 = Sing(X_2)$ satisfies $\mathrm{depth}_{I(Z_2)} D_2 \geq 4$. Hence combining (4.14), Lemma 4.9 and Theorem 4.5, we are done. $\square$

To apply Theorem 4.5 in the codimension $c = 5$ case, it suffices to prove that

$$\mathrm{Ext}^1_{D_3}(I_{D_3} \otimes I_3^*, I_3)_\nu = \mathrm{Ext}^1_{D_3}(I_{D_3} \otimes M_3, M_3^*(-a_{t+2} - a_{t+3}))_\nu = 0$$

for $\nu \leq 0$. Since $\mathrm{depth}_{I(Z_3)} D_3 \geq 3$ and I_3 is a maximal Cohen-Macaulay D_3-module, we have by (3.9)

$$\mathrm{Ext}^1_{D_3}(I_{D_3} \otimes M_3, M_3^*) \cong H^1_*(U_3, \mathcal{H}om(\mathcal{I}_{X_3} \otimes \tilde{M}_3, \tilde{M}_3^*)) \cong$$
$$H^1_*(U_3, \mathcal{H}om(\mathcal{I}_{X_3} \otimes S_2(\tilde{M_3}), \mathcal{O}_{X_3})) \cong \mathrm{Ext}^1_{D_3}(I_{D_3} \otimes S_2(M_3), D_3)$$

where $U_3 = X_3 - Z_3$. Since by Proposition 2.2(iii) $K_{D_3}(n+c+1-\ell_3) \cong S_2(M_3)$ with $\ell_3 = \sum_{j=0}^{t+1} a_j - \sum_{i=1}^{t} b_i$, we get (letting $B := D_3$)

$$\mathrm{Ext}^1_{D_3}(I_{D_3} \otimes I_3^*, I_3) = \mathrm{Ext}^1_B(I_B \otimes K_B(n+1+c-\ell_3), B(-a_{t+2} - a_{t+3})) \tag{4.20}$$
$$\cong \mathrm{Ext}^1_B(I_B \otimes K_B(n+1+c), B)(\ell_3 - a_{t+2} - a_{t+3}).$$

Lemma 4.11. *Let $R \to B = R/I_B$ be a codimension 3 good determinantal quotient, let $X \hookrightarrow \mathbb{P}^{n+c}$ be the corresponding embedding, and let $Z = Sing(X)$.*

(a) *If $\mathrm{depth}_{I(Z)} B \geq 4$ then there is an exact sequence*

$$0 \to \mathrm{Ext}^1_B(I_B \otimes K_B(n+c+1), B) \to I_B/I_B^2 \to (I_B/I_B^2)^{**}$$

which preserves the grading. In particular,

(a1) $\mathrm{Ext}^1_B(I_B \otimes K_B(n+c+1), B)(\ell_3 - a_{t+2} - a_{t+3})_\nu = 0$ *for* $\nu < a_{t+3} + a_{t+2} - a_{t+1} - a_t$.

(a2) *If $Char(k) = 0$, then* $\mathrm{Ext}^1_B(I_B \otimes K_B(n+c+1), B)(\ell_3 - a_{t+2} - a_{t+3})_\nu = 0$ *for* $\nu \leq a_{t+3} + a_{t+2} - a_{t+1} - a_t$.

(b) *If $\mathrm{depth}_{I(Z)} B \geq 5$ then there is an exact sequence*

$$I_B/I_B^2 \to (I_B/I_B^2)^{**} \to \mathrm{Ext}^2_B(I_B \otimes K_B(n+c+1), B) \cong H^1_{I(Z)}(I_B/I_B^2) \to 0$$

which preserves the grading.

Remark 4.12. Note that (a2) shows the desired vanishing because we in Theorem 4.5 have assumed $a_0 \le a_1 \le \cdots \le a_{t+3}$.

Proof. (a) The Eagon-Northcott resolution associated to $\varphi_3 : F \to G_3 = \oplus_{j=0}^{t+1} R(a_j)$ leads to

$$0 \to F_3 := \wedge^{t+2} G_3^* \otimes S_2 F \otimes \wedge^t F \to F_2 := \wedge^{t+1} G_3^* \otimes S_1 F \otimes \wedge^t F \to \tag{4.21}$$

$$\to F_1 := \wedge^t G_3^* \otimes \wedge^t F \to I_B \to 0.$$

Applying $\mathrm{Hom}_R(.,R)$ we get the exact sequence

$$0 \to R \to F_1^* \to F_2^* \to F_3^* \to \mathrm{Ext}_R^2(I_B, R) \cong K_B(n+1+c) \to 0.$$

Tensoring with $. \otimes_R B$ leads to a complex

$$0 \to (I_B/I_B^2)^* \to F_1^* \otimes B \to F_2^* \otimes B \xrightarrow{\psi} F_3^* \otimes B \to K_B(n+c+1) \to 0 \tag{4.22}$$

which is exact except in the middle where we have the homology $I_B \otimes K_B(n+c+1) \cong \mathrm{Tor}_1^R(K_B(n+c+1), B)$. Indeed this easily follows from the right exactness of $. \otimes_B B$ and the left exactness of $\mathrm{Hom}_R(.,B)$ (applied to (4.21)). Since (4.21) also implies

$$0 \to H_1' := \ker(\rho) \to F_1 \otimes_R B \xrightarrow{\rho} I_B/I_B^2 \to 0 \tag{4.23}$$

(observe that ${H'}_1$ is quite close to the Koszul homology H_1). By [21], Lemma 35, we have $\mathrm{depth}_{\mathfrak{m}} (I_B/I_B^2)^* \ge \mathrm{depth}_{\mathfrak{m}} B - 1$ and hence by (3.9),

$$\mathrm{Ext}_B^1(I_B/I_B^2, B) = 0.$$

Dualizing (4.23), it follows that

$$0 \to (I_B/I_B^2)^* \to F_1^* \otimes B \to {H'}_1^* \to 0 \tag{4.24}$$

(and, if desirable, one may see $H_1^* \cong {H'}_1^*$). Since we know the homology "in the middle" of (4.22), we get the exact sequences

$$0 \to {H'}_1^* \to \ker(\psi) \to I_B \otimes K_B(n+c+1) \to 0, \tag{4.25}$$

$$0 \to \ker(\psi) \to F_2^* \otimes B \xrightarrow{\psi} F_3^* \otimes B \to K_B(n+c+1) \to 0. \tag{4.26}$$

Now we have the set-up to prove that $\mathrm{Ext}_B^1(I_B \otimes K_B(n+c+1), B) \cong \mathcal{K} := \ker(I_B/I_B^2 \to (I_B/I_B^2)^{**})$. Firstly note that dualizing (4.24) once more and comparing with (4.23) in obvious way, we see that

$$0 \to {H'}_1 \to {H'}_1^{**} \to \mathcal{K} \to 0$$

by the snake-lemma. Now we apply $\mathrm{Hom}(.,B)$ to (4.25) and the left part of (4.26). We get a commutative diagram

$$\begin{array}{ccccccc}
F_2 \otimes B & \to & {H'}_1 & \to & 0 & & \\
\downarrow & & \downarrow & & & & \\
\mathrm{Hom}(\ker(\psi), B) & \to & \mathrm{Hom}({H'}_1^*, B) & \to & \mathrm{Ext}_B^1(I_B \otimes K_B(n+c+1), B) & \to & \mathrm{Ext}_B^1(\ker(\psi), B) \\
\downarrow & & & & & & \\
\mathrm{Ext}_B^1(\mathrm{im}(\psi), B) & & & & & &
\end{array}$$

from which we deduce the exact sequence

$$\operatorname{Ext}^1_B(\operatorname{im}(\psi), B) \to \mathcal{K} \to \operatorname{Ext}^1_B(I_B \otimes K_B(n+c+1), B) \to \operatorname{Ext}^1_B(\ker(\psi), B). \tag{4.27}$$

