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THE HILBERT-FLAG SCHEME, ITS PROPERTIES AND ITS CONNECTION WITH THE HILBERT SCHEME. APPLICATIONS TO CURVES IN 3-SPACE

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2.3. The technique of linkage (liaison).

In the preceding section we studied problems related to the non-singularity of the Hilbert scheme Hilb^{p_1} at a rational point $(X \subseteq \mathbb{P})$ where $\mathbb{P} = \mathbb{P}^3_k$. It should not be difficult to understand that this is in general a rather hard problem. Already the question of existence a (smooth connected) curve X_1 with a given degree d_1 and arithmetic genus g_1, sometimes with additional requirements such as $h^0(\underline{I}_X(s)) > 0$ for a given s, is a non-trivial one. We often solve it by linkage (liaison), and we link X_1 (think that X_1 exists) to a curve X_2 of low degree d_2 and arithmetic genus g_2 where we know existence. Reversing this process, we may prove the existence of $X_1 \subseteq \mathbb{P}$.

Now instead of considering the problem of the non-singularity of Hilb^{p_1} at $(X_1 \subseteq \mathbb{P})$ directly as we did in Section 2.2, we may seek for a connection between the non-singularity of Hilb^{p_1} at $(X_1 \subseteq \mathbb{P})$ and the non-singularity of Hilb^{p_2} at $(X_2 \subseteq \mathbb{P})$ where X_1 and X_2 are linked by a global complete intersection Y. In this section we answer this problem, and also when the linked curve $X_2 \subseteq \mathbb{P}$ of a "generic" curve $X_1 \subseteq \mathbb{P}$ of some irreducible component of Hilb^{p_1} is itself a "generic" curve for some component of Hilb^{p_2}. To be precise we will be studying how deformations of $X_1 \subseteq \mathbb{P}$ correspond to deformations of $X_2 \subseteq \mathbb{P}$. It is not surprising that one rather have to study the connection between deformations of $X_1 \subseteq Y \subseteq \mathbb{P}$ and of $X_2 \subseteq Y \subseteq \mathbb{P}$ where $Y = X_1 \cup X_2$ is a global complete intersection of two surfaces of degree f_1 and f_2. So we are situated on the Hilbert-flag scheme $D(p_i;f_1,f_2)$, and here the connection is very nice. Indeed if $D(p;\underline{f})_{CM}$ is the open subscheme of $D(p;\underline{f})$ consisting of objects $(X \subseteq Y \subseteq \mathbb{P})$ such that X is Cohen Macaulay and equi-dimensional and such that Y is a global complete intersection in $\mathbb{P}$ of type $\underline{f} = (f_1,f_2)$, see (1.3.11), then

$$D(p_1;\underline{f})_{CM} \simeq D(p_2;\underline{f})_{CM}$$

are isomorphic. Now if Hilb^p_{CM} is the Hilbert scheme of curves in $\mathbb{P}$ (2.2.7), we know that

$$pr_1 : D(p;\underline{f})_{CM} \to \mathrm{Hilb}^p_{CM}$$

is smooth at $(X \subseteq Y \subseteq \mathbb{P})$ under some conditions (1.3.4) or (1.3.14), and in these cases Hilb^{p_1} and Hilb^{p_2} are closely related. In particular $(X_1 \subseteq \mathbb{P})$ is a non-singular point, resp a "generic" point of a component, of Hilb^{p_1} iff $(X_2 \subseteq \mathbb{P}) \in \mathrm{Hilb}^{p_2}$ is correspondingly. As byproducts we also prove

$$\dim_k \operatorname{coker} \alpha_{X_1 \subseteq Y} = \dim_k \operatorname{coker} \alpha_{X_2 \subseteq Y},$$

and the exact sequence of (1.3.10) will therefore, in some cases, make it easy to find $h^1(\underline{N}_{X_1})$ provided we know $h^1(\underline{N}_{X_2})$. Finally we introduce what we call the postulated dimension of a reduced component V of the Hilbert scheme which is the number

$$4d + \delta^2$$

where δ^2, defined in (2.2.7), belongs to a sufficiently general curve $(X \subseteq \mathbb{P})$ of V. Then we prove that V has postulated dimension, i.e. $\dim V = 4d + \delta^2$, iff the "linked" component has postulated dimension (2.3.16).

To begin with, we recall some basic facts about the notion of liaison as proved by Peskine and Szpiro in [P.S].

Let k be a field and let $X_1 \subseteq \mathbb{P}^n_k$ and $X_2 \subseteq \mathbb{P}^n_k$ be closed subschemes whose union $Y = X_1 \cup X_2$ is a global complete intersection in $\mathbb{P}^n_k$. If X_1 and X_2 are equidimensional, without embedded components and without common irreducible components, then X_1 and X_2 are <u>geometrically linked</u> by Y. It follows that

$$(1) \qquad \underline{I}_{X_1/Y} = O^{\vee}_{X_2}, \quad \underline{I}_{X_2/Y} = O^{\vee}_{X_1}$$

where $(-)^{\vee} = \underline{\operatorname{Hom}}_{O_Y}(-, O_Y)$. See [P.S.,(1.1)]. Moreover if X_1 and X_2 are Cohen Macaulay, one knows by Gorenstein duality that

$$(2) \qquad \underline{\operatorname{Ext}}^1_{O_Y}(O_{X_i}, O_Y) = 0 \quad \text{for } i = 1,2.$$

Then dualizing the exact sequence

$$0 \to \underline{I}_{X_i/Y} \to O_Y \to O_{X_i} \to 0,$$

it follows from (1) and (2) that

$$(3) \qquad \underline{I}^{\vee}_{X_1/Y} = O_{X_2}, \quad \underline{I}^{\vee}_{X_2/Y} = O_{X_1} .$$

Now let $Y \subseteq \mathbb{P}^n_k$ be a global complete intersection and let $X_i \subseteq Y$ be closed subschemes for $i = 1,2$ such that X_1 and X_2 are equidimensional and without embedded components. Then by definition X_1 and X_2 are <u>algebraically linked</u> by Y if (1) holds [P.S.,§2]. Again we deduce (2) and (3) provided the X_i are Cohen Macaulay.

Furthermore we have the following important result [P.S.,(1.3)].

<u>Proposition 2.3.1</u>. Let $X \hookrightarrow Y$ be a closed embedding of equidimensional projective k-schemes of the same dimension, and let X be Cohen Macaulay and Y be Gorenstein. If $X' \hookrightarrow Y$ is defined by the sheaf of ideals $\underline{I}_{X'/Y} = O^{\vee}_X$ in O_Y, then X' is Cohen Macaulay and equidimensional of dimension $\dim Y$. Moreover (1), (2) and (3) holds if we replace X_1 by X and X_2 by X'.

Now we will state and prove the main theorem of this section. For this we need to have a notion of liaison satisfying (3), and we will therefore make the following definition of linkage used in this paper. First note that if we define $X' \subseteq Y$ by $\underline{I}_{X'/Y} = O^{\vee}_X = \underline{\mathrm{Hom}}_{O_Y}(O_X, O_Y)$ where $X \subseteq Y$ is given and if $X'' = (X')'$, then by dualizing

$$0 \to \underline{I}_{X/Y} \to O_Y \to O_X \to 0$$

we find a morphism

$$O_{X'} \hookrightarrow \underline{I}^{\vee}_{X/Y}$$

whose cokernel is $\underline{\mathrm{Ext}}^1_{O_Y}(O_X,O_Y)$. Correspondingly there is a morphism

$$O_{X''} \hookrightarrow \underline{I}^{\vee}_{X'/Y}$$

with cokernel $\underline{\mathrm{Ext}}^1_{O_Y}(O_{X'},O_Y)$, and since there are natural morphisms

$$\underline{I}_{X/Y} \to \underline{I}^{\vee\vee}_{X/Y} \to O^{\vee}_{X'} = \underline{I}_{X''/Y},$$

it follows that

$$X'' \subseteq X.$$

<u>Definition 2.3.2</u>. Let $X \hookrightarrow Y$ be a closed embedding of projective k-schemes, and define $X' \hookrightarrow Y$ by $\underline{I}_{X'/Y} = O^{\vee}_X$ as above.

i) We say that $X \hookrightarrow Y$ is linkable if the natural morphism

$$O_{X'} \hookrightarrow \underline{I}^{\vee}_{X/Y}$$

and the composition of natural morphisms

$$O_X \twoheadrightarrow O_{X''} \hookrightarrow \underline{I}^{\vee}_{X'/Y}$$

are isomorphisms. Under these conditions we also say that $X \hookrightarrow Y$ and $X' \hookrightarrow Y$ are linked, or link, or that X and X' are linked by Y.

ii) If any $x \in \mathrm{Ass}(O_X)$ satisfies

$$O_{X,x} = O_{Y,x},$$

we say that $X \hookrightarrow Y$ is geometrically linkable, or that $X \hookrightarrow Y$ and $X' \hookrightarrow Y$ link geometrically.

Note that the definition (2.3.2i) is just (3) of (2.3.1), from which (2) and (1) follows easily. Moreover by (2.3.1), $X \hookrightarrow Y$

is linkable provided X is Cohen Macaulay and equidimensional, Y is Gorenstein and equidimensional and $\dim X = \dim Y$. And under these conditions, together with requiring that Y is a global complete intersection, the notion of linkable, resp. geometrically linkable, of (2.3.2) coincides with the algebraic, resp. geometric notion of liaison of [P.S.].

Before stating the theorem, we will see how the Hilbert polynomials of X and X' correspond.

Lemma 2.3.3. i) Let $Y = V(F_1,\ldots,F_r) \subseteq \mathbb{P}^n_k$ be a global complete intersection of type $\underline{f} = (f_1,\ldots,f_r)$ with Hilbert polynomial q, and let $X \hookrightarrow Y$ satisfy the conditions of (2.3.1). If X, resp. the linked scheme X', has Hilbert polynomial p, resp. p', and if $f = \sum_{i=1}^{r} f_i$, then

$$p(\nu) + (-1)^{n-r} p'(f-n-1-\nu) = q(\nu).$$

ii) If $n = 3$ and $r = 2$, and if the degree, resp. arithmetic genus, of X and X' is d and d', resp. g and g', then

$$d + d' = f_1 f_2$$

$$g - g' = (d-d')\,\frac{f_1+f_2-4}{2}.$$

Moreover in this case

$$h^0(\underline{I}_{X/Y}(\nu)) = h^1(O_{X'}(f_1+f_2-4-\nu)),$$

$$h^1(\underline{I}_X(\nu)) = h^1(\underline{I}_{X'}(f_1+f_2-4-\nu)),$$

$$h^1(O_X(\nu)) = h^0(\underline{I}_{X'/Y}(f_2+f_2-4-\nu)).$$

Proof. Indeed the exact sequence

$$0 \to \underline{I}_{X/Y} \to O_Y \to O_X \to 0$$

together with

$$\underline{I}_{X/Y} = \underline{\mathrm{Hom}}_{O_Y}(O_{X'}, O_Y) = \underline{\mathrm{Hom}}_{O_Y}(O_{X'}, \omega_Y) \otimes \omega_Y^{-1} \simeq \omega_{X'} \underset{O_{X'}}{\otimes} \omega_Y^{-1},$$

see [A.K.,I,(2.3)], imply that

$$\chi(O_X(\nu)) = \chi(O_Y(\nu)) - \chi(\omega_{X'} \otimes \omega_Y^{-1}(\nu)).$$

Using that the dualizing sheaf ω_Y on Y satisfies $\omega_Y \simeq O_Y(f-n-1)$, and using duality on X', we find that

$$\chi(\omega_{X'} \otimes \omega_Y^{-1}(\nu)) = \chi(\omega_{X'}(n+1-f+\nu)) = (-1)^{n-r}\chi(O_{X'}(f-n-1-\nu)).$$

Then we easily obtain the relationship between the Hilbert polynomials p and p' as above, and also the formulas for the degree and genus. Moreover the expressions for $h^0(\underline{I}_{X/Y}(\nu))$ and $h^1(O_X(\nu))$ are indeed easy.

Finally to prove

$$h^1(\underline{I}_X(\nu)) = h^1(\underline{I}_{X'}(f_1+f_2-4-\nu)),$$

we use the exact sequence

$$0 \to \underline{I}_Y \to \underline{I}_X \to \underline{I}_{X/Y} \to 0$$

together with $H^1(\underline{I}_Y(\nu)) = 0$ and $H^2(\underline{I}_Y(\nu)) \simeq H^1(O_Y(\nu))$, and we deduce that

$$0 \to H^1(\underline{I}_X(\nu)) \to H^1(\underline{I}_{X/Y}(\nu)) \to H^1(O_Y(\nu))$$

is exact. We have already seen that

$$\underline{I}_{X/Y} \simeq \omega_{X'}(4-f_1-f_2) \quad \text{and} \quad O_Y \simeq \omega_Y(4-f_1-f_2),$$

and so by dualizing the exact sequence above,

$$H^0(O_Y(f_1+f_2-4-\nu)) \to H^0(O_{X'}(f_1+f_2-4-\nu)) \to H^1(\underline{I}_X(\nu))^{\vee} \to 0$$

is exact. However the cokernel of

$$H^0(O_Y(\nu')) \to H^0(O_{X'}(\nu'))$$

is $H^1(\underline{I}_{X'}(\nu'))$ because $H^0(O_{\mathbb{P}}(\nu')) \to H^0(O_Y(\nu'))$ is surjective. Thus

$$H^1(\underline{I}_X(\nu))^{\vee} \simeq H^1(\underline{I}_{X'}(f_1+f_2-4-\nu))$$

as required.

We now come to the theorem

<u>Theorem 2.3.4.</u> If q is the Hilbert polynomial of a global complete intersection of type $\underline{f} = (f_1,\ldots,f_r)$ in $\mathbb{P}^n_k$ and if p and p' are polynomials satisfying

$$p(\nu) + (-1)^{n-r} p'(\Sigma f_i - n-1-\nu) = q(\nu),$$

then there is an isomorphism

$$D(p;\underline{f})_{CM} \simeq D(p';\underline{f})_{CM}$$

Recall that $D(p;\underline{f}) \subseteq D(p,q)$ is an open subscheme (1.3.11) and that the K-points of $D(p;\underline{f})_{CM}$, $k \hookrightarrow K$ a field extension, are objects $(X \subseteq Y \subseteq \mathbb{P} \times \mathrm{Spec}(K))$ with X Cohen Macaulay and equidimensional. Thus $D(p;\underline{f})_{CM} \subseteq D(p,q)$ is open.[1)]

The key lemma of the proof of (2.3.4) is this

<u>Lemma 2.3.5.</u> Let $\mathbb{P}$ be a projective scheme over a field k, let $(X \subseteq Y \subseteq \mathbb{P} \times S)$ be an S-point of $D(\mathbb{P})$ and assume for all

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1) by well-known depth and dimension formulas. See the proof of (2.3.6)

$s \in S$ that the corresponding $X_s \subseteq Y_s$ of the fibers is linkable. If $X' \hookrightarrow Y$ is defined by $\underline{I}_{X'/Y} = O_X^{\vee}$, then X' is S-flat and for any $s \in S$ the fiber morphisms

$$X_s \hookrightarrow Y_s \quad \text{and} \quad (X')_s \hookrightarrow Y_s$$

are linked, i.e. $(X')_s = (X_s)'$ where $\underline{I}_{(X_s)'/Y} = O_{X_s}^{\vee}$.
Furthermore

$$X'' = X.$$

<u>Proof</u>. Let $s \in S$ and let m be the maximal ideal of $O = O_{S,s}$. Put $O_i = O/m^i$, $J_i = \ker(O_i \to O_{i-1})$ and

$$X_i = X \underset{S}{\times} \operatorname{Spec}(O_i) \subseteq Y_i = Y \underset{S}{\times} \operatorname{Spec}(O_i).$$

Thus $O_1 = k(s)$ and $X_1 = X_s \subseteq Y_1 = Y_s$. Moreover let $O_{X_i}^{\vee} = \underline{\mathrm{Hom}}_{O_{Y_i}}(O_{X_i}, O_{Y_i})$ and $O_X^{\vee} = \underline{\mathrm{Hom}}_{O_Y}(O_X, O_Y)$.

First we prove that

$$O_{X_i}^{\vee} \underset{O_i}{\otimes} O_{i-1} \xrightarrow{\sim} O_{X_{i-1}}^{\vee}$$

is an isomorphism. To see this we consider the following diagram of exact horizontal sequences

$$\begin{array}{ccccccc}
O_{X_i}^{\vee} \underset{O_i}{\otimes} J_i & \longrightarrow & O_{X_i}^{\vee} & \longrightarrow & O_{X_i}^{\vee} \underset{O_i}{\otimes} O_{i-1} & \longrightarrow & 0 \\
\downarrow & \circ & \downarrow\simeq & \circ & \downarrow & & \\
0 \to \underline{\mathrm{Hom}}_{O_{Y_i}}(O_{X_i}, O_{Y_i} \otimes J_i) & \to & \underline{\mathrm{Hom}}_{O_{Y_i}}(O_{X_i}, O_{Y_i}) & \to & \underline{\mathrm{Hom}}_{O_{Y_i}}(O_{X_i}, O_{Y_{i-1}}) & \to & 0,
\end{array}$$

recalling that J_i is a $k(s)$-module, so

$$\underline{\mathrm{Ext}}^1_{O_{Y_i}}(O_{X_i}, O_{Y_i} \otimes J_i) \simeq \underline{\mathrm{Ext}}^1_{O_{Y_1}}(O_{X_1}, O_{Y_1}) \underset{k(s)}{\otimes} J_i = 0.$$

Now the vertical arrow to the right in the diagram above is surjective for any i. It follows that the vertical arrow to the left is surjective, which therefore implies that the vertical arrow to the right is indeed an isomorphism.

Next we prove that

$$O^\vee_X \otimes_{O_X} k(s) \simeq O^\vee_{X_s}$$

is an isomorphism. Note that $\underline{I}_{X'/Y} = O^\vee_X$ and $\underline{I}_{(X_s)'/Y_s} = O^\vee_{X_s}$. So the isomorphism above implies that $X' \to S$ is flat at $s \in S$ and that the fiber morphism $(X')_s \subseteq Y_s$ of $X' \subseteq Y$ at $s \in S$ coincides with $(X_s)' \subseteq Y_s$, i.e. that $(X_s)' = (X_s)'$. Now let $x \in X$ map to $s \in S$ via the structure morphism $X \to S$. Then $x \in Y$ and abusing the language, $x \in X_i$ and $x \in Y_i$ as well. Put

$$A = O_{X,x},\quad B = O_{Y,x},\quad A_i = O_{X_i,x},\quad B_i = O_{Y_i,x},$$
$$\hat{B} = \varprojlim B_i,\quad \hat{A} = \varprojlim A_i,\quad A^\vee = \mathrm{Hom}_B(A,B) \text{ and } A_i^\vee = \mathrm{Hom}_{B_i}(A_i,B_i).$$

Then it will be sufficient to show

$$A^\vee \otimes_B B_1 \xrightarrow{\sim} A_1^\vee .$$

Since we already know

$$A_i^\vee \otimes_{B_i} B_{i-1} = A_i^\vee \otimes_{O_i} O_{i-1} \simeq A_{i-1}^\vee$$

we deduce that [EGA, 0_I, (7.2.9)]

$$(\varprojlim_i A_i^\vee) \otimes_{\hat{B}} B_1 \simeq A_1^\vee .$$

Moreover one knows that [EGA, 0_I, (7.2.10)]

$$\varprojlim_i A_i^\vee = \operatorname{Hom}_{\hat{B}}(\hat{A},\hat{B}) = A^\vee \otimes_B \hat{B}$$

and we conclude as expected.

Finally it remains to show that

$$X'' = X.$$

We know $X'' \subseteq X$, so let

$$0 \to \underline{I}_{X''/X} \to O_X \to O_{X''} \to 0$$

be exact. Since X'' is S-flat and $(X'')_s = (X_s)'' = X_s$,

$$\underline{I}_{X''/X} \otimes k(s) = 0,$$

and so by Nakayamas lemma, $X'' = X$, as required.

Note that we can easily continue the proof and we will see that, with assumptions as in the lemma, X and X' are linked by Y.

Proof of (2.3.4). If $S = D(p;\underline{f})_{CM}$ and if $(X \subseteq Y \subseteq \mathbb{P} \times S)$ is the restriction of the universal object of $D(p,q)$ to S, then by (2.3.5) there is an object $(X' \subseteq Y \subseteq \mathbb{P} \times S) \in \underset{\sim}{D}(\mathbb{P})(S)$ which by (2.3.3) and (2.3.1) factors via

$$D(p;\underline{f})_{CM} = S \to D(p';\underline{f})_{CM}.$$

Starting with $S = D(p';\underline{f})_{CM}$ and using (2.3.5), we have an inverse, and the proof is complete.

Note that it is not necessary for the theorem (2.3.4) to deal only with global complete intersection $Y \subseteq \mathbb{P}^n_k$. Indeed let $\mathbb{P}$ be any projective k-scheme and let $D(p,q)_{CMG}$ be the open[1] subscheme of $D(p,q)$ consisting of points $(X \subseteq Y \subseteq \mathbb{P})$ with $X \hookrightarrow Y$ as in (2.3.1). If $(X \subseteq Y \subseteq \mathbb{P})$ is a given k-point of $D(p,q)_{CMG}$,

1) Use for instance [H4, V, § 9].

and if the corresponding linked object $X' \hookrightarrow Y$ has Hilbert polynomial p', then the connected component of $D(p,q)_{CMG}$ containing $(X \subseteq Y \subseteq \mathbb{P})$ and the connected component of $D(p',q)_{CMG}$ containing $(X' \subseteq Y \subseteq \mathbb{P})$ are isomorphic. This follows easily from (2.3.5) and (2.3.1)

Now we come to the corollaries, and we restrict to the case of curves in $\mathbb{P} = \mathbb{P}^3_k$. In the following $D(p,q)_{CM}$ is the open subscheme of $D(p,q)$ whose K-points are $(X \subseteq Y \subseteq \mathbb{P}^3_K)$ with X a curve over K (2.2.7), so $D(p;\underline{f})_{CM} \subseteq D(p,q)_{CM}$ is open, and abusing the language, we usually look upon the first projection morphism pr_1 as defined on $D(p;\underline{f})_{CM}$, i.e. the morphism

$$pr_1 : D(p;\underline{f})_{CM} \to \mathrm{Hilb}^p_{CM}$$

is the composition of $D(p;\underline{f})_{CM} \hookrightarrow D(p,q)_{CM}$ and the first projection morphism $D(p,q)_{CM} \to \mathrm{Hilb}^p_{CM}$. Note that pr_1 as above is no longer in general a projective morphism. Since we work with curves in $\mathbb{P}^3_k$, we sometimes write

$$\mathrm{Hilb}^p_{CM} = H(d,g)_{CM},$$

$$D(p;\underline{f})_{CM} = D(d,g;\underline{f})_{CM}$$

where $p(\nu) = d\nu + 1 - g$. If d' and g' are as in (2.3.3ii), we let

$$pr_1' : D(d',g';\underline{f})_{CM} \to H(d',g')_{CM}.$$

<u>Corollary 2.3.6.</u> Let f_1, f_2, d, g, d' and g' be numbers satisfying the relations of (2.3.3), let $U \subseteq H(d,g)$ be the open subscheme whose K-points $(X \subseteq \mathbb{P}^3_K)$ satisfy

$$H^1(\underline{I}_X(f_i)) = 0 \quad \text{and} \quad H^1(\underline{I}_X(f_i - 4)) = 0$$

for $i = 1,2$, and let $U' \subseteq H(d',g')$ be defined by the same conditions. Then the schemes

$$U(\underline{f}) = U \cap pr_1(D(d,g;\underline{f})_{CM})$$

$$U'(\underline{f}) = U' \cap pr_1'(D(d',g';\underline{f})_{CM})$$

are open subschemes of $H(d,g)_{CM}$ and $H(d',g')_{CM}$ respectively, and there is a commutative diagram

$$\begin{array}{ccc} D(p;\underline{f})_{CM} & \simeq & D(p';\underline{f})_{CM} \\ \uparrow & \circ & \uparrow \\ pr_1^{-1}(U(\underline{f})) & \simeq & (pr_1')^{-1}(U'(\underline{f})) \\ \downarrow pr_1 & & \downarrow pr_1' \\ U(\underline{f}) & & U'(\underline{f}) \end{array}$$

where the restricted projection morphisms pr_1 and pr_1' of the diagram are irreducible, surjective and smooth of relative dimension

$$\dim_x pr_1^{-1}(U(\underline{f}))_{pr_1(x)} = \sum_{i=1}^{2} h^o(\underline{I}_{X/Y}(f_i)),$$

for any $x = (X \subseteq Y \subseteq \mathbb{P}) \in U(\underline{f})$,

$$\dim_{x'}(pr_1')^{-1}(U'(\underline{f}))_{pr_1'(x')} = \sum_{i=1}^{2} h^o(\underline{I}_{X'/Y'}(f_i)),$$

for any $x' = (X' \subseteq Y' \subseteq \mathbb{P}) \in U'(\underline{f})$.

In particular the irreducible, resp. embedded, resp. connected components of $U(\underline{f})$ and $U'(\underline{f})$ are an one-to-one correspondence.

We observe by (2.3.6) that the k-points of the "linked" family $U'(\underline{f})$ is given by

$$U'(\underline{f}) = \left\{ (X' \subseteq \mathbb{P}) \in H(d',g')_{CM} \;\middle|\; \begin{array}{l} \text{there exists } Y \text{ of type } \underline{f} \text{ and a} \\ \text{curve } (X \subseteq \mathbb{P}) \text{ of } U \text{ containing } Y \\ \text{such that } X' \text{ and } X \text{ are linked by } Y \end{array} \right\}$$

Moreover if $X \subseteq \mathbb{P}$ is a curve of $U(\underline{f})$ contained in a global complete intersection Y of type $\underline{f}$, and if $X' \subseteq \mathbb{P}$ is the linked curve of X by Y, then we easily deduce by (2.3.6) that the local ring $O_{H,X}$ of $H = H(d,g)$ at $(X \subseteq \mathbb{P})$ is non-singular (i.e. as always smooth), resp. a complete intersection, resp. satisfies the condition R_k, resp. the condition S_k, resp. is generically reduced (i.e. R_o), resp. is without embedded components (i.e. S_1), resp. reduced, resp. normal (by Serre's criterion), resp. Cohen Macaulay, resp. regular, resp. an integral domain iff the local ring $O_{H',X'}$ of $H' = H(d',g')$ at $(X' \subseteq \mathbb{P})$ is non-singular, resp. a complete intersection etc. See for instance [A.K.,VII,(4.9)]. Furthermore $X \subseteq \mathbb{P}$ is a "generic" curve for some irreducible component of $H(d,g)_{CM}$ iff $(X' \subseteq \mathbb{P})$ is a "generic" for some component of $H(d',g')_{CM}$. Indeed we will call a sufficiently general point $(X \subseteq \mathbb{P})$ of an irreducible non-embedded component V of $H(d,g)_{CM}$ a "generic" curve for V.

