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THE HILBERT-FLAG SCHEME, ITS PROPERTIES AND ITS CONNECTION WITH THE HILBERT SCHEME. APPLICATIONS TO CURVES IN 3-SPACE

by

Jan O. Kleppe

Inst. of Math., University of Oslo

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Contents.

Introduction.

The purpose of this paper is to study the Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}^3)$, parametrizing curves in the projective 3-space with Hilbert polynomial $p(x) = dx+1-g$. A well known example of Mumford shows that $\mathrm{Hilb}^p(\mathbb{P}^3)$ may have some rather startling singularities. In fact for $d = 14$ and $g = 24$, one may find a non-reduced component of $\mathrm{Hilb}^p(\mathbb{P}^3)$ consisting generically of non-singular curves, sitting on surfaces of degree 3. This example of Mumford is the starting point of the research presented here.

In what follows k denotes an algebraically closed field, and $\mathbb{P}$ is any projective k-scheme. Let $\underline{p} =(p_1,\ldots,p_r)$ be an r-tuple of polynomials, $p_i \in \mathbb{Q}[x]$. We shall show that the "flags" of closed subschemes

$$X_1 \subseteq X_2 \subseteq \ldots \subseteq X_r \subseteq \mathbb{P}$$

of $\mathbb{P}$, X_i having Hilbert polynomial p_i, are parametrized by a projective k-scheme $D(\mathbb{P};\underline{p}) = D(\mathbb{P};p_1,\ldots,p_r)$. If $r = 1$, we clearly have

$$D(\mathbb{P};p) = \mathrm{Hilb}^p(\mathbb{P}).$$

We shall usually suppress mentioning the space $\mathbb{P}$ both in $D(\mathbb{P};\underline{p})$ and in $\mathrm{Hilb}^p(\mathbb{P})$. To simplify further, we sometimes write

$$D(\mathbb{P};\underline{p}) = D(\underline{p}) = D\,,$$

$$\mathrm{Hilb}^p(\mathbb{P}) = \mathrm{Hilb}^p = H(p) = H\,.$$

Consider the obvious projection morphisms

$$pr_i\colon D(\mathbb{P};\underline{p}) \to \mathrm{Hilb}^{p_i}(\mathbb{P}),$$

and pick a closed point $x = (X_1 \subseteq X_2 \subseteq \ldots \subseteq X_r \subseteq \mathbb{P})$. We may compute the tangent spaces of $D(\mathbb{P};\underline{p})$ at x and of $\mathrm{Hilb}^{p_i}(\mathbb{P})$ at $pr_i(x)$, together with the tangent maps p_i^1 at x. Let $\underline{d}^r$ be the category associated to a flag $x = (X_1 \subseteq X_2 \subseteq \ldots \subseteq \mathbb{P})$, the objects of which are the morphisms $X_i \hookrightarrow \mathbb{P}$. There are corresponding cohomology groups of algebras $A^q(\underline{d}^r, O_{\underline{d}^r})$ (see (1.2)) such that

i) $A^1(\underline{d}^r, O_{\underline{d}^r})$ is the tangent space of $D(\mathbb{P};\underline{p})$ at x.

ii) If $r = 1$ and $x = (X \subseteq \mathbb{P})$, then $A^1(\underline{d}^1, O_{\underline{d}^1}) = H^0(X, \underline{N}_X)$ where $\underline{N}_X$ is the normal bundle of $X \hookrightarrow \mathbb{P}$. If we by definition let

$$A^2(h, O_X) = A^2(\underline{d}^1, O_{\underline{d}^1}),$$

then $A^2(h, O_X) = H^1(X, \underline{N}_X)$ provided $h : X \hookrightarrow \mathbb{P}$ is locally a complete intersection.

iii) If $a^q = \dim_k A^q(\underline{d}^r, O_{\underline{d}^r})$, then

$$a^1 - a^2 \leq \dim O_{D,x} \leq a^1.$$

Moreover $D = D(\mathbb{P}, \underline{p})$ is non-singular at x iff (if and only if) $\dim O_{D,x} = a^1$.

Now restrict to the case $r = 2$. Let $\underline{d} = \underline{d}^2$, $x = (X \overset{f}{\hookrightarrow} Y \overset{g}{\hookrightarrow} \mathbb{P})$, and suppose that $g : Y \subseteq \mathbb{P}$ is locally a complete intersection. We shall have the following two situations in mind

a) $Y = V(F)$ is a surface of degree $s = \deg F$ in $\mathbb{P} = \mathbb{P}_k^3$ with Hilbert polynomial p_2, and X is a curve on Y of degree d and genus g with Hilbert polynomial p_1. Put

$$D(d,g;s) = D(\mathbb{P}^3; p_1, p_2) \text{ and } H(d,g) = \mathrm{Hilb}^{p_1}(\mathbb{P}^3).$$

b) Y is a global complete intersection in $\mathbb{P}_k^3$ of two surfaces of degree f_1 and f_2 and $X \subseteq Y$ is a curve.

An explicite description of $A^q(\underline{d},O_{\underline{d}})$ for $q = 1,2$ may be given as follows. Let $\underline{I}_X,\underline{I}_Y$ be the sheaves of ideals which define $X \hookrightarrow \mathbb{P}$ and $Y \hookrightarrow \mathbb{P}$ respectively, and let $\underline{N}_X = \underline{\mathrm{Hom}}_{O_{\mathbb{P}}}(\underline{I}_X,O_X)$, $\underline{N}_Y = \underline{\mathrm{Hom}}_{O_{\mathbb{P}}}(\underline{I}_Y,O_Y)$ be the normal bundles.