Hence it suffices to show

$$\operatorname{Ext}^1_B(\operatorname{im}(\psi), B) = 0 = \operatorname{Ext}^1_B(\ker(\psi), B).$$

By (4.26) we have

$$\operatorname{Ext}^1_B(\operatorname{im}(\psi), B)(n+c+1) \cong \operatorname{Ext}^2_B(K_B, B) \cong \operatorname{Ext}^2_B(K_B \otimes K_B, K_B)$$

$$\operatorname{Ext}^1_B(\ker(\psi), B)(n+c+1) \cong \operatorname{Ext}^3_B(K_B, B) \cong \operatorname{Ext}^3_B(K_B \otimes K_B, K_B)$$

where the rightmost isomorphism is a consequences of the spectral sequence used in [17], Satz 1.2 because we have $\operatorname{depth}_{I(Z)} B \geq 3$. By [7]; Corollary 3.4, we know that $\operatorname{depth}_{\mathfrak{m}} S_2(K_B) \geq \operatorname{depth}_{\mathfrak{m}} B - 1$. Hence by Gorenstein duality $\operatorname{Ext}^i_B(S_2(K_B), K_B) = 0$ for $i \geq 2$. Defining $\wedge$ by

$$0 \to \wedge \to K_B \otimes K_B \to S_2(K_B) \to 0$$

and noting that $\tilde{\wedge}|_{\operatorname{Proj}(B)-Z} = 0$, we get $\operatorname{Ext}^i_B(\wedge, K_B) = 0$ for $i \leq 3$ by (3.9) and the assumption $\operatorname{depth}_{I(Z)} B \geq 4$. Combining we get $\operatorname{Ext}^i_B(K_B \otimes K_B, K_B) \cong \operatorname{Ext}^i_B(S_2(K_B), K_B) = 0$ for $i = 2$ and 3 as required, i.e. $\mathcal{K} \cong \operatorname{Ext}^1_B(I_B \otimes K_B(n+c+1), B)$ by (4.27).

Now it is a triviality to see (a1) because the smallest degree of a minimal generator of I_B is $\ell_3 - a_t - a_{t+1}$.

(a2) Since $\operatorname{depth}_{I(Z)} B \geq 2$, we get $(I_B/I_B^2)^{**} \cong H^0(X - Z, \mathcal{I}_X/\mathcal{I}_X^2)$ and hence that $\mathcal{K}$ is isomorphic to $H^0_{I(Z)}(B)$. Similarly we prove that the kernel of the "universal" derivation $d : I_B/I_B^2 \to \Omega_{R/k} \otimes_R B$ is $H^0_{I(Z)}(B)$ which by [11], Theorem 3, is isomorphic to $I_B^{(2)}/I_B^2$ where $I_B^{(2)}$ is the second symbolic power of I_B. Hence we have a grading-preserving isomorphism

$$\operatorname{Ext}^1_B(I_B \otimes K_B(n+c+1), B) \cong I_B^{(2)}/I_B^2. \tag{4.28}$$

Now, in characteristic zero, $I_B^{(2)} \subset \mathfrak{m} I_B$ by [11]; Proposition 13, which shows that the smallest degree of the minimal generators of $I_B^{(2)}$ is at least one less than the smallest degree of the generators of I_B, i.e. we have

$$(I_B^{(2)})_{\ell_3 - a_{t+2} - a_{t+3} + \nu} = 0 \text{ for } \nu \leq 0$$

and we conclude by (4.28).

(b) Again since $\operatorname{depth}_{I(Z)} B \geq 2$, we have $(I_B/I_B^2)^{**} \cong H^0_*(X - Z, \mathcal{I}_X/\mathcal{I}_X^2)$ and hence $\operatorname{coker}[I_B/I_B^2 \to (I_B/I_B^2)^{**}] \cong H^1_{I(Z)}(I_B/I_B^2) \cong H^2_{I(Z)}(H_1')$, cf. (4.23) for the last isomorphism. Using (4.25), we get the exact sequence

$$\operatorname{Ext}^1_B(\ker(\psi), B) \to \operatorname{Ext}^1_B(H_1'^{*}, B) \to \operatorname{Ext}^2_B(I_B \otimes K_B(n+c+1), B) \to \operatorname{Ext}^2_B(\ker(\psi), B).$$

As argued in (4.27) and after (4.27), we see that $\operatorname{Ext}^1_B(\ker(\psi), B) = 0$ and

$$\operatorname{Ext}^2_B(\ker(\psi), B)(n+c+1) \cong \operatorname{Ext}^4_B(K_B \otimes K_B, K_B) \cong \operatorname{Ext}^4_B(S_2(K_B), K_B) = 0$$

where the last isomorphism to the second symmetric power follows from the fact that $\operatorname{depth}_{I(Z)} B \geq 5$ implies $\operatorname{Ext}^i_B(\wedge, K_B) = 0$ for $i \leq 4$, and the vanishing to the right follows from $\operatorname{depth}_{\mathfrak{m}} S_2(K_B) \geq \operatorname{depth}_{\mathfrak{m}} B - 1$. Since by (3.9),

$$\operatorname{Ext}^1_B(H_1^{'*}, B) \cong H^1_*(U, \mathcal{H}om(\tilde{H_1^{'*}}, \tilde{B})) \cong H^1_*(U, \tilde{H_1'}) \cong H^2_{I(Z)}(H_1')$$

we are done. $\square$

Remark 4.13. For generic determinantal schemes one knows that $\operatorname{depth}_{I(Z)}(I_B/I_B^2) \geq 2$ by [5]. So the vanishing of $\operatorname{Ext}^1_B(I_B \otimes K_B(n+c+1), B)_\nu$ under reasonable genericity assumptions is expected (for any ν).

Corollary 4.14. *Let $W(\underline{b}; \underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \geq 1$ and $c = 5$. If $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$ and $Char(k) = 0$, then*

$$\dim W(\underline{b}; \underline{a}) = \lambda_5 + K_3 + K_4 + K_5.$$

Proof. It follows from Remark 2.7 and the assumption $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$ that the set $Z_j = Sing(X_j)$ has $\operatorname{depth}_{I(Z_j)} D_j \geq 4$ for $j = 2$ and 3 provided $X = \operatorname{Proj}(A)$ is chosen general in $W(\underline{b}; \underline{a})$. By (4.14), Lemma 4.9, (4.20), Lemma 4.11(a2) and Remark 4.12 the assumptions of Theorem 4.5 are fulfilled and we conclude by applying it. $\square$

Now we state the last Corollaries of this section which shows that the upper bound of $\dim W(\underline{b}; \underline{a})$ given in Theorem 3.5 is indeed equal to $\dim W(\underline{b}; \underline{a})$ for *all* $c \geq 3$ and most values of $a_0, a_1, \cdots, a_{t+c-2}; b_1, \cdots, b_t$. Our result is based upon Remark 4.4 and the proof of Theorem 4.5. Indeed, we have seen that

$$\dim W(\underline{b}; \underline{a}) = \lambda_c + K_3 + K_4 + \cdots + K_c$$

provided

$$\dim W(F, G_{c-1}) = \lambda_{c-1} + K_3 + K_4 + \cdots + K_{c-1}$$

and

$$\operatorname{Hom}_R(I_{D_{c-2}}, I_{c-1})_0 = 0. \tag{4.29}$$

Corollary 4.15. *Let $W(\underline{b}; \underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \geq 0$ and $c \geq 6$. Assume $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$, $Char(k) = 0$ and*

(i_6) : $a_{t+4} > a_{t-1} + a_t + a_{t+1} + a_{t+2} - a_0 - a_1 - a_2$,

(i_7) : $a_{t+5} > a_{t-1} + a_t + a_{t+1} + a_{t+2} + a_{t+3} - a_0 - a_1 - a_2 - a_3$,

....