Proof. Note that U is an open subscheme of $H(d,g)$ by semicontinuity [H 1, III,(12.8)], and that the restriction of the projection $pr_1 : D(p,f)_{CM} \to Hilb^p_{CM}$ to $pr_1^{-1}(U)$ is smooth by (1.3.4). It follows that $U(\underline{f})$, which indeed is equal to $pr_1(pr_1^{-1}(U))$, is open, and that

$$pr_1 : pr_1^{-1}(U(\underline{f})) = U \to U(\underline{f})$$

is smooth and surjective. By (1.3.13) it is irreducible, the same arguments apply to $pr_1' : (pr_1')^{-1}(U'(\underline{f})) \to U'(\underline{f})$ as well.

To see that we have

$$pr_1^{-1}(U(\underline{f})) \simeq (pr_1'(U'(\underline{f}))$$

under the isomorphism $D(p;\underline{f})_{CM} \simeq D(p';\underline{f})_{CM}$ of (2.3.4), it will be sufficient to show that

$$H^1(\underline{I}_X(f_i)) = 0 \quad \text{and} \quad H^1(\underline{I}_X(f_i-4)) = 0$$

for $i = 1,2$ iff

$$H^1(\underline{I}_{X'}(f_i)) = 0 \quad \text{and} \quad H^1(\underline{I}_{X'}(f_i-4)) = 0$$

for $i = 1,2$, where X and X' are linked by some Y of type $\underline{f}$. This equivalence follows from (2.3.3). Moreover the dimension formulas for the fibers are a direct consequence of (1.3.12).

Finally for any irreducible morphism $p : D \to H$ of finite type of noetherian schemes, we easily see by the discussion of (1.3.13) that the inverse image of a decomposition of H into connected components, resp. a topological decomposition of H into irreducible components gives a corresponding decomposition of D. Moreover if we take the inverse image of a decomposition of H into irreducible and embedded components, i.e. $H = \cup H_i$ as a scheme, we obtain a decomposition

$$D = \cup p^{-1}(H_i)$$

into irreducible subschemes where the inverse image of the non-embedded components are non-embedded by the topological argument. However, if p is smooth and if $x \in D$ is the generic point of $p^{-1}(H_i)$ where H_i is an embedded component, we deduce by the depth-formula

$$\operatorname{depth} O_{D,x} = \operatorname{depth} O_{H,p(x)} + \operatorname{depth}(O_{D,x} \otimes k(p(x)))$$

and the corresponding dimension formula that

$$\operatorname{depth} O_{D,x} < \dim O_{D,x}$$

because $\operatorname{depth}(O_{D,x} \otimes k(p(x))) = \dim(O_{D,x} \otimes k(p(x)))$, and we are done.

Sometimes we just want to know when we can deduce the non-singularity of $H(d',g')$ at $(X' \subseteq \mathbb{P})$ from the non-singularity of $H(d,g)$ at $(X \subseteq \mathbb{P})$, and we have

Remark 2.3.7. Let $X \subseteq \mathbb{P} = \mathbb{P}^3_k$ be a curve contained in a Y of type $\underline{f}$ such that the morphisms γ of (1.3.1C),

$$\gamma_{X \subseteq Y} : H^0(\underline{N}_X) \to \bigoplus_{i=1}^{2} H^1(\underline{I}_X(f_i))$$

is surjective, and the corresponding morphism of the linked curve

$$\gamma_{X' \subseteq Y} : H^0(\underline{N}_{X'}) \to \bigoplus_{i=1}^{2} H^1(\underline{I}_{X'}(f_i))$$

is the zero map. Then if $H(d,g)$ is non-singular (i.e. as always smooth) at $(X \subseteq \mathbb{P})$, then $H(d',g')$ is non-singular at $(X' \subseteq \mathbb{P})$. This follows easily from (1.3.3), (2.3.4) and from [EGA,IV,(17.11.1)] since, according to the exact sequence of (1.3.1C), $\gamma_{X' \subseteq Y} = 0$ iff the tangent map of pr'_1 at $(X' \subseteq Y \subseteq \mathbb{P})$ is surjective.

Before giving examples, we observe that we via (2.3.3ii) can replace $U(\underline{f})$ and $U'(\underline{f})$ by some smaller open subschemes and still conclude as in (2.3.6). Indeed if

$$e,f,g : \mathbb{Z} \to \mathbb{Z}_+$$

are maps where $\mathbb{Z}_+$ is the positive integers, we can consider the open subscheme U_1 of $U(\underline{f})$ of curves $X \subseteq \mathbb{P}$ where

$$h^0(\underline{I}_{X/Y}(\nu)) \leq e(\nu)\,, \quad h^1(\underline{I}_X(\nu)) \leq f(\nu) \quad \text{and} \quad h^1(O_X(\nu)) \leq g(\nu)$$

for all ν. Then by (2.3.3ii) we transform these conditions into corresponding conditions for the linked curves, and these define a subscheme $U_1' \subseteq U'(\underline{f})$. Then (2.3.6) holds if we replace $U(\underline{f})$ and $U(\underline{f})$ by U_1 and U_1' respectively.

Note also that if $X \subseteq \mathbb{P} = \mathbb{P}^3_k$ is a curve which is generically a complete intersection over an algebraically closed field of characteristic zero, then by [P.S.,(4.1)] there is a global complete intersection Y of type $\underline{f} = (f_1, f_2)$ containing X with $f_j \leq \max n_{1i}$ for $j = 1,2$ such that the linked curve X' of X by Y is reduced and the linkage is geometric. Moreover X' is non-singular, resp. locally a complete intersection, provided X is non-singular, resp. locally a complete intersection. Observe that

$$\max_{1 \leq i \leq r_1} n_{1i} \leq \max\{e(X)+3, c(X)+2\}.$$

Indeed $c(X) > e(X)$ implies $\max n_{1i} \leq \max n_{3i} - 2 = c(X)+2$, and $c(X) \geq e(X)$ implies $\max n_{1i} \leq \max n_{2i} - 1 = e(X)+3$.

<u>Examples 2.3.8.</u> We consider the Hilbert scheme $H(9,8)_S$ over an algebraically closed field k of characteristic zero, and we review the family of (2.2.10i) which is the subset $U_1 \cap H(9,8)_S$ where $U_1 \subseteq H(9,8)_{CM}$ consists of curves $(X \subseteq \mathbb{P})$ satisfying

$$h^1(\underline{I}_X(\nu)) = \begin{cases} 1 & \text{for } \nu = 2 \\ 0 & \text{for } \nu \neq 2 \end{cases}$$

$$h^1(O_X(\nu)) = 0 \quad \text{for } \nu \geq 2.$$

Since $\chi(\underline{I}_X(2)) = -1$, $h^1(\underline{I}_X(2)) \geq 1$, and so U_1 is open by semi-continuity. We have in (2.2.10i) seen that for any smooth curve $X \subseteq \mathbb{P}$ of U_1, $h^1(\underline{N}_X) = 0$, and this is equivalent to saying that $U_1 \cap H(9,8)_S$ is smooth of dimension $4d = 36$. We will now illustrate (2.3.6) by giving another proof for this, from which we also deduce some further informations.

Since the resolution of $I = \oplus H^0(\underline{I}_X(\nu))$ must be of the form as in (2.2.10i), any $X \subseteq \mathbb{P}$ of U_1 is contained in a global complete intersection Y of type $(4,4)$ where $4 = \max n_{1i}$. This gives by (2.3.3) that the linked curves $X' \subseteq \mathbb{P}$ satisfy

$$d' = 7, \quad g' = 4,$$

$$h^1(\underline{I}_{X'}(\nu)) = h^1(\underline{I}_X(4-\nu)) = \begin{cases} 1 & \text{for } \nu = 2 \\ 0 & \text{otherwise,} \end{cases}$$

$$h^0(\underline{I}_{X'/Y}(\nu)) = h^1(O_X(4-\nu)) = 0 \quad \text{for} \quad \nu \leq 2,$$

and that by [P.S.,(4.1)], if X is non-singular, there exists Y of type $(4,4)$ such that X' is non-singular. Now we consider the open set U_1' of $H(7,4)_{CM}$ of curves $(X' \subseteq \mathbb{P})$ satisfying

$$h^0(\underline{I}_{X'}(\nu)) = 0 \quad \text{for} \quad \nu \leq 2$$

$$h^1(\underline{I}_{X'}(\nu)) = \begin{cases} 1 & \text{for } \nu = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and we observe that for any $(X' \subseteq \mathbb{P}) \in U_1' \cap H(7,4)_S$ there is a Y of type $(4,4)$ such that the linked curve $X \subseteq \mathbb{P}$ of $X' \subseteq \mathbb{P}$ by Y is non-singular. This follows from

$$\max(c(X')+2, e(X')+3) = 4$$

and [P.S.,(4.1)]. Then by (2.3.6) $U_1 \cap H(9,8)_S$ is smooth, resp. irreducible, iff $U_1' \cap H(7,4)_S$ is smooth, resp. irreducible. And it is easy to see that $U_1' \cap H(7,4)_S$ is smooth of dimension $4d' = 28$ because $d' > 2g'-2$ implies $H^1(O_{X'}(1)) = 0$. Hence $H^1(\underline{N}_{X'}) = 0$. Thus by (2.3.6)

$$\dim(U_1 \cap H(9,8)_S) + 2h^0(\underline{I}_{X/Y}(4)) = \dim(U_1' \cap H(7,4)_S) + 2h^0(\underline{I}_{X'/Y}(4)),$$

and since $h^0(\underline{I}_{X'/Y}(4)) = h^1(O_X) = 8$ by (2.3.3 ii),

$$\dim(U_1 \cap H(9,8)_S) = 36.$$

Furthermore if we accept that $U_1' \cap H(7,4)_{S'}$ is irreducible (Recall $d' \geq g' + 3$ should imply $H(7,4)_S$ integral, see [N,§2]), it follows that $U_1 \cap H(9,8)_S$ is smooth and connected.

Example 2.3.9. Let k be algebraically closed. Then there is a non-embedded non-reduced component V of $H(14,24)$ and an open subset U of $H(14,24)$, $U \subseteq V$, of smooth connected curves lying on smooth cubic surfaces. See (3.2.4) or [M2]. By (3.1.3),

$$H^1(\underline{I}_X(\nu)) = 0 \quad \text{for} \quad \nu \notin \{3,4,5\}$$

for any $X \subseteq \mathbb{P}$ of U, and one may also prove (use (3.1.6 iii))

$$h^1(\underline{I}_X(3)) = 1, \; h^1(\underline{I}_X(4)) = 2, \; h^1(\underline{I}_X(5)) = 1.$$

Using $\chi(\underline{I}_X(\nu)) = \chi(O_{\mathbb{P}}(\nu)) - \chi(O_X(\nu))$ and Riemann-Roch, we find that any $X \subseteq \mathbb{P}$ of U is contained in a global complete intersection Y of type $(6,6)$. By (2.3.3ii) the linked curve $X' \subseteq \mathbb{P}$ is of degree $d' = 22$ and arithmetic genus $g' = 56$. Moreover any $X \subseteq \mathbb{P}$ of U is contained in $U(6,6)$ of (2.3.6) since

$$H^1(\underline{I}_X(6)) = 0 \quad \text{and} \quad H^1(\underline{I}_X(2)) = 0,$$

and (2.3.6) applies. Thus the linked curves $X' \subseteq \mathbb{P}$ belong to a non-embedded non-reduced component V' of $H(22,56)$. Finally by using [P.S.,(4.1)], V' contains smooth connected curves. Indeed it will be sufficient to prove $\max n_{1i} = 6$ where the n_{ji} belong to the graded resolution of $I = \oplus H^0(\underline{I}_X(v))$ for $X \subseteq \mathbb{P}$ in U. We omit proving this.

Our next corollary is concerned with a family V of Hilb^p and its corresponding "linked" family V' of $\mathrm{Hilb}^{p'}$, and it relates the dimension of V' to the dimension of V. First to define V', we will here just consider those closed irreducible families V of Hilb^p, which appear as the image, via the first projection morphism pr_1, of some irreducible non-embedded component $W \subseteq \overline{D(p;\underline{f})_{CM}}$ where $\overline{D(p;\underline{f})_{CM}}$ is the closure of $D(p;\underline{f})_{CM}$ in $D(p,q)$. Now start with such an irreducible closed subset V of Hilb^p and give V the reduced scheme structure unless V is an irreducible component of Hilb^p in which case we always endow V with the scheme structure induced from the scheme structure of Hilb^p. Anyway by generic flatness [M 1, Lect 8] and by the smoothness and connectedness of the fibers of $pr_1 : D(p;\underline{f})_{CM} \to \mathrm{Hilb}^p_{CM}$, see (1.3.12), there is an open subscheme $U \subseteq V_{red}$ such that the restriction of pr_1 to $pr_1^{-1}(U) \subseteq D(p;\underline{f})_{CM}$,

$$pr_1 : pr_1^{-1}(U) \to U$$

is irreducible and smooth of relative dimension

$$\dim pr_1^{-1}(U) - \dim U = \sum_{i=1}^{2} h^o(\underline{I}_{X/Y}(f_i))$$

where $(X \subseteq Y \subseteq \mathbb{P}) \in pr_1^{-1}(U)$. In particular there is only one irreducible component W of $\overline{D(p;\underline{f})_{CM}}$ whose image via the first projection morphism is V, and using the isomorphism $D(p;\underline{f})_{CM} \simeq D(p';\underline{f})_{CM}$ of (2.3.4), W corresponds to an irreducible component W' of $\overline{D(p';\underline{f})_{CM}}$. Then we define the "linked" family V' of $\mathrm{Hilb}^{p'}$ by

$$V' = pr_1'(W')$$

where $pr_1' : D(p',q) \to \mathrm{Hilb}^{p'}$, and we give V' the reduced scheme structure unless it is an irreducible component of $\mathrm{Hilb}^{p'}$. Now starting with V', then since there is only one irreducible component W' of $\overline{D(p;\underline{f})_{CM}}$ whose image is V', we deduce

$$(V')' = V.$$

Moreover there is an open subset $U' \subseteq V'$ such that

$$\dim W' - \dim V' = \sum_{i=1}^{2} h^o(\underline{I}_{X'/Y}(f_i)) = \sum_{i=1}^{2} h^1(O_X(f_i-4))$$

for any $(X' \subseteq Y \subseteq \mathbb{P})$ of U' where $X' \hookrightarrow Y$ and $X \hookrightarrow Y$ are linked. Note that when we talk about (irreducible) components of Hilb^p, or of $D(p,q)$, or of some open subschemes of these, we always mean, unless explicitely mentioning the contrary, a non-embedded irreducible component endowed with the scheme structure inherited from the scheme of which it is a component. It follows that the dimension of the corresponding local ring at the generic point of W is zero, and such a component is reduced iff this local ring is a field.

Corollary 2.3.10. i) Let $V \subseteq \mathrm{Hilb}^p_{CM}$ be a closed irreducible subset of Hilb^p_{CM} which is the image[1] of some irreducible component of $D(p;\underline{f})_{CM}$. Then there is a well-defined "linked" irreducible closed subset V' of $\mathrm{Hilb}^{p'}_{CM}$ such that $(V')' = V$ and such that

$$\dim V' = \dim V + \sum_{i=1}^{2} [h^o(\underline{I}_{X/Y}(f_i)) - h^1(O_X(f_i-4))]$$

where $X \subseteq \mathbb{P}$ is a sufficiently general point of V and where Y is a global complete intersection of type $\underline{f}$ containing X.

ii) Moreover if

$$H^1(\underline{I}_X(f_i-4)) = 0$$

for $i = 1,2$, then V' is an irreducible component of Hilb^p, and in this case, if V itself is a reduced component, then so is V'.

Proof. ii) By (2.3.3ii), $H^1(\underline{I}_{X'}(f_i)) = 0$ for $i = 1,2$, and so V' is a component by (1.3.5). Now if V is reduced, then the component W of $\overline{D(p,f)_{CM}}$ which maps to V via the first projection is reduced. Indeed this follows from the smoothness of $pr_1 : pr_1^{-1}(U) \to U$. Thus W' is reduced, and so is V' since pr_1' is smooth at points $(X' \subseteq Y \subseteq \mathbb{P})$ satisfying $H^1(\underline{I}_{X'}(f_i)) = 0$. This proves (ii).

Our final corollary is concerned with the relationship between the algebra cohomology associated to $x = (X \subseteq Y \subseteq \mathbb{P})$ and the corresponding linked object $x' = (X' \subseteq Y \subseteq \mathbb{P})$.

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1) More precisely, the closure of the image.

Corollary 2.3.11. Let X be a curve in $\mathbb{P} = \mathbb{P}^3_k$, let Y be a global complete intersection of type (f_1, f_2) containing X, and let X' be linked to X by Y. Then

$$\dim \operatorname{coker} \alpha_{\underline{X \subseteq Y}} - \dim \operatorname{coker} 1^2_{\underline{X \subseteq Y}} = \dim \operatorname{coker} \alpha_{\underline{X' \subseteq Y}} - \dim \operatorname{coker} 1^2_{\underline{X' \subseteq Y}}$$

where α and 1^2 are as in (1.3.10). Moreover if the linkage is geometric, then

$$\operatorname{coker} 1^2_{\underline{X \subseteq Y}} = 0 = \operatorname{coker} 1^2_{\underline{X' \subseteq Y}}.$$

Proof. If $\underline{d}(x)$ is the category associated to $x = (X \subseteq Y \subseteq \mathbb{P})$, see the discussion before (1.2.5), and if $x' = (X' \subseteq Y \subseteq \mathbb{P})$, then by (2.2.14)

$$\chi(\underline{d}(x)) = \chi(\underline{d}(x'))$$

because $(4-f_1-f_2)d + 2g = (4-f_1-f_2)d' + 2g'$ by (2.3.3ii). Using that the isomorphism $D(p;\underline{f})_{CM} \simeq D(p';\underline{f})_{CM}$ induces an isomorphism of tangent spaces, we have the first part of the conclusion of (2.3.11). Finally suppose that $f : X \hookrightarrow Y$ is generically an isomorphism, and let $g : Y \hookrightarrow \mathbb{P}$ be the embedding. To show $\operatorname{coker} 1^2_{\underline{X \subseteq Y}} = 0$, it will, by the big diagram of (1.3.1), be sufficient to show that $A^3(f, O_X) \to A^3(gf, O_X)$ is injective. According to the spectral sequence of (1.2.3), there is a commutative diagram

$$\begin{array}{ccc} A^3(f,O_X) & \longrightarrow & A^3(gf,O_X) \\ \downarrow \cong & \circ & \downarrow \\ H^0(\underline{A}^3(f,O_X)) & \longrightarrow & H^0(\underline{A}^3(gf,O_X)) \end{array}$$

where the vertical morphism to the left is an isomorphism because $\underline{A}^1(f, O_X)$ and $\underline{A}^2(f, O_X)$ have support in $X \cap X'$. Moreover the local version of the exact sequence of (1.2.3) shows

$$\underline{A}^3(f,O_X) \simeq \underline{A}^3(gf,O_X)$$

and we are done.

We may use (2.3.11) to compute $h^1(\underline{N}_X)$, and for this we consider the exact sequence of (1.3.1C);

$$\xrightarrow{\gamma_{X\subseteq Y}} \oplus H^1(\underline{I}_X(f_i)) \to \operatorname{coker}\sigma_{X\subseteq Y} \to H^1(\underline{N}_X) \xrightarrow{1^2_{X\subseteq Y}} \oplus H^1(O_X(f_i)) \to ,$$

and we review (2.2.10ii) where now k is an algebraically closed field of characteristic zero.

Example 2.3.12. Let X be as in (2.2.10ii). We want to prove $H^1(\underline{N}_X) = 0$. Since

$$\max n_{1i} \leq \max(c(X)+2,\ e(X)+3) = 5$$

and since $s(X) = 3$, we may link X by a global complete intersection Y of type $(3,5)$ such that the linked curve X' is reduced.(Indeed since X is smooth, we may essentially by [P.S.,(4.1)] find a Y of type (f_1,f_2) such that the linked curve X' is reduced, where $f_2 = \max n_{1i}$ or larger and where f_1 is the degree of a surface $V(F_1)$ for which $X \hookrightarrow V(F_1)$ is generically a divisor.) Using (2.3.3ii) it follows that $\gamma_{X'\subseteq Y}$ is surjective because

$$h^1(\underline{I}_{X'}(3)) = h^1(\underline{I}_X(1)) = 0,\ h^1(\underline{I}_{X'}(5)) = h^1(\underline{I}_X(-1)) = 0,$$

and that $H^1(\underline{N}_{X'}) = 0$ because X' is reduced and $h^1(O_{X'}(1)) = h^0(\underline{I}_{X/Y}(3)) = 0$. Thus $\operatorname{coker}\alpha_{X'\subseteq Y} = 0$ by the exact sequence of (1.3.1C), and by (2.3.11), $\operatorname{coker}\alpha_{X\subseteq Y} = 0$. Then again by the exact sequence of (1.3.1C), $\gamma_{X\subseteq Y}$ is surjective and

$$H^1(\underline{N}_X) \simeq H^1(O_X(3)) \oplus H^1(O_X(5)) = 0.$$

The remaining part of this section deals with the problem of finding a good formula for the dimension of a reduced component V of Hilb^p_{CM} which holds in most cases. Indeed the formula of (2.2.13) is not always easy to use. So we postulate a dimension formula of V, and our main contribution is that, under some conditions, the dimension of V is as postulated iff the dimension of the "linked" component V' is as postulated. Technically the whole point is to generalize (2.3.11) together with the exact sequence of (1.3.10), i.e. we will construct an exact sequence

$$0 \to \bigoplus_1^2 H^0(\underline{I}_{X/Y}(f_i)) \to A^1(\underline{d},0_{\underline{d}}) \to H^0(\underline{N}_X) \xrightarrow{\gamma_{X\subseteq Y}} \bigoplus_1^2 H^1(\underline{I}_X(f_i)) \to$$

$$C(X\subseteq Y) \to H^1(\underline{N}_X) \to {}_0\mathrm{Hom}_R(I,H^3_m(I)) \to {}_0\mathrm{Ext}^2_R(I,H^2_m(I)) \to 0$$

where $C(X\subseteq Y) \subseteq \mathrm{coker}\,\alpha_{X\subseteq Y}$ and where $A = R/I$ is the minimal cone of X in $\mathbb{P}$, and we will show that $C(X\subseteq Y)$ and ${}_0\mathrm{Ext}^2_R(I,H^2_m(I))$ is invariant under linkage. Now if $X\subseteq \mathbb{P}$ is a sufficiently general curve of a reduced component, one knows that

$$\dim V = 4d + h^1(\underline{N}_X).$$

The number

$$4d + \dim {}_0\mathrm{Hom}_R(I,H^3_m(I))$$

which we can show is equal to $4d+\delta^2$ with δ^2 as in (2.2.7), is the postulated dimension of V. Therefore V has postulated dimension provided

$$C(X\subseteq Y) = 0 \quad\text{and}\quad {}_0\mathrm{Ext}^2_R(I,H^2_m(I)) = 0.$$

We expect these conditions to be weak for a "generic" point $X\subseteq\mathbb{P}$ of a reduced component.

To motivate we consider the problem of whether it is reasonable to expect $\operatorname{coker}\alpha_{\underline{X\subseteq Y}} = 0$ if $Y = V(F_1,F_2)$ is chosen in the following minimal way:

$$H^0(\underline{I}_X(f_1-1)) = 0 \qquad \text{for } f_1 = \deg F_1$$
$$H^0(\underline{I}_{X/V(F_1)}(f_2-1)) = 0 \quad \text{for } f_2 = \deg F_2.$$

Indeed (2.3.11) makes it easy to produce special examples where $\operatorname{coker}\alpha_{\underline{X\subseteq Y}} \neq 0$ as proposed by M. Noether [N,§ 12.4]. On the other hand we may prove that the conditions

$$s_{\mu,\nu} = \sigma_{\mu,\nu} \quad (\mu = f_1 \text{ and } \nu = f_2)$$

appearing in [N] is equivalent to

$$\operatorname{coker}\alpha_{\underline{X\subseteq Y}} = 0$$

provided $D(p,q)$ is non-singular at $(X\subseteq Y\subseteq \mathbb{P})$. And the whole list of curves in [N] satisfies $s_{\mu,\nu} = \sigma_{\mu,\nu}$, except for those of [N, § 12.4]. So we expect that $\operatorname{coker}\alpha_{\underline{X\subseteq Y}} = 0$ is a weak claim for large classes of curves provided f_1 and f_2 are small, and that $C(X\subseteq Y) = 0$ for large classes without requiring f_1 and f_2 small.

Example 2.3.13. There are curves X of degree $d = 21$ and genus $g = 54$ in $\mathbb{P} = \mathbb{P}^3_k$ whose ideals $\underline{I}_X$ possess a resolution of the form

$$0 \to O_{\mathbb{P}}(-9) \oplus O_{\mathbb{P}}(-6) \to \bigoplus_{i=1}^{3} O_{\mathbb{P}}(-5) \to \underline{I}_X \to 0.$$

Thus for any such curve, $e(X) = \max n_{2i} - 4 = 5$, and since the cone of X is Cohen Macaulay,

$$H^1(\underline{N}_X) \simeq {}_0\operatorname{Hom}_R(I,H^3_{\mathfrak{m}}(I)) \simeq H^1(O_X(5))^{\oplus 3}$$

by (2.2.9). Therefore if Y is a global complete intersection of type $(5,5)$ containing X, then the exact sequence

$$0 \to \operatorname{coker} \alpha_{\underline{X \subseteq Y}} \to H^1(\underline{N}_X) \xrightarrow{1^2} \bigoplus_{i=1}^{2} H^1(O_X(5)) \to 0$$

implies

$$a^2_{res} = \dim \operatorname{coker} \alpha_{\underline{X \subseteq Y}} = 1.$$

Note that to construct curves X as above we take a plan curve X' with $d' = 4$, $g' = 3$ and a global complete intersection Y of type $(5,5)$ containing X', and we let X be the linked curve of X' by Y. Since X' is of type $(1,4)$, $H^1(\underline{N}_{X'}) \simeq H^1(O_{X'}(1)) \oplus H^1(O_{X'}(4)) \simeq k$ because $\omega_{X'} \simeq O_{X'}(1)$. Thus

$$\dim \operatorname{coker} \alpha_{\underline{X' \subseteq Y}} = h^1(\underline{N}_{X'}) = 1$$

by (1.3.10) and by (2.3.11)

$$\dim \operatorname{coker} \alpha_{\underline{X \subseteq Y}} = 1.$$

Moreover using (2.3.3), $d = 21$ and $g = 54$.