Corresponding to the diagram of projections

$$\begin{tikzcd} D(p_1,p_2) \arrow[r, "pr_2"] \arrow[d, dashed, "pr_1"'] & \mathrm{Hilb}^{p_2} \\ \mathrm{Hilb}^{p_1} & \end{tikzcd}$$

there is, on the tangent space level at $x = (X \subseteq Y \subseteq \mathbb{P})$, a cartesian diagram

$$\begin{tikzcd} A^1(\underline{d},O_{\underline{d}}) \arrow[r, "p_2^1"] \arrow[d, "p_1^1"'] \arrow[dr, phantom, "\square"] & H^o(Y,\underline{N}_Y) \arrow[d, "m^1"] \\ H^o(X,\underline{N}_X) \arrow[r, "l^1"'] & H^o(X,f^*\underline{N}_Y) \end{tikzcd}$$

This means that $A^1(\underline{d},O_{\underline{d}}) = \ker(l^1,m^1)$. Here l^1 and

$$m^i : H^{i-1}(Y,\underline{N}_Y) \to H^{i-1}(X,f^*\underline{N}_Y)$$

are deduced from the natural morphisms $\underline{N}_X \to f^*\underline{N}_Y$ and $\underline{N}_Y \to f_*f^*\underline{N}_Y$ respectively. Moreover let $\underline{N}_{X/Y}$ be the normal bundle of $X \hookrightarrow Y$. Essentially from the exact sequence

$$0 \to \underline{N}_{X/Y} \to \underline{N}_X \to f^*\underline{N}_Y$$

we deduce a long exact sequence

$$0 \to H^o(X,\underline{N}_{X/Y}) \to H^o(X,\underline{N}_X) \xrightarrow{l^1} H^o(X,f^*\underline{N}_Y) \xrightarrow{\delta^1}$$

$$A^2(f,O_X) \to A^2(gf,O_X) \xrightarrow{l^2} H^1(X,f^*\underline{N}_Y) \to$$

Let

$$\alpha = \alpha_{X \subseteq Y} : H^0(Y, \underline{N}_Y) \to A^2(f, O_X)$$

be defined by $\alpha = \delta^1 \circ m^1$ and let

$$\gamma = \gamma_{X \subseteq Y} : H^0(X, \underline{N}_X) \to \operatorname{coker} m^1$$

be the composition of l^1 with the natural map $H^0(X, f^* \underline{N}_Y) \twoheadrightarrow \operatorname{coker} m^1$. Then there is a diagram of exact horizontal sequences (see (1.3))

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker}\gamma \arrow[r] \arrow[dr, phantom, "\circ"] & A^2(\underline{d}, O_{\underline{d}}) \arrow[r] \arrow[dr, phantom, "\circ"] & \ker(l^2, m^2) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{coker}\gamma \arrow[r] \arrow[u, equal] & \operatorname{coker}\alpha \arrow[r] \arrow[u, hook] & \ker l^2 \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

In the first chapter we prove the following results

(1.3.2): If $\mathrm{Hilb}^{p_2}(\mathbb{P})$ is non-singular at $pr_2(x) = (Y \subseteq \mathbb{P})$, then

(1) $$a^1 - \dim_k \operatorname{coker}\alpha \leq \dim O_{D,x} \leq a^1$$

In particular if α is surjective, $D = D(\mathbb{P}; p_1, p_2)$ is non-singular at $x = (X \subseteq Y \subseteq \mathbb{P})$.

(1.3.4): Let $\mathrm{Hilb}^{p_2}(\mathbb{P})$ be non-singular at $pr_2(x)$. If $m^1 : H^0(Y, \underline{N}_Y) \to H^0(X, f^* \underline{N}_Y)$ is surjective, then

$$pr_1 : D(\mathbb{P}; p_1, p_2) \to \mathrm{Hilb}^{p_1}(\mathbb{P})$$

is smooth at $x = (X \subseteq Y \subseteq \mathbb{P})$. If in addition γ is surjective, the converse is true.

(1.3.6): If m^1 is surjective and if $(Y \subseteq \mathbb{P}) \in \mathrm{Hilb}^{p_2}(\mathbb{P})$ is non-singular, the results above imply that

$$\operatorname{coker}\alpha = \ker l^2 \subseteq A^2(gf, O_X)$$

and moreover that

(2) $$h^o(\underline{N}_X) - \dim_k \operatorname{coker}\alpha \leq \dim O_{H(p_1),pr_1(x)} \leq h^o(\underline{N}_X)$$

where $h^o(\underline{N}_X) = \dim_k H^o(X,\underline{N}_X)$. In particular if 1^2 is injective, $H(p_1) = \mathrm{Hilb}^{p_1}(\mathbb{P})$ is non-singular at $pr_1(x) = (X \subseteq \mathbb{P})$.

Now $\mathrm{Hilb}^{p_2}(\mathbb{P})$ is non-singular at $(Y \subseteq \mathbb{P})$ if Y is as in (a) or in (b), and m^1 is surjective iff

$H^1(\mathbb{P}, \underline{I}_X(s)) = 0$ in the case (a) and

$H^1(\mathbb{P}, \underline{I}_X(f_i)) = 0$ for $i = 1,2$ in case (b).

Using the theory of deformations of graded algebras we can improve upon the inequalities (1) and (2), at least in some special cases. In fact let $X \hookrightarrow Y \hookrightarrow \mathbb{P}$ correspond to surjections $R \twoheadrightarrow B \twoheadrightarrow A$ of graded k-algebras, see (1.4.1), and suppose that $\dim Y \geq 1$. Let $I_B = (F_1,\ldots,F_r) \subseteq R$ be a graded ideal such that $R/I_B = B$, and suppose $\{F_1,F_2,\ldots,F_r\}$ is an R-regular sequence of homogeneous elements. Let $I = (F_1,\ldots,F_r, F_{r+1},\ldots,F_t)$ be a graded ideal such that $R/I = A$ and let $f_i = \deg F_i$ for $i = 1,\ldots,t$. The algebra cohomology $H^i(R,A,A)$ associated to the canonical morphism $R \to A$, are graded A-modules; the submodule of $H^i(R,A,A)$ of elements of degree zero is denoted by ${}_oH^i(R,A,A)$.