(i_c) : $a_{t+c-2} > \sum_{j=t-1}^{t+c-4} a_j - \sum_{j=0}^{c-4} a_j$.

Then,

$$\dim W(\underline{b}; \underline{a}) = \lambda_c + K_3 + \cdots + K_c.$$

Proof. By the Eagon-Northcott resolution the largest possible degree of a generator of $I_{D_{c-2}}$ is $\ell_c - \sum_{j=0}^{c-4} a_j - a_{t+c-3} - a_{t+c-2}$ where $\ell_c = \sum_{j=0}^{t+c-2} a_j - \sum_{i=0}^{t} b_i$ and the smallest possible degree of a generator of $I_{c-1} \cong I_{D_c}/I_{D_{c-1}}$ is $\ell_c - \sum_{j=t-1}^{t+c-3} a_j$ because $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$. Hence if the latter is strictly larger than the former, i.e. if

$$a_{t+c-2} > \sum_{j=t-1}^{t+c-4} a_j - \sum_{j=0}^{c-4} a_j$$

then $\operatorname{Hom}(I_{D_{c-2}}, I_{c-1})_0 = 0$ and we conclude using the argument of (4.29) and Corollaries 4.7, 4.10 and 4.14. □

Remark 4.16. (1) If we want to skip the characteristic zero assumption, we can avoid the use of Corollary 4.14 by introducing the assumption

$$(i_5): \quad a_{t+3} > a_{t-1} + a_t + a_{t+1} - a_0 - a_1.$$

We still get $\dim W(\underline{b}; \underline{a}) = \lambda_c + K_3 + \cdots + K_c$, supposing $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$ and (i_5), (i_6),...,(i_c).

(2) We can further weaken $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$ to $a_{i-min(2,t)} \geq b_i$ for $min(2,t) \leq i \leq t$ by avoiding Corollary 4.10 and assuming in addition

$$(i_4): \quad a_{t+2} > a_{t-1} + a_t - a_0.$$

Remark 4.17. While Corollaries 4.7, 4.10 and 4.14 do not apply to the case when $W(\underline{b}; \underline{a})$ is the locus of zero-dimensional determinantal schemes, Corollary 4.15 and Remark 4.16 do apply to the zero-dimensional case. In particular, using Remark 4.16 (1) (resp. (2)) for $c = 5$ (resp. $c = 4$), we get a single assumption, namely (i_5) (resp. (i_4)) in addition to $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$ (resp. $a_{i-min(2,t)} \geq b_i$ for $min(2,t) \leq i \leq t$) which suffices for having $\dim W(\underline{b}; \underline{a})$ equal to the upper bound given in Theorem 3.5 for the zero schemes as well.

It is worthwhile to point out that this last remark on zero-schemes works also in the codimension $c = 3$ case, and here the (i_3) assumption is very weak. We have

Corollary 4.18. *Let $W(\underline{b}; \underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+3}$ of codimension 3. If $a_{i-min(2,t)} \geq b_i$ for $min(2,t) \leq i \leq t$ and if in addition*

$$(i_3): \quad a_{t+1} > a_{t-1}$$

then $\dim W(\underline{b}; \underline{a}) = \lambda_3 + K_3$.

Proof. Slightly extending Remark 2.7 by introducing the determinantal hypersurface $X_1 = \operatorname{Proj}(D_1)$ we have $\operatorname{depth}_{I(Z_1)} D_1 \geq 3$ and $\operatorname{depth}_{I(Z_2)} D_2 \geq 2$ by choosing $X = \operatorname{Proj}(A) \in W(\underline{b}; \underline{a})$ general. It follows that (4.4) is exact also for $i = 1$, and since $a_{i-min(2,t)} \geq b_i$ for $min(2,t) \leq i \leq t$ implies $\operatorname{Hom}_R(I_{D_1}, I_2)_0 = 0$ we get

$$\hom(I_{D_2}, I_2)_0 \cong \dim(D_2)_{a_t - a_{t+1}}.$$

Hence Proposition 4.1 (ii) for $c = 3$ applies to explicitly get a lower bound of $\dim W(\underline{b}; \underline{a})$, which combining (4.10) and (4.9) turns out to be $\lambda_3 + K_3$. Hence, $\dim W(\underline{b}; \underline{a}) = \lambda_3 + K_3$ by Theorem 3.5. □

5. Unobstructedness of determinantal schemes

In this section we keep the notation introduced in sections 3 and 4 and we consider the problem of when the closure of $W(\underline{b}; \underline{a})$ is an irreducible component of $\operatorname{Hilb}^p(\mathbb{P}^{n+c})$ and when $\operatorname{Hilb}^p(\mathbb{P}^{n+c})$ is smooth or, at least, generically smooth along $W(\underline{b}; \underline{a})$.

Through this section we *always* assume $n \geq 1$ and $c \geq 2$. The following result is crucial to our work in this section:

Theorem 5.1. *Let $X \subset \mathbb{P}^{n+c}$ be a good determinantal scheme of dimension $n \geq 1$, let $X = X_c \subset X_{c-1} \subset ... \subset X_2 \subset \mathbb{P}^{n+c}$ be the flag obtained by successively deleting columns from the right hand side and let $Z_i \subset X_i$ be some closed subset such that $X_i - Z_i \subset \mathbb{P}^{n+c}$ is a local complete intersection. Let $p_i \in \mathbb{Q}[s]$ be the Hilbert polynomial of X_i.*

(i) If $\operatorname{depth}_{I(Z_i)} D_i \geq 3$ for $2 \leq i \leq c-1$, and if $\operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i)_0 \hookrightarrow \operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, D_i)_0$ for $i = 2, ..., c-1$, then X (and each X_i) is unobstructed, and

$$\dim_{X_{i+1}} \operatorname{Hilb}^{p_{i+1}}(\mathbb{P}^{n+c}) = \dim_{X_i} \operatorname{Hilb}^{p_i}(\mathbb{P}^{n+c}) + \dim(N_{D_{i+1}/D_i})_0 - \hom(I_{D_i}, I_i)_0$$

for $i = 2, 3, ..., c-1$.

(ii) If $a_0 \leq a_1 \leq ... \leq a_{t+c-2}$, $b_1 \leq ... \leq b_t$ and $a_{i-min(2,t)} \geq b_i$ for $min(2,t) \leq i \leq t$, and if a general $X \in W(\underline{b}; \underline{a})$ satisfies

$$\operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i)_0 = 0 \text{ for } i = 2, ..., c-1$$

then $\overline{W(\underline{b}; \underline{a})}$ is a generically smooth irreducible component of $\operatorname{Hilb}^p(\mathbb{P}^{n+c})$.

Proof. (i) First of all we claim that there are two short exact sequences, the vertical and the horizontal one, fitting into a commutative diagram (whose square is cartesian)

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \operatorname{Hom}_R(I_{D_i}, I_i)_0 \arrow[d] & \\
 & & A^1 \arrow[r, "T_{pr_2}"] \arrow[d, "T_{pr_1}"'] \arrow[dr, phantom, "\square"] & \operatorname{Hom}_R(I_{D_i}, D_i)_0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Hom}(I_i, D_{i+1})_0 \arrow[r] & \operatorname{Hom}(I_{D_{i+1}}, D_{i+1})_0 \arrow[r] & \operatorname{Hom}_R(I_{D_i}, D_{i+1})_0 \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd} \tag{5.1}$$

where A^1 is the tangent space of the Hilbert flag scheme $D(p_{i+1}, p_i)$ at $(X_{i+1} \subset X_i)$ and T_{pr_i} the tangent maps of the projections pr_i (see Proposition 4.1 for details). Since the vertical sequence is exact by assumption and the tangent space description of the Hilbert flag scheme and its projections are well known ([19], Chapter 6, and note that the zero piece of the graded Hom's above and the corresponding global sections of their sheaves of [19] coincide by (3.9)), we only have to prove the short-exactness of the horizontal sequence. Hence it suffices to prove that T_{pr_2} is surjective. To see this, it suffices to slightly generalize the argument in the proof of Proposition 4.1 where we showed that the dimension of the fiber is $\geq m_c(0)$. We skip the details since [19], Theorem 10.13 shows more (see also Remark 6.3 and note that the new hypothesis (*) of Remark 6.3 corresponds to the depth assumption of Theorem 5.1). Indeed, it contains a deformation theoretic argument which shows that pr_2 is not only dominating but also "infinitesimal dominating or surjective" (i.e. smooth at $(X_{i+1} \subset X_i)$). In particular, we have that the tangent map T_{pr_2} is surjective, cf. Remark 5.2 for another argument.