This example of $\operatorname{coker} \alpha_{\underline{X \subseteq Y}} \neq 0$ is minimal with respect to the number $s(X)$ among the curves whose cone is Cohen Macaulay. However by (2.3.14) and (2.2.9), all curves satisfying the conditions of (2.2.9) have $C(X \subseteq Y) = 0$. In particular all curves whose cone is Cohen Macaulay satisfy $C(X \subseteq Y) = 0$.

<u>Lemma 2.3.14.</u> Let $X \subseteq \mathbb{P} = \mathbb{P}^3_k$ be a curve of degree d and let Y be a global complete intersection of type (f_1, f_2) containing X such that $X \hookrightarrow Y$ is generically an isomorphism (as in (2.3.2ii)). Then there is a group $C(X \subseteq Y)$

and exact sequences

$$\to H^0(\underline{N}_X) \xrightarrow{\gamma_{X\subseteq Y}} \bigoplus_{i=1}^{2} H^1(\underline{I}_X(f_i)) \to C(X\subseteq Y) \to$$

$$\to H^1(\underline{N}_X) \to {}_0\mathrm{Hom}_R(I,H^3_m(I)) \to {}_0\mathrm{Ext}^2_R(I,H^2_m(I)) \to 0,$$

and

$$0 \to C(X\subseteq Y) \to \operatorname{coker}\alpha_{X\subseteq Y} \to {}_0\mathrm{Hom}_R(I/I_B,H^3_m(I)) \to {}_0\mathrm{Ext}^2_R(I,H^2_m(I)) \to 0$$

where $A = R/I$ and $B = R/I_B$ is the minimal cone of X and Y in $\mathbb{P}$ respectively. Moreover all groups of the second exact sequence are invariant under linkage, and if δ^2 is the number introduced in (2.2.7),

$$4d + \delta^2 + \sum_{i=1}^{2} [h^0(\underline{I}_{X/Y}(f_i)) - h^1(\underline{I}_X(f_i))]$$

is also invariant under linkage.

__Proof.__ We consider the diagram of exact horizontal and vertical sequences

$$\xrightarrow{\gamma_{X\subseteq Y}} \oplus H^1(\underline{I}_X(f_i)) \to \operatorname{coker}\alpha_{X\subseteq Y} \to H^1(\underline{N}_X) \xrightarrow{1^2} \oplus H^1(O_X(f_i)) \to 0$$

$$H^1(\underline{N}_X) \downarrow \qquad \oplus H^1(O_X(f_i)) \,\big|\, \cong$$

$$0 \to {}_0\mathrm{Hom}_R(I/I_B,H^3_m(I)) \to {}_0\mathrm{Hom}_R(I,H^3_m(I)) \to \oplus H^1(O_X(f_i))$$

$$\downarrow$$

$${}_0\mathrm{Ex}^2_R(I,H^2_m(I))$$

$$\downarrow$$

$$0 .$$

For the vertical sequence, see the spectral sequence of (2.1.2), and combine with $H^1(\underline{N}_X) \twoheadrightarrow {}_0\mathrm{Ext}^3_m(I,I)$ surjective. It follows easily that there is a well-defined morphism

$$\operatorname{coker} \alpha_{X\subseteq Y} \to {}_0\mathrm{Hom}_R(I/I_B, H^3_m(I))$$

whose cokernel is ${}_0\mathrm{Ext}^2_R(I, H^2_m(I))$ by the five-lemma . Let $C(X\subseteq Y)$ be the kernel. Then both sequences of (2.3.14) is easily seen to be exact, and it will therefore be sufficient to prove that

$${}_0\mathrm{Hom}_B(I_{B/A}, H^3_m(I)) \quad \text{and} \quad {}_0\mathrm{Ext}^2_R(I, H^2_m(I))$$

are invariant under linkage where $I_{B/A} = I/I_B$. To see this, let $A' = R/I'$ be the minimal cone of the linked curve X' of X by Y. Then by Gorenstein duality the pairing

$$H^2_m(A) \times \mathrm{Hom}_B(A,B) \to H^2_m(B)$$

induces isomorphisms

$$H^2_m(A) \simeq I_{B/A'}^{\vee} \quad \text{and} \quad H^2_m(A)^{\vee} \simeq \hat{I}_{B/A'}$$

because $I_{B/A'} = \oplus H^0(\underline{I}_{X'/Y}(\nu)) = \oplus \underline{\mathrm{Hom}}_{O_Y}(O_X, O_Y(\nu)) = \mathrm{Hom}_B(A,B)$. ($(-)^{\wedge}$ is completion with respect to the maximal ideal of B). Correspondingly

$$H^2_m(A') \simeq I_{B/A}^{\vee} \quad \text{and} \quad H^2_m(A')^{\vee} \simeq \hat{I}_{B/A}.$$

Recalling that

$$H^3_m(I) \simeq H^2_m(A) \quad \text{and} \quad H^3_m(I') \simeq H^2_m(A'),$$

we deduce isomorphisms

$${}_0\mathrm{Hom}_B(I_{B/A}, H^3_m(I)) \simeq {}_0\mathrm{Hom}_B(I_{B/A'}, H^3_m(I'))$$

since to any B-linear map

$$I_{B/A} \to H^3_m(I)$$

there is a map

$$I_{B/A'} \hookrightarrow H^3_m(I)^\vee \to I_{B/A}^\vee \simeq H^3_m(I')$$

and conversely.

Next, to prove that

$${}_0\mathrm{Ext}^2_R(I,H^2_m(I)) \simeq {}_0\mathrm{Ext}^2_R(I',H^2_m(I'))$$

we use the duality theorem (2.1.5), and it will therefore be sufficient to prove

$${}_{-4}\mathrm{Ext}^2_m(H^2_m(I),I) \simeq {}_{-4}\mathrm{Ext}^2_m(H^2_m(I'),I').$$

Now the spectral sequence of (2.1.2) implies that

$${}_{-4}\mathrm{Ext}^2_m(H^2_m(I),I) \simeq {}_{-4}\mathrm{Hom}_R(H^2_m(I),H^2_m(I))$$

and correspondingly for ${}_{-4}\mathrm{Ext}^2_m(H^2_m(I'),I')$. So it suffices to prove that there is a perfect pairing

$$H^2_m(I) \times H^2_m(I') \to H^2_m(B).$$

This follows from the usual Gorenstein duality concerning

$$H^2_m(I_{B/A}) \times \mathrm{Hom}_B(I_{B/A},B) \to H^2_m(B)$$

if we combine with the exact sequences

$$0 \to H^2_m(I) \to H^2_m(I_{B/A}) \to H^2_m(B),$$

$$B \to \mathrm{Hom}_B(I_{B/A},B) \to H^2_m(I') \to 0,$$

see the proof of $H^1(\underline{I}_X(\nu)) \simeq H^1(\underline{I}_{X'}(f_1+f_2-4-\nu))^\vee$ of (2.3.3ii) which is similar. Thus

$$H^2_m(I) \simeq H^2_m(I')^\vee$$

as required.

Finally using the first exact sequence of (2.3.14), we get that

$$a^1 - \dim C(X \subseteq Y) + \dim {}_o\mathrm{Ext}^2_R(I, H^2_m(I)) =$$

$$\Sigma[h^o(\underline{I}_{X/Y}(f_i)) - h^1(\underline{I}_X(f_i))] + 4d + \dim {}_o\mathrm{Hom}_R(I, H^3_m(I)).$$

Therefore it will be sufficient to prove

$$\delta^2 = \dim {}_o\mathrm{Hom}_R(I, H^3_m(I))$$

since it then will follow that

$$\Sigma[h^o(\underline{I}_{X/Y}(f_i)) - h^1(\underline{I}_X(f_i))] + 4d + \delta^2$$

is invariant under linkage by the part of this lemma which already is proved. Now using the graded resolution of I appearing in (2.1.6), we find a complex

$$0 \to {}_o\mathrm{Hom}_R(I, H^3_m(I)) \to \bigoplus_{i=1}^{r_1} H^1(O_X(n_{1i})) \to \bigoplus_{i=1}^{r_2} H^1(O_X(n_{2i})) \to \bigoplus_{i=1}^{r_3} H^1(O_X(n_{3i})) \to 0,$$

and it is enough to prove that this complex is exact. So we must prove

$${}_o\mathrm{Ext}^1_R(I, H^3_m(I)) = 0 \quad \text{and} \quad {}_o\mathrm{Ext}^2_R(I, H^3_m(I)) = 0.$$

Now by the duality theorem (2.1.5) we easily see

$${}_o\mathrm{Ext}^4_m(I, I) = 0 \quad \text{and} \quad {}_o\mathrm{Ext}^5_m(I, I) = 0 ,$$

and so by one of the spectral sequences of (2.1.2), we conclude as required.

Then we make the following definition.

Definition 2.3.15. Let V be a reduced component of $H(d,g)_{CM}$ and let $X \subseteq \mathbb{P}$ be a sufficiently general curve of V. With δ^2 as in (2.2.7), we say that V has postulated dimension provided

$$\dim V = 4d + \delta^2.$$

Then we easily prove

Proposition 2.3.16. Let V be a reduced component of $H(d,g)_{CM}$ and suppose there is a curve $(X \subseteq \mathbb{P})$ of V and a global complete intersection Y of type (f_1, f_2) containing X such that

$$H^1(\underline{I}_X(f_i)) = 0 \quad \text{and} \quad H^1(\underline{I}_X(f_i-4)) = 0$$

for $i = 1,2$. If V' is the linked component, then V' has postulated dimension iff V has postulated dimension.

Proof. If $(X_1 \subseteq \mathbb{P})$ is the "generic" curve of V, then

$$H^1(\underline{I}_{X_1}(f_i)) = 0 = H^1(\underline{I}_{X_1}(f_i-4))$$

for $i = 1,2$ by semicontinuity and by the assumption on $(X \subseteq \mathbb{P})$. If Y_1 is a global complete intersection of type (f_1, f_2) containing X_1, then by (2.3.6)

$$\dim V + \Sigma\, h^0(\underline{I}_{X_1/Y_1}(f_i)) = \dim V' + \Sigma\, h^0(\underline{I}_{X_1'/Y_1}(f_i)),$$

and we conclude by the last part of (2.3.14).

We have introduced the notion of postulated dimension only for reduced components. For the non-reduced components we usually expect

$$\dim V < 4d + \delta^2$$

since we for a sufficiently general curve $X \subseteq \mathbb{P}$ of V expect $h^1(\underline{N}_X) = \delta^2$. See (2.2.9) and the discussion after the proof. Anyway we will in Section 3.2 give classes of non-reduced components, all of which satisfy

$$\dim V = d + g + 18 < 4d + h^1(O_X(3)) = 4d + \delta^2 .$$

Once having such a non-reduced component, we can find other non-reduced components by using (2.3.6) as illustrated in (2.3.9). Combining the conditions of (2.3.6) with the last part of the conclusion of (2.3.14) we see that

$$\dim V < 4d + \delta^2$$

for all the non-reduced components obtained in this way.

Observe also that the number δ^2 is easily found provided $s(X) = \min n_{1i}$ is small. Indeed if X is a smooth connected curve and if $s = s(X)$, then

1) $s \leq 3$ implies $\delta^2 = h^1(O_X(s))$,

2) $s = 4$ implies $\delta^2 = h^1(O_X(s))$

unless X is a global complete intersection in which case $\delta^2 = h^1(O_X(s)) + 1$,

3) $s = 5$ implies $\delta^2 = h^1(O_X(s))$

unless the cone of $X \subseteq \mathbb{P}$ is Cohen Macaulay. In fact if X is a global complete intersection, then δ^2 is immediately found, and otherwise we link the curve X to a curve X' by a global complete intersection Y of type (s,f) where f is as small as possible. By (2.3.3 ii)

$$h^1(O_X(f)) = h^0(\underline{I}_{X'/Y}(s-4)) = 0$$

unless X' is plane and $s=5$ which implies that the cone of $X \subseteq \mathbb{P}$ is Cohen Macaulay by (2.3.3 ii). See also (2.3.13).

Remark 2.3.17. (Components of the Hilbert-flag scheme of postulated dimension.)

Let W be a reduced component of the Hilbert-flag scheme $D(p,q)_{CM} = D(d,g;\underline{f})_{CM}$. We say that W has postulated dimension provided

$$\dim W = 4d + \delta^2 + \sum_{i=1}^{r} [h^0(\underline{I}_{X/Y}(f_i) - h^1(\underline{I}_X(f_i))]$$

where $X \subseteq Y \subseteq \mathbb{P}$ is a sufficiently general point of W and where Y is a global complete intersection of type $(f_1,\ldots,f_r)$, $r \leq 2$. In view of (2.2.14) we deduce that W has postulated dimension iff

$$a^2_{res} - \text{coker } 1^2 = \delta^2 - \sum_{i=1}^{r} h^1(O_X(f_i).$$

Compare with (2.2.9 ii). If W has postulated dimension, we observe that

$$\dim pr_1(W) = 4d + \delta^2 - \sum_{i=1}^{r} h^1(\underline{I}_X(f_i)).$$

Moreover if $r = 2$ we get by (2.3.14) that a reduced component $W \subseteq D(d,g;f_1,f_2)_{CM}$ has postulated dimension iff the "linked" component $W' \subseteq D(d',g';f_1,f_2)_{CM}$ has postulated dimension. If $r = 1$, say $f_1 = s$, we find that W

has postulated dimension iff

$$\dim W = 4d+\delta^2+h^0(\underline{I}_X(s))-h^1(\underline{I}_X(s)) = (4-s)d + \binom{s+3}{3} + g-2+\delta^2-h^1(O_X(s))$$

by (2.2.14). In particular for small s,

$$\dim W = 2d+g+8 \quad \text{if} \quad s = s(X) = 2,$$

$$\dim W = d+g+18 \quad \text{if} \quad s = s(X) = 3,$$

and unless X is a global complete intersection,

$$\dim W = g+33 \quad \text{if} \quad s = s(X) = 4.$$

We prove the validity of these formulas in Section 3.1 under some conditions. In fact the dimension of W is as postulated provided X is a smooth connected curve which is a divisor on Y where $(X \subseteq Y \subseteq \mathbb{P})$ is the "generic" point of W.

Finally observe that a reduced component V of the Hilbert scheme $H(d,g)$ has postulated dimension in the following cases. First if $X \subseteq \mathbb{P}$ is a sufficiently general point of V, we know that V has postulated dimension provided the curve $X \subseteq \mathbb{P}$ satisfies the conditions of (2.2.9 i). Next we claim that if X is a smooth connected curve which is a divisor on some surface Y of degree $s \leq 4$, then V has postulated dimension and $H^1(\underline{I}_X(s)) = 0$. To see this, let $W \subseteq D(d,g;s)$ be a component such that $pr_1(W) = V$. Then pr_1, restricted to $pr_1^{-1}(V)$, is generically smooth by generic flatness and by the smoothness of the fibers of pr_1 (1.3.12). In particular W is a reduced component of postulated dimension, and the tangent map of pr_1,

$$p_1^1 : A^1(\underline{d}, O_{\underline{d}}) \to H^0(\underline{N}_X)$$

is surjective. This implies the injectivity of

$$H^1(\underline{I}_X(s)) \hookrightarrow A^2(\underline{d},O_{\underline{d}})_{res} = \operatorname{coker}\alpha_{X\subseteq Y}$$

by using the exact sequence of (1.3.1 C). By the discussion of (3.1.1) which essentially uses (1.3.9 B), $\operatorname{coker}\alpha_{X\subseteq Y} = 0$, unless $s = 4$ and X is a global complete intersection in $\mathbb{P}$. Combining with

$$\dim pr_1(W) = 4d + \delta^2 - h^1(\underline{I}_X(s))$$

we conclude easily.

Example 2.3.18. We claim that there is a reduced component $V' \subseteq H(10,14)$ of dimension 43 which does not have postulated dimension, and we will indicate why. In fact there is a reduced component V of $H(4,-1)$ whose general member $X \subseteq \mathbb{P}^3$ is a disjoint union of two conics. (Apply (3.1.10) to the component $W(\delta,\underline{m}) = W(2,2,0,0,0,0,0)$ of $D(4,-1;3)$ and let $V = pr_1(W(\delta,\underline{m}))$). Now if $W \subseteq D(4,-1;2,7)$ is a component such that $V = pr_1(W)$, then there is a "linked" irreducible subset V' of $H(10,14)$ which by (2.3.10) is a reduced component of $H(10,14)$ of dimension 43. The number $4d + \delta^2$ which belongs to V', is

$$4d + \delta^2 = 4d + h^1(O_{X'}(2)) = 4d + h^0(\underline{I}_{X/Y}(3)) = 42$$

where X' and X are linked by a sufficiently general Y of type $(2,7)$. The dimension of the groups $C(X' \subseteq Y)$ is seen to be 1, and V' contains reduced curves.

3.1. The Hilbert-flag scheme of curves and low degree surfaces.

In this chapter k is algebraically closed, and the components are irreducible and non-embedded unless explicitely stating "connected component" or "embedded component".

Let $D(d,g;s)_S$ be the Hilbert-flag scheme of smooth connected curves X of degree d and arithmetic genus g and surfaces Y of degree s, $X \subseteq Y \subseteq \mathbb{P} = \mathbb{P}^3_k$. We describe in this section the irreducible components W of $D(d,g;s)_S$ for $s \leq 3$ which contain points $(X \subseteq Y \subseteq \mathbb{P})$ where X is a divisor on Y. Indeed $D(d,g;2)_S$ is a smooth connected scheme, and it contains points $x = (X \subseteq Y \subseteq \mathbb{P})$ where X is a divisor on a smooth quadric surface Y if it is non-empty. And the main theorem of this section (3.1.4) implies that there is a one-to-one correspondence between components of $D(d,g;3)_S$ as above and tuples $(\delta,\underline{m}) = (\delta,m_1,\ldots,m_6) \in \mathbb{Z}^{\oplus 7}$ satisfying

$$\delta \geq m_1+m_2+m_3\,,\quad m_1 \geq m_2 \geq \ldots \geq m_6 \geq 0\,,$$

$$d = 3\delta - \sum_{i=1}^{6} m_i \quad \text{and} \quad g = \binom{\delta-1}{2} - \sum_{i=1}^{6} \binom{m_i}{2}\,,$$

with two exceptions. If $(\delta,\delta,0,0,0,0,0)$ for $\delta \neq 1$ is a solution, then the corresponding component does not contain smooth connected curves, and $D(1,0;3)_S$ corresponds to $(0,0,0,0,0,0,-1)$.

Let $W(\delta,m)$ be the component of $D(d,g;3)_S$ corresponding to a solution $(\delta,\underline{m})$ of the system above, and let $S(\delta,\underline{m}) \subseteq W(\delta,\underline{m})$ consist of curves X and smooth surfaces Y, $X \subseteq Y \subseteq \mathbb{P}$. Then $S(\delta,\underline{m})$ is an open smooth subscheme of $D(d,g;3)_S$, hence irreducible, and we can describe $S(\delta,\underline{m})$ as follows. To each geometric K-point $(X \subseteq Y \subseteq \mathbb{P}^3_K)$ of $S(\delta,\underline{m})$, $k \subseteq K$ a field extension, there are six mutually skew lines $E_1,\dots,E_6$ on Y inducing an isomorphism $\mathrm{Pic}(Y) \simeq \mathbb{Z}^{\oplus 7}$ under which the invertible sheaf $\mathcal{O}_Y(X)$ maps to $(\delta,\underline{m})$.

As corollaries we study the image in $H(d,g)_S$ of the components W of $D(d,g;s)_S$ via the first projection morphism

$$pr_1 : D(d,g;s)_S \to H(d,g)_S \ .$$

According to conventions earlier made in this paper, the image is the scheme-theoretic one unless $pr_1(W)$ is a non-embedded irreducible (resp. connected) component in which case $pr_1(W)$ has a scheme structure inherited from the scheme structure of $H(d,g)_S$, i.e.

$$\mathcal{O}_{pr_1(W),t} = \mathcal{O}_{H(d,g),t}$$

for most points $t \in pr_1(W)$ (resp. for all points $t \in pr_1(W)$). Then we show that $pr_1(D(d,g;2)_S)$ is a smooth connected component of $H(d,g)_S$ provided $g \neq 0$ and $g \neq d-3$ The exceptional cases are treated separately. If $s = 3$, then (3.1.10) states that $pr_1(W(\delta,\underline{m}))$ is a reduced irreducible component of $H(d,g)_S$ provided

$$H^1(\underline{I}_X(3)) = 0$$

for some $(X \subseteq Y \subseteq \mathbb{P}) \in S(\delta,\underline{m})$. The condition $H^1(\underline{I}_X(3)) = 0$ is usually equivalent to $m_6 \geq 3$, see (3.1.3) for precise information. Moreover if

$$H^1(\mathcal{O}_X(3)) = 0,$$

then $pr_1(W(\delta,\underline{m})) \subseteq H(d,g)_S$ is a closed subscheme of codimension $h^1(\underline{I}_X(3))$, and $H(d,g)_S$ is non-singular along $pr_1(S(\delta,\underline{m}))$. Finally for the remaining cases where we in particular have

$H^1(\underline{I}_X(3)) \neq 0$, we prove that if $pr_1(W(\delta,\underline{m}))$ is an irreducible component of $H(d,g)_S$, it is necessarily non-reduced. A necessary condition for $pr_1(W(\delta,\underline{m}))$ to be a non-reduced component is seen to be

$$0 \neq h^1(\underline{I}_X(3)) \leq h^1(O_X(3)).$$

We conjecture that this condition is sufficient, and Section 3.2 is devoted to a closer study of what happens in this case.

Furthermore we determine the dimension of the components of $D(d,g;s)_S$ for $s \leq 4$ and of $H(d,g)_S$ obtained as above, and we can see that the components involved have postulated dimension (2.3.15) and (2.3.17).

We will begin by determining the dimension of the irreducible components of $D(d,g;s)$ for $s \leq 4$.

Let X be a divisor on a surface Y, and let $x = (X \subseteq Y \subseteq \mathbb{P}) \in D(d,g;s)$. Recall (1.3.9) that if X is reduced and $s \leq 3$, then

$$\operatorname{coker} \alpha = 0 \quad \text{and} \quad \operatorname{coker} 1^2 = 0.$$

It follows that $D = D(d,g;s)$ is non-singular at $x = (X \subseteq Y \subseteq \mathbb{P})$ and that

$$\dim O_{D,x} = a^1 = (4-s)d + \binom{s+3}{3} + g-2,$$

see (1.2.9) and (2.2.14). In particular if $W \subseteq D(d,g;s)$ is any irreducible component containing x, then W is reduced and

$$\dim W = \begin{cases} 2d+g+8 & \text{for } s=2, \\ d+g+18 & \text{for } s=3. \end{cases}$$

If X is integral and $s = 4$, then

$$\dim W = g+34$$

if W contains a point $x = (X \subseteq Y \subseteq \mathbb{P})$ where X is a global complete intersection of Y and some other surface, and

$$\dim W = g + 33 \qquad \text{otherwise.}$$

To see this, we will use a result of Noether, here stated as follows.

<u>Lemma 3.1.1.</u> Let W be an irreducible component of $D(d,g;s)$ with $s \geq 4$. Then the following conditions are equivalent

i) There is a closed point $x = (X \subseteq Y \subseteq \mathbb{P})$ of W where $\alpha = 0$ and $D(d,g;s)$ is non-singular.

ii) There is a closed point $x_1 = (X_1 \subseteq Y_1 \subseteq \mathbb{P})$ of W where X_1 is a global complete intersection of Y_1 with some other surface.

Now the dimension formula for W in case $s = 4$ follows from (3.1.1). Indeed if $x = (X \subseteq Y \subseteq \mathbb{P})$ is a sufficiently general point of W, then

$$\alpha : H^0(\underline{N}_Y) \to H^1(\underline{N}_{X/Y})$$

is either zero or surjective since $H^1(\underline{N}_{X/Y}) \simeq H^0(O_X)^\vee \simeq k$ by (1.3.9). Consider the following three cases

1) α surjective,

2) $\alpha = 0$ and $D(d,g;4)$ is singular at x,

3) $\alpha = 0$ and $D(d,g;4)$ is non-singular at x.

Using (1.2.9) and (2.2.14) and observing that

$$\operatorname{coker} 1^2 = 0$$

by the discussion of (1.3.9C), we find

$$g + 33 \leq \dim W \leq a^1 = g + 33 + a^2_{res} \,.$$

Since the inequality to the right is strict provided $D(d,g;4)$ is singular at x and since $a^2_{res} = \dim \operatorname{coker} \alpha \leq 1$, we deduce $\dim W = g+33$ in the first two cases and $\dim W = g+34$ in the case (3). Moreover (3.1.1) characterizes the case(3) in a nice way, thus giving the desired description of when $\dim W = g+34$.

Proof of (3.1.1). Suppose (i). Since the second projection

$$pr_2 : D(d,g;s) = D(p,q) \to \mathrm{Hilb}^q$$

is smooth at x (1.3.7), and since Hilb^q is irreducible, $pr_2(W) = \mathrm{Hilb}^q$. Using a theorem of Noether, see [Le, page 396] or [SGA, 7 II, exp XIX], there is a smooth surface Y_1 of degree s with

$$\mathrm{Pic}(Y_1) \simeq \mathbb{Z}.$$

We deduce that every effective divisor on Y_1 is a global complete intersection as done in [SGA, 7II, exp XIX], and (ii) follows from $pr_2(W) = \mathrm{Hilb}^q$ because $(Y_1 \subseteq \mathbb{P}) \in \mathrm{Hilb}^q$.

Conversely if $x_1 = (X_1 \subseteq Y_1 \subseteq \mathbb{P})$ is as in (ii), then $D(d,g;s)$ is non-singular at x_1 by (1.4.7), and $\alpha_{X_1 \subseteq Y_1} = 0$ by the fact that the sequence

$$0 \to \underline{N}_{X_1/Y_1} \to \underline{N}_{X_1} \to \mathcal{O}_{X_1}(s) \to 0$$

is split. So (i) holds.