(1.4.6): If $H^1(\mathbb{P}, \underline{I}_X(f_i)) = 0$ for $i = r+1,\ldots,t$, then there is an injective map

$$ {}_oH^2(R,A,A) \hookrightarrow \operatorname{coker}\alpha $$

and

(3) $$a^1 - \dim_k {}_oH^2(R,A,A) \leq \dim O_{D,x} \leq a^1.$$

In particular if ${}_oH^2(R,A,A) = 0$, then $D = (\mathbb{P};p_1,p_2)$ is non-singular at $x = (X \subseteq Y \subseteq \mathbb{P})$.

(1.4.8): If $H^1(\mathbb{P}, \underline{I}_X(f_i)) = 0$ for all $i = 1,2,\ldots,t$, then

$$ {}_oH^2(R,A,A) \subseteq \operatorname{coker}\alpha \subseteq A^2(gf, O_X) $$

and

(4) $$ h^o(\underline{N}_X) - \dim_k {}_oH^2(R,A,A) \leq \dim O_{H(p_1),pr_1(x)} \leq h^o(\underline{N}_X). $$

Therefore if ${}_oH^2(R,A,A) = 0$, then $H(p_1) = \mathrm{Hilb}^{p_1}(\mathbb{P})$ is non-singular at $pr_1(x) = (X \subseteq \mathbb{P})$.

Now, to apply these results, we should like to know how to compute $\operatorname{coker}\alpha$ or ${}_oH^2(R,A,A)$.

First we describe ${}_oH^2(R,A,A)$. Let X be locally Cohen Macaulay and equidimensional, and let $X \hookrightarrow \mathbb{P} = \mathbb{P}^n$ be generically a complete intersection of codimension 2. Then there is an isomorphism of graded A-modules

$$ H^i(R,A,A) \simeq \mathrm{Ext}^i_R(I,I) \qquad \text{for } i = 1,2 $$

where $I = \ker(R \twoheadrightarrow A)$. See (2.2). To compute $\mathrm{Ext}^2_R(I,I)$ we establish the following duality theorem. Suppose R is of dimension $n+1$, and let M and N be graded R-modules of finite type and of finite projective dimension. Let $\mathrm{Ext}^i_m(M,-)$ be the right derived functor of $\Gamma_m \circ \mathrm{Hom}_R(M,-)$ where Γ_m means sections with support in the irrelevant maximal ideal $m \subseteq R$. If R is Gorenstein, there is an integer p given by the dualizing sheaf $\omega_{\mathbb{P}}$ of $\mathbb{P} = \mathrm{Proj}(R)$ such that $\omega_{\mathbb{P}} = O_{\mathbb{P}}(-p)$. By (2.1.5) there is a perfect pairing

$$ {}_\nu\mathrm{Ext}^{i+1}_m(M,N) \times {}_{-\nu-p}\mathrm{Ext}^{n-i}_R(N,M) \to k . $$

This theorem modulo obvious modifications is also valid if (R,m) is a local Gorenstein ring. If $M = R$ we obtain the usual Gorenstein duality theorem under the restriction "N of finite projective dimension".

Now if $X = \mathrm{Proj}(A) \hookrightarrow \mathbb{P} = \mathbb{P}^3 = \mathrm{Proj}(R)$ is a curve such that ${}_oH^2(R,A,A) \simeq {}_o\mathrm{Ext}^2_R(I,I)$, we deduce by the duality theorem above that

$$ {}_oH^2(R,A,A)^\vee \simeq {}_{-4}\mathrm{Hom}_R(I,H^2_m(I)) \subseteq \bigoplus_{i=1}^{t} H^1(\underline{I}_X(f_i-4)) . $$

where $H^1(\underline{I}_X(\nu)) = H^1(\mathbb{P}, \underline{I}_X(\nu))$. And if ${}_oH^2(R,A,A) = 0$, we can find the dimension of $H^1(\underline{N}_X)$ under some additional requirements (2.2.9). Moreover if we combine the vanishing of ${}_oH^2(R,A,A)$ with (3) or (4), we deduce criterions for the non-singularity of $D(p_1,p_2)$ and $H(p_1)$ at $x = (X \subseteq Y \subseteq \mathbb{P})$ and $(X \subseteq \mathbb{P})$ respectively, in which case we also find $\dim O_{D,x}$ and $\dim O_{H,pr_1(x)}$. Finally if we can compute $\dim {}_oH^2(R,A,A)$ for a sufficiently general curve $X \subseteq \mathbb{P}$ of a reduced component $V \subseteq H(p_1)$, then $\dim V$ is found in (2.2.13).

For the case (b), let X be Cohen Macaulay and equidimensional. Since Y is a global complete intersection of dimension 1 which contains X, there is a linked curve $X' \subseteq Y$ defined by $\underline{I}_{X'}/\underline{I}_Y = \underline{\mathrm{Hom}}_{O_Y}(O_X,O_Y)$. Then $x' = (X' \subseteq Y \subseteq \mathbb{P}^3) \in D(p_1',p_2) = D'$. If $X \hookrightarrow Y$ is generically an isomorphism, i.e. if the linkage is geometric, we prove in (2.3.11) that

(5) $$\dim \operatorname{coker} \alpha_{X \subseteq Y} = \dim \operatorname{coker} \alpha_{X' \subseteq Y}.$$

This result enables us to compute $\dim \operatorname{coker} \alpha_{X \subseteq Y}$ provided we can by linkage obtain a simpler situation. In fact (5) is an easy consequence of the isomorphism

$$D(p_1;f_1,f_2)_{CM} \simeq D(p_1';f_1,f_2)_{CM} ,$$

see (2.3) for notations. In particular $O_{D,x} \simeq O_{D',x'}$, valid also for algebraic linkage. Combining with (1.3.4) we have the following result.