By the proof of Theorem 10.13 of [19], $D(p_{i+1}, p_i)$ is smooth at $(X_{i+1} \subset X_i)$ provided $\operatorname{Hilb}^{p_i}(\mathbb{P}^{n+c})$ is smooth at X_i (see Remark 5.2 for an easy argument). Since the tangent map of the first projection $pr_1 : D(p_{i+1}, p_i) \to \operatorname{Hilb}^{p_{i+1}}(\mathbb{P}^{n+c})$ is surjective, we get that

$\mathrm{Hilb}^{p_{i+1}}(\mathbb{P}^{n+c})$ is smooth at X_{i+1}. By induction X (and each X_i) is unobstructed since X_2 is unobstructed [12], and the two exact sequences of (5.1) easily lead to the dimension of $\dim_{X_{i+1}} \mathrm{Hilb}^{p_{i+1}}(\mathbb{P}^{n+c})$ because $N_{D_{i+1}/D_i} = \mathrm{Hom}(I_i, D_{i+1})$.

(ii) To prove that $\overline{W(\underline{b};\underline{a})}$ is an irreducible component, we use the notation of Proposition 4.1 and we may by induction suppose that $\overline{W(F,G_{c-1})}$ is an irreducible component of $\mathrm{Hilb}^{p_{c-1}}(\mathbb{P}^{n+c})$ since $\overline{W(F,G_2)}$ is an irreducible component by [12]. We have

$$\dim D(F,G_c,G_{c-1}) \geq \dim W(F,G_{c-1}) + m_c(0) \tag{5.2}$$

by Proposition 4.1 (ii) while for an irreducible component V of $D(p_c,p_{c-1})$ containing $D(F,G_c,G_{c-1})$ we must have

$$\dim V \leq \dim W(F,G_{c-1}) + \dim(N_{D_c/D_{c-1}})_0 \tag{5.3}$$

because $\dim(N_{D_c/D_{c-1}})_0$ is the fiber dimension of pr_2 at $(X_c \subset X_{c-1})$. Since $\mathrm{depth}_{I(Z_{c-1})} D_{c-1} \geq 3$, we have by (3.8)

$$\dim(N_{D_c/D_{c-1}})_0 = m_c(0).$$

Combining (5.2) and (5.3) we get $\dim D(F,G_c,G_{c-1}) \geq \dim V$ and hence $\overline{D(F,G_c,G_{c-1})}$ is an irreducible component of $D(p_c,p_{c-1})$. Since the first projection $pr_1 : D(p_c,p_{c-1}) \to \mathrm{Hilb}^{p_c}(\mathbb{P}^{n+c})$ is smooth at $(X_{i+1} \subset X_i)$ by the surjectivity of T_{pr_1} and the smoothness of $D(p_{i+1},p_i)$ at $(X_{i+1} \subset X_i)$, we get that $\overline{W(\underline{b};\underline{a})}$ is an irreducible component, which necessarily is generically smooth because $\mathrm{Hilb}^{p_c}(\mathbb{P}^{n+c})$ is smooth at a general point X by the first part of the proof and by Remark 2.7. □

Remark 5.2. If, in Theorem 5.1, we suppose $\mathrm{depth}_{I(Z_i)} D_i \geq 4$ we may easily see the surjectivity of T_{pr_2} in the following way. Using (3.9), we get $\mathrm{Ext}^1_{D_i}(I_i,R_i) = \mathrm{Ext}^2_{D_i}(I_i,I_i) = 0$ by the depth condition above. Applying $\mathrm{Hom}_{D_i}(I_i,.)$ to the exact sequence $0 \to I_i \to D_i \to D_{i+1} \to 0$, we get $\mathrm{Ext}^1_{D_i}(I_i,D_{i+1}) = 0$ and the lower horizontal sequence of (5.1) is short exact and we easily conclude. Finally using the vanishing of $\mathrm{Ext}^1_{D_i}(I_i,D_{i+1})_0$, it follows from (3.9) that $H^1(U_i,\tilde{N}_{D_{i+1}/D_i}) \cong \mathrm{Ext}^1_{D_{i+1}}(I_i/I_i^2,D_{i+1})_0 = 0$. Then it is not difficult to see that $D(p_{i+1},p_i)$ is smooth at $(X_{i+1} \subset X_i)$ provided $\mathrm{Hilb}^{p_i}(\mathbb{P}^{n+c})$ is smooth at X_i.

To apply Theorem 5.1(ii) in the codimension $c = 3$ case, it suffices to prove that

$$\mathrm{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_2)_0 = 0.$$

By (3.9) and (4.12) we see that $(U_2 = X_2 - Z_2)$

$$\begin{aligned} \mathrm{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_2) &\cong H^1_*(U_2, \mathcal{H}om(\mathcal{I}_{X_2}, \tilde{K}_{D_2}(n+4), \mathcal{O}_{X_2})(\ell_2 - a_{t+1})) \\ &\cong H^1_*(U_2, \mathcal{I}_{X_2}/\mathcal{I}^2_{X_2}(\ell_2 - a_{t+1})) \end{aligned} \tag{5.4}$$

and we consider two cases:

If $\mathrm{depth}_{I(Z_2)} D_2 \geq 4$, we get $\mathrm{depth}_{I(Z_2)} I_{D_2}/I^2_{D_2} \geq 3$ [1] and the group in (5.4) vanishes.

If $\mathrm{depth}_{I(Z_2)} D_2 = 3$ (e.g. X_2 is smooth and 2 dimensional), the group, in degree zero, is clearly $H^1(U_2, \mathcal{I}_{X_2}/\mathcal{I}^2_{X_2}(\ell_2 - a_{t+1}))$, and we have to suppose it vanishes in order to conclude that $\overline{W(\underline{b};\underline{a})}$ is a generically smooth component of $\mathrm{Hilb}^p(\mathbb{P}^{n+3})$ of dimension $\lambda_3 + K_3$. All this is essentially [19], Corollary 10.15 (ii).

The case $c = 4$ is also straightforward. In this case it suffices to see that (5.4) vanishes and that

$$\operatorname{Ext}^1_{D_3}(I_{D_3}/I^2_{D_3}, I_3) = 0. \tag{5.5}$$

If we suppose

$$\operatorname{depth}_{I(Z_2)} D_2 \geq 4 \text{ and } \operatorname{depth}_{I(Z_3)} D_3 \geq 4 \tag{5.6}$$

we claim that both groups vanish. We only need to prove (5.5). Since $\operatorname{depth}_{I(Z_2)} D_2 \geq 4$ it follows from (3.9) that (4.4) is short-exact for $i = 3$. Using Lemma 4.9 and (4.14), we see that (4.5) is short-exact for $i = 3$ as well, i.e. we have exact sequences

$$\begin{array}{c} 0 \to D_3(a) \to \operatorname{Hom}_R(I_{D_3}, I_3) \to \operatorname{Hom}_R(I_{D_2}, I_3) \to 0 \\ \hfill \| \hspace{6em} \\ 0 \to \operatorname{Hom}(I_{D_2} \otimes I_2^*, I_2(a)) \to \operatorname{Hom}_R(I_{D_2}, I_2(a)) \to \operatorname{Hom}_{D_2}(I_{D_2} \otimes I_2^*, D_3(a)) \to 0 \end{array} \tag{5.7}$$

where $a = a_{t+1} - a_{t+2}$. By Lemma 4.9, the codepth of $\operatorname{Hom}(I_{D_2} \otimes I_2^*, I_2(a))$ is at most 1 while (4.12) shows the same conclusion for $\operatorname{Hom}(I_{D_2}, I_2(a))$. The lower exact sequence of (5.7) therefore shows that the codepth of $\operatorname{Hom}_{D_2}(I_{D_2} \otimes I_2^*, D_3(a))$ is at most 1 as a D_3-module. The upper sequence shows that

$$\operatorname{depth}_{\mathfrak{m}} \operatorname{Hom}_R(I_{D_3}, I_3) \geq \operatorname{depth}_{\mathfrak{m}} D_3 - 1. \tag{5.8}$$