Now we want to describe the components of $D(d,g;s)$ for $s \leq 3$. We begin with $s = 3$ and a remark which describe the curves X which are divisors on a fixed smooth cubic surface Y.

Remark 3.1.2. (Y a smooth cubic surface). Recall that any smooth cubic surface is obtained by blowing up six points $P_1, \dots, P_6 \in \mathbb{P}^2_k$ in "general position" [H 1, V, § 4]. Fix six points $P_1,\dots,P_6 \in \mathbb{P}^2_k$, let $E_1,\dots,E_6$ be the exceptional lines, and let H be the inverse image of a line in $\mathbb{P}^2$ via the blowing up morphism $\pi : Y \to \mathbb{P}^2$. If $h,e_1,\dots,e_6 \in \mathrm{Pic}(Y)$ are the linear equivalence classes of $H,E_1,\dots,E_6$ respectively, then $\{h,e_1,\dots,e_6\}$ is a $\mathbb{Z}$-basis for Pic(Y). In the following we will always identify Pic(Y) and $\mathbb{Z}^{\oplus 7}$ via the isomorphism

$$\beta : \mathbb{Z}^{\oplus 7} \to \mathrm{Pic}(Y)$$

given by

$$\beta(\delta,m_1,\dots,m_6) = \delta h - \sum_{i=1}^{6} m_i e_i .$$

Moreover if X is a given curve on Y, then there exist six mutually skew lines $E_1,\dots,E_6$ giving rise to an isomorphism $\beta : \mathbb{Z}^{\oplus 7} \simeq \mathrm{Pic}(Y)$ such that the tuple $(\delta,\underline{m}) = (\delta,m_1,\dots,m_6)$ corresponding to $\underline{L} = O_Y(X)$ satisfies

$$(*)\begin{cases} d = 3\delta - \sum_{i=1}^{6} m_i , \quad g = \binom{\delta-1}{2} - \sum_{i=1}^{6} \binom{m_i}{2} , \\ \delta \geq m_1+m_2+m_3 , \quad \delta \geq m_1^+ \text{ and } m_1 \geq m_2 \geq \dots \geq m_6 \end{cases}$$

where $m_1^+ = \max(0,m_1)$. This follows from the proof of (4.2) in [H 1, V, § 4], and as in Peskines lectures (University of Oslo, 1978), we call the corresponding basis $\{h,e_1,\dots,e_6\}$ of Pic(Y) on adequate basis with respect to $\underline{L} = O_Y(X)$. Different adequate basis define the same tuple $(\delta,\underline{m})$. Conversely given six mutually skew lines on Y, i.e. an isomorphism $\beta : \mathbb{Z}^{\oplus 7} \simeq \mathrm{Pic}(Y)$, and a tuple $(\delta,m) \in \mathbb{Z}^{\oplus 7}$ satisfying (*) <u>and</u> the additional condition $m_6 \geq 0$, then the corresponding invertible sheaf $\underline{L} \in \mathrm{Pic}(Y)$ has sections, and if we exclude tuples of the form $(\delta,\delta,0,0,0,0,0)$ where $\delta \neq 1$, then there are irreducible non-singular curves among the sections of $\underline{L}$. These tuples together with $(0,0,0,0,0,0,-1)$ give precisely the tuples whose corresponding $\underline{L}$'s have non-singular irreducible curves among its sections. For a proof of this, see [H 1, V,(4.13)] and the exercise (4.8).

Now if X is a reduced curve on Y, it is easy to see that the tuple $(\delta,\underline{m})$ which corresponds to $\underline{L} = O_Y(X)$ must satisfy

$$(**)\begin{cases} d = 3\delta - \sum_{i=1}^{6} m_i \,, \quad g = \binom{\delta-1}{2} - \sum_{i=1}^{6} \binom{m_i}{2} \\ \delta \geq m_1 + m_2 + m_3 \,, \quad \delta \geq m_1^+ \quad \text{and} \quad m_1 \geq m_2 \geq \dots \geq m_6 \geq -1 \,. \end{cases}$$

Indeed if $m_6 \leq -2$, then X will contain E_6 at least 2 times. And conversely one may prove that an invertible sheaf $\underline{L}$ with a tuple $(\delta,\underline{m})$ as in (**) has sections among which there are reduced curves.

For later use we will include the following result, pointed out to us by Peskine-Gruson, and indicate the proof.

Proposition 3.1.3. Let X be an effective divisor on a smooth cubic surface Y, and let $\underline{L} = O_Y(X)$ correspond to a tuple $(\delta,\underline{m})$ where

$$\delta \geq m_1 + m_2 + m_3 \,, \quad m_1 \geq m_1 \geq \dots \geq m_6 \,.$$

i) Then

$$H^1(\underline{I}_X(n)) \neq 0 \iff n \in \langle m_6, 2\delta - \sum_{i=2}^{6} m_i - 1 \rangle$$

provided $(\delta,\underline{m})$ is not of the form

$$(\delta,\underline{m}) = (\lambda+3t, \lambda+t, t, t, t, t, t) \quad \text{for some} \quad \lambda \geq 2$$

in which case

$$H^1(\underline{I}_X(n)) \neq 0 \iff n \in [m_6, 2\delta - \sum_{i=2}^{6} m_i - 1\rangle,$$

or not of the form

$$(\delta,\underline{m}) = (3t,t,t,t,t,t,t-\lambda) \quad \text{for some} \quad \lambda \geq 2$$

in which case

$$H^1(\underline{I}_X(n)) \neq 0 \iff n \in \langle m_6, 2\delta - \sum_{i=2}^{6} m_i - 1]$$

ii) Moreover

$$h^1(O_X(n)) = h^0(\underline{L}(-n-1)) \quad \text{for} \quad n \geq 0 .$$

<u>Proof</u>. ii) Indeed there is an exact sequence

$$0 \to O_Y \to \underline{L} \to \underline{N}_{X/Y} \to 0$$

and an isomorphism (1.3.9)

$$\underline{N}_{X/Y} \simeq \omega_X(1) ,$$

and we conclude easily by duality on X.

i) Assume $H^1(\underline{I}_X(\nu)) \neq 0$ for some integer ν.

<u>Step 1</u>. Computing the intersection numbers $X \cdot E$ for every line $E \subseteq Y$, we find

$$m_6 = X \cdot E_6 = \min(X \cdot E) ,$$
$$2\delta - \sum_{i=2}^{6} m_i = \max(X \cdot E) .$$

If we can prove

$$\min\{n \mid H^1(\underline{I}_X(n)) \neq 0\} = m_6 + 1$$

(except in one special case), we deduce by duality on Y and the expression of $\max(X \cdot E)$ above, recalling $H^1(\underline{I}_X(n)) = H^1(\underline{I}_{X/Y}(n))$, that

$$\max\{n \mid H^1(\underline{I}_X(n)) \neq 0\} = 2\delta - \sum_{i=2}^{6} m_i - 2$$

(except in one special case).

Step 2. We claim that it will be sufficient to find $\min\{n \mid H^1(\underline{I}_X(n)) \neq 0\}$ if $\underline{L} = O_Y(X)$ satisfies

$$H^0(\underline{L}) \neq 0 \quad \text{and} \quad H^0(\underline{L}(-1)) = 0.$$

Indeed if $\underline{L}$ is as in (3.1.3), then there is an integer ν such that

$$H^0(\underline{L}(-\nu)) \neq 0 \quad \text{and} \quad H^0(\underline{L}(-\nu-1)) = 0.$$

If D is a section of $\underline{L}(-\nu)$, considered as a curve, then

$$h^1(\underline{I}_X(n)) = h^1(\underline{L}(-n) \otimes \omega_Y) = h^1(\underline{L}(-\nu)(-n+\nu) \otimes \omega_Y) = h^1(\underline{I}_D(n-\nu)).$$

So

$$\min\{n \mid H^1(\underline{I}_D(n)) \neq 0\} = \min\{n \mid H^1(\underline{I}_X(n)) \neq 0\} - \nu.$$

Since $\underline{L} = O_Y(X)$ and $\underline{L}(-\nu) = O_Y(D)$ correspond to $(\delta, m_1, \ldots, m_6)$ and $(\delta-3\nu, m_1-\nu, \ldots, m_6-\nu)$ respectively, the claim follows easily.

Step 3. Since $m_6 - 1 \geq 0$ implies $H^0(\underline{L}(-1)) \neq 0$ by the discussion of (3.1.2), $m_6 \leq 0$. We can by (3.1.2) find a section $\tilde{X}$ of the invertible sheaf $\underline{L}^+$ defined by $(\delta, m_1^+, \ldots, m_6^+)$ which is a smooth connected curve unless $(\delta, \underline{m}^+) = (\lambda, \lambda, 0, 0, 0, 0, 0)$ for some $\lambda \neq 1$. If $\lambda \geq 2$, then $\tilde{X}$ is a smooth non-connected curve. Since

$$X \equiv \tilde{X} + \sum_{i=t}^{6} \nu_i E_i$$

are linearly equivalent where $\nu_i = -m_i$ and where the number t is defined by

$$m_{t-1} \geq 0 \quad \text{and} \quad m_t < 0$$

($t = 1$ if $m_i < 0$ for all i and $t = 7$ if $m_i \geq 0$ for all i), we deduce

$$H^1(\underline{I}_X(n)) \simeq H^0(O_X(n)) \simeq H^0(O_{\tilde{X}}(n)) \ni H^0(O_{\Sigma\nu_iE_i}(n)) \simeq \bigoplus_{i=t}^{6} H^0(O_{\nu_iE_i}(n))$$

if $n < 0$ and a slightly modified result if $n = 0$. Now let $E \subseteq Y$ be a line and let $\nu \geq 1$ be an integer. Using that there is an exact sequence

$$0 \to O_E(\nu-1) \to O_{\nu E} \to O_{(\nu-1)E} \to 0\ ,$$

we find

$$\min\{n \mid H^0(O_{\nu E}(n)) \neq 0\} = -\nu + 1\ .$$

Thus if $m_6 \leq -1$, then

$$\min\{n \mid H^1(\underline{I}_X(n)) \neq 0\} = m_6 + 1\ ,$$

and if $m_6 = 0$, we do have $X \equiv \tilde{X}$, i.e. $\underline{L} \simeq \underline{L}^+$. Since $\tilde{X}$ is smooth and connected (resp. non-connected if $\underline{L}$ corresponds to $(\lambda,\lambda,0,0,0,0,0)$ for $\lambda \geq 2$), $H^1(\underline{I}_X) = 0$ (resp. $H^1(\underline{I}_X) \neq 0$), i.e.

$$\min\{n \mid H^1(\underline{I}_X(n)) \neq 0\} \geq 1 = m_6 + 1$$

$$(\text{resp. } \min\{n \mid H^1(\underline{I}_X(n)) \neq 0\} = 0 = m_6)\ .$$

Moreover $X \equiv \tilde{X}$ are rational curves by (3.1.3ii), and it follows that

$$\min\{n \mid H^1(\underline{I}_X(n)) \neq 0\} = 1 = m_6 + 1\ .$$

Finally, reviewing the proof, we will see that $H^1(\underline{I}_X(n)) = 0$ for every integer n iff $\underline{L}(-\nu) \simeq O_Y$ (step 2), or $X = E_6$ ($m_6 = -1$ of step 3) or $X \equiv \tilde{X}$ are rational curves of degree $d \leq 3$ ($m_6 = 0$ of step 3). The conclusion of (3.1.3i) holds for these cases as well. In fact for these cases $\underline{L}$ corresponds to $(3\nu,\nu,\nu,\nu,\nu,\nu,\nu)$, $(3\nu,\nu,\nu,\nu,\nu,\nu,\nu-1)$, $(3\nu+1,\nu,\nu,\nu,\nu,\nu,\nu)$ or $(3\nu+1,\nu+1,\nu,\nu,\nu,\nu,\nu)$.

Now we aim at describing the irreducible components of $D(d,g;3)$. We restrict, however, to those components containing points $(X\subseteq Y\subseteq \mathbb{P})$ where X is a reduced curve and a divisor on Y. In fact let

$$U(d,g;3) = \left\{(X\subseteq Y\subseteq \mathbb{P}) \in D(d,g;3) \,\middle|\, \begin{array}{l} X \text{ is a divisor on } Y \\ \text{and } H^0(O_X(-1)) = 0 \end{array}\right\} \text{ and}$$

$$S(d,g;3) = \{(X\subseteq Y\subseteq \mathbb{P}) \in U(d,g;3) \mid Y \text{ is a smooth surface}\}.$$

Then $S(d,g;3) \subseteq U(d,g;3)$ are both open in $D(d,g;3)$ and the composition

$$U(d,g;3) \to D(d,g;3) \xrightarrow{pr_2} \mathrm{Hilb}^q$$

is smooth because $H^1(\underline{N}_{X/Y}) \simeq H^0(O_X(-1))^{\vee} = 0$. See (1.3.7). It follows that $U(d,g;3)$ is non-singular, so there is a decomposition

$$U(d,g;3) \simeq \coprod_{i\in I} U_i$$

into connected components. By the smoothness of pr_2, $pr_2(U_i)$ is open in Hilb^q, and $pr_2(U_i)$ will therefore contain a smooth cubic surface. Thus $S_i = S(d,g;3) \cap U_i$ is smooth and non-empty and

$$S(d,g;3) \simeq \coprod_{i\in I} S_i .$$

Take the closure W_i of S_i in $D(d,g;3)$. Then W_i is a reduced (i.e. generically smooth) irreducible component of $D(d,g;3)$. Since for reduced curves X, $H^0(O_X(-1)) = 0$, all irreducible components containing points $(X\subseteq Y\subseteq \mathbb{P})$ with X reduced and $X \hookrightarrow Y$ a divisor are among the W_i's (and there are no more W_i's.

We skip the details since otherwise we can redefine $S(d,g;3)$ by throwing away the S_i's which do not contain reduced curves).
Now the main theorem of this section determines the index set I, and describes the S_i in a nice way.

<u>Theorem 3.1.4.</u> i) Any irreducible component $W \subseteq D(d,g;3)$ containing points $(X \subseteq Y \subseteq \mathbb{P})$ where X is reduced and where X is a divisor on Y is a reduced component of dimension

$$\dim W = d + g + 18 .$$

ii) There is a one-to-one correspondence between the components of (i) and tuples $(\delta,\underline{m}) \in \mathbb{Z}^{\oplus 7}$ satisfying (3.1.2**). Put

$$W(\delta,\underline{m}) = W \quad \text{and} \quad S(\delta,\underline{m}) = S_i$$

if W corresponds to $(\delta,\underline{m})$ and $W = \overline{S}_i$. Then to each geometric K-point $(X \subseteq Y \subseteq \mathbb{P}^3_K)$ of $S(\delta,\underline{m})$, there is an isomorphism $\beta : \mathbb{Z}^{\oplus 7} \simeq \mathrm{Pic}(Y)$ such that the invertible sheaf $\underline{L} = O_Y(X)$ corresponds to $(\delta,\underline{m})$.

<u>Proof.</u> i) is already proved, and for ii) we will construct a nice scheme $T = T(d,g;3)$ and a smooth surjective morphism

$$\psi : T \to S(d,g;3)$$

where, over T, the pullback $X_T \subseteq Y_T \subseteq \mathbb{P} \times T$ of the universal object of $D(d,g;3)$ <u>and</u> the family of six mutually skew lines are defined. Using intersection number theory we can describe easily the connected components of T as we now shall see.

First we will define T and ψ. Let

$$R' = S(1,0;3) \times_{\mathrm{Hilb}^q} \cdots \times_{\mathrm{Hilb}^q} S(1,0;3) \qquad \text{(six times)}$$

Then any point of R' is just six times $E_1,\ldots,E_6$ and a smooth cubic surface Y, $E_i \subseteq Y$ for $i = 1,\ldots,6$. If we denote by Hilb_S^q the scheme of smooth surfaces in Hilb^q, then the obvious morphism $R' \twoheadrightarrow \mathrm{Hilb}_S^q$ is smooth and surjective (it is in fact étale) since $pr_2 : S(1,0;3) \twoheadrightarrow \mathrm{Hilb}_S^q$ is. Moreover since the intersection numbers $E_i \cdot E_j$ is defined in terms of Hilbert polynomials [M 1, Lect 12], and since these Hilbert polynomials are "constant on the connected components of R'", there is a subscheme R og R', open and closed in R', defined by

$$E_i \cdot E_j = 0 \qquad\qquad \text{for } i \neq j,$$

i.e. a point of R is just six mutually skew lines $E_1,\ldots,E_6$ and a surface Y, $E_i \subseteq Y$ for $i = 1,\ldots,6$. Then we define $T = T(d,g;3)$ by the cartesian diagram

$$\begin{array}{ccc} T & \xrightarrow{p_2} & R \\ {\scriptstyle \psi}\downarrow & \square & \downarrow \\ S(d,g;3) & \xrightarrow[pr_2]{} & \mathrm{Hilb}_S^q \end{array}$$

and the morphisms of this diagram are smooth, hence dominating provided $S(d,g;3)$ is non-empty. They are surjective if we can show that

$$pr_2 : S(d,g;3) \to \mathrm{Hilb}_S^q$$

is surjective. To see this, pick $(X \subseteq Y \subseteq \mathbb{P}) \in S(d,g;3)$ and let $(Y' \subseteq \mathbb{P}) \in \mathrm{Hilb}_S^q$, be arbitrary. To prove that there is a point $(X' \subseteq Y' \subseteq \mathbb{P}) \in S(d,g;3)$, choose isomorphisms

$$\mathrm{Pic}(Y) \simeq \mathbb{Z}^{\oplus 7} \simeq \mathrm{Pic}(Y')$$

as in (3.1.2**), and let $\underline{L} = \mathcal{O}_Y(X) \in \mathrm{Pic}(Y)$ map to $\underline{L}' \in \mathrm{Pic}(Y')$.

Then $\underline{L}$ and $\underline{L}'$ correspond to the same tuple $(\delta,\underline{m})$, a tuple which satisfies (3.1.2**). So $\underline{L}'$ has sections, and a section X' of $\underline{L}'$ considered as a divisor on Y' defines a point $(X' \subseteq Y' \subseteq \mathbb{P}) \in S(d,g;3)$.

To study the components of T, let for any $t \in T$

$$m_i(t) = X \cdot E_i$$

$$\delta(t) = \frac{1}{3}(d - \sum_{i=1}^{6} m_i(t))$$

where the $t = \mathrm{Spec}(k(t))$ - point of T is the six lines $E_1,\ldots,E_6$ and the curve X of degree d and the smooth surface Y, $E_i \subseteq Y$ and $X \subseteq Y$. Again the intersection numbers $X \cdot E_i$ are constant on each connected component T_j of T. Put

$$(\delta(T_j),\underline{m}(T_j)) = (\delta(t),\underline{m}(t)) \in \mathbb{Z}^{\oplus 7}$$

for some $t \in T_j$. Moreover we claim that different components of T correspond to different tuples. To see this, let $T(\delta,\underline{m})$ be the disjoint union of those connected components $T_j \subseteq T$ such that

$$(\delta(T_j),\underline{m}(T_j)) = (\delta,\underline{m}) .$$

Then consider the geometric fibers of the composition

$$T(\delta,\underline{m}) \subseteq T \xrightarrow{p_2} R .$$

If $r \in R$ is a given geometric K-point of R, which means that a smooth surface Y and six mutually skew lines $E_1,\ldots,E_6$ over an algebraically closed field K are given, then the fiber $p_2^{-1}(r) \cap T(\delta,\underline{m})$ consists of curves X on Y satisfying $X \cdot E_i = m_i$ for all $i = 1,2,\ldots,6$. This is a linear system, see [M1, Lect 13]. Thus $p_2^{-1}(r) \cap T(\delta,\underline{m})$ is connected. Since the composition

$$T(\delta,\underline{m}) \subseteq T \xrightarrow{p_2} R$$

is smooth and surjective, it is also connected. Using the next lemma which states that R is connected, we deduce that $T(\delta,\underline{m})$ is connected.

We have now a smooth surjective morphism

$$\psi : T \to S(d,g;3)$$

of smooth schemes. For each connected component $S_i \subseteq S(d,g;3)$, $\psi^{-1}(S_i)$ consists of a certain number of the connected components of T. Choosing a geometric K-point $x = (X \subseteq Y \subseteq \mathbb{P}^3_K)$ of S_i, we see by the construction of T that the fiber $\psi^{-1}(x) \subseteq T$ is x together with all possible choices of six mutually skew lines on Y. Observing that (3.1.2**) is obtained by a special choice of six mutually skew lines, we easily see that among the components of T which map to S_i, there is a component $T(\delta,\underline{m})$ whose corresponding tuple $(\delta,\underline{m})$ satisfies (3.1.2**). **This** $(\delta,\underline{m})$ is unique. Moreover

$$T(\delta,\underline{m}) \to S_i$$

is a **surjective morphism** since the composition $T(\delta,\underline{m}) \to T \xrightarrow{p_2} R$ is surjective. Let this S_i be $S(\delta,\underline{m})$. Now putting all this together we easily get **the theorem.**

<u>Lemma 3.1.5.</u> R is a connected scheme.

<u>Proof.</u> Let r and r' be k-points of R corresponding to two choices of six mutually skew lines on smooth surfaces, $E_1,\dots,E_6$ on $Y \subseteq \mathbb{P}^3$ and $E'_1,\dots,E'_6$ on $Y' \subseteq \mathbb{P}^3$, respectively. If we can prove that there are connected schemes

$$X_0, X_1, \dots, X_n$$

over R and morphisms

$$X_i \dashrightarrow X_{i+1} \quad \underline{\text{or}} \quad X_i \leftarrow X_{i+1}$$

<u>over R</u> for each $i = 0,\ldots,n-1$, such that

$$X_o = \operatorname{Spec}(k) \to R \quad \text{and} \quad X_n = \operatorname{Spec}(k) \to R$$

are the k-points r and r', then R is connected. See [H 3, Chap I] or use that each time there is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & R & \end{array}$$

where X and Y are connected schemes, then the images of X and Y in R are contained in the same connected component of R.

Recall that the lines $E_1,\ldots,E_6$ on $Y \subseteq \mathbb{P}^3$ are obtained by blowing up six points $P_1,\ldots,P_6 \in \mathbb{P}^2$ in "general position", and that if $\tilde{\mathbb{P}}^2$ is the blowing up of $\mathbb{P}^2$ along $Z = P_1 \cup \ldots \cup P_6$, then the linear system of curves of degree 3 in $\mathbb{P}^2$ passing through $P_1,\ldots,P_6$ defines an embedding

$$\tilde{\mathbb{P}}^2 \to \mathbb{P}^3$$

whose image is $Y \subseteq \mathbb{P}^3$. If $\underline{I}_Z = \ker(O_{\mathbb{P}^2} \to O_Z)$, then the linear system above is given as the k-vector space $H^o(\underline{I}_Z(3))$. And it is a choice of a basis of $H^o(\underline{I}_Z(3))$ which defines $\tilde{\mathbb{P}}^2 \to \mathbb{P}^3$, or more precisely, if

$$\pi : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$$

is the blowing up morphism, we know that the inverse ideal sheaf $\tilde{\underline{I}}_Z \subseteq O_{\tilde{\mathbb{P}}^2}$ given by $\pi^{-1}\underline{I}_Z \cdot O_{\tilde{\mathbb{P}}^2}$ is invertible. Indeed

$$\tilde{\underline{I}}_Z = O_{\tilde{\mathbb{P}}^2}(-E_1) \otimes \ldots \otimes O_{\tilde{\mathbb{P}}^2}(-E_6) .$$

Hence $\underline{\tilde{I}}_Z \otimes \pi^* O_{\mathbb{P}^2}(3)$ is invertible, and any choice of a basis of $H^o(\underline{\tilde{I}}_Z \otimes \pi^* O_{\mathbb{P}^2}(3))$ defines an embedding $\tilde{\mathbb{P}}^2 \to \mathbb{P}^3$. However

$$H^o(\underline{I}_Z(3)) \simeq H^o(\underline{\tilde{I}}_Z \otimes \pi^* O_{\mathbb{P}^2}(3)) .$$

In the same way the lines $E_1', \ldots, E_6'$ correspond to six points $P_1', \ldots, P_6' \in \mathbb{P}^2$ in "general position" etc.

To prove that R is connected, it suffices, heuristically speaking, to prove that the scheme W whose k-points are ordered tuples $(P_1, \ldots, P_6, \underline{s})$ where $(P_1, \ldots, P_6)$ are points in "general position" and where $\underline{s} = \{s_o, s_1, s_2, s_3\}$ is a choice of a basis of $H^o(\underline{I}_Z(3))$, is connected. We shall prove that W is connected (or since we do not prove representability, we will prove that the corresponding functor is connected). Hence R is a connected scheme since $W \to R$ is surjective.

To be precise, the ordered tuples $x = (P_1, \ldots, P_6)$ and $x' = (P_1', \ldots, P_6')$ are k-points of

$$V = \mathbb{P}^{2^\vee} \underset{k}{\times} \mathbb{P}^{2^\vee} \underset{k}{\times} \ldots \underset{k}{\times} \mathbb{P}^{2^\vee} \qquad \text{(six times)}$$

where $\mathbb{P}^{2^\vee}$ is the Hilbert scheme of points in $\mathbb{P}^2$. If

$$A = O_{V,x} \quad \text{and} \quad A' = O_{V,x'}$$

and if K is the quotient field of A or of A', then, over $\mathrm{Spec}(A)$, there exists an ordered tuple $(P_{1A}, \ldots, P_{6A})$ of A-flat schemes $P_{iA} \subseteq \mathbb{P}^2 \times \mathrm{Spec}(A) = \mathbb{P}^2_A$ whose fiber at the closed point of $\mathrm{Spec}(A)$ is $P_i \in \mathbb{P}^2$, $1 \leq i \leq 6$. By simply blowing up $\mathbb{P}^2_A$ along $Z_A = P_{1A} \cup \ldots \cup P_{6A}$, we claim that there is a flat family of six mutually skew lines over $\mathrm{Spec}(A)$ and an embedding of A-flat schemes

$$\tilde{\mathbb{P}}^2_A \to \mathbb{P}^3 \times \mathrm{Spec}(A) = \mathbb{P}^3_A$$

whose image is $Y_A \subseteq \mathbb{P}^3_A$. It will follow that there exists a map

$$\psi : \mathrm{Spec}(A) \to R$$

such that $\psi(\mathrm{Spec}(k)) = r$. Correspondingly there will be a map $\psi' : \mathrm{Spec}(A') \to R$ such that $\psi'(\mathrm{Spec}(k)) = r'$.