(2.3.6): Suppose

$$H^1(\underline{I}_X(f_i)) = 0 \quad \text{for } i = 1,2, \text{ and}$$

$$H^1(\underline{I}_X(f_i-4)) = 0 \quad \text{for } i = 1,2.$$

Then $H(p_1')$ is non-singular at $(X' \subseteq \mathbb{P}^3)$ iff $H(p_1)$ is non-singular at $(X \subseteq \mathbb{P}^3)$. Moreover $(X' \subseteq \mathbb{P}^3)$ is a "generic point" of a non-embedded component of $H(p_1')$ iff $(X \subseteq \mathbb{P}^3)$ is a "generic point" of a non-embedded component of $H(p_1)$.

More generally (2.3.10) if $V \subseteq H(p_1)$ is an irreducible component, there is a "linked" irreducible closed subset $V' \subseteq H(p_1')$ which is a component of $H(p_1')$ provided $H^1(\underline{I}_X(f_i-4)) = 0$ for $i = 1,2$. Moreover under this condition, if V is a reduced component, then so is V'. Finally we also introduce the notion of expected or postulated dimension of a reduced component V of $H(p_1)$ (2.3.15). Under the conditions of (2.3.6), $\dim V$ is equal to the postulated dimension iff $\dim V'$ is, provided the linkage is geometric.

In the case (a) we shall assume that X is a local complete intersection in $Y = V(F)$. If $s = \deg F \leq 3$ and if X is reduced, we easily prove that $\operatorname{coker} \alpha = 0$ and therefore that D is non-singular at $x = (X \subseteq Y \subseteq \mathbb{P}^3)$. Let W be a non-embedded irreducible component of $D = D(d,g;s)$ which contains x. Then W is reduced and

$$\dim W = \begin{cases} 2d+g+8 & \text{if } s = 2, \\ d+g+18 & \text{if } s = 3, \end{cases}$$

If $s = 4$ and if X is a smooth connected curve, then

$$\dim W = g + 33$$

provided W does not contain any closed point $(X_1 \subseteq Y_1 \subseteq \mathbb{P}^3)$ where X_1 is a global complete intersection in Y_1. If W does, we find

$$\dim W = g + 34 .$$

See (3,1). These dimension formulas are found in Noethers fundamental paper on space curves [N].

Moreover let $H(d,g)_S$ be the open subscheme of $H(d,g)$ consisting of smooth connected curves, and let $D(d,g;s)_S = pr_1^{-1}(H(d,g)_S)$. Then we prove that $D(d,g;2)_S$ a smooth connected scheme, and if we combine with (1.3.4), we find that $pr_1(D(d,g;2)_S)$ is a <u>smooth</u>[1)] <u>connected component</u> of $H(d,g)_S$ provided $g \neq 0$ and $g \neq d-3$. Furthermore we describe in (3.1.4) all irreducible components W of $D(d,g;3)_S$ which contain points $(X \subseteq Y \subseteq \mathbb{P}^3)$ where X is a divisor on Y. Indeed if $d > 2$, there is a one-to-one correspondence between components as above and tuples $(\delta,\underline{m}) = (\delta,m_1,\ldots,m_6) \in Z^{\oplus 7}$ satisfying

$$\delta \geq m_1 + m_2 + m_3 \ , \quad \delta > m_1 \geq m_2 \geq \ldots \geq m_6 \geq 0,$$

$$d = 3\delta - \sum_{i=1}^{6} m_i \quad \text{and} \quad g = \binom{\delta-1}{2} - \sum_{i=1}^{6} \binom{m_i}{2} .$$

Now if

$$H^1(\underline{I}_X(3)) = 0 \quad \text{for some} \quad (X \subseteq Y \subseteq \mathbb{P}^3) \in W,$$

1) i.e. $H(d,g)_S$ is smooth along $pr_1(D(d,g;2)_S)$

we deduce by (1.3.4) that $pr_1(W)$ is a reduced [1] irreducible component of $H(d,g)_S$. And if

$$H^1(O_X(3)) = 0 \quad \text{for some} \quad (X \subseteq Y \subseteq \mathbb{P}^3) \in W,$$

then $H(d,g)_S$ is generically non-singular along $pr_1(W)$, and the codimension of $pr_1(W)$ in $H(d,g)_S$ is $h^1(\underline{I}_X(3))$. We divide the remaining cases into

A) $0 \neq h^1(\underline{I}_X(3)) \leq h^1(O_X(3))$ for some $(X \subseteq Y \subseteq \mathbb{P}) \in W$,

B) $0 \neq h^1(O_X(3)) < h^1(\underline{I}_X(3))$ for some $(X \subseteq Y \subseteq \mathbb{P}) \in W$.

In both cases the tangent map p_1^1 of

$$pr_1 : D(d,g;s)_S \to H(d,g)_S$$

is not surjective. We deduce that if $pr_1(W)$ is an irreducible component of $H(d,g)_S$, it is necessarily non-reduced. Moreover we easily prove that if $pr_1(W)$ is an irreducible non-reduced component of $H(d,g)_S$, then (A) holds. We conjecture the converse, and Section 3.2 is devoted to a study of the conjecture. In fact we prove the conjecture for all components W of $D(d,g;3)_S$ of "maximal genus under the condition (A)", see (3.2.2). There exist such components W for every degree $d \geq 14$, and the corresponding genus is

$$g = \left[\frac{d^2-4}{8}\right]$$

where $[\nu]$ is the greatest integer such that $[\nu] \leq \nu$. The example of the lowest degree $d = 14$ gives $g = 24$, and this is the example of Mumford [M 2]. Observe also that if $V \subseteq H(p_1) = H(d,g)$ is the closure in $H(d,g)$ of one of the non-reduced components

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1) i.e. $H(d,g)_S$ is generically non-singular along $pr_1(W)$.