Now since $\operatorname{depth}_{I(Z_3)} D_3 \geq 4$, we get $\operatorname{depth}_{I(Z_3)} \operatorname{Hom}_{D_3}(I_{D_3}/I^2_{D_3}, I_3) \geq 3$ and hence by (3.9) that (5.5) holds. By Remark 2.7, we see that (5.6) holds for a general $X \in W(\underline{b}; \underline{a})$ provided $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$. Combining with Corollary 4.7 and Corollary 4.10 we get

Corollary 5.3. *Let $W(\underline{b}; \underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \geq 2$ and $c = 3$ or $c = 4$. If $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$, then $\overline{W(\underline{b}; \underline{a})}$ is a generically smooth, irreducible component of* $\operatorname{Hilb}^p(\mathbb{P}^{n+c})$ *of dimension $\lambda_c + K_3 + ... + K_c$.*

Remark 5.4. If $c = 4$, then the assumption (5.6) excludes the interesting case when $W(\underline{b}; \underline{a})$ parameterizes good determinantal curves in $\mathbb{P}^5$. To consider this case we will weaken (5.6) and only suppose $\operatorname{depth}_{I(Z_2)} D_2 \geq 4$. Recalling that (4.12) leads to

$$H^1_*(U_2, \mathcal{H}om(\mathcal{I}_{X_2} \otimes \mathcal{I}_2^*, \mathcal{O}_{X_2})) \cong H^1_*(U_2, \mathcal{I}_{X_2}/\mathcal{I}^2_{X_2}(\ell_2 - a_{t+1})) = 0,$$

we have by (4.7) injections

$$\begin{array}{ccccc} \operatorname{Ext}^1_{D_3}(I_{D_3}/I^2_{D_3}, I_3)_0 & \hookrightarrow & H^1(U_2, \mathcal{H}om(\mathcal{I}_{X_2} \otimes \mathcal{I}_2^*, \mathcal{O}_{X_3}(a))) & \hookrightarrow & H^2(U_2, \mathcal{H}om(\mathcal{I}_{X_2} \otimes \mathcal{I}_2^*, \mathcal{I}_2(a))) \\ & & \| & & \| \\ & & H^1(U_2, \mathcal{I}_{X_2}/\mathcal{I}^2_{X_2} \otimes \mathcal{O}_{X_3}(\ell_2 - a_{t+2})) & \hookrightarrow & H^2(U_2, \mathcal{I}_{X_2}/\mathcal{I}^2_{X_2} \otimes \tilde{K}^*_{D_2}(a')) \end{array} \tag{5.9}$$

where $a = a_{t+1} - a_{t+2}$ and $a' = 2\ell_2 - 6 - a_{t+1} - a_{t+2}$. In the interesting case $X = X_4 \subset X_3 \subset X_2 \subset \mathbb{P}^5$ where X_2 is smooth, then $U_2 = X_2$. In particular if one of the groups of (5.9) vanishes, then $\overline{W(\underline{b}; \underline{a})}$ is still a generically smooth, irreducible component of $\operatorname{Hilb}^p(\mathbb{P}^5)$ of dimension $\lambda_4 + K_3 + K_4$.

As a Corollary of the first part of Theorem 5.1 we also get the unobstructedness and the vanishing of some $H^i_*(X,\mathcal{N}_X)$ for good determinantal schemes $X \in \mathbb{P}^{n+c}$ of codimension $3 \le c \le 4$. For $c = 3$, the unobstructedness is essentially proved in [19], Corollary 10.15 and the vanishing of $H^i_*(X,\mathcal{N}_X)$ is shown in [21], Lemma 35. Notice that the Corollary really gives additional information about the generically smooth component $\overline{W(b;a)}$ of Corollary 5.3, because it tells more precisely where Hilb is smooth. Finally note also that there exists obstructed good determinantal reduced curves in $\mathbb{P}^4$, cf. [19]; Remark 9.12, so some kind of limitations on the singular locus of X is expected to get unobstructedness.

Corollary 5.5. *Let $X = \mathrm{Proj}(A) \subset \mathbb{P}^{n+c}$ be a good determinantal scheme of dimension $n \ge 2$ for which there is a flag satisfying $\mathrm{depth}_{I(Z_i)}(D_i) \ge 4$ for $2 \le i \le c-1$. If $3 \le c \le 4$, then X is unobstructed and the normal module $N_A := Hom_R(I_A, A)$ satisfies $\mathrm{depth}_{\mathfrak{m}}(N_A) \ge n-1$. In particular*

$$H^i_*(X,\mathcal{N}_X) = 0 \text{ for } 1 \le i \le n-2.$$

Proof. Due to the vanishing of (5.4) and (5.5), the unobstructedness of X follows at once from Theorem 5.1(i). Moreover exactly as we managed to show that the exact sequences of (5.7) implied (5.8), we may see that the graded exact horizontal and vertical sequences of (5.1) for $i = 2$ (and not only the degree zero piece of these sequences) imply

$$\mathrm{depth}_{\mathfrak{m}} \mathrm{Hom}(I_{D_3}, D_3) \ge \dim D_3 - 1 \tag{5.10}$$

because we have $\mathrm{depth}_{I(Z_2)}(D_2) \ge 4$ and hence $\mathrm{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_2) = 0$ by (5.4). This shows what we want for $c = 3$. Finally the same argument for $i = 3$, assuming (5.6) and using (5.8) and (5.10), leads to $\mathrm{depth}_{\mathfrak{m}} \mathrm{Hom}_R(I_{D_4}, D_4) \ge \dim D_4 - 1$ and we are done. ☐

In the following example we will see that Corollary 5.3 does not always extend to determinantal curves $C \subset \mathbb{P}^5$, i.e. the closure of $W(\underline{b};\underline{a})$ is not necessarily an irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^5)$ although by Corollary 4.10 we know that $\dim W(\underline{b};\underline{a})$ is indeed $\lambda_4 + K_3 + K_4$.

Example 5.6. Let $C \subset \mathbb{P}^5$ be a smooth good determinantal curve of degree 15 and arithmetic genus 10 defined by the maximal minors of a 3×6 matrix with linear entries. The closure of $W(\underline{b};\underline{a}) = W(0,0,0;1,1,1,1,1,1)$ inside $\mathrm{Hilb}^{15t-9}(\mathbb{P}^5)$ is not an irreducible component. In fact, let $H_{15,10} \subset \mathrm{Hilb}^{15t-9}(\mathbb{P}^5)$ be the open subset parameterizing smooth connected curves of degree $d = 15$ and arithmetic genus $g = 10$. It is well known that any irreducible component of $H_{15,10}$ has dimension $\ge \chi(N_C) = 6d + 2(1-g) = 72$ (cf. [25] §11b); while by Corollary 4.10, $\dim W(\underline{b};\underline{a}) = 64$.