Now define

$$\underline{C} = \bigoplus_{d \geq 0} \underline{I}^d_Z \quad \text{and} \quad \underline{C}_A = \underset{d \geq 0}{} \underline{I}^d_{Z_A}$$

where $\underline{I}^d_Z$ is the ith power of $\underline{I}_Z$ and $\underline{I}^0_Z = O_{\mathbb{P}^2}$.
Then let

$$\tilde{\mathbb{P}}^2 = \underline{\mathrm{Proj}}(\underline{C}) \quad \text{and} \quad \tilde{\mathbb{P}}^2_A = \underline{\mathrm{Proj}}(\underline{C}_A).$$

By the universal property of blowing up, there is a commutative diagram

$$\begin{array}{ccc} \tilde{\mathbb{P}}^2_A & \xrightarrow{\pi_A} & \mathbb{P}^2_A \\ \uparrow & \circ & \uparrow \\ \tilde{\mathbb{P}}^2 & \xrightarrow{\pi} & \mathbb{P}^2 \end{array}$$

Moreover $\tilde{\mathbb{P}}^2_A$ is A-flat since the sheaf $\underline{I}_{Z_A}$ is.

To prove that there is an embedding

$$\tilde{\mathbb{P}}^2_A \longrightarrow \mathbb{P}^3_A$$

whose fiber at the closed point of Spec(A) is $\tilde{\mathbb{P}}^2 \to \mathbb{P}^3$, we claim that

$$H^0(\underline{I}_{Z_A}(3)) \otimes_A k \simeq H^0(\underline{I}_Z(3))$$

and that $H^0(\underline{I}_{Z_A}(3))$ is A-free. Indeed counting dimensions of the vector space appearing in the exact sequence

$$0 \to H^0(\underline{I}_Z(3)) \to H^0(O_{\mathbb{P}^2}(3)) \to H^0(O_Z(3))$$

we get that $H^0(O_{\mathbb{P}^2}(3)) \to H^0(O_Z(3))$ is surjective, so

$$H^1(\underline{I}_Z(3)) = 0$$

and we conclude as claimed by base change theorem.

Next if

$$\tilde{\underline{I}}_{Z_A} = \pi_A^{-1}\underline{I}_{Z_A} \cdot O_{\tilde{\mathbb{P}}_A^2}$$

then we claim that

$$H^0(\tilde{\underline{I}}_{Z_A} \otimes \pi_A^* O_{\mathbb{P}_A^2}(3)) \underset{A}{\otimes} k \simeq H^0(\underline{I}_Z \otimes \pi^* O_{\mathbb{P}^2}(3))$$

and that

$$H^0(\tilde{\underline{I}}_{Z_A} \otimes \pi_A^* O_{\mathbb{P}_A^2}(3)) \simeq H^0(\underline{I}_{Z_A}(3)).$$

Indeed there is a commutative diagram

$$\begin{array}{ccc} H^0(\underline{I}_{Z_A}(3)) & \to & H^0(\tilde{\underline{I}}_{Z_A} \otimes \pi_A^* O_{\mathbb{P}_A^2}(3)) \\ \downarrow\!\!\downarrow & & \downarrow \\ H^0(\underline{I}_Z(3)) & \simeq & H^0(\tilde{\underline{I}}_Z \otimes \pi^* O_{\mathbb{P}^2}(3)) \end{array}$$

so the vertical map to the right is surjective. We conclude by base change theorem and Nakayama's lemma. Now since the morphism $\tilde{\mathbb{P}}^2 \to \mathbb{P}^3$ is defined by a basis of $H^0(\tilde{\underline{I}}_Z \otimes \pi^* O_{\mathbb{P}^2}(3))$, we can lift the basis to a basis of $H^0(\tilde{\underline{I}}_{Z_A} \otimes \pi_A^* O_{\mathbb{P}_A^2}(3))$, thus defining a morphism $\tilde{\mathbb{P}}_A^2 \to \mathbb{P}_A^3$ making a commutative diagram

$$\begin{array}{ccc} \tilde{\mathbb{P}}^2_A & \longrightarrow & \mathbb{P}^3_A \\ \uparrow & \circ & \uparrow \\ \tilde{\mathbb{P}}^2 & \longrightarrow & \mathbb{P}^3 \end{array}$$

Moreover $\tilde{\underline{I}}_{Z_A} \otimes \pi_A^* O_{\mathbb{P}^2_A}(3)$ is very ample because $\tilde{\underline{I}}_Z \otimes \pi^* O_{\mathbb{P}^2}(3)$ is.

If $\eta : \mathrm{Spec}(K) \to R$ is the composition of the natural map $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$ with ψ, and if η' is correspondingly defined, then we have commutative diagrams

$$\begin{array}{ccccc} \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(K) \\ & {\scriptstyle r}\searrow \;\circ & \downarrow{\scriptstyle\psi} & \circ\; \swarrow{\scriptstyle\eta} & \\ & & R & & \end{array} \qquad \begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(A') & \longleftarrow & \mathrm{Spec}(k) \\ & {\scriptstyle\eta'}\searrow \;\circ & \downarrow{\scriptstyle\psi'} & \circ\; \swarrow{\scriptstyle r'} & \\ & & R & & \end{array}$$

Moreover there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(A) & \longleftarrow & \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(A') \\ & \searrow \;\circ & \downarrow & \circ\; \swarrow & \\ & & V & & \end{array}$$

The latter diagram implies that the restriction of the tuples $(P_{1A},\ldots,P_{6A})$ and $(P'_{1A'},\ldots,P'_{6A'})$ to $\mathrm{Spec}(K)$ are the same tuple $(P_{1K},\ldots,P_{6K})$. So

$$\underline{I}_{Z_A} \otimes_A K = \underline{I}_{Z_{A'}} \otimes_{A'} K = \underline{I}_{Z_K},$$

and we deduce

$$\tilde{\mathbb{P}}^2_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K) = \tilde{\mathbb{P}}^2_{A'} \times_{\mathrm{Spec}(A')} \mathrm{Spec}(K) = \tilde{\mathbb{P}}^2_K.$$

However the restriction of the embeddings

$$\tilde{\mathbb{P}}^2_A \longrightarrow \mathbb{P}^3_A \quad \text{and} \quad \tilde{\mathbb{P}}^2_{A'} \longrightarrow \mathbb{P}^3_{A'}$$

to Spec(K) need not be the same. Over Spec(K) they correspond to two choices $\{s_o, \ldots, s_3\}$ and $\{s'_o, \ldots, s'_3\}$ of a basis of $H^o(\underline{I}_{Z_K}(3))$. We therefore let

$$B = K[t]$$

be a polynomial ring in one variable, and if we lift $(P_{1K}, \ldots, P_{6K})$ and $\mathbb{P}^2_K$ trivially to Spec(B), i.e.

$$P_{iB} = P_{iK} \times_{\mathrm{Spec}(K)} \mathrm{Spec}(B) \subseteq \mathbb{P}^2_K \times_{\mathrm{Spec}(K)} \mathrm{Spec}(B) = \mathbb{P}^2_B \text{ and}$$

$$\tilde{\mathbb{P}}^2_B = \tilde{\mathbb{P}}^2_K \times \mathrm{Spec}(B)$$

we can define a morphism

$$\tilde{\mathbb{P}}^2_B \longrightarrow \mathbb{P}^3_B$$

using the basis $\{s_o + t(s'_o - s_o), \ldots, s_3 + t(s'_3 - s_3)\}$ of $H^o(\underline{I}_{Z_B}(3)) = H^o(\underline{I}_{Z_K}(3)) \otimes_K B$. It follows that there is a morphism

$$\mathrm{Spec}(B) = \mathbb{A}^1_K \longrightarrow R ,$$

and the K-points $t = 0$ and $t = 1$ of Spec(B) composed with $\mathrm{Spec}(B) \to R$ are just the K-points η and η' of R. Thus the diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \xrightarrow{t=0} & \mathrm{Spec}(B) & \xleftarrow{t=1} & \mathrm{Spec}(K) \\ & {\scriptstyle\eta}\searrow & \downarrow & \swarrow{\scriptstyle\eta'} & \\ & & R & & \end{array}$$

commutes, and the proof is complete.

<u>Examples 3.1.6.</u> i) Solving (3.1.2**) for $d = 9$ and $g = 8$, there is only one solution

$$(7,3,2,2,2,2,1).$$

Thus $D(9,8;3)$ contains only one irreducible component of the form described in (3.1.4), and its dimension is $d+g+18=35$. Compare with (2.2.10ii) and (2.3.12).

ii) Solving (3.1.2**) for $d=8$ and $g=5$ there are two solutions $(5,2,1,1,1,1,1)$ and $(6,2,2,2,2,2,0)$. So $D(8,5;3)$ contains two irreducible components of the form described in (3.1.4), both of dimension 31. Compare with (2.2.16)

iii) Solving (3.1.2**) for $d=14$ and $g=24$ under the condition $m_6 \geq 0$, there are two solutions $(11,4,3,3,3,3,3)$ and $(12,4,4,4,4,4,2)$. The corresponding components are of dimension 56. For later use we will compute $h^1(O_X(3))$. Indeed if $\underline{L}$ corresponds to $(\delta,m_1,\dots,m_6)$, then $\underline{L}(n)$ corresponds to $(\delta+3n,m_1+n,\dots,m_6+n)$ since $O_Y(1)$ corresponds to $(3,1,1,1,1,1,1)$. Thus if $\underline{L}$ is given by $(11,4,3,3,3,3,3)$, then

$$h^1(O_X(3)) = h^o(\underline{L}(-4)) = 0,$$

by (3.1.3), and if $\underline{L}$ is given by $(12,4,4,4,4,4,2)$, then

$$h^1(O_X(3)) = h^o(\underline{L}(-4)) = 1,$$

iv) For $d=15$ and $g=27$ there are three solutions

$(11,3,3,3,3,3,3)$, $(12,5,4,3,3,3,3)$ and $(12,4,4,4,4,3,2)$

satisfying $m_6 \geq 0$. Computing $h^1(O_X(3))$ as in (iii), we find

$$h^1(O_X(3)) = \begin{cases} 0 & \text{for the first two cases} \\ 1 & \text{for the case } (12,4,4,4,4,3,2). \end{cases}$$

So there are three components of $D(15,27;3)$ of dimension 60.

v) For $d = 16$ and $g = 31$ there are two components of dimension 65 given by (12,4,4,3,3,3,3) and (13,5,4,4,4,4,2) containing smooth connected curves. Moreover

$$h^1(O_X(3)) = \begin{cases} 1 & \text{in the case } (12,4,4,3,3,3,3) \\ 2 & \text{in the case } (13,5,4,4,4,4,2) \end{cases}$$

vi) For $d = 16$ and $g = 29$ we find three components of dimension 63 containing smooth connected curves. They correspond to tuples

(11,3,3,3,3,3,2), (12,5,4,3,3,3,2) and (12,4,4,4,4,2,2).

This time

$$h^1(O_X(3)) = \begin{cases} 0 & \text{in the first two cases} \\ 1 & \text{in the case } (12,4,4,4,4,2,2). \end{cases}$$

If we consider the irreducible components $W \subseteq D(d,g;2)$ containing points $(X \subseteq Y \subseteq \mathbb{P})$ where X is a reduced curve and a divisor on Y, we can by the discussion right before (3.1.4) conclude that W contains points where in addition Y is a smooth quadric surface. Moreover, slightly modifying the proof of (3.1.4) we claim that there is a one-to-one correspondence between such components and tuples $(q_1,q_2) \in \mathbb{Z}^{\oplus 2}$ satisfying

$$d = q_1 + q_2, \quad g = (q_1-1)(q_2-1) \quad \text{and} \quad 0 \leq q_1 \leq q_2 .$$

Indeed this time a k-point of R is two intersecting lines E_1, E_2 and a smooth quadric surface Y, $E_i \subseteq Y$. And to see that R is connected, which is the main point of a proof of the claim above, we use that to any k-point of R, there is a morphism

$$\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow[\varphi_1]{} \mathbb{P}^3 \xrightarrow[\varphi_2]{\sim} \mathbb{P}^3$$

where φ_1 is the morphism of Segree and φ_2 an automorphism of $\mathbb{P}^3$, $\varphi = \varphi_2\varphi_1$, such that $\varphi(\mathbb{P}^1 \times \mathbb{P}^1) = Y$ and

$$\varphi^*E_1 \equiv \mathbb{P}^1 \times \{pt\}\ ,\ \ \varphi^*E_2 \equiv \{pt\} \times \mathbb{P}^1$$

for some point $\{pt\} \in \mathbb{P}^1$. Furthermore since deformations obtained via automorphism of $\mathbb{P}^3$ lie in the same connected component of R, see the last part of the proof of (3.1.5), we conclude that R is connected.

Since there is at most one solution of the system above, there is only one component $W \subseteq D(d,g;2)$. Note also that a solution (q_1,q_2) of positive integers together with the solution $(0,1)$ if $d = 1$ corresponds to a component which contain irreducible non-singular curves on smooth quadrics.

If we also study $D(d,g;2)$ at points $x = (X \subseteq Y \subseteq \mathbb{P})$ where X is smooth and connected, but not necessarily a divisor on Y, we can determine the structure of $D(d,g;2)_S$ completely. Indeed we claim that $D(d,g;2)_S$ is non-singular at such points x. To see this, we use [H 1, IV, (6.4.1)] which states that if X is a smooth connected curve on a singular quadric surface Y, then $q_1 = q_2$ or $q_1 = q_2 - 1$. If $q_1 = q_2$, then X is a global complete intersection, and in the final case, the cone of $X \subseteq \mathbb{P}^3$ is Cohen Macaualy. $D(d,g;2)_S$ will therefore be non-singular at x by (1.4.7) and (2.2.8). Finally if W is a component of $D(d,g;2)_S$ which contains x, then W contains points $(X' \subseteq Y' \subseteq \mathbb{P})$ where Y' is a smooth quadric surface. Indeed if we make the deformations of the cone of $X \subseteq \mathbb{P}^3$ explicitely which is easy since the cone is determinantiel, we will see that there is a deformation X' which is contained in a smooth quadric surface $Y' \subseteq \mathbb{P}^3$. This gives

<u>Proposition 3.1.7</u>. $D(d,g;2)_S$ is a <u>smooth connected scheme</u> of dimension $2d+g+8$ if it is non-empty. Moreover $D(d,g;2)_S$, in case $d>1$, is non-empty iff there is a tuple (q_1,q_2) satisfying

$$d = q_1 + q_2, \quad g = (q_1-1)(q_2-1) \quad \text{and} \quad 0 < q_1 \leq q_2$$

in which case $D(d,g;2)_S$ contain points $x = (X \subseteq Y \subseteq \mathbb{P})$ where Y is a smooth quadric surface. Finally to any such point x there are two intersection lines E_1, E_2 contained in Y inducing an isomorphism

$$\mathrm{Pic}(Y) \simeq Z^{\oplus 2}$$

which maps $O_Y(X)$ onto (q_1,q_2).

As applications of (3.1.4) and (3.1.7) we will study the Hilbert scheme $H(d,g)$ and certain subfamilies which correspond to the image of some component of $D(d,g;s)$ via the morphism

$$pr_1 : D(d,g;s) \to H(d,g).$$

Start with $s = 2$ and $D(d,g;2)_S$ non-empty. Then we have a well-defined tuple (q_1,q_2) and there are two cases to consider.

1) $H^1(\underline{I}_X(2)) = 0$ for some $(X \subseteq Y \subseteq \mathbb{P}) \in D(d,g;2)_S$

2) $H^1(\underline{I}_X(2)) \neq 0$ for all $(X \subseteq Y \subseteq \mathbb{P}) \in D(d,g;2)_S$.

Note that <u>any</u> point $(X \subseteq Y \subseteq \mathbb{P}) \in D(d,g;2)_S$ satisfies $H^1(\underline{I}_X(2)) = 0$

if <u>one</u> point does. This follows from the fact that we can characterize the vanishing of $H^1(\underline{I}_X(2))$ in terms of the integers q_1 and q_2, see (1.3.10) if Y is a smooth quadric surface and observe that $H^1(\underline{I}_X(2)) = 0$ if Y is singular. Using Künneth's formula as we did in (1.3.10), we can easily see that

1') $H^1(\underline{I}_X(2)) = 0$ iff $q_1 \geq 3$ or $q_2 \leq 3$

2') $H^1(\underline{I}_X(2)) \neq 0$ iff $q_1 < 3$ and $q_2 > 3$

in which case

$$h^1(\underline{I}_X(2)) = (q_2-3)(3-q_1).$$

In general

$$pr_1 : D(d,g;2)_S \to H(d,g)_S$$

is proper, hence closed. Moreover in the first case (1), pr_1 is a smooth morphism by (1.3.4), hence open. We deduce that $pr_1(D(d,g;2)_S)$ is a <u>smooth connected component</u> of $H(d,g)_S$ of dimension

$$\begin{cases} 4d + (q_1-3)(q_2-3) = 2d+g+8 & \text{if} \quad q_1 \geq 3 \\ 4d & \text{if} \quad q_2 \leq 3 \,, \end{cases}$$

again by (1.3.10). Note that the dimension of $pr_1(D(d,g;2)_S)$ is not given by $2d+g+8$ if $q_2 < 3$, i.e. if $d \leq 4$. Otherwise it is, see (1.3.7) and use that $h^0(\underline{I}_X(2)) = 1$ for $d > 4$.

In the last case (2) or (2')

$$H^1(\underline{N}_X) \simeq H^1(O_X(2)) = 0$$

for any $(X \subseteq Y \subseteq \mathbb{P}) \in D(d,g;2)_S$ by (1.3.10) since Y is necessarily smooth. In particular $H(d,g)_S$ is non-singular along $pr_1(D(d,g;2)_S)$, and we have the following corollary in which (i) corresponds to (1') with $q_2 < 3$, and (ii) and (iii) correspond to $q_1 = 1$ and $q_1 = 2$ of (2') respectively.

Corollary 3.1.8. The scheme $pr_1(D(d,g;2)_S)$ of smooth connected curves which are contained in some surface of degree 2 form a smooth connected component of $H(d,g)_S$ of dimension

$$2d + g + 8$$

except in the following three cases

i) $d \leq 4$ in which case $pr_1(D(d,g;2)_S)$ is a smooth connected component of $H(d,g)_S$ of dimension $4d$

ii) $d \geq 5$ and $g = 0$ in which case $pr_1(D(d,0;2)_S) \simeq D(d,0,2)_S$ is a smooth connected scheme which is of codimension $2(d-4)$ in $H(d,0)_S$.

iii) $d \geq 6$ and $g = d-3$ in which case $pr_1(D(d,g;2)_S) \simeq D(d,g;2)_S$ is a smooth connected scheme of codimension $d-5$ in $H(d,g)_S$, and $H(d,g)_S$ is non-singular along $D(d,g;2)_S$.

Compare (3.1.8 iii) with (2.2.16E).

Using the corollary above, we find that $H(d,g)_S$ is sometimes disconnected. A classical example is $H(9,10)_S$ which is a smooth scheme consisting of two connected components, both of dimension 36. See [N, § 15] or [H 1, IV, (6.4.3)] or [T]. Another example is the following.

Example 3.19. $H(10,12)_S$ is a smooth scheme consisting of two connected components, both of dimension 40. Indeed $D(10,12;2)_S$ is non-empty since $q_1 = 3$ and $q_2 = 7$ is a solution of

$$10 = q_1 + q_2 , \quad 12 = (q_1-1)(q_2-1).$$

By (3.1.8), $pr_1(D(10,12;2)_S)$ is a smooth connected component of $H(10,12)_S$. Moreover any curve $X \subseteq \mathbb{P}^3$ of $H(10,12)_S$ for which $s(X) \geq 3$ is easily seen to be contained in a global complete intersection Y of type $(3,4)$. The linked curve $X' \subseteq Y$ is therefore a plane curve since $\chi(\underline{I}_X(2)) = 1$ implies $h^1(O_X(2)) \neq 0$ and since

$$h^0(\underline{I}_{X'/Y}(1)) = h^1(O_X(2)) \neq 0$$

by (2.3.3). The cone of $X \subseteq \mathbb{P}^3$ is Cohen Macaulay, again by (2.3.3), so

$$H(10,12)_S - pr_1(D(10,12;2)_S)$$

is smooth and connected by (2.3.6) of dimension 40 by (2.2.9) because $3d > 2g-2$ implies $\delta^2 = 0$.

We now study the images of the components $W(\delta,\underline{m}) \subseteq D(d,g;3)$ appearing in (3.1.4) via the first projection pr_1, and there are three cases to consider.

1) $H^1(\underline{I}_X(3)) = 0$ for some $x = (X \subseteq Y \subseteq \mathbb{P}) \in W(\delta,\underline{m})$,

2) $H^1(\underline{I}_X(3)) \neq 0$ for any $x = (X \subseteq Y \subseteq \mathbb{P}) \in W(\delta,\underline{m})$ and $H^1(O_X(3)) = 0$ for some $x \in W(\delta,\underline{m})$,

3) $H^1(\underline{I}_X(3)) \neq 0$ and $H^1(O_X(3)) \neq 0$ for any $x \in W(\delta,\underline{m})$.

The case 3) did not occur in the discussion of $s = 2$ since $H^1(\underline{I}_X(2)) \neq 0$ implied $H^1(O_X(2)) = 0$. Observe that if $H^1(\underline{I}_X(3)) = 0$, resp. $H^1(O_X(3)) = 0$, for some $x \in W(\delta,\underline{m})$, then the groups vanish for any $x = (X \subseteq Y \subseteq \mathbb{P}) \in S(\delta,\underline{m})$, i.e. for any $(X \subseteq Y \subseteq \mathbb{P})$ such that X is a divisor on a smooth cubic surface Y. This is a consequence of (3.1.3) and of the last part of the theorem (3.1.4).

Moreover what is the codimension of $pr_1(W(\delta,\underline{m}))$ in $H(d,g)$? To answer this, let $V \supseteq pr_1(W(\delta,\underline{m}))$ be any irreducible component of $H(d,g)$. Since the fiber of pr_1 is of dimension $h^0(\underline{I}_{X/Y}(3))$ by (1.3.7) for $(X \subseteq Y \subseteq \mathbb{P}) \in S(\delta,\underline{m})$, we deduce

$$\dim pr_1(W(\delta,\underline{m})) = \dim W(\delta,\underline{m}) - h^0(\underline{I}_{X/Y}(3))$$
$$= d + g + 18 - h^0(\underline{I}_{X/Y}(3))$$

as in the discussion of (2.3.10 i). Using (2.2.14) we find

$$\dim pr_1(W(\delta,\underline{m})) = 4d + \gamma(3)$$

where $\gamma(3) = h^1(O_X(3)) - h^1(\underline{I}_X(3))$. Combining with

$$4d \leq \dim V \leq h^0(\underline{N}_X) = 4d + h^1(O_X(3))$$

(recall $h^1(\underline{N}_X) = h^1(O_X(3))$, see (1.3.9 C)) and observing that the inequality to the right is strict iff $H(d,g)$ is singular at $(X \subseteq \mathbb{P})$, see (1.2.9), we get that

$$3d - g - 18 - h^0(\underline{I}_{X/Y}(3)) = -\gamma(3) \leq \dim V - \dim pr_1(W(\delta,\underline{m})) \leq h^1(\underline{I}_X(3))$$

where the inequality to the right is strict iff $H(d,g)$ is singular along $pr_1(W(\delta,\underline{m}))$. This gives

Corollary 3.1.10. i) If

$$H^1(\underline{I}_X(3)) = 0 \quad \text{for some} \quad x = (X \subseteq Y \subseteq \mathbb{P}) \in S(\delta,\underline{m}),$$

then $pr_1(W(\delta,\underline{m}))$ is a reduced irreducible component of $H(d,g)$ of dimension

$$d + g + 18 - h^0(\underline{I}_{X/Y}(3)) .$$

Indeed $pr_1(S(\delta,\underline{m}))$ is an open smooth subscheme of $H(d,g)$, and if $d \geq 10$, $s(X) = 3$ (2.2.7) and if X is integral, then

$$h^0(\underline{I}_{X/Y}(3)) = 0.$$

ii) If

$$H^1(O_X(3)) = 0 \quad \text{for some} \quad x \in S(\delta,\underline{m}) ,$$

then $pr_1(W(\delta,\underline{m}))$ is a reduced subscheme of $H(d,g)$ of codimension

$$h^1(\underline{I}_X(3)) ,$$

and $H(d,g)$ is non-singular along $pr_1(S(\delta,\underline{m}))$.

iii) If V is an irreducible component of $H(d,g)$ containing $pr_1(W(\delta,\underline{m}))$, then

$$h^1(\underline{I}_X(3)) - h^1(O_X(3)) \leq \dim V - \dim pr_1(W(\delta,\underline{m})) \leq h^1(\underline{I}_X(3))$$

for any $x = (X \subseteq Y \subseteq \mathbb{P}) \in S(\delta,\underline{m}))$. Moreover $H(d,g)$ is singular along $pr_1(W(\delta,\underline{m}))$ iff the inequality to the right is strict, and we have

$$h^1(\underline{I}_X(3)) - h^1(O_X(3)) = 3d - g - 18 + h^0(\underline{I}_{X/Y}(3)).$$

iv) In particular if $pr_1(W(\delta,\underline{m}))$ is an irreducible component, then

$$h^1(\underline{I}_X(3)) \leq h^1(O_X(3)),$$

and $pr_1(W(\delta,\underline{m}))$ is non-reduced iff $h^1(\underline{I}_X(3)) \neq 0$.

We now ask:

Does there always exist an irreducible component $V \subseteq H(d,g)$ containing $pr_1(W(\delta,\underline{m}))$ such that

$$\dim V - \dim pr_1(W(\delta,\underline{m})) = [h^1(\underline{I}_X(3)) - h^1(O_X(3))]^+$$

where $(X \subseteq Y \subseteq \mathbb{P}) \in S(\delta,\underline{m})$ and where $m^+ = \max(0,m)$ for $m \in \mathbb{Z}$.

We know by (3.1.10 i,ii) that the answer is yes if $H^1(\underline{I}_X(3)) = 0$ or $H^1(O_X(3)) = 0$. Suppose therefore that

$$H^1(\underline{I}_X(3)) \neq 0 \quad \text{and} \quad H^1(O_X(3)) \neq 0$$

which is equivalent to

$$[h^1(\underline{I}_X(3)) - h^1(O_X(3))]^+ < h^1(\underline{I}_X(3)).$$

Note that if the answer to the question above is positive, then $H(d,g)$ is singular along $pr_1(W(\delta,\underline{m}))$. We divide into two cases

A) $0 \neq h^1(\underline{I}_X(3)) \leq h^1(O_X(3))$, $(X \subseteq Y \subseteq \mathbb{P}) \in S(\delta,\underline{m})$.