$pr_1(W) \subseteq H(d,g)_S$ above, then, under the conditions of (2.3.6), the corresponding "linked" component $V' \subseteq H(p_1')$ is non-reduced.

In the situation of (B) we easily prove that $pr_1(W)$ can not be an irreducible component of $H(d,g)_S$. We expect, however, that $H(d,g)_S$ is singular along $pr_1(W)$, and in Section 3.3 we give an example of this phenomenon. In fact we find a singular subscheme $Z = pr_1(W)$ of $H(16,29)$ for some $W \subseteq D(16,29;3)$ satisfying (B) which is of dimension $4d-1 = 63$ such that if V is any component of $H(16,29)$ which contains Z, then V is a reduced component of dimension $4d$.

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March 14, 1981 Jan O. Kleppe
Oslo

While typing this paper, the article [T 1] appeared covering at least (1.3.10).

Chapter 1.

1.1. The representability of the Hilbert-flag functor.

In this section we shall define the Hilbert-flag functor and show its representability.

Notations and definitions 1.1.1. Let k be a field, let $\mathbb{P}$ be a projective k-scheme, $\mathbb{P} \subseteq \mathbb{P}^n_k$, and let $\underline{F}$ be a coherent $O_{\mathbb{P}}$-Module. Put

$$H^i(\underline{F}) = H^i(\mathbb{P}, \underline{F}),\ h^i(\underline{F}) = \dim_k H^i(\underline{F})$$

and let

$$\chi(\underline{F}) = \sum_{i=0}^{\infty} (-1)^i h^i(\underline{F})$$

be the Euler-Poincaré characteristic. If ν is an integer, $\chi(\underline{F}(\nu))$ is a polynomial, called the Hilbert polynomial of $\underline{F}$, and $\chi(O_{\mathbb{P}}(\nu))$ is called the Hilbert polynomial of $\mathbb{P}$ or of $\mathbb{P} \subseteq \mathbb{P}^n_k$. See [EGA, III, (2.5)].

Let $\underline{Sch}/k$ be the category of locally noetherien k-schemes and let $\underline{Sets}$ be the category of sets. If $f: X \to S$ is a morphism of schemes, the fiber of f at $s \in S$ is the scheme $X_s = X \times_S \operatorname{Spec}(k(s))$ where $k(s)$ is the residue field of s, i.e. of $O_{S,s}$.

Definition 1.1.2. Let $\mathbb{P}$ be a projective k-scheme, $\mathbb{P} \subseteq \mathbb{P}^n_k$, and let r be an integer. If $S \in \operatorname{ob} \underline{Sch}/k$ we define $D^r(\mathbb{P}; S)$ to be the set

$$\left\{ (X_1 \subseteq X_2 \subseteq \ldots \subseteq X_r \subseteq \mathbb{P} \times_k S) \;\middle|\; \begin{array}{l} \text{all inclusions are closed} \\ \text{embeddings of } S\text{-schemes and} \\ \text{each } X_i \text{ is } S\text{-flat.} \end{array} \right\}$$

If $\varphi : S' \to S$ is a morphism in $\underline{Sch/k}$, we let

$$D^r(\mathbb{P};\varphi) : D^r(\mathbb{P};S) \to D^r(\mathbb{P};S')$$

be the map given by

$$D^r(\mathbb{P};\varphi)(X_1 \subseteq \ldots \subseteq X_r \subseteq \mathbb{P}\times S) = (X_1 \times_S S' \subseteq \ldots \subseteq X_r \times_S S' \subseteq \mathbb{P}\times S').$$

The <u>r-th Hilbert-flag functor of $\mathbb{P}$</u>

$$\underset{\sim}{D}^r(\mathbb{P}) : \underline{Sch/k} \to \underline{Sets}$$

is defined by

$$\underset{\sim}{D}^r(\mathbb{P})(S) = D^r(\mathbb{P};S)\ , \qquad S \in \operatorname{ob} \underline{Sch/k},$$

$$\underset{\sim}{D}^r(\mathbb{P})(\varphi) = D^r(\mathbb{P};\varphi)\ , \qquad \varphi \in \operatorname{Mor} \underline{Sch/k}.$$

Let $\underline{p} = (p_1,\ldots,p_r)$ be a tuple of polynomials in one variable with rational coefficients. The <u>Hilbert-flag functor of $\mathbb{P}$ with Hilbert polynomials</u> $\underline{p}$, denoted by $\underset{\sim}{D}(\mathbb{P};\underline{p}) = \underset{\sim}{D}(\mathbb{P};p_1,\ldots,p_r)$, is the subfunctor given by

$$\underset{\sim}{D}(\mathbb{P};\underline{p})(S) = \left\{ (X_1 \subseteq \ldots \subseteq X_r \subseteq \mathbb{P}\times_k S) \;\middle|\; \begin{array}{l}\text{for all } i=1,\ldots,r, \text{ the}\\ \text{fibers } (X_i)_s \text{ have Hilbert}\\ \text{polynomial } p_i \text{ for any } s\in S\end{array} \right\}$$