For the codimension $c = 5$ case we have

Corollary 5.7. *Let $W(\underline{b};\underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \ge 1$ and $c = 5$. If $a_{i-min(3,t)} \ge b_i$ for $min(3,t) \le i \le t$ and if $W(\underline{b};\underline{a})$ contains a determinantal scheme $X = \mathrm{Proj}(D_5)$ whose flag $R \to D_2 \to D_3 \to D_4 \to D_5$ obtained by deleting columns of "largest possible degree" satisfies (with $Z_i = Sing(X_i)$), $\mathrm{depth}_{I(Z_2)} D_2 \ge 4$, $\mathrm{depth}_{I(Z_3)} D_3 \ge 5$ and $H^1(X_3 - Z_3, \mathcal{I}^2_{X_3}(\ell_3 - 2a_{t+3})) = 0$, then $\overline{W(\underline{b};\underline{a})}$ is a generically* *smooth, irreducible component of* $\mathrm{Hilb}^p(\mathbb{P}^{n+5})$ *(of dimension* $\lambda_5 + K_3 + K_4 + K_5$ *provided* $Char(k) = 0$*).*

Proof. First of all note that by Remark 2.7, if we choose $X \in W(\underline{b};\underline{a})$ general, we have $\operatorname{depth}_{I(Z_2)} D_2 \geq 4$ and $\operatorname{depth}_{I(Z_3)} D_3 \geq 5$. By Theorem 5.1 and the conclusion of (5.6), it suffices to show

$$\operatorname{Ext}^1_{D_4}(I_{D_4}/I^2_{D_4}, I_4)_0 = 0. \tag{5.11}$$

By Lemma 4.2(b) and (5.5) we must show that $\operatorname{Ext}^2_{D_3}(I_{D_3} \otimes I_3^*, I_3)_{a_{t+2}-a_{t+3}} = 0$.

Looking to (4.20), this group is isomorphic to

$$\operatorname{Ext}^2_{D_3}(I_{D_3} \otimes K_{D_3}(n+c+1), D_3)_{\ell_3 - 2a_{t+3}}$$

which by Lemma 4.11 (b) is further isomorphic to $H^1_{I(Z_3)}(I_{D_3}/I^2_{D_3})_{\ell_3-2a_{t+3}}$. Since $X_3 = \operatorname{Proj}(D_3)$ is Cohen-Macaulay, the cohomology sequence associated to

$$0 \to I^2_{D_3} \to I_{D_3} \to I_{D_3}/I^2_{D_3} \to 0$$

gives us

$$H^1_{I(Z_3)}(I_{D_3}/I^2_{D_3})_{\ell_3-2a_{t+3}} \cong H^2_{I(Z_3)}(I^2_{D_3})_{\ell_3-2a_{t+3}} \cong H^1(X_3 - Z_3, \mathcal{I}^2_{X_3}(\ell_3 - 2a_{t+3}))$$

and we get (5.11). $\square$

We will now give two examples. The first one will be a smooth determinantal surface $S \subset \mathbb{P}^7$ whose flag $R \to D_2 \to D_3 \to D_4 \to D_5$ obtained by deleting columns of "largest possible degree" satisfies all hypothesis required in Corollary 5.7 and hence $\overline{W(\underline{b};\underline{a})}$ is a generically smooth, irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^6)$ of dimension $\lambda_5 + K_3 + K_4 + K_5$ ($Char(k) = 0$). The second one will be a smooth determinantal curve $C \subset \mathbb{P}^6$ hence the condition $\operatorname{depth}_{I(Z_3)} D_3 \geq 5$ is not fulfilled and in this case we will see that the closure of $W(\underline{b};\underline{a})$ is not an irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^6)$ although by Corollary 4.14 we know that $\dim W(\underline{b};\underline{a})$ is indeed $\lambda_5 + K_3 + K_4 + K_5$ ($Char(k) = 0$).

Example 5.8. (1) Let $S \subset \mathbb{P}^7$ be a smooth good determinantal surface of degree 6 defined by the maximal minors of a 2×6 matrix with general linear entries. Let $R \to D_2 \to D_3 \to D_4 \to D_5$ be the flag obtained by deleting columns from the right hand side. With the computer program Macaulay [2] we check that all hypothesis required in Corollary 5.7 are satisfied, i.e., $\operatorname{depth}_{I(Z_2)} D_2 \geq 4$, $\operatorname{depth}_{I(Z_3)} D_3 \geq 5$ and $H^1(X_3 - Z_3, \mathcal{I}^2_{X_3}(\ell_3 - 2a_{t+3})) = H^1(X_3, \mathcal{I}^2_{X_3}(2)) = 0$. By Corollary 5.7 the closure of $W(\underline{b};\underline{a})$ inside $\mathrm{Hilb}^p(\mathbb{P}^7)$ is a generically smooth irreducible component of dimension 57.

(2) Let $C \subset \mathbb{P}^6$ be a smooth good determinantal curve of degree 21 and arithmetic genus 15 defined by the maximal minors of a 3×7 matrix with linear entries. Since $\dim(C) = 1$, we have $\dim(X_3) = 3$ and hence $\operatorname{depth}_{I(Z_3)} D_3 \leq 4$. The closure of $W(\underline{b};\underline{a})$ inside $\mathrm{Hilb}^{21t-14}(\mathbb{P}^6)$ is not an irreducible component. In fact, let $H_{21,15} \subset \mathrm{Hilb}^{21t-14}(\mathbb{P}^6)$ be the open subset parameterizing smooth connected curves of degree $d = 21$ and arithmetic genus $g = 15$. It is well known that any irreducible component of $H_{21,15}$ has dimension $\geq 7d + 3(1-g) = 105$; while by Corollary 4.14, $\dim W(\underline{b};\underline{a}) = 90$.

Our final Corollaries are similar to Corollaries 4.15 and 4.18. To apply the final part of Theorem 5.1 we must show that $\mathrm{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i)_0 = 0$ for $i = 2, ..., c-1$. Using, however, the upper sequence of (4.8) it suffices to show that $\mathrm{Ext}^1_{D_{i-1}}(I_{D_{i-1}}/I^2_{D_{i-1}}, I_i)_0 = 0$ provided $\mathrm{depth}_{I(Z_{i-1})} D_{i-1} \geq 4$. This vanishing is fulfilled if

$$\mathrm{Ext}^1_R(I_{D_{i-1}}, I_i)_0 = 0 \text{ for } i = 4, ..., c-1 \tag{5.12}$$

(since $\mathrm{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i)_0 = 0$ for $i = 2, 3$ provided $\dim_{I(Z_i)} D_i \geq 4$ by (5.4) and (5.5)).

Corollary 5.9. *Let $W(\underline{b};\underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \geq 1$ and $c \geq 5$, or $n \geq 2$ and $3 \leq c \leq 4$. Assume $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$. Moreover if $c \geq 5$, then*

$(j_5) : a_{t+3} > a_{t-1} + a_t + a_{t+1} - a_0 - b_1,$
$(j_6) : a_{t+4} > a_{t-1} + a_t + a_{t+1} + a_{t+2} - a_0 - a_1 - b_1,$
$....$
$(j_c) : a_{t+c-2} > \sum_{j=t-1}^{t+c-4} a_j - \sum_{j=0}^{c-5} a_j - b_1.$

Then $\overline{W(\underline{b};\underline{a})}$ is a generically smooth, irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^{n+c})$ *of dimension $\lambda_c + K_3 + ... + K_c$.*

Proof. The relation of $I_{D_{c-2}}$ of the largest possible degree is $\ell_c - \sum_{j=0}^{c-5} a_j - b_1 - a_{t+c-3} - a_{t+c-2}$ and the smallest possible degree of a generator of I_{c-1} is $\ell_c - \sum_{j=t-1}^{t+c-3} a_j$. Hence $\mathrm{Ext}^1_R(I_{D_{c-2}}, I_{c-1})_0 = 0$ if $\ell_c - \sum_{j=0}^{c-5} a_j - b_1 - a_{t+c-3} - a_{t+c-2} < \ell_c - \sum_{j=t-1}^{t+c-3} a_j$ or, equivalently, $a_{t+c-2} > \sum_{j=t-1}^{t+c-4} a_j - \sum_{j=0}^{c-5} a_j - b_1$ which is our assumption (j_c).