If $pr_1(W(\delta,\underline{m}))$ is a non-reduced component of $H(d,g)$, then (A) holds by (3.1.10 iv) ; the question above deals with the converse which we think is true:

<u>Conjecture 3.1.11.</u> $pr_1(W(\delta,\underline{m}))$ is a non-reduced irreducible component iff (A) holds.

Also in the case

B) $h^1(\underline{I}_X(3)) > h^1(O_X(3)) \neq 0$, $(X \subseteq Y \subseteq \mathbb{P}) \in S(\delta,\underline{m})$

we expect that the answer to the question above is positive.

In the next section (3.2) we discuss the conjecture, and (B) is considered in the last section of this paper.

Remark 3.1.12. If $W \subseteq D(d,g;s)$ is an irreducible component satisfying

$$\operatorname{coker} \alpha_{\underline{X}\subseteq\underline{Y}} = 0 \quad \text{and} \quad \operatorname{coker} 1^2_{\underline{X}\subseteq\underline{Y}} = 0$$

for some $(\underline{X} \subseteq \underline{Y} \subseteq \mathbb{P})$, then (3.1.10) is true with obvious modifications, i.e.

i) If $H^1(\underline{I}_X(s)) = 0$, then $pr_1(W)$ is a reduced component of $H(d,g)$ of dimension

$$(4-s)d + g + \binom{s+3}{3} - 2 - h^0(\underline{I}_{X/Y}(s)).$$

ii) If $H^1(O_X(s)) = 0$, then $pr_1(W)$ is a reduced subscheme of $H(d,g)$ of codimension $h^1(O_X(s))$, and $H(d,g)$ is generically non-singular along $pr_1(W)$.

iii) $$h^1(\underline{I}_X(s)) - h^1(O_X(s)) \leq \dim V - \dim pr_1(W) \leq h^1(\underline{I}_X(s))$$

for any component V of $H(d,g)$ containing $pr_1(W)$ etc.

3.2. Non-reduced components of H(d,g).

In the preceding section we conjectured that $pr_1(W(\delta,\underline{m}))$ was a non-reduced irreducible component of $H(d,g)$ iff

A) $0 \neq h^1(\underline{I}_X(3)) \leq h^1(O_X(3))$, $(X \subseteq Y \subseteq \mathbb{P}^3_k) \in S(\delta,\underline{m})$.

One way is true by (3.1.10 iv), the unproven part is whether (A) implies that $pr_1(W(\delta,\underline{m}))$ is an irreducible component of $H(d,g)$. If so, $pr_1(W(\delta,\underline{m}))$ is automatically non-reduced (3.1.10 iv).

We are for the time being not able to prove the conjecture, so we illustrate by considering examples. These leads to a partially proof of the conjecture, namely that the images $pr_1(W(\delta,\underline{m}))$ of the components $W(\delta,\underline{m}) \subseteq D(d,g;3)$ of "maximal genus under the condition (A)", see (3.2.2), are non-reduced irreducible components of $H(d,g)$. We end this section by a short discussion.

To begin with, we give bounds for the degree d and genus g for curves X which satisfy (A).

<u>Lemma 3.2.1</u>. Let $(X \subseteq Y \subseteq \mathbb{P}^3_k) \subseteq S(\delta,\underline{m})$ and suppose that X is a smooth connected curve of degree d and arithmetic genus g. Then the following conditions are equivalent

A) $\qquad 0 \neq h^1(\underline{I}_X(3)) \leq h^1(O_X(3))$

A') $\qquad d \geq 14, \quad 3d-18 \leq g \leq \left[\frac{d^2-4}{8}\right]$,

$\qquad H^0(\underline{I}_{X/Y}(3)) = 0$ and $H^1(\underline{I}_X(3)) \neq 0$

where $[m]$ denotes the greatest integer such that $[m] \leq m$.

<u>Proof</u>. Assume (A). Then $h^1(\underline{I}_X(3)) \neq 0$ implies that $s(X) \geq 2$ since the cone of a plane curve is Cohen Macaulay. Moreover we must have $s(X) \neq 2$ since the implication

$$H^1(\underline{I}_X(3)) \neq 0 \Rightarrow H^1(O_X(3)) = 0$$

is true for smooth connected curves on surfaces of degree 2. Indeed the discussion for $s = 2$ just before (3.1.8) reveals that if $s(X) = 2$, then X lies on a smooth quadric surface $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$ because $H^1(\underline{I}_X(3)) \neq 0$. We easily deduce $H^1(O_X(3)) = 0$ by Künneth's formula, see (1.3.10).

We now have $s(X) = 3$ and $e(X) \geq 3$. Using the theorem of speciality of Peskine-Gruson [G.P] which states that

$$e(X) + 4 \leq s(X) + \frac{d}{s(X)} ,$$

we deduce $d \geq 12$. It follows that

$$H^0(\underline{I}_{X/Y}(3)) = 0 \qquad \text{by } (3.1.10\,i) \text{ and}$$

$$g \geq 3d - 18 \qquad \text{by } (3.1.10\,iii) \text{ and } (A).$$

To prove

$$g \leq \left[\frac{d^2-4}{8}\right]$$

it will be sufficient to find the maximum of the genus g of curves of degree d satisfying $H^1(\underline{I}_X(3)) \neq 0$ which are contained in a smooth cubic surface. Consider the related problem of determining the largest genus g for which the degree d where $d \geq 12$ and the number m_6 are constants (under the relations of (3.1.2*)). Let therefore

$$g : \mathbb{R}^{\oplus 5} \to \mathbb{R} ,$$

$\mathbb{R}$ the real numbers, be defined by

$$g(x_1,\ldots,x_5) = \tfrac{1}{2}(\delta-1)(\delta-2) - \tfrac{1}{2}\sum_{i=1}^{5} x_i(x_i-1) - \tfrac{1}{2}\alpha(\alpha-1)$$

where

$$\delta = \tfrac{1}{3}\left(d + \sum_{i=1}^{5} x_i + \alpha\right)$$

and where α is a constant. Straightforward calculas shows that $g(x)$ has a maximum at the point $(x_1,\ldots,x_5)$ given by

$$x_1 = \ldots = x_5 = \tfrac{1}{4}(d+\alpha) .$$

Moreover

$$g(\underline{x})_{max} = \frac{1}{8}((d+\alpha-2)^2 + 4 - 4\alpha(\alpha-1)) .$$

Varying α between 0 and 2, we find that $\alpha = 2$ gives the largest $g(\underline{x})_{max}$. This number is

$$\frac{1}{8}(d^2-4) .$$

In view of (3.1.3) we find that if X is any curve of degree $d \geq 12$ satisfying $H^1(\underline{I}_X(3)) \neq 0$, then $m_6 \leq 2$ or $O_Y(X)$ corresponds to $(\delta+9,\delta+3,3,3,3,3,3)$ for some $\delta \geq 2$. If $m_6 \leq 2$, then

$$g \leq \left[\frac{d^2-4}{8}\right]$$

by the maximum of the function $g(x)$, and in the second case

$$g = \frac{1}{2}(5d-25).$$

Since $\frac{1}{2}(5d-25) \leq \frac{1}{8}(d^2-4)$ provided $d \geq 12$, we find

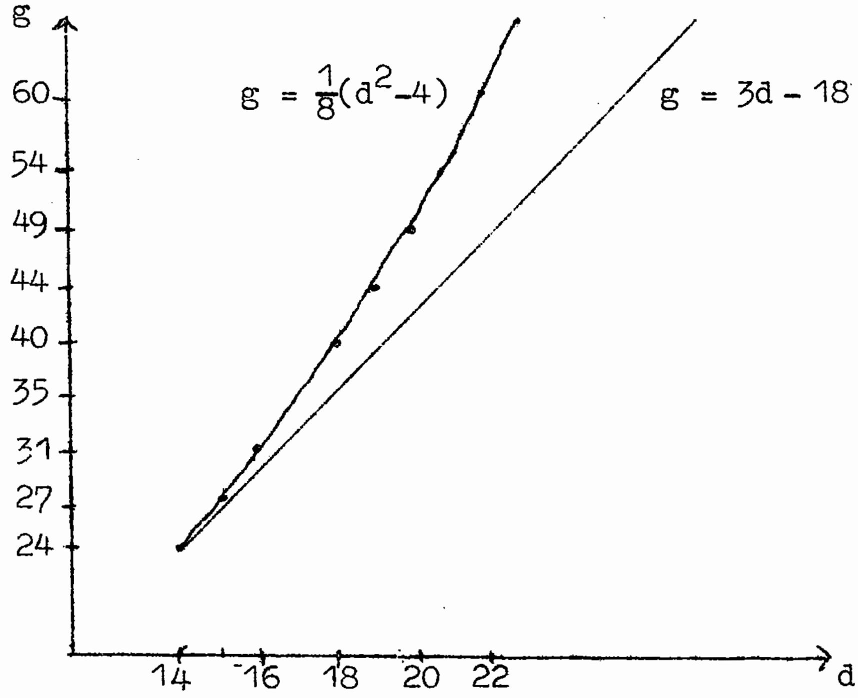
$$g \leq \left[\frac{d^2-4}{8}\right]$$

in both cases. Thus

$$3d - 18 \leq g \leq \left[\frac{d^2-4}{8}\right].$$

Finally solving the inequality $3d - 18 \leq \frac{1}{8}(d^2-4)$, we deduce $d \geq 14$, and (A') is proved. For the converse, we use (3.1.10 iii).

Geometric picture of $3d-18 \leq g \leq \frac{1}{8}(d^2-4)$.



In particular the g's which satisfy (A') for some given d's are:

$d = 14$: $g = 24$

$d = 15$: $g = 27$

$d = 16$: $30 \leq g \leq 31$

$d = 17$: $33 \leq g \leq 35$.

Remark 3.2.2. For every $d \geq 14$ the maximum $g = \left[\frac{d^2-4}{8}\right]$ is achieved by some smooth connected curve satisfying (A'). So there is a component $W(\delta,\underline{m}) \subseteq D(d,[\frac{1}{8}(d^2-4)];3)$ containing points $(X \subseteq Y \subseteq \mathbb{P})$ satisfying (A'). Letting

$$d = 4\alpha + r \quad \text{where} \quad \alpha \geq 4 \quad \text{and} \quad r = -2,-1,0,1,$$

we have the following four types of components $W(\delta,\underline{m})$ of "maximal genus under the condition (A)"

i) $r = -2$, $(\delta,\underline{m}) = (3\alpha,\alpha,\alpha,\alpha,\alpha,\alpha,2)$ and $g = \frac{1}{8}(d^2-4)$

ii) $r = -1$, $(\delta,\underline{m}) = (3\alpha,\alpha,\alpha,\alpha,\alpha,\alpha-1,2)$ and $g = \frac{1}{8}(d^2-9)$

iii) $r = 0$, $(\delta,\underline{m}) = (3\alpha+1,\alpha+1,\alpha,\alpha,\alpha,\alpha,2)$ and $g = \frac{1}{8}(d^2-8)$

iv) $r = 1$, $(\delta,\underline{m}) = (3\alpha+1,\alpha,\alpha,\alpha,\alpha,\alpha,2)$ and $g = \frac{1}{8}(d^2-9)$.

Next we consider the unproved part of the conjecture, i.e. the problem of determining whether $pr_1(W(\delta,\underline{m}))$ is an irreducible component of $H(d,g)$. For these considerations, classical in nature, we can as well suppose $H^0(\underline{I}_{X/Y}(3)) = 0$ in view of (3.2.1 A').

More generally fix an irreducible component $W \subseteq D(d,g;s)$ containing points where X is a divisor on Y and where $H^0(\underline{I}_{X/Y}(s)) = 0$, and let V be any irreducible component containing $pr_1(W)$. Using (1.3.2), (1.3.12) and (2.2.14) we get

$$a^1 - a^2_{res} \leq \dim W = \dim pr_1(W) \leq \dim V$$

where

$$a^1 - a^2_{res} = 4d + h^1(O_X(s)) - h^1(\underline{I}_X(s)) = (4-s)d + g - 2 + \binom{s+3}{3}$$

for $(X \subseteq Y \subseteq \mathbb{P})$ sufficiently general in W. Now suppose that $pr_1(W)$ is <u>not a component</u> of $H(d,g)$. Then we claim that

$$s(X_1) > s$$

for some point $(X_1 \subseteq \mathbb{P}) \in V$, i.e. that $H^0(\underline{I}_{X_1}(s)) = 0$. To see this, suppose $h^0(\underline{I}_{X_1}(s)) \geq 1$ for all $(X_1 \subseteq \mathbb{P}) \in V$. Then there is an irreducible component W' of $D(d,g;s)$ such that

$$pr_1(W') = V.$$

By assumption, $h^0(\underline{I}_{X/Y}(s)) = 0$, i.e. $h^0(\underline{I}_X(s)) = 1$ for some $(X \subseteq Y \subseteq \mathbb{P}) \in W$, so by semicontinuity, there is an open subset $U \subseteq W$

such that

$$h^{o}(\underline{I}_X(s)) = 1 \quad \text{for all} \quad (X \subseteq Y \subseteq \mathbb{P}) \in U.$$

It follows that there is only one surface Y of degree s containing X provided $(X \subseteq Y \subseteq \mathbb{P}) \in U$, and this combined with $pr_1(W') = V$ leads to the inclusion

$$U \subseteq W'$$

of subsets of $D(d,g;s)$. So $W \subseteq W'$ and since W is an irreducible component of $D(d,g;s)$, $W = W'$. We deduce

$$pr_1(W) = pr_1(W') = V$$

contradicting $pr_1(W) \subsetneq V$.

In particular if Y_1 is a surface of degree r containing the "generic point" $(X_1 \subseteq \mathbb{P})$ of V, then

$$r > s,$$

and we deduce easily

$$\dim V = \dim W(r) - h^{o}(\underline{I}_{X_1/Y_1}(r))$$

for some component $W(r) \subseteq D(d,g;r)$, see the discussion of (2.3.10 i). This leads to

<u>Lemma 3.2.3.</u> i) Let $W \subseteq D(d,g;s)$ be an irreducible component containing points $(X \subseteq Y \subseteq \mathbb{P})$ where X is a smooth connected curve and a divisor on Y and where

$$h^{o}(\underline{I}_X(s)) = 1.$$

Then $pr_1(W) \subseteq H(d,g)$ is an irreducible component provided the inequality

$$4d + h^1(O_X(s)) - h^1(\underline{I}_X(s)) \geq \dim V$$

holds for those components $V \subseteq H(d,g)_S$ whose"generic point" $(X_1 \subseteq \mathbb{P})$ satisfies

$$s(X_1) > s$$

$$h^i(\underline{I}_{X_1}(\nu)) \leq h^i(\underline{I}_X(\nu)) \qquad \text{for all } i \text{ and } \nu ,$$

$$h^i(\underline{N}_{X_1}(\nu)) \leq h^i(\underline{N}_X(\nu)) \qquad \text{for all } i \text{ and } \nu .$$

Moreover

$$4d + h^1(O_X(s)) - h^1(\underline{I}_X(s)) = (4-s)d + g - 2 + \binom{s+3}{3} .$$

ii) If all curves of V, V a component of $H(d,g)$, are contained in some surface of degree r, then there is a component $W(r) \subseteq D(d,g;r)$ satisfying $pr_1(W(r)) = V$ such that

$$\dim V = \dim W(r) - h^o(\underline{I}_{X_1/Y_1}(r))$$

where $(X_1 \subseteq Y_1 \subseteq \mathbb{P})$ is a sufficiently general point of $W(r)$.

iii) If V contains an open subset $U \subseteq V$ of curves $X_1 \subseteq \mathbb{P}$ which are contained in **global** complete intersections Y_1 of type (f_1,f_2), then

$$\dim V = \dim W(f_1,f_2) - \sum_{i=1}^{2} h^o(\underline{I}_{X_1/Y_1}(f_1))$$

where $W(f_1,f_2)$ is some component of $D(d,g;\underline{f})$ satisfying $pr_1(W(\underline{f})) = V$, and $(X_1 \subseteq Y_1 \subseteq \mathbb{P})$ is "generic" in $W(\underline{f})$.

To use lemma (3.2.3) for $s = 3$ and $W = W(\delta,\underline{m})$ in the situation of (A) or (A') to see that $pr_1(W(\delta,\underline{m}))$ is a component of $H(d,g)$, it will be sufficient to prove

$$4d + h^1(O_X(3)) - h^1(\underline{I}_X(3)) \geq h^o(\underline{N}_{X_1})$$

or equivalently

$$h^1(\underline{N}_{X_1}) \leq h^1(O_X(3)) - h^1(\underline{I}_X(3)) = -3d + g + 18$$

for any smooth connected curve $X_1 \subseteq \mathbb{P}$ satisfying

$$s(X_1) \geq 4 \quad \text{and} \quad h^i(\underline{I}_{X_1}(\nu)) \leq h^i(\underline{I}_X(\nu)), \qquad i \geq 0 \quad \text{and all } \nu.$$

We now give four examples of components $W(\delta,\underline{m})$ satisfying (A) where we prove $H^1(\underline{N}_{X_1}) = 0$.

Example 3.2.4. [M2]. Let $d = 14$ and $g = 24$. In view of (3.1.6 iii) there are two components of $D(14,24;3)$ of the form $W(\delta,\underline{m})$. The image $pr_1(W(\delta,\underline{m}))$ of the component $W(\delta,\underline{m})$ where $(\delta,\underline{m}) = (11,4,3,3,3,3,3)$ is a reduced irreducible component of $H(14,24)$ by (3.1.3) and (3.1.10 i). For $(\delta,\underline{m}) = (12,4,4,4,4,4,2)$ we claim that the image $pr_1(W(\delta,\underline{m}))$ is an irreducible component of $H(14,24)$, hence non-reduced.

To prove this, let $(X_1 \subseteq \mathbb{P}) \in H(14,24)_S$ be any curve satisfying $s(X_1) \geq 4$ and $e(X_1) \leq 3$. It will be sufficient to prove $H^1(\underline{N}_{X_1}) = 0$ by (3.2.3 i). By Riemann-Roch,

$$\chi(\underline{I}_{X_1}(\nu)) = \binom{\nu+3}{3} - (14\nu+1-24),$$

so $\chi(\underline{I}_{X_1}(3)) = 1$ and $\chi(\underline{I}_{X_1}(4)) = 2$ and we deduce

$$h^1(O_{X_1}(3)) \neq 0 \quad \text{and} \quad h^0(\underline{I}_{X_1}(4)) \geq 2.$$

If Y_1 is a global complete intersection of type (4.4) containing X_1, then the linked curve X_1' is a plane curve because

$$h^0(\underline{I}_{X_1'/Y_1}(1)) = h^1(O_{X_1}(3)) \neq 0$$

by (2.3.3). The cone of $X_1' \subseteq \mathbb{P}$ will therefore be Cohen Macaulay, hence the cone of $X_1 \subseteq \mathbb{P}$ will also be, again by (2.3.3). We deduce $H^1(\underline{N}_{X_1}) = 0$ by (2.2.9), and we are done. (Once having that the cone of $X_1 \subseteq \mathbb{P}$ is Cohen Macaulay, we conclude easily. Without using this, we give another partially independent proof to illustrate (3.2.3 ii). With notations as in (3.2.3) it will be sufficient to show $\dim V \leq d+g+18 = 56$. Since $h^0(\underline{I}_{X_1}(4)) \geq 2$, (3.2.3 ii) applies with $r = 4$, and we get

$$\dim V = \dim W(4) - h^0(\underline{I}_{X_1/Y_1}(4)) \leq \dim W(4) - 1 .$$

Moreover since X_1 is a divisor on Y_1 for some surface Y_1 of degree 4 (see [M 2] for a short proof),

$$\dim W(4) = g + 33 = 57 ,$$

see the discussion of (3.1.1), and we are done).

Example 3.2.5. Let $d = 15$ and $g = 27$ and observe that the image $pr_1(W(\delta,\underline{m}))$ of the components $W(\delta,\underline{m})$ corresponding to $(\delta,\underline{m}) = (11,3,3,3,3,3,3)$ and $(\delta,\underline{m}) = (12,5,4,3,3,3,3)$, see (3.1.6 iv), form reduced and irreducible components of $H(15,27)$ by (3.1.3) and (3.1.10 i). The final component to consider corresponds to $(12,4,4,4,4,3,2)$. We claim that the image og this component is an irreducible non-reduced component of $H(15,27)$.

To see this we apply (3.1.3) and (3.1.6 iv), and we get

$$h^1(\underline{I}_X(\nu)) = 0 \text{ for } \nu \notin \{3,4,5\},\ h^1(O_X(3)) = 1 \text{ and } e(X) = 3.$$

In view of (3.2.3 i), let $(X_1 \subseteq \mathbb{P}) \in H(15,27)_S$ be any curve satisfying

$$s(X_1) \geq 4 \quad \text{and} \quad h^i(\underline{I}_{X_1}(\nu)) \leq h^i(\underline{I}_X(\nu)) \quad \text{for} \quad i,\nu \in \mathbb{Z}$$

where $(X \subseteq \mathbb{P}) \in pr_1(S(\delta,\underline{m}))$. We need to prove $h^1(\underline{N}_{X_1}) = 0$. First we claim

$$h^1(\underline{I}_{X_1}(\nu)) = 0 \qquad \text{for} \quad \nu \neq 5.$$

In fact using Riemann-Roch's theorem, we get

$$\chi(\underline{I}_{X_1}(\nu)) = \binom{\nu+3}{3} - (15\nu+1-27),$$

so $\chi(\underline{I}_{X_1}(3)) = 1$ and $\chi(\underline{I}_{X_1}(4)) = 1$. Since $s(X_1) \geq 4$, $\chi(\underline{I}_{X_1}(3)) = -h^1(\underline{I}_{X_1}(3)) + h^1(\mathcal{O}_{X_1}(3)) = 1$, and since $h^1(\mathcal{O}_{X_1}(3)) \leq h^1(\mathcal{O}_X(3)) = 1$, it follows that

$$h^1(\underline{I}_{X_1}(3)) = 0.$$

Moreover since $\chi(\underline{I}_{X_1}(4)) = h^0(\underline{I}_{X_1}(4)) - h^1(\underline{I}_{X_1}(4)) = 1$, we find $h^0(\underline{I}_{X_1}(4)) \geq 1$, and we must in fact have

$$h^0(\underline{I}_{X_1}(4)) = 1$$

from which we deduce $h^1(\underline{I}_{X_1}(4)) = 0$. Indeed if $h^0(\underline{I}_{X_1}(4)) \geq 2$, then there is a global complete intersection of type (4,4) containing X_1. By (2.3.3) the linked curve has degree 1 and arithmetic genus -1, and such a curve does not exist. The claim follows therefore from $h^1(\underline{I}_X(\nu)) = 0$ for $\nu \notin \{3,4,5\}$.

Next since $\chi(\underline{I}_{X_1}(5)) = 7$ there is a global complete intersection Y_1 of type $(4,5)$ containing X_1. The linked curve X_1' is of degree 5 and genus 2 and satisfies

$$h^1(\underline{I}_{X_1'}(\nu)) = 0 \qquad \text{for } \nu \neq 0$$

by (2.3.3). Then we claim that the cone of $X_1' \subseteq \mathbb{P}$ is Cohen Macaulay. Suppose not. Since $\chi(\underline{I}_{X_i'}(2)) = 1$, we deduce $h^0(\underline{I}_{X_1'}(2)) = 1$. It follows that $\min n_{2i} \geq 4$ where the integers n_{ji} belong to the minimal reduction of $I_1' = \oplus H^0(\underline{I}_{X_1'}(\nu))$. So $\min n_{3i} \geq 5$ which contradicts

$$\max n_{3i} = c(X_1') + 4 = 4,$$

and the claim follows.

Finally, since the cone of $X_1' \subseteq \mathbb{P}$ is Cohen Macaulay, the cone of $X_1 \subseteq \mathbb{P}$ will also be by (2.3.3). We deduce $H^1(\underline{N}_{X_1}) = 0$ by (2.2.9) because $e(X_1) \leq 3$. (Using (2.2.9) we can prove $H^1(\underline{N}_{X_1}) = 0$ directly in an easier way without using liaison. Indeed we must also in this case prove $H^1(\underline{I}_{X_1}(3)) = 0$ and $h^0(\underline{I}_{X_1}(4)) = 1$. Then (2.2.9) applies without difficulties if we observe that $h^0(\underline{I}_{X_1}(4)) = 1$ implies $\min n_{2i} \geq 6$ with n_{ji} as in (2.2.9)).

Before giving the last examples we want to add a remark which we will use in the following and frequently in Section 3.3.

Remark 3.2.6. Let $R = k[X_0,\ldots,X_3]$, $m \subseteq R$ be the irrelevant maximal ideal, and let $X \subseteq \mathbb{P} = \mathbb{P}_k^3$ be a curve. Put $I = \oplus H^0(\underline{I}_X(\nu))$, and consider the graded resolution (2.1.6) of I which we now suppose is minimal. It is easy to spond to K, then up to sign we have

$$M_{K,i} = N_K F_i .$$

Example 3.2.8. Let $d = 16$ and $g = 31$, and there are two components $W(\delta,\underline{m})$ of $D(16,31;3)$ given by $(12,4,4,3,3,3,3)$ and $(13,5,4,4,4,4,2)$ by (3.1.6 v). The first one has an image in $H(16,31)$ which is a reduced irreducible component by (3.1.3) and (3.1.10 i). We claim that the image of the second one is an irreducible non-reduced component of $H(16,31)$.