If $r = 1$, put

$$\underset{\sim}{Hilb}(\mathbb{P}) = \underset{\sim}{D}^1(\mathbb{P})$$

$$\underset{\sim}{Hilb}^p(\mathbb{P}) = \underset{\sim}{D}^1(\mathbb{P};p).$$

The Hilbert-flag functor of $\mathbb{P}$ with Hilbert polynomials $\underline{p}$ is well-defined since Hilbert polynomials are stable under base field extensions. Let $(X_1 \subseteq \ldots \subseteq X_r \subseteq \mathbb{P}\times S) \in \underset{\sim}{D}^r(\mathbb{P})(S)$ and let $p_i(s)$ be the Hilbert polynomial of X_i at the fiber $s \in S$. By base change theorem [M 1, Lect 7] we deduce that $p_i(s)$ is locally constant on S. See also [EGA, III, (7.9.11)]. Therefore if

$S \in \mathrm{ob}\,\underline{\mathrm{Sch}}/k$ is connected, there is a decomposition

$$\underset{\sim}{D}^r(\mathbb{P})(S) = \coprod_{\underline{p}} \underset{\sim}{D}(\mathbb{P};\underline{p})(S).$$

It follows that $\underset{\sim}{D}^r(\mathbb{P})$ is representable if $\underset{\sim}{D}(\mathbb{P};\underline{p})$ is, and that its representing object $D^r(\mathbb{P})$ is given by

$$D^r(\mathbb{P}) = \coprod_{\underline{p}} D(\mathbb{P};\underline{p})$$

where $\underset{\sim}{D}(\mathbb{P};\underline{p}) = \mathrm{Mor}(-,D(\mathbb{P};\underline{p}))$. Usually we omit mentioning $\mathbb{P}$ in $\underset{\sim}{D}^r(\mathbb{P})$, $D^r(\mathbb{P})$, $\underset{\sim}{D}(\mathbb{P};\underline{p})$, $D(\mathbb{P};\underline{p})$, $\underset{\sim}{\mathrm{Hilb}}(\mathbb{P})$ etc and sometimes also the Hilbert polynomials $\underline{p}$ and p_i.

<u>Theorem 1.1.3.</u> $\underset{\sim}{D}(\mathbb{P};\underline{p})$ is representable and its representing object $D(\mathbb{P};\underline{p})$ is a projective k-scheme.

In [M 1, Lect 15] there is a proof for the representability of the functor $\mathrm{Curves}^p(\mathbb{P})$. As Mumford remarks in his introduction, the proof which is Grothendieck's with some modifications [SB, exp 221], may be used without any change to prove that the Hilbert functor is representable. Using this proof it is easy to deduce (1.1.3) as we now shall see.

<u>Proof</u> Let α_i be positive integers, $1 \leq i \leq r$, and let $\underline{\alpha} = (\alpha_1,\ldots,\alpha_r)$. If a finite k-vector space V is given, we may define the functor

$$\underset{\sim}{\mathrm{Flag}}(V;\underline{\alpha}) : \underline{\mathrm{Sch}}/k \to \underline{\mathrm{Sets}}$$

by

$$\underset{\sim}{\mathrm{Flag}}(V;\underline{\alpha})(S) = \left\{ (V \underset{k}{\times} O_S \to \underline{E}_r \to \cdots \to \underline{E}_1) \;\middle|\; \begin{array}{l} \underline{E}_i \text{ are locally free} \\ O_S\text{-Modules of rank } \alpha_i, \\ \text{all morphisms are sur-} \\ \text{jections.} \end{array} \right\}$$

One knows that $\underset{\sim}{\mathrm{Flag}}(V;\underline{\alpha})$ is representable and that its representing object $\mathrm{Flag}(V;\underline{\alpha})$ is a projective k-scheme. See [C,exp.12]. If $r = 1$ we let

$$\underset{\sim}{\mathrm{Grass}}^{\alpha}(V) = \underset{\sim}{\mathrm{Flag}}(V;\alpha).$$

Step 1. Let $(X_1 \subseteq \dots \subseteq X_r \subseteq \mathbb{P}\times S) \in \underset{\sim}{D}(\underline{p})(S)$, let $\underline{I}_{X_i} = \ker(O_{\mathbb{P}\times S} \to O_{X_i})$ and let $\pi : \mathbb{P}\times S \to S$ be the projection. By [M 1, Lect 14] there is an integer m_o, depending only on $\underline{p} = (p_1,\dots,p_r)$, such that if $m \geq m_o$, then

$$R^j\pi_*\underline{I}_{X_i}(m) = 0 \quad \text{for} \quad j>0, \quad i\in[1,r] \quad \text{and}$$

$$\pi^*\pi_*\underline{I}_{X_i}(m) \to \underline{I}_{X_i}(m) \quad \text{is surjective for} \quad i\in[1,r].$$

Choose $m_1 \geq m_o$ such that $H^o(O_{\mathbb{P}}(m_1)) = 0$. Since

$$0 \to \underline{I}_{X_i} \to O_{\mathbb{P}\times S} \to O_{X_i} \to 0$$

is exact, we deduce

$$R^1\pi_*O_{X_i}(m_1) \simeq R^1\pi_*O_{\mathbb{P}\times S}(m_1) = 0$$

for all i. If $X_{r+1} = \mathbb{P}\times S$, $\underline{I}_{X_{r+1}} = 0$, and if $\underline{I}_{X_i/X_{i+1}} = \ker(O_{X_{i+1}} \to O_{X_i})$ for $i\in[1,r]$, there is an exact sequence

$$0 \to \underline{I}_{X_{i+1}} \to \underline{I}_{X_i} \to \underline{I}_{X_i/X_{i+1}} \to 0.$$