Similarly we get $\mathrm{Ext}^1_R(I_{D_{i-1}}, I_i)_0 = 0$ if (j_{i+1}) holds. Since we by Remark 2.7 and the hypothesis $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$ know that a general $X \in W(\underline{b};\underline{a})$ satisfies $\mathrm{depth}_{I(Z_i)} D_i \geq 4$ for $2 \leq i \leq c-2$, we conclude by (5.12). For the dimension formula we use Remark 4.16 (1). ☐

Since Corollary 5.9 does not apply to $n = 1$ and $3 \leq c \leq 4$, we include one more result to cover these cases. For $c = 3$, the result is known ([19], Corollary 10.11).

Corollary 5.10. *Let $W(\underline{b};\underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ of dimension $n \geq 1$. If either*

(1) *$c = 3$, $a_{i-min(2,t)} \geq b_i$ for $min(2,t) \leq i \leq t$ and $a_{t+1} > a_{t-1} + a_t - b_1$, or*
(2) *$c = 4$, $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$ and $a_{t+2} > a_{t-1} + a_t - b_1$,*

then $\overline{W(\underline{b};\underline{a})}$ is a generically smooth, irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^{n+c})$ *of dimension $\lambda_c + K_3 + ... + K_c$.*

Proof. Let $c = 3$. To see the vanishing of $\mathrm{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_2)_0$ of Theorem 5.1, it suffices to prove $\mathrm{Ext}^1_R(I_{D_2}, I_2)_0 = 0$. As in the proof of Corollary 5.9, we find the minimal degree of relations of I_{D_2} to be $\ell_2 - b_1$, and we get the vanishing of the Ext^1_R-group above by assuming $\ell_2 - b_1 = \ell_3 - a_{t+1} - b_1 < \ell_3 - \sum_{j=t-1}^{t} a_j$, i.e.

$$(j_3') : \quad a_{t+1} > a_{t-1} + a_t - b_1.$$

If $c = 4$, it suffices to prove that $\mathrm{Ext}^1_R(I_{D_2}, I_3)_0 = 0$ by the argument of (5.12). Indeed, (5.4) vanishes and we know that $\mathrm{depth}_{I(Z_2)} D_2 \geq 4$ implies an injection

$$\mathrm{Ext}^1_{D_3}(I_{D_3}/I^2_{D_3}, I_3)_0 \hookrightarrow \mathrm{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_3)_0$$

by (4.8) and that the latter Ext^1-group vanishes if $\mathrm{Ext}^1_R(I_{D_2}, I_3)_0 = 0$. Now exactly as in the first part of the proof of Corollary 5.9, we have $\mathrm{Ext}^1_R(I_{D_2}, I_3)_0 = 0$ provided

$$(j_4): \quad a_{t+2} > a_{t-1} + a_t - b_1.$$

and we conclude by Theorem 5.1. For the dimension formulas, we use Remark 4.16(2) and Corollary 4.18. ☐

Remark 5.11. Looking to the proofs of Corollaries 5.9 and 5.10 we get the following. Let $U \subset \overline{W(\underline{b};\underline{a})}$ be the subset where $\mathrm{Hilb}^p(\mathbb{P}^{n+c})$ is smooth, and assume $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$ and $b_1 \leq \cdots \leq b_t$. Then U contains every X for which the flag

$$X = X_c \subset \cdots \subset X_i = Proj(D_i) \subset \cdots \subset X_2 \subset \mathbb{P}^{n+c}$$

of Theorem 5.1 satisfies $\mathrm{depth}_{I(Z_i)} D_i \geq 4$ for $2 \leq i \leq c-2$, $\mathrm{depth}_{I(Z_{c-1})} D_{c-1} \geq 3$ and
(1) (j_3') if $c = 3$,
(2) (j_4) if $c = 4$,
(3) (j_5) to (j_c) if $c \geq 5$.
Moreover, if $3 \leq c \leq 4$, we can drop (j_3') and (j_4) provided we increase the depth assumption to $\mathrm{depth}_{I(Z_i)} D_i \geq 4$ for $2 \leq i \leq c-1$.

Corollaries 4.15 and 5.9 and Remark 4.16(1) can be improved a little bit if we increase $\mathrm{depth}_{I(Z_i)} D_i$. In fact, by Remark 2.7 we know that under the assumption $a_{i-min(3,t)} \geq b_i$ for $min(3,t) \leq i \leq t$, we can suppose $\mathrm{depth}_{I(Z_i)} D_i \geq 5$ for $i \geq 3$, letting $Z_i = Sing(X_i)$. Since $Z_i \subset Z_{i-1}$, if we suppose

$$\mathrm{depth}_{I(Z_{i-2})} D_{i-2} \geq 5 \tag{5.13}$$

we get $\mathrm{depth}_{I(Z_{i-2})} D_{i-1} \geq 4$, and hence $\mathrm{depth}_{I(Z_{i-1})} D_{i-1} \geq 4$, as well as $\mathrm{depth}_{I(Z_{i-2})} D_i \geq 3$. As in (4.3), we see that

$$\mathrm{Ext}^1_{D_{i-2}}(I_{i-2}, I_i) \cong H^1_*(U_{i-2}, \mathcal{H}om_{\mathcal{O}_{X_{i-2}}}(\mathcal{I}_{i-2}, \mathcal{I}_i)) \cong$$
$$H^1_*(U_{i-2}, \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{I}_{i-2} \otimes \mathcal{I}_i^*, \mathcal{O}_{X_i})) \cong H^1_*(U_{i-2}, \mathcal{O}_{X_i}(a_{t+i-3} - a_{t+i-2})) = 0.$$

Arguing as in (4.7) and combining with (4.8) we get injections

$$\mathrm{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i) \hookrightarrow \mathrm{Ext}^1_{D_{i-1}}(I_{D_{i-1}}/I^2_{D_{i-1}}, I_i) \hookrightarrow \mathrm{Ext}^1_{D_{i-2}}(I_{D_{i-2}}/I^2_{D_{i-2}}, I_i). \tag{5.14}$$

In particular, if $\mathrm{Ext}^1_R(I_{D_{i-2}}, I_i)_0 = 0$ and if (5.13) hold, then $\mathrm{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i)_0 = 0$. Now looking to the proof of Corollary 5.9, we easily see that $\mathrm{Ext}^1_R(I_{D_{c-3}}, I_{c-1})_0 = 0$ provided

$$a_{t+c-2} > \sum_{j=t-1}^{t+c-5} a_j - \sum_{j=0}^{c-6} a_j - b_1. \tag{5.15}$$

Hence if (j_{c-1}) holds, then (5.15) holds because $a_{t+c-2} \ge a_{t+c-3}$, and it is superfluous to assume (j_c) in Corollary 5.9. The argument requires $\operatorname{depth}_{I(Z_{c-3})} D_{c-3} \ge 5$, i.e. $c \ge 6$, and arguing slightly more general, we see that for $6 \le i \le c$, then (j_i) is superfluous provided (j_{i-1}) holds.

In particular the conclusion of Corollary 5.9 holds if (j_i) holds for any *odd* number i such that $5 \le i \le c$.

Remark 5.12. (1) In Corollary 5.9, the assumption (j_i) is superfluous if (j_{i-1}) holds, $6 \le i \le c$.

(2) Increasing $\operatorname{depth}_{I(Z_i)} D_i$ even more (say by assuming $a_{i-min(4,t)} \ge b_i$ for $min(4,t) \le i \le t$, cf. Remark 2.7), we can weaken (j_k), resp. (i_k), conditions of Corollary 5.9, resp. Corollary 4.15, further.