To prove this, let $(X_1 \subseteq \mathbb{P}) \in H(16,31)_S$ be any curve satisfying $s(X_1) \geq 4$ and

$$h^i(\underline{I}_{X_1}(\nu)) \leq h^i(\underline{I}_X(\nu)) \qquad \text{for } i,\nu \in \mathbb{Z}$$

where $(X \subseteq \mathbb{P}) \in pr_1(S(\delta,\underline{m}))$. We must prove $h^1(\underline{N}_{X_1}) \leq 1$ by (3.2.3). First we prove

$$H^1(\underline{I}_{X_1}(\nu)) = 0 \qquad \text{for } \nu \notin \{5,6\}$$

in exactly the same way as we did in (3.2.5). Moreover since $\chi(\underline{I}_{X_1}(5)) \geq 6$ there is a global complete intersection Y_1 of type $(4,5)$ containing X_1, and the linked curve X_1' which is of degree 4 and genus 1 satisfies

$$H^1(\underline{I}_{X_1'}(\nu)) = 0 \qquad \text{for } \nu \notin \{1,0\} \text{ and } e(X_1') = 0$$

by (2.3.3). Then we claim that the minimal resolution of $I_1' = \oplus H^0(\underline{I}_{X_1'}(\nu))$ must be of the form

$$0 \to R(-4)^{\oplus y} \xrightarrow[N]{} R(-4)^{\oplus 1+y} \oplus R(-3)^{\oplus x} \to R(-3)^{\oplus x} \oplus R(-2)^{\oplus 2} \to I_1' \to 0$$

for some non-negative integers x and y. Indeed $c(X_1') \leq 0$

and $c(X_1') \leq e(X_1') = 0$ implies

$$\max n_{3i} = c(X_1') + 4 \leq 4 ,$$
$$\max n_{2i} = e(X_1') + 4 = 4 ,$$

see the discussion after (2.2.7). Moreover since the resolution is minimal,

$$\min n_{1i} < \min n_{2i} < \min n_{3i}$$

and combining with Riemann-Roch which implies

$$h^0(\underline{I}_{X_1'}(1)) = 0, \quad h^0(\underline{I}_{X_1'}(2)) = 2, \quad h^0(\underline{I}_{X_1'}(3)) = 8,$$

we find a resolution as required.

Suppose $y = 0$. Then the cone of $X_1' \subseteq \mathbb{P}$ and therefore the cone of $X_1 \subseteq \mathbb{P}$ will be Cohen Macaulay. We deduce $H^1(\underline{N}_{X_1}) = 0$ by (2.2.9) because $e(X_1) \leq 3$.

Suppose $y \geq 1$, and let $(0,\dots,0,H_1,\dots,H_x)$ be the transpose of say the first column-vector of N. By (3.2.6) it follows that $r((H_1,\dots,H_x)) = (X_0,\dots,X_3)$ which implies $x \geq 4$. This is impossible since if we consider the resolution of I_1' above, we observe that x is the number of the relations among the generators of I_1' of degree 2. Since there are just two such generators, $x \leq 1$.

Example 3.2.9. Let $d = 17$ and $g = 35$. We claim that $pr_1(W(\delta,\underline{m}))$ where $(\delta,\underline{m}) = (13,4,4,4,4,4,2)$ is an irreducible non-reduced component of $H(17,35)$. To see this, we observe that

$$e(X) = 3, \quad H^1(\underline{I}_X(\nu)) = 0 \quad \text{for} \quad \nu < 3$$

for $(X \subseteq \mathbb{P}) \in pr_1(S(\delta,\underline{m}))$ by (3.1.3). If $(X_1 \subseteq \mathbb{P}) \in H(17,35)_S$ satisfies

$$s(X_1) \geq 4 \text{ and } h^i(\underline{I}_{X_1}(\nu)) \leq h^i(\underline{I}_X(\nu)) \quad \text{for all } i,\nu \in \mathbb{Z},$$

then we easily prove that $H^1(\underline{I}_{X_1}(\nu)) = 0$ for $\nu < 5$, and that there is a global complete intersection Y_1 of type $(4,5)$ containing X_1 in exactly the same way as we did in (3.2.5). The linked curve X_1' which is of degree 3 and genus 0 satisfies

$$c(X_1') \leq 0 \quad \text{and} \quad e(X_1') \leq 0 ,$$

and arguing as in (3.2.8), we find that $I_1' = \oplus H^0(\underline{I}_{X_1'}(\nu))$ admits a resolution of the form

$$0 \to R(-4)^{\oplus y} \xrightarrow{N} R(-4)^{\oplus y} \oplus R(-3)^{\oplus 2+x} \xrightarrow{M} R(-3)^{\oplus x} \oplus R(-2)^{\oplus 3} \to I_1' \to 0.$$

Again if we can prove $y = 0$, we deduce $H^1(\underline{N}_{X_1}) = 0$ as in (3.2.8) and we are done. Assume therefore $y \geq 1$ and let

$$N = \begin{bmatrix} 0 \\ N' \end{bmatrix}$$

where 0 is the zero matrix and N' is a $(2+x) \times y$ matrix. N' induces a map

$$R(-4)^{\oplus p} \xrightarrow{N'} R(-3)^{\oplus q}$$

where $p = y$ and $q = 2+x$ and we have $I^0(N) = I^0(N')$, say equal to I. Since $\text{depth}_I R = 4$ by (3.2.6), we deduce $x \geq y+1$ by using the formula

$$\text{depth}_I R \leq (p-y+1)(q-y+1) ,$$

see [E.N]. Taking a non-vanishing y-minor N_K of N and one of the three generators F of I_1' of degree 2, then up to sign we have

$$M_{K'} = N_K \cdot F$$

for some minor $M_{K'}$ of M (3.2.7). Since there are matrices A, B, C, O where O is the zero matrix of size $x \times (2+x)$ such that

$$M = \begin{bmatrix} A & O \\ B & C \end{bmatrix},$$

we find, after throwing away y columns and one of the last three rows and taking the determinant, that $M_{K'} = 0$ because $x \geq y+1$. This gives a contradiction.

The four examples of non-reduced components we now have considered correspond to $\alpha = 4$ in (3.2.2). We claim that, extending the analysis of these examples, it will cover all the components of "maximal genus for a non-vanishing $H^1(\underline{I}_X(3))$" of (3.2.2). Hence

<u>Theorem 3.2.10.</u> The image $pr_1(W(\delta,\underline{m}))$ of any component $W(\delta,\underline{m})$ described in (3.2.2) is a non-reduced irreducible component of $H(d,g)$ of dimension $d+g+18$ where $g = \left[\frac{d^2-4}{8}\right]$.

<u>Proof.</u> Let $d = 4\alpha + r$ where $\alpha \geq 4$ and $r = -2,-1,0,1$. We have four types of components $W(\delta,\underline{m})$ described in (3.2.2), and the components which correspond to $\alpha = 4$ is already treated.

Start with $r = -2$, $d = 4\alpha - 2$ and the component $W(\delta,\underline{m})$ given by $(\delta,\underline{m}) = (3\alpha,\alpha,\alpha,\alpha,\alpha,\alpha,2)$. It follows that

$$g = \tfrac{1}{8}(d^2-4) = 2\alpha^2 - 2\alpha .$$

We will first, for $(X \subseteq Y \subseteq \mathbb{P}) \in S(\delta, \underline{m})$, prove that

$$h^1(O_X(\nu)) = \begin{cases} 2(\alpha-1-\nu)(\alpha-\nu)+1 & \text{for } 1 \leq \nu < \alpha \\ 0 & \text{for } \nu = \alpha . \end{cases}$$

Indeed by using (3.1.3 ii),

$$h^1(O_X(\nu)) = h^0(\underline{L}(-\nu-1))$$

where $\underline{L} = O_Y(X)$. As explained in (3.1.6 iii) we see that $\underline{L}(-\nu-1)$ corresponds to $(3n,n,n,n,n,n,1-\nu)$ for $n = \alpha-\nu-1$, and if $\underline{L}'$ corresponds to $(3n,n,n,n,n,n,0)$, then we clearly have

$$h^0(\underline{L}(-\nu-1)) = h^0(\underline{L}') \quad \text{for} \quad \nu \geq 1.$$

Moreover one knows that

$$h^0(\underline{L}'') = \binom{\delta+2}{2} - \sum_{i=1}^{6} \binom{m_i+1}{2}$$

provided $\underline{L}''$ corresponds to a tuple $(\delta, \underline{m})$ satisfying (3.1.2**). In fact combining the exact sequence of the proof of (3.1.3 ii) with Riemann-Roch, we get

$$h^0(\underline{L}'') = 1 + h^0(\omega_D(1)) = 1 - \chi(O_D(-1)) = d(D) + g(D)$$

provided D is a section of $\underline{L}''$ which we can consider as a reduced curve of degree $d(D)$ and arithmetic genus $g(D)$. The dimension formula for $h^0(\underline{L}'')$ follows easily from

$$d(D) = 3\delta - \Sigma m_i , \quad g(D) = \binom{\delta-1}{2} - \Sigma\binom{m_i}{2} .$$

Using this formula we get

$$h^0(\underline{L}') = \binom{3n+2}{2} - 5\binom{n+1}{2} = 2n^2 + 2n + 1 ,$$

and we deduce easily the required formula for $h^1(\mathcal{O}_X(\nu))$.

Next we claim that if $(X_1 \subseteq \mathbb{P}) \in H(d,g)_S$ is any curve satisfying $s(X_1) \geq 4$ and $h^i(\underline{I}_{X_1}(\nu)) \leq h^i(\underline{I}_X(\nu))$ for all i and ν, then there is a global complete intersection Y_1 of type $(4,\alpha)$ containing X_1. Indeed

$$h^1(\mathcal{O}_{X_1}(4)) \leq h^1(\mathcal{O}_X(4)) = \begin{cases} 2(\alpha-4)(\alpha-5)+1 = 2\alpha^2-18\alpha+41 & \text{for } \alpha>4 \\ 0 & \text{for } \alpha=4, \end{cases}$$

and since

$$\chi(\underline{I}_{X_1}(4)) = \binom{7}{3} - (4d+1-g) = 2\alpha^2 - 18\alpha + 42$$

we deduce

$$h^0(\underline{I}_{X_1}(4)) \geq \begin{cases} 1 & \text{for } \alpha > 4 \\ 2 & \text{for } \alpha = 4 \end{cases}$$

i.e. that there is a surface Z of degree 4 containing X_1. Now it will be sufficient to prove $h^0(\underline{I}_{X_1/Z}(\alpha)) > 0$. For this, we use the exactness of

$$0 \to \underline{I}_Z \to \underline{I}_{X_1} \to \underline{I}_{X_1/Z} \to 0$$

together with $\underline{I}_Z \simeq \mathcal{O}_{\mathbb{P}}(-4)$, and we deduce

$$h^0(\underline{I}_{X_1/Z}(\alpha)) = h^0(\underline{I}_{X_1}(\alpha)) - \binom{\alpha-1}{3}.$$

Finally we find that $h^0(\underline{I}_{X_1}(\alpha)) > \binom{\alpha-1}{3}$ because

$$h^0(\underline{I}_{X_1}(\alpha)) \geq \chi(\underline{I}_{X_1}(\alpha)) = \binom{\alpha+3}{3} - (\alpha d+1-g) > \binom{\alpha-1}{3},$$

where the first inequality follows from $h^1(\mathcal{O}_{X_1}(\alpha)) = 0$ and the second from

$$\binom{\alpha+3}{3} - \binom{\alpha-1}{3} = 2\alpha^2+2, \quad \alpha d + 1 - g = 2\alpha^2+1.$$

If we now follow the analysis for $\alpha = 4$ in (3.2.4), then it is straightforward to see that $pr_1(W(\delta,\underline{m}))$ is a component of $H(d,g)$. Indeed the linked curve X_1' of X_1 by Y_1 is a plane curve because the degree $d' = 2$ and the genus $g' = 0$, or because

$$h^0(\underline{I}_{X_1'/Y_1}(1)) = h^1(O_{X_1}(\alpha-1)) \neq 0$$

by (2.3.3). It follows that the cone of $X_1 \subseteq \mathbb{P}$ is Cohen Macaulay. Therefore any component $V \subseteq H(d,g)_S$ as in (3.2.3 i), i.e. with "generic point" $(X_1 \subseteq \mathbb{P})$, must have postulated dimension (2.3.15) by (2.2.9). **(By (2.2.12), V** is a reduced component.) Hence

$$\dim V = g + 33 - h^0(\underline{I}_{X_1/Z}(4))$$

by (2.3.17). Since $\dim pr_1(W(\delta,\underline{m})) \geq \dim V$ we deduce from (3.2.3 i) that $pr_1(W(\delta,\underline{m}))$ is a component, hence non-reduced.

Next let $r = -1$, $d = 4\alpha - 1$ and consider the component $W(\delta,\underline{m})$ given by $(\delta,\underline{m}) = (3\alpha,\alpha,\alpha,\alpha,\alpha,\alpha-1,2)$. In this case $g = \frac{1}{8}(d^2-9) = 2\alpha^2 - \alpha - 1$. If $(X \subseteq Y \subseteq \mathbb{P}) \in S(\delta,\underline{m})$, then we first prove

$$h^1(O_X(\nu)) = (\alpha-\nu)(2(\alpha-\nu)+1) \qquad \text{for } 1 \leq \alpha \leq \nu$$

in exactly the same way as for the case $r = -2$, $d = 4\alpha - 2$. Moreover if $(X_1 \subseteq \mathbb{P})$ is the "generic point" of a component $V \subseteq H(d,g)_S$ satisfying

$$s(X_1) \geq 4 \text{ and } h^i(\underline{I}_{X_1}(\nu)) \leq h^i(\underline{I}_X(\nu)) \qquad \textbf{for any } i \text{ and } \nu,$$

then we prove that X_1 is contained in a global complete intersection Y_1 of type $(4,\alpha+1)$ by the same proof as for the case $r = -2$. Furthermore we claim that

$$H^1(\underline{I}_{X_1}(\nu)) = 0 \qquad \text{for } \nu \leq \alpha .$$

To prove this, let Z be a surface of degree 4 containing X_1 and consider the exact sequence

$$0 \to \underline{I}_Z \to \underline{I}_{X_1} \to \underline{I}_{X_1/Z} \to 0$$

where $\underline{I}_Z \simeq O_{\mathbb{P}}(-4)$. We deduce

$$h^0(\underline{I}_{X_1}(\nu)) = h^0(O_{\mathbb{P}}(\nu-4)) + h^0(\underline{I}_{X_1/Z}(\nu)) = \binom{\nu-1}{3} \quad \text{for} \quad 1 \leq \nu \leq \alpha$$

since we easily prove $h^0(\underline{I}_{X_1/Z}(\nu)) = 0$ for $\nu \leq \alpha$ by liaison. This gives

$$h^0(\underline{I}_{X_1}(\nu)) + h^1(O_{X_1}(\nu)) \leq \binom{\nu-1}{3} + (\alpha-\nu)(2(\alpha-\nu)-1).$$

On the other hand we have the identities

$$\chi(\underline{I}_{X_1}(\nu)) = \binom{\nu+3}{3} - (d\nu+1-g) = \binom{\nu-1}{3} + (\alpha-\nu)(2(\alpha-\nu)-1),$$

the last equality is seen by using $d = 4\alpha - 1$, $g = 2a^2 - 2\alpha$ and $\binom{\nu+3}{3} - \binom{\nu-1}{3} = 2\nu^2 + 2$. Combining we get

$$H^1(\underline{I}_{X_1}(\nu)) = 0 \qquad \text{for} \quad 1 \leq \nu \leq \alpha$$

as required.

If we now follow the analysis for $\alpha = 4$ in (3.2.5), then we easily prove that $pr_1(W(\delta,\underline{m}))$ is a component of $H(d,g)$. Indeed the linked curve X_1' of X_1 by Y_1 is of degree 5 and genus 2 and satisfies $c(X_1') \leq 0$ by (2.3.3). Moreover

$$h^0(\underline{I}_{X_1'}(2)) = h^0(\underline{I}_{X_1'/Y_1}(2)) = h^1(O_{X_1}(\alpha-1)) = 1,$$

and by arguing as in (3.2.5), the cone of $X_1 \subseteq \mathbb{P}$ is Cohen Macaulay. We deduce that $pr_1(W(\delta,\underline{m}))$ is a component by the last part of the proof of the case $r = -2$.

The remaining cases $r = 0$ and $r = -1$ is treated in a similar way as the case $r = -1$, using the same proof. Indeed with $(X_1 \subseteq \mathbb{P}) \in V$ as in (3.2.3 i) we prove

1) $X_1 \subseteq Y_1$ for some global complete intersection Y_1 of type $(4, \alpha + 1)$.

2) $H^1(\underline{I}_{X_1}(\nu)) = 0$ for $\nu \leq \alpha$.

And as a byproduct of the proof of (2),

3) $H^0(\underline{I}_{X_1/Z}(\alpha - 1)) = 0$

where Z is the surface of degree 4 containing X_1. Then the linked curve X_1' must satisfy

$$c(X_1') \leq 0 \quad \text{and} \quad e(X_1') \leq 0 \ .$$

We prove this by combining (2) and (3) with (2.3.3). If $r = 1$, X_1' is of degree 3 and genus 0, and we have precisely the same situation as in (3.2.9) from which we deduced that the cone of $X_1 \subseteq \mathbb{P}$ was Cohen Macaulay. Moreover if $r = 0$, X_1' is of degree 4 and genus 1, and since the genus is positive, $e(X_1') = 0$. Again we have precisely the same numerical situation as in (3.2.8). The cone of $X_1 \subseteq \mathbb{P}$ is therefore Cohen Macaulay. It follows that

$$\dim V = g + 33$$

by (2.3.17), and $pr_1(W(\delta, \underline{m}))$ is a component by (3.2.3 i), hence non-reduced. The proof is now complete.

Recall the theorem on the majorization of the genus of curves of degree d stated in [G.P]. It says that the genus g of a smooth connected curve X_1 of degree d which is not contained in a surface of degree $<s$, is given by

$$g \leq g(s)_{max} = 1 + \frac{d}{2}(s + \frac{d}{s} - 4) - \frac{r(s-r)(s-1)}{2s}, \text{ provided } d > s(s-1),$$

where $0 \leq r < s$ and $d + r \equiv 0$ (modulo s). Moreover $g = g(s)_{max}$ iff X_1 is linked to a plane curve of degree r by a global complete intersection of type $(s, \frac{d+r}{s})$. Using this for $s = 4$ and $r = 2$, i.e. for $d = 4a - 2$ where $a \geq 4$, we find

$$g(4)_{max} = \frac{d^2-4}{8} .$$

Compare with (3.2.2 i). Now if $V \not\supseteq pr_1(W(3a,a,a,a,a,a,2))$ is any component with "generic point" $X_1 \subseteq \mathbb{P}$, then $s(X_1) > 3$ by (3.2.3 i), and by the theorem [G,P] above it follows that the cone of $X_1 \subseteq \mathbb{P}$ is Cohen Macaulay. As in the proof of (3.2.10), if Y_1 is a surface of degree 4, $Y_1 \supseteq X_1$, we easily deduce

$$\dim V = g + 33 - h^0(\underline{I}_{X_1/Y_1}(4)).$$

In view of (3.2.3 i) we thus have a simple proof of (3.2.10) for one of the four classes of components of $D(d,g;3)$ of "maximal genus under the condition (A)". However we have not been able to find such a simple proof for the other three classes, and the details of the proof of (3.2.10) seem therefore necessary at least for these cases.

Observe one more fact which follows from the theorem in [G.P]. Indeed solving the inequality

$$\left[\frac{d^2-4}{8}\right] > g(5)_{max} ,$$

we find $d \geq 22$. The curve $X_1 \subseteq \mathbb{P}$ appearing in the proof of

(3.2.10) is therefore contained in a surface of degree 4 provided $d \geq 22$ (confirming with what we proved in (3.2.10)). We think it should be "easy" to prove the conjecture for all components $W(\delta,\underline{m}) \subseteq D(d,g;3)$ satisfying (A) where $g > g(5)_{max}$. Indeed in this case, with V as in (3.2.3 i), $V = pr_1(W(4))$ for some component $W(4) \subseteq D(d,g;4)$, and we know that $W(4)$ has postulated dimension (2.3.17) by (3.1.1) in "most" cases. And if $W(4)$ has postulated dimension, then it is easy to prove the conjecture by using (3.2.3 i). Even if $\dim W(4) = g + 33 + \xi$ for some integer $\xi \leq d - 15$, (3.2.3 i) applies, and the conjecture follows. We have worked out the details for one more class of components, namely the components of the form $W(\delta,\underline{m}) = W(3\alpha+2,\alpha+1,\alpha+1,\alpha,\alpha,\alpha,2)$ for $\alpha \geq 4$. In this case if $V \supsetneqq pr_1(W(\delta,\underline{m}))$ is a component of $H(d,\frac{d^2-4}{8}-1)$ with "generic" point $X_1 \subseteq \mathbb{P}$, then $s(X_1) = 4$, and the analysis follows the lines of the proof of (3.2.10). We find that the conjecture holds for this class as well. (We can also prove the conjecture for the component $W(13,5,4,4,4,3,2) \subseteq D(17,34;3)$ where we also need to consider components $V \supsetneqq pr_1(W(\delta,\underline{m}))$ satisfying $s(X_1) = 5$, and this makes the computations more complicated. Compare with the example of section 3.3).

Therefore trying to produce counterexamples to the conjecture, one should perhaps consider components where g is not far from $3d - 18$. For instance if $g = 3d - 18$ and if we want to apply (3.2.3), we must prove $\dim V = 4d$ or $H^1(\underline{N}_{X_1}) = 0$. However, if $g = 3d - 18$, $e = e(X_1)$ is small because of $ed \leq 2g - 2$ which implies $e \leq 5$. In many cases $e = 3$, and at least for these cases $\dim V = 4d$ provided V has postulated dimension, see (2.3.15) and (2.3.17). So having confidence to the assertion that components usually have postulated dimension, it should not be easy to find counterexamples near the line $g = 3d - 18$ either. Moreover if e is 4 or 5, it looks like $s(X_1) > e(X_1)$ if $(X_1 \subseteq \mathbb{P})$ is a "generic curve". If so, and if V has postulated dimension, $\dim V = 4d$ for all the cases where $g = 3d - 18$. (In fact there is only a finite number of components $W(\delta,\underline{m})$ having $g = 3d - 18$ since $e(X) \geq 3$ implies $d \leq 90$, see the discussion at the end of Section 3.3) .

3.3. Singularities of codimension 1 of H(16,29).[1)]

In Section 3.2 we found non-reduced (i.e. generically non-smooth components of the Hilbert scheme $H(d,g)$ which were of the form $pr_1(W(\delta,\underline{m}))$ for $(\delta,\underline{m})$ suitably chosen. They satisfied

A) $0 \neq h^1(\underline{I}_X(3)) \leq h^1(O_X(3))$ for some $(X \subseteq Y \subseteq \mathbb{P}) \in S(\delta,\underline{m})$.

As mentioned in the discussion of (3.1.11), we might also in the case

B) $0 \neq h^1(O_X(3)) < h^1(\underline{I}_X(3))$ for some $(X \subseteq Y \subseteq \mathbb{P}) \in S(\delta,\underline{m})$,

expect that $pr_1(W(\delta,\underline{m}))$ consists of singular points of $H(d,g)$. Note that if X is a smooth connected curve, then (B) is equivalent to

B') $g < 3d - 18$, $H^0(\underline{I}_{X/Y}(3)) = 0$ and $H^1(O_X(3)) \neq 0$

by the first part of the proof of (3.2.1). We can establish

1) The characteristic of the field is zero.

some further inequalities for the degree and genus deduced from (B), see the discussion at the end of this section.

The following example deals with the Hilbert scheme $H = H(16,29)$. In fact we shall prove that

(3.3) there is an irreducible closed subset $Z \subseteq H(16,29)$ of dimension $4d-1 = 63$ of the form $pr_1(W(\delta,\underline{m}))$ where $(\delta,\underline{m}) = (12,4,4,4,4,2,2)$, consisting of singular points of $H = H(16,29)$. Moreover if V is any irreducible component containing Z, then V is a reduced component of dimension $4d = 64$, and a sufficiently general point $X_1 \subseteq \mathbb{P}$ of V satisfies

$$s(X_1) = 5 \quad , \quad e(X_1) = 2$$

$$h^1(\underline{I}_{X_1}(\nu)) = \begin{cases} 1 & \text{if } \nu = 4 \\ 0 & \text{if } \nu \neq 4. \end{cases}$$

Recall the question from the discussion of (3.1.11): Does there always exist a component $V \subseteq H(d,g)$ containing $pr_1(W(\delta,\underline{m}))$ for which

$$\dim V - \dim pr_1(W(\delta,\underline{m})) = [h^1(\underline{I}_X(3)) - h^1(O_X(3))]^+ ?$$

Since $h^1(\underline{I}_X(3)) - h^1(O_X(3)) = 3d-g-18 = 1$, we conclude that the answer is the affirmative in this case, not only for one component $V \subseteq H(16,29)$, but for any component $V \supseteq Z$.

Unfortunately we have not been able to prove completely the (at least for this example) expected assertion that there is only one component V of $H(16,29)$ which contains Z. If there are two or more components, then by the characterization of the "generic point" $X_1 \subseteq \mathbb{P}$ of V appearing in (3.3), it is impossible to distinguish their "generic points" by the dimension of the groups $H^i(\underline{I}_{X_1}(\nu))$ for any i and ν. Indeed if V_1 and V_2 are two components which contains Z with "generic points" $X_1 \subset \mathbb{P}$ and $X_2 \subseteq \mathbb{P}$ respectively, then

$$h^i(\underline{I}_{X_1}(\nu)) = h^i(\underline{I}_{X_2}(\nu)) \qquad \text{for any } i \text{ and } \nu.$$

It follows that the corresponding Hilbert functions $h^0(O_{X_1}(\nu))$ and $h^0(O_{X_2}(\nu))$ are the same. In view of (2.3.6) we will further motivate why we expect that Z is contained in a unique component $V \subseteq H(16,29)$. Indeed if V_1 and V_2 are different components containing Z, then there are different components V_1' and V_2' of $H(9,8)$ obtained by liaison. Moreover if $X_1' \subseteq \mathbb{P}$ and $X_2' \subseteq \mathbb{P}$ are the "generic points" of V_1' and V_2' respectively, then

$$h^i(\underline{I}_{X_1'}(\nu)) = h^i(\underline{I}_{X_2'}(\nu)) \qquad \text{for all } i \text{ and } \nu .$$

and the resolutions of $I_1' = \oplus H^0(\underline{I}_{X_1'}(\nu))$ and $I_2' = \oplus H^0(\underline{I}_{X_2'}(\nu))$ are numerically the same, see (2.2.10 i) and (2.3.8). Furthermore one may prove, by using [P.S.,(4.1)] as explained in (2.3.12), that V_1' and V_2' contain reduced curves. By further liaison, there are two components of $H(7,4)$ containing reduced curves as well. We certainly do not think $H(7,4)$, or equivalently $H(9,8)$, contains two such non-distinguishable components.

We shall prove 3.3 as follows. First with $(\delta,\underline{m}) = (12,4,4,4,4,2,2)$ we observe that if $(X \subseteq Y \subseteq \mathbb{P}) \in S(\delta,\underline{m})$ where X is a smooth connected curve, then $d = 16$ and $g = 29$ by (3.1.2*), $H^0(\underline{I}_{X/Y}(3)) = 0$ by (3.1.10 i) and $h^1(O_X(3)) = 1$, $e(X) = 3$

by (3.1.6 vi). Since $g < 3d - 18$, (B') holds. In fact, by (3.1.10 iii), since

$$h^1(\underline{I}_X(3)) - h^1(O_X(3)) = 3d - g - 18 = 1,$$

$h^1(\underline{I}_X(3)) = 2$, and $H = H(16,29)$ is singular along $pr_1(W(\delta,\underline{m})) = Z$ if there is an irreducible component $V \subseteq H$, $Z \subseteq V$ such that

$$\dim V - \dim Z = h^1(\underline{I}_X(3)) - h^1(O_X(3)) = 1.$$

We know that

$$\dim pr_1(W(\delta,\underline{m})) = \dim W(\delta,\underline{m}) = d + g + 18 = 63,$$

and it will therefore be sufficient to show $\dim V = 4d = 64$. Indeed if $V \subseteq H$ is <u>any</u> component which contains Z, and if $X_1 \subseteq \mathbb{P}$ is a sufficiently geheral point of V, then we will show

$$H^1(\underline{N}_{X_1}) = 0 .$$

It follows that V is a reduced component of dimension $4d$. Furthermore we can by the proof of (3.2.3) suppose $s(X_1) \geq 4$ in which case there is an irreducible component $W(r) \subseteq D(d,g;r)$ for $r = s(X_1) \geq 4$ such that $V = pr_1(W(r))$. Since $\chi(\underline{I}_{X_1}(5)) = 4$, $s(X_1) \leq 5$. The proof which now follows is long and technical, and we will therefore first give the ideas.

In 1) we discuss the case $s(X_1) = 5$, i.e. we study components V of H of the form $V = pr_1(W(5))$ which is not of the form $V = pr_1(W(4))$, and we divide into two cases

i) $e(X_1) \leq 2$

ii) $e(X_1) = 3$.

In (i) we quickly see $e(X_1) = 2$, and to show $H^1(\underline{N}_{X_1}) = 0$, we use (2.2.9) after having proved

$$h^1(\underline{I}_{X_1}(\nu)) = \begin{cases} 1 & \text{if} \quad \nu = 4 \\ 0 & \text{if} \quad \nu \neq 4 \end{cases}$$

by liaison and by using (3.2.6). In the case (ii) we first prove

$$h^1(\underline{I}_{X_1}(\nu)) = 0 \quad \text{for} \quad \nu \notin \{3,4\}$$

by liaison and (3.2.6). Then we use (2.3.6) which implies that if Y_1 is a global complete intersection of type (5,5) containing X_1, then the linked curve X_1' is a "generic point" of some component of $H(9,8)$. We then prove that this is impossible, and we have a contraction. We have by this proved that the family of curves given by (1,ii), if it exists, does not form a dense subset of any component V of $H(16,29)$, $V \supseteq Z$.

In 2) we analyse the case $V = pr_1(W(4))$, and we consider three subcases

i) $c(X_1) \leq 4$

ii) $c(X_1) > 4$ and $e(X_1) \leq 2$

iii) $c(X_1) > 4$ and $e(X_1) = 3$

For all three cases we prove that $\dim pr_1(W(4)) < 4d$ which implies that there are no component V of $H(16,29)$ which contains Z and satisfies $s(X_1) = 4$. This contradicts $V = pr_1(W(4))$.

1) As always $(X \subseteq \mathbb{P}) \in pr_1(S(\sigma(\underline{m})) \subseteq Z$ and $X_1 \subseteq \mathbb{P}$ is a sufficiently general point of some component V containing Z. Using (3.1.3) and (3.1.6 vi)

$$h^1(\underline{I}_X(\nu)) = 0 \ \text{for} \ \nu \notin \{3,4,5,6\}, \ h^1(O_X(3)) = 1 \ \text{and} \ e(X) = 3.$$

Combining with

$$H^0(\underline{I}_{X_1}(4)) = 0, \quad \chi(\underline{I}_{X_1}(2)) = 6, \quad \chi(\underline{I}_{X_1}(3)) = 0 \text{ and } \chi(\underline{I}_{X_1}(4)) = -1,$$

and for later use $\chi(\underline{I}_{X_1}(5)) = 4$, we deduce

$$H^1(\underline{I}_{X_1}(\nu)) = 0 \quad \text{for} \quad \nu \notin \{3,4,5,6\}$$

$$h^1(\underline{I}_{X_1}(3)) = h^1(O_{X_1}(3)) \leq 1$$

$$h^1(\underline{I}_{X_1}(4)) = 1 \quad \text{and} \quad e(X_1) \geq 2 .$$

1,i) Suppose $e(X_1) = 2$ and let Y_1 be a global complete intersection of type $(5,5)$ containing X_1. The linked curve $X_1' \hookrightarrow Y_1$ is of degree $d' = 9$ and $g' = 8$ and satisfies

$$h^0(\underline{I}_{X_1'/Y_1}(\nu)) = h^1(O_{X_1}(6-\nu)), \quad h^1(\underline{I}_{X_1'}(\nu)) = h^1(\underline{I}_{X_1}(6-\nu)) \quad \text{and}$$

$$h^1(O_{X_1'}(\nu)) = h^0(\underline{I}_{X_1/Y_1}(6-\nu)).$$

We deduce $s(X_1') = 4$, $c(X_1') = 2$, $e(X_1') = 1$, and knowing this we find by the arguing of (2.2.10 i) that the resolution of the ideal $I_1' = \oplus H^0(\underline{I}_{X_1'}(\nu))$ must be

$$0 \to R(-6) \xrightarrow{N} R(-5)^{\oplus 6} \longrightarrow R(-4)^{\oplus 6} \to I_1' \to 0 ,$$

where $R = k[X_0, X_1, X_2, X_3]$ is a polynomial ring. If the transpose of N is

$${}^t\mathbf{N} = [L_1, \ldots, L_6],$$

then by (3.2.6) and by the fact that $\deg L_i = 1$ for all i, we deduce

$$(X_0, \ldots, X_3) = r((L_1, \ldots, L_6)) = (L_1, \ldots, L_6).$$

It follows that the cokernel of

$$R_{1-\nu}^{\oplus 6} \xrightarrow{{}^t N} R_{2-\nu}$$

is zero except for $\nu = 2$, and by (3.2.6), this cokernel is precisely $H^1(\underline{I}_{X_1'}(\nu))^\vee$. Thus

$$0 = h^1(\underline{I}_{X_1'}(\nu)) = h^1(\underline{I}_{X_1}(6-\nu))$$

for $\nu \leq 1$ and $c(X_1) = 4$. We deduce $H^1(\underline{N}_{X_1}) = 0$ by (2.2.9).

1,ii) Suppose $h^1(\underline{I}_{X_1}(3)) = h^1(O_{X_1}(3)) = 1$. If $Y_1 = V(F_1, F_2) \supseteq X_1$ is a global complete intersection such that $\deg F_i = 5$ for $i = 1,2$ and if $X_1' \hookrightarrow Y_1$ is the linked curve, then $I_1' = \oplus H^0(\underline{I}_{X_1'}(\nu))$ has a resolution of the following form

$$0 \to R(-7) \oplus R(-6)^{\oplus y} \xrightarrow{N} R(-6)^{\oplus y+3} \oplus R(-5)^{\oplus x} \to R(-5)^{\oplus x} \oplus R(-4)^{\oplus 2} \oplus R(-3) \to I_1' \to 0$$

for some non-negative integers x and y. We deduce such a resolution from $s(X_1') = 3$, $c(X_1') = 3$ and $e(X_1') = 1$ and by computing $\chi(\underline{I}_{X_1}(\nu))$ for different ν's.

Suppose $y = 0$ and let ${}^tN = [L_1, L_2, L_3, S_1, \ldots, S_x]$ where $\deg L_i = 1$ for $i \in [1,3]$ and $\deg S_i = 2$ for all $i \in [1,x]$. Since

$$R_{2-\nu}^{\oplus 3} \oplus R_{1-\nu}^{\oplus x} \xrightarrow{{}^t N} R_{3-\nu} \to H^1(\underline{I}_{X_1'}(\nu))^\vee \to 0$$

is exact by (3.2.6) and since $h^1(\underline{I}_{X'}(2)) = 1$, it follows that the codimension of the vector space $(L_1, L_2, L_3)_1$ in R_1 is 1, where $(L_1, L_2, L_3)_1$ is graded piece of the ideal $(L_1, L_2, L_3) \subseteq R$ of degree 1. Moreover by (3.2.6)

$$r((L_1, L_2, L_3, S_1, \ldots, S_x)) = (X_o, X_1, X_2, X_3)$$

which proves that

$$(S_1,\dots,S_x)_2 \not\subseteq (L_1,L_2,L_3)_2$$

considered as subvector spaces of R_2. Since the k-vector space $(L_1,L_2,L_3)_2$ is 9 dimensional and R_2 is 10 dimensional, we deduce

$$(L_1,L_2,L_3)_2 + (S_1,\dots,S_x)_2 = R_2$$

which proves that $h^1(\underline{I}_{X_1'}(\nu)) = 0$ for $\nu \leq 1$, i.e. that $c(X_1) = 4$. Note that $h^1(\underline{I}_{X_1'}(2)) = 1$ implies $y \leq 1$. In fact since the sequence

$$(*) \qquad R_{2-\nu}^{\oplus y+3} \oplus R_{1-\nu}^{\oplus x} \xrightarrow{t_N} R_{3-\nu} \oplus R_{2-\nu}^{\oplus y} \to H^1(\underline{I}_{X_1'}(\nu))^{\vee} \to 0$$

is exact and since the composition

$$R_{2-\nu}^{\oplus y+3} \oplus R_{1-\nu}^{\oplus x} \xrightarrow{t_N} R_{3-\nu} \oplus R_{2-\nu}^{\oplus y} \xrightarrow{P} R_{2-\nu}^{\oplus y}$$

is the zero map for $\nu = 2$ where P is the projection onto its y last factors, it follows that

$$h^1(\underline{I}_{X_1'}(2)) \geq \dim_k R_0^{\oplus y} = y.$$

Suppose $y = 1$ and let

$$t_N = \begin{bmatrix} L_1 L_2 L_3 L_4 S_1 \dots S_x \\ 0\ 0\ 0\ 0\ H_1 \dots H_x \end{bmatrix}$$

where $\deg L_i = 1 = \deg H_i$ and $\deg S_i = 2$ for all i. Using $(*)$ for $y = 1$ and $\nu = 2$ we deduce $(L_1,L_2,L_3,L_4)_1 = R_1$. Moreover by (3.2.6) and the linearity of the H_i's,

$$(X_0,X_1,X_2,X_3) = r((H_1,\dots,H_x)) = (H_1,\dots,H_x)$$

from which the surjectivity of the morphism t_N for $1-\nu \geq 0$ is easily deduced. It follows that

$$0 = h^1(\underline{I}_{X_1'}(\nu)) = h^1(\underline{I}_{X_1}(6-\nu))$$

for $\nu \leq 1$, i.e. that $c(X_1) = 4$.

Now since $H^1(\underline{I}_{X_1}(5)) = 0$ and $H^1(\underline{I}_{X_1}(1)) = 0$, (2.3.6) applies. It follows that $X_1' \subseteq \mathbb{P}$ is a "generic point" of some component $V' \subseteq H(9,8)$, and since $s(X_1') = 3$, $V' = pr_1(W)$ for some irreducible component $W \subseteq D(9,8;3)$. Observe that it is not clear that we can use (3.1.10 ii) since we do not know whether the surface Y_1 of degree 3 which contains X_1' is non-singular. However we can apply (3.1.12 ii) because

$$\operatorname{coker} \alpha_{X_1' \subseteq Y_1} = 0 \quad \text{and} \quad \operatorname{coker} 1^2_{X_1' \subseteq Y_1} = 0$$

by (2.2.9). It follows that $pr_1(W)$ is not a component of $H(9,8)$ because $H^1(\underline{I}_{X_1'}(3)) \neq 0$. Hence X_1' is not "generic", and we have a contradiction (or we can use the discussion concerning the example (2.2.10 ii) appearing right before (2.2.13) to see that X_1' is not "generic").

2) Let V be an irreducible component of $H(16,29)$ containing Z, and suppose that $V = pr_1(W(4))$ for some component $W(4) \subseteq D(16,29;4)$. This time $h^0(\underline{I}_{X_1}(3)) = 0$ and $h^0(\underline{I}_{X_1}(4)) = 1$, so

$$h^1(\underline{I}_{X_1}(4)) = 2 \quad \text{and} \quad h^1(\underline{I}_{X_1}(3)) = h^1(\mathcal{O}_{X_1}(3)) \leq 1.$$

Moreover let Y_1 be a surface of degree 4 which contains X_1.

2,i) Suppose $c(X_1) \leq 4$. By the discussion just before (2.3.8),

$$\max n_{1i} \leq \max(c(X_1) + 2, e(X_1) + 3)) \leq 6,$$

and the conditions of (2.2.9) are satisfied. Thus

$$\operatorname{coker}\alpha_{X_1\subseteq Y_1} = 0 \quad \text{and} \quad \operatorname{coker} 1^2_{X_1\subseteq Y_1} = 0 .$$

Using (3.1.12 ii), we find

$$\dim \operatorname{pr}_1(W(4)) = \dim V - h^1(\underline{I}_{X_1}(4)),$$

contradicting $\operatorname{pr}_1(W(4)) = V$.

2,ii) Suppose $c(X_1) > 4$ and $e(X_1) \leq 2$. Since the surjectivity of

$$H^1(\underline{I}_{X_1}(\nu)) \otimes_k H^0(O_{\mathbb{P}}(1)) \to H^1(\underline{I}_{X_1}(\nu+1))$$

is implied by the surjectivity of

$$H^0(O_{X_1}(\nu)) \otimes_k H^0(O_{\mathbb{P}}(1)) \to H^0(O_{X_1}(\nu+1))$$

which is true by Castelnuovo 's lemma, if $H^1(O_{X_1}(\nu-1)) = 0$, i.e. if $\nu - 1 > e(X_1)$, see [M1, Lect 14], we deduce that if $H^1(\underline{I}_{X_1}(5)) = 0$, then $H^1(\underline{I}_{X_1}(\nu)) = 0$ for $\nu \geq 5$. Hence since $c(X_1) > 4$, it follows that $H^1(\underline{I}_{X_1}(5)) \neq 0$, and since $\chi(\underline{I}_{X_1}(5)) = 4$, $h^0(\underline{I}_{X_1}(5)) \geq 5$ which proves that there is a global complete intersection $Y_1' = V(F_1, F_2)$ containing X_1 such that $\deg F_1 = 4$ and $\deg F_2 = 5$. The linked curve $X_1' \leftrightarrow Y_1'$ is of degree $d' = 4$ and genus $g' = -1$ and satisfies $e(X_1') \leq 0$ and

$$h^1(\underline{I}_{X_1'}(\nu)) = h^1(\underline{I}_{X_1}(5-\nu)) = \begin{cases} 2 & \text{for } \nu = 1 \\ 0 & \text{for } \nu \notin \{-1,0,1\} \end{cases}$$

by (2.3.3). This gives $c(X_1') = 1 > e(X_1')$, so

$$c(X') + 4 = 5 = \max n_{3i} > \max n_{2i} > \max n_{1i}$$

where the integers n_{ji} belong to the resolution of $I_1' = \oplus H^0(\underline{I}_{X_1'}(\nu))$.

In particular using [P.S.,(4.1)] there is a global complete intersection $Y_1'' = V(G_1,G_2)$ containing X_1' such that $\deg G_i = 3$ for $i = 1,2$ and such that the linked curve X_1'' by Y_1'' is reduced and of degree $d'' = 5$ and genus $g'' = 0$. Moreover

$$H^1(\underline{I}_{X_1''}(\nu)) = 0 \quad \text{for} \quad \nu \notin \{1,2,3\}$$

and $h^1(\underline{I}_{X_1''}(1)) = 2$. Now a result of Castelnuovo says that $h^1(\underline{I}_{X_1''}(\nu))$ is decreasing for $\nu \geq [\frac{d''}{2}] - 1 = 1$ and strictly decreasing for $\nu > [\frac{d''}{2}] - 1 = 1$. We deduce

$$2 = h^1(\underline{I}_{X_1''}(1)) \geq h^1(\underline{I}_{X_1''}(2)) > h^1(\underline{I}_{X_1''}(3)).$$

And this is all we need; the general theory will now produce a contradiction. In fact since X_1'' is reduced and since $e(X_1'') < 1$, it follows that $H^1(\underline{N}_{X_1''}) = 0$. By the exact sequence of (1.3.1 C), see the discussion right before (2.2.14) for further details, we have a surjective map

$$\gamma : H^1(\underline{I}_{X_1''}(3))^{\oplus 2} \longrightarrow \operatorname{coker} \alpha_{X_1'' \subseteq Y_1''}$$

and this combined with (2.3.11) leads to

$$\dim \operatorname{coker} \alpha_{X_1'' \subseteq Y_1''} = \dim \operatorname{coker} \alpha_{X_1' \subseteq Y_1''} \leq 2.$$

Using $c(X_1') \leq 2$ and $e(X_1') \leq 2$ and the exact sequence of (1.3.1 C), we deduce that

$$\operatorname{coker} \alpha_{X_1' \subseteq Y_1''} \Rightarrow H^1(\underline{N}_{X_1'}) \Leftarrow \operatorname{coker} \alpha_{X_1' \subseteq Y_1'} .$$

Thus

$$\dim \operatorname{coker} \alpha_{X_1 \subseteq Y_1'} \leq 2.$$

Observing that there is an irreducible component $W(4,5) \subseteq D(16,29;4,5)$ such that $pr_1(W(4,5)) = V = pr_1(W(4))$ we get

$$\dim pr_1(W(4,5)) = \dim W(4,5) - h^0(\underline{I}_{X_1/Y_1'}(4)) - h^0(\underline{I}_{X_1/Y_1'}(5))$$

by (3.2.3 iii). Moreover if $A^1(\underline{d}, O_{\underline{d}})$ is the tangent space of $W(4,5)$ at $(X_1 \subseteq Y_1' \subseteq \mathbb{P})$, then by (2.2.14)

$$\dim A^1(\underline{d}, O_{\underline{d}}) - h^0(\underline{I}_{X_1/Y_1'}(4)) - h^0(\underline{I}_{X_1/Y_1'}(5)) =$$

$$4d - h^1(\underline{I}_{X_1}(4)) - h^1(\underline{I}_{X_1}(5)) + \dim \operatorname{coker} \alpha_{X_1 \subseteq Y_1'} \leq 4d - 1$$

which gives a contradiction.

2,iii) The final case is $c(X_1) > 4$ and $e(X_1) = 3$. As in (2,ii) we conclude that $H^1(\underline{I}_{X_1}(5)) \neq 0$, so there is a global complete intersection $Y_1' = V(F_1, F_2) \supseteq X_1$ such that $\deg F_1 = 4$ and $\deg F_2 = 5$. The linked curve $X_1' \Leftrightarrow Y_1'$ is of degree $d' = 4$ and genus $g' = -1$ and satisfies

$$h^1(\underline{I}_{X_1'}(2)) = 1,\ h^1(\underline{I}_{X_1'}(1)) = 2,\ H^1(\underline{I}_{X_1'}(\nu)) = 0 \quad \text{for} \quad \nu \notin \{-1,0,1,2\}.$$

Easy computations show that the resolution of $I_1' = \oplus H^0(\underline{I}_{X_1'}(\nu))$ must be of the form

$$0 \to R(-6) \oplus R(-5)^{\oplus y} \underset{N}{\to} R(-5)^{\oplus 2+y} \oplus R(-4)^{\oplus 1+x} \underset{M}{\to} R(-4)^{\oplus x} \oplus R(-3)^{\oplus 2} \oplus R(-2) \to I_1' \to 0.$$

Assume $y = 0$ and let ${}^tN = [L_1, L_2, S_1, \ldots, S_{1+x}]$ where $\deg L_i = 1$ and $\deg S_i = 2$ for all i. Since $h^1(\underline{I}_{X_1'}(1)) = 2$, the k-vector space $(L_1, L_2)_1 \subseteq R_1$ is of dimension 2, and since we know that $r((L_1, L_2, S_1, \ldots, S_{1+x})) = (X_0, X_1, X_2, X_3)$ by (3.2.6), $x \geq 1$. However if $x \geq 2$, the $(2+x)$-minor $M_{1,3+x}$, see (3.2.7) for notations, given as

$$\det \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \dots & 0 \\ 2 & 1 & \dots & 1 \\ 2 & 1 & \dots & 1 \end{bmatrix},$$

is zero (the numbers in the matrix above denote the degrees of the elements at that place, the elements of degree zero is however zero since the resolution is minimal). By (3.2.7) this is impossible. So $x = 1$ and since

$$r((L_1,L_2,S_1,S_2)) = (X_o,X_1,X_2,X_3)$$

it follows that $\{\overline{S}_1,\overline{S}_2\}$ form a regular sequence in the ring $R/(L_1,L_2)$. Hence

$$\dim_k(\overline{S}_1,\overline{S}_2)_2 = 2 \quad \text{and} \quad \dim_k(\overline{S}_1,\overline{S}_2)_3 = 4 .$$

This gives

$$h^1(\underline{I}_{X_1'}(-1)) = \dim_k R_3/(L_1,L_2,S_1,S_2)_3 = 0 \quad \text{and} \quad h^1(\underline{I}_{X_1'}) = 1.$$

Unfortunately the assumption $H^1(\underline{I}_{X_1'}(n_{1i}-4)) = 0$ of (2.2.9) is not satisfied. We used this assumption in the proof of (2.2.9) to conclude that ${}_o\mathrm{Ext}^2_R(I_1',I_1') = 0$. However combining (2.1.6) with $x = 1$, we find

$$\dim {}_o\mathrm{Ext}^1_R(I_1',I_1') \leq 1 .$$

Following the first part of the proof of (2.2.9), we find

$$h^1(\underline{N}_{X_1'}) \leq \dim {}_o\mathrm{Ext}^1_R(I_1',I_1') + \delta^2$$

because $c(X_1') < \min n_{2i}$. Thus $h^1(\underline{N}_{X_1'}) \leq 1$, and now the arguments at the end of (2,ii) apply and we get

$$\dim\operatorname{coker}\alpha_{X_1\subseteq Y_1'} = \dim\operatorname{coker}\alpha_{X_1'\subseteq Y_1'} = h^1(\underline{N}_{X_1'}) \leq 1,$$

and

$$\dim \mathrm{pr}_1(W(4)) = \dim \mathrm{pr}_1(W(4,5)) \leq 4d-2$$

which gives the contradiction.

Suppose $y \geq 1$ and let

$$t_N = \begin{bmatrix} L & S \\ 0 & N' \end{bmatrix}$$

where $L, S, 0$ and N' are matrices of size $1\times(2+y)$, $1\times(1+x)$, $y\times(2+y)$ and $y\times(1+x)$ respectively. Since $h^1(\underline{I}_{X_1'}(1)) = 2$, $L = [L_1,\ldots,L_{1+x}] \neq 0$, say $L_1 \neq 0$. Observe that the ideal $I^0(N)$ generated by the $(1+y)$-minors of N and the ideal generated by the y-minors of N' satisfy $I^0(N) \subseteq I^0(N')$ and also

$$\operatorname{depth}_{I^0(N')} R = 4$$

by (3.2.6). On the other hand it is well known that

$$\operatorname{depth}_{I^0(N')} R \leq (1+x-y+1)(y-y+1) = x-y+2,$$

see (3.2.9). Thus $x \geq y+2$. Now choose a non-vanishing y-minor $N'_{K'}$ of N', and let N_K be the $(1+y)$-minor given by

$$N_K = L_1 N'_{K'} .$$

If F is a generator of I_1' of degree 2, then

$$M_{K,x+3} = \pm N_K F .$$

However if we throw away the last row, the first column and y of the last $(1+x)$ columns of M, and take the determinant, we will see $M_{K,x+3} = 0$ because $x \geq y+2$ (the matrix M has too many entries which are zero). This gives a contradiction, i.e. curves X_1'

as above having $y \geq 1$ do not exist, and the proof of the claims of (3.3) is complete.

Without going into the details, we will mention that if $W(\delta,\underline{m})$ is any component of $D(d,g;3)$ which contains smooth connected curves such that

$$H^1(\underline{I}_X(3)) \neq 0 \quad \text{and} \quad H^1(O_X(3)) \neq 0$$

for some $(X \subseteq Y \subseteq \mathbb{P}) \in S(\delta,\underline{m})$, then

$$g \geq \frac{7}{2}(d-18)$$

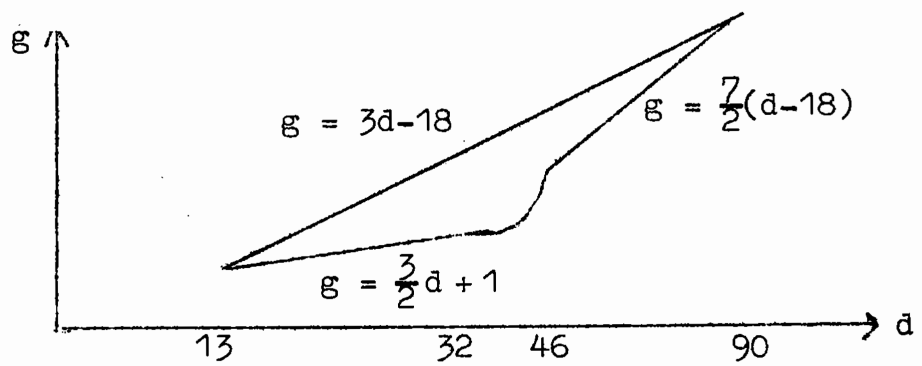
provided $d \geq 46$. There are a few exceptions to this lower bound in the range $33 \leq d \leq 45$,, and they satisfy

$$12 \leq \delta \leq 15 - \sum_{i=1}^{6} m_i .$$

Moreover since $H^1(O_X(3)) \neq 0$ implies $3d \leq 2g-2$, we also have

$$g \geq \frac{3}{2}d + 1 ,$$

and this gives a better bound provided $d \leq 32$. Using these inequalities, we find that if $W(\delta,\underline{m})$ is any component which contains smooth connected curves and which satisfies the condition (B), then the degree d and the genus g must belong to the closed region indicated by the following diagram



In fact the component $W(12,4,4,4,4,4,1) \subseteq D(15,25;3)$ is minimal under the condition (B) both with respect to the degree and the genus.

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1) Part II with Deligne, P. and Katz, N.