It follows that

$$R^j\pi_*\underline{I}_{X_i/X_{i+1}}(m_1) = 0 \quad \text{for} \quad j>0, \quad i\in[1,r],$$

which implies that the morphism

$$\pi_*O_{X_{i+1}}(m_1) \to \pi_*O_{X_i}(m_1)$$

is surjective. By base change theorem and by $R^1\pi_*O_{X_i}(m_1) = 0$, we deduce that $\pi_*O_{X_i}(m_1)$ is locally free. We have therefore constructed a sequence of surjections

$$H^0(O_{\mathbb{P}}(m_1)) \underset{k}{\otimes} O_S = \pi_*O_{X_{r+1}}(m_1) \to \pi_*O_{X_r}(m_1) \to \dots \to \pi_*O_{X_1}(m_1)$$

of locally free O_S-Modules, i.e. we have defined a map

$$h(m_1)(S): \underset{\sim}{D}(\underline{p})(S) \to \underset{\sim}{\mathrm{Flag}}(H^0(O_{\mathbb{P}}(m_1)); \underline{\alpha})(S)$$

where $\alpha_i = \operatorname{rank} \pi_*O_{X_i}(m_1)$. This morphism is functorial in S by base change theorem.

<u>Step 2.</u> Define

$$\underset{\sim}{D}(\mathbb{P}; \underline{p})(S) \to \prod_{i=1}^{r} \underset{\sim}{\mathrm{Hilb}}^{p_i}(\mathbb{P})(S)$$

by sending $(X_1 \subseteq \dots \subseteq X_r \subseteq \mathbb{P}\times S)$ to $\{(X_i \subseteq \mathbb{P}\times S)\}$ and define

$$\underset{\sim}{\mathrm{Flag}}(V, \underline{\alpha}) \to \prod_{i=1}^{r} \underset{\sim}{\mathrm{Grass}}^{\alpha_i}(V)$$

correspondingly. Then there is a commutative diagram of functors

$$\begin{array}{ccc}
\underset{\sim}{D}(\mathbb{P}; \underline{p}) & \to & \prod_{i=1}^{r} \underset{\sim}{\mathrm{Hilb}}^{p_i}(\mathbb{P}) \\
{\scriptstyle h(m_1)}\downarrow & \circ & \downarrow{\scriptstyle \pi h^i(m_1)} \\
\underset{\sim}{\mathrm{Flag}}(V, \underline{\alpha}) & \to & \prod_{i=1}^{r} \underset{\sim}{\mathrm{Grass}}^{\alpha_i}(V)
\end{array}$$

where $V = H^0(O_{\mathbb{P}}(m_1))$ and where the vertical arrow to the right $\pi h^i(m_1)$ is defined in the same way as $h(m_1)$. It follows from the proof of Mumford in [M 1, Lect 15] that

$$h^i(m_1): \underset{\sim}{\mathrm{Hilb}}^{p_i}(\mathbb{P}) \to \underset{\sim}{\mathrm{Grass}}^{\alpha_i}(V)$$

is a morphism of representable functors, and that the corresponding morphism of representing objects

$$\mathrm{Hilb}^{p_i}(\mathbb{P}) \to \mathrm{Grass}^{\alpha_i}(V)$$

is a closed embedding. Now if we prove that the commutative diagram above is cartesian, it will follow that $\underset{\sim}{D}(\mathbb{P};\underline{p})$ is representable and that the morphism of representing objects

$$D(\mathbb{P},\underline{p}) \to \mathrm{Flag}(V,\underline{\alpha})$$

is a closed embedding and the theorem follows. To prove that the diagram is cartesian, let $V\otimes O_S \to \underline{E}_r \to \dots \to \underline{E}_1$ and let $X_i \subseteq \mathbb{P}\times S$ for $1\leq i\leq r$ be given such that $V\otimes O_S \to \underline{E}_i$ and $V\otimes O_S \to \pi_* O_{X_i}(m_1)$ coincide. The morphisms

$$\underline{E}_{i+1} = \pi_* O_{X_{i+1}}(m_1) \to \underline{E}_i = \pi_* O_{X_i}(m_1)$$

induce morphisms

$$\pi_*\underline{I}_{X_{i+1}}(m_1) \to \pi_*\underline{I}_{X_i}(m_1),$$

and since the sheaf homomorphisms

$$(\pi^*\pi_*\underline{I}_{X_i}(m_1))(-m_1) \to \underline{I}_{X_i}$$

are surjective, we easily deduce

$$\underline{I}_{X_{i+1}} \subseteq \underline{I}_{X_i},$$

and we are done.

The scheme $D^r(\mathbb{P})$ is called the <u>r-th Hilbert-flag scheme of $\mathbb{P}$</u> and $D(\mathbb{P};\underline{p})$ is the <u>Hilbert-flag scheme of $\mathbb{P}$ with Hilbert polynomials</u> $\underline{p}$. Moreover the projection morphisms

$$\underset{\sim}{D}(\mathbb{P};\underline{p}) \to \underset{\sim}{\mathrm{Hilb}}^{p_i}(\mathbb{P})$$

defined in step 2, induce morphisms

$$pr_i : D(\mathbb{P},\underline{p}) \to \mathrm{Hilb}^{p_i}(\mathbb{P})$$

of projective k-schemes. pr_i is therefore projective, and we call it the <u>projection morphism</u> from the Hilbert-flag scheme onto its i-th factor.

<u>Notations and terminology 1.1.4.</u> Since $\underset{\sim}{\underline{D}}(\underline{p})$ is representable, there is a canonical element

$$\underline{X}_D = (X_{1D} \subseteq \dots \subseteq X_{rD} \subseteq \mathbb{P} \times D) \in \underset{\sim}{\underline{D}}(\underline{p})(D) = \mathrm{Mor}(D,D)$$

which corresponds to the identity of $D = D(\underline{p})$. We call this element the <u>universal object</u> of D. Usually we simplify the notations and we write

$$(X_1 \subseteq \dots \subseteq X_r \subseteq \mathbb{P} \times S) = (X_1, \dots, X_r) = \underline{X}$$

where $(X_1 \subseteq \dots \subseteq X_r \subseteq \mathbb{P} \times S) \in \underset{\sim}{\underline{D}}(\underline{p})(S)$ is an <u>S-point</u> or an S-valued point <u>of D</u>. To any S-point $\underline{X}$ of D there is a morphism

$$\varphi_{\underline{X}} : S \to D$$

such that

$$D(\underline{p};\varphi_{\underline{X}})(\underline{X}_D) = \underline{X},$$

and there are ideals $\underline{I}_{X_i} = \ker(O_{\mathbb{P}\times S} \to O_{X_i})$, $\underline{I}_{X_i/X_{i+1}} = \ker(O_{X_{i+1}} \to O_{X_i})$ and normal bundles $\underline{N}_{X_i} = \underline{\mathrm{Hom}}_{O_{\mathbb{P}\times S}}(\underline{I}_{X_i}, O_{X_i})$, $\underline{N}_{X_i/X_{i+1}} = \underline{\mathrm{Hom}}_{O_{X_{i+1}}}(\underline{I}_{X_i/X_{i+1}}, O_{X_i})$.

A <u>K-point</u> $\underline{X}$ <u>of D</u> is an object of $\underset{\sim}{\underline{D}}(\underline{p})(\mathrm{Spec}(K))$ where $k \hookrightarrow K$ is a field extension. So to any point $x \in D$ there is a $k(x)$-point $\underline{X}$ of D, and to any K-point $\underline{X}$ of D there is a point $x \in D$ and a field extension $k(x) \hookrightarrow K$. In both cases we let

$$O_{D,\underline{X}} = O_{D,x}.$$

By <u>a point</u> $\underline{X}$ <u>of</u> D we mean a k-point of D. If a morphism $\psi : S \to D$ in <u>Sch</u>/k is given, then <u>a K-point</u> $\underline{X}$ <u>of</u> S is a morphism $\varphi_{\underline{X}} : \mathrm{Spec}(K) \to S$ such that

$$D(\underline{p};\psi\varphi_{\underline{X}})(\underline{X}_D) = \underline{X}.$$

1.2. The local study of the Hilbert-flag scheme. Preliminaries.

Let k/<u>Sch</u>/k be the category of pointed k-schemes (locally noetherian) and let $\underline{1} \subset$ k/<u>Sch</u>/k be the full subcategory consisting of affine schemes $S = \mathrm{Spec}(A)$ where A is an artinian local k-algebra with maximal ideal m, endowed with an isomorphism $\eta : A/m \cong k$. The point $\mathrm{Spec}(k) \hookrightarrow S = \mathrm{Spec}(A)$ is defined by $A \twoheadrightarrow A/m \underset{\eta}{\simeq} k$, and abusing the language we also denote by S the morphism $\mathrm{Spec}(k) \hookrightarrow S$ considered as an object of $\underline{1}$.

<u>Definition 1.2.1.</u> Let $\underline{X} = (X_1,\ldots,X_r) \in \underset{\sim}{D}(\mathbb{P};\underline{p})(\mathrm{Spec}(k))$ and define the <u>local Hilbert-flag functor</u> at $\underline{X}$

$$\underset{\sim}{D}_{\underline{X}} = \underset{\sim}{D}_{X_1 \subseteq \ldots \subseteq X_r} : \underline{1} \to \underline{\mathrm{Sets}}$$

by

$$\underset{\sim}{D}_{\underline{X}}(\mathrm{Spec}(k) \xrightarrow{\varphi} S) = \{\underline{X}_S \in \underset{\sim}{D}(\underline{p})(S) \mid D(\underline{p};\varphi)(\underline{X}_S) = \underline{X}\}.$$

We let $\underset{\sim}{D}_{\underline{X}}(S) = \underset{\sim}{D}_{\underline{X}}(\varphi)$ and if $\psi : S \to S'$ is a morphism in $\underline{1}$, we define

$$\underset{\sim}{D}_{\underline{X}}(\psi) : \underset{\sim}{D}_{X}(S') \to \underset{\sim}{D}_{X}(S)$$

to be the map induced by

$$D(\underline{p};\psi) : \underset{\sim}{D}(\underline{p})(S') \to \underset{\sim}{D}(\underline{p})(S).$$

If $r = 1$ and $\underline{X} = (X)$, let

$$\underset{\sim}{\mathrm{Hilb}}_X = \underset{\sim}{D}_{(X)}$$

be the <u>local Hilbert functor</u> at $X = (X \subseteq \mathbb{P})$.

Notice that we shall sometimes regard $\underset{\sim}{D}_{\underline{X}}$ as defined on the dual category $\underline{l}^o$.

<u>Remark 1.2.2.</u> Since $\underset{\sim}{D}(\underline{p})$ is represented by $D = D(\underline{p})$, any $\underline{X} = (X_1,\dots,X_r) \in \underset{\sim}{D}(\underline{p})(\mathrm{Spec}(k))$ defines a morphism $\mathrm{Spec}(k) \to D$ factorizing via $\mathrm{Spec}(k) \to \mathrm{Spec}(\hat{O}_{D,\underline{X}})$. Clearly $\underset{\sim}{D}_{\underline{X}}(S) \simeq \mathrm{Mor}(S,\mathrm{Spec}(\hat{O}_{D,\underline{X}})) \simeq \mathrm{Mor}(\hat{O}_{D,\underline{X}},A)$ for any $S = \mathrm{Spec}(A) \in \mathrm{ob}\,\underline{l}$. In particular $\hat{O}_{D,\underline{X}}$ is a hull and in fact a noetherian prorepresenting object