6. Conjecture

We now state a Conjecture raised by this paper. In fact Theorem 3.5, Example 3.8 (i)-(iv), Proposition 3.13, and Corollaries 4.7, 4.10, 4.14, 4.15 and 4.18 suggest - and prove in many cases - the following conjecture:

Conjecture 6.1. *Given integers $a_0 \le a_1 \le ... \le a_{t+c-2}$ and $b_1 \le ... \le b_t$, we set $\ell := \sum_{j=0}^{t+c-2} a_j - \sum_{i=1}^{t} b_i$ and $h_i := 2a_{t+1+i} + a_{t+2+i} + \cdots + a_{t+c-2} - \ell + n + c$, for $i = 0, 1, ..., c-3$. Assume $a_{i-min([c/2]+1,t)} \ge b_i$ for $min([c/2]+1,t) \le i \le t$. Then we have*

$$\dim W(\underline{b};\underline{a}) = \sum_{i,j} \binom{a_i - b_j + n + c}{n+c} - \sum_{i,j} \binom{a_i - a_j + n + c}{n+c} -$$

$$\sum_{i,j} \binom{b_i - b_j + n + c}{n+c} + \sum_{j,i} \binom{b_j - a_i + n + c}{n+c} + \binom{h_0}{n+c} + 1 +$$

$$\sum_{i=1}^{c-3} \left(\sum_{r+s=i} \sum_{\substack{0 \le i_1 < \ldots < i_r \le t+i \\ 1 \le j_1 \le \cdots \le j_s \le t}} (-1)^{i-r} \binom{h_i + a_{i_1} + \cdots + a_{i_r} + b_{j_1} + \cdots + b_{j_s}}{n+c} \right).$$

In particular, we would like to know if the Conjecture 6.1 is at least true when the entries of $\mathcal{A}$ all have the same degree. More precisely,

Conjecture 6.2. *Let $W(\underline{0};\underline{d})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ of codimension c given by the maximal minors of a $t \times (t+c-1)$ matrix with entries homogeneous forms of degree d. Then,*

$$\dim W(\underline{0};\underline{d}) = t(t+c-1)\binom{d+n+c}{n+c} - t^2 - (t+c-1)^2 + 1.$$

Finally, since the results of this paper deal with and extend the results [19]; §10, we take the opportunity to mention an inaccuracy in [19], (10.12) and correct it.

Remark 6.3. We propose to substitute for the hypothesis [19], (10.12):

Given $C \subset \mathbb{P}^{n+c}$ a good standard determinantal scheme of dimension n, there always exists a flag $C = X_c \subset X_{c-1} \subset \ldots \subset X_2 \subset \mathbb{P}^{n+c}$ such that for each $i < c$, the closed embedding $X_i \hookrightarrow \mathbb{P}^{n+c}$ and $X_{i+1} \hookrightarrow X_i$ are local complete intersection (l.c.i.) outside some set Z_i of codimension 1 in X_{i+1} ($\operatorname{depth}_{Z_i} \mathcal{O}_{X_{i+1}} \geq 1$).

the following corrected hypothesis (*):

Given $C \subset \mathbb{P}^{n+c}$ a good standard determinantal scheme of dimension n, we will assume that there exists a flag $C = X_c \subset X_{c-1} \subset \ldots \subset X_2 \subset \mathbb{P}^{n+c}$ such that for each $i < c$, the closed embedding $X_{i+1} \hookrightarrow X_i$ is l.c.i. outside some set Z_i of codimension 2 in X_{i+1} ($\operatorname{depth}_{Z_i} \mathcal{O}_{X_{i+1}} \geq 2$). Moreover, we suppose $X_2 \hookrightarrow \mathbb{P}^{n+c}$ is a l.c.i. in codimension ≤ 1.

The reason of increasing the depth related to $X_{i+1} \hookrightarrow X_i$ by 1 is that the exactness of [19], (10.15) in the proof of [19], Proposition 10.12 is straightforward to see if (*) holds (by e.g. (3.9)) while it is doubtful with the original hypothesis. Hence in [19], Proposition 10.12, Theorem 10.13, Remarks 10.6 and 10.14, Example 10.16, Corollary 10.17 ($n = 1$) and Example 10.18 ($n = 1$) we should suppose (*) instead of [19], (10.12). So in [19] Corollary 10.17 ($n = 1$) and Example 10.18 ($n = 1$) we need to suppose $C \subset S$ to be Cartier instead of generically Cartier and S to be G_1, while [19], Corollary 10.15 need no change because $C \subset S$ is supposed to be Cartier outside a sufficiently small Z. In [19], Corollary 10.15 we *may* replace [19], (10.12) by (*) and hence by "S is G_1", and hence we point out that [19], Propositions 10.7 and 1.12 are valid as stated.

Now we consider the 0-dimensional case. Looking closer to the proof of [19], Proposition 10.12, we need the graded version of [19] (10.15) to be exact in degree zero. Therefore still assuming (*) the proof is only complete for a flag $R \twoheadrightarrow D_2 \twoheadrightarrow \ldots \twoheadrightarrow D_c = A$ satisfying $\dim A \geq 2$ or $\dim A = 1$ and $M_c(a_{t+c-2})_0 \cong (N_{D_c/D_{c-1}})_0$, cf. (3.9). Hence in [19], Theorem 10.13 the H-unobstructedness (when $\dim C = 0$) does not necessarily follow from [19], Proposition 10.12. Fortunately, [19], Theorem 9.6, makes explicit an assumption which implies the H-unobstructedness of C ((iii) below), and we can weaken (iii) further to $\operatorname{Ext}^1_A(I_c/I_c^2, A)_0 = 0$ by the proof of [19], Theorem 9.6 because $H^2(D_{c-1}, A, A) \cong \operatorname{Ext}^1_A(I_c/I_c^2, A)$. Summing up, in the $\dim C = 0$ case of [19], Theorem 10.13 and Corollary 10.17, if we assume (*) instead of [19], (10.12), and in addition at least one of the following conditions

(i) C is H-unobstructed (e.g.$\operatorname{Ext}^1_A(I_c/I_c^2, A)_0 = 0$),
(ii) $M_c(a_{t+c-2})_0 \cong (N_{A/D_{c-1}})_0$ (i.e. (3.8) extends to a short exact sequence in degree zero for $i = c - 1$),
(iii) $\operatorname{Hom}_R(I_c, H^2_{\mathfrak{m}}(I_A))_0 = 0$,

then the conclusions hold. The only example in which $\dim C = 0$ is [19], Example 10.18. In this example (iii) holds because the degree of all minimal generators of I_C are $m - t - 1$ and $H^2_{\mathfrak{m}}(I_A)_\nu \cong H^1(\mathcal{I}_C(\nu)) = 0$ for $\nu > m + t - 3$, cf. [19], Example 7.5 for details. Similarly, in the situation of [19], Corollary 10.17, using the order $b_1 \geq \ldots \geq b_t$ and $a_0 \geq a_1 \geq \ldots \geq a_{t+1}$ appearing in [19], Corollary 10.17 we see that (iii) holds provided $a_1 + a_2 \leq 3 + 2b_t$, a rather strong condition. The condition (i) might however be week.

So while the substitution of [19] (10.12) by (*) in the case $\dim C \geq 1$ is relatively innocent (i.e. $X_{i+1} \hookrightarrow X_i$ has to be Cartier in codimension ≤ 1 instead of generically Cartier), the 0-dimensional case leads to an extra assumption. Therefore, it is probably a more natural approach in the 0-dimensional case to just find $\dim W(\underline{b};\underline{a})$ without trying to prove that $W(\underline{b};\underline{a})$ is an irreducible component of GradAlg and if $\dim C \geq 1$, it is natural both to find $\dim W(\underline{b};\underline{a})$ and to show that $W(\underline{b};\underline{a})$ is a generically smooth, irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^{n+c})$, as has been the strategy of this paper.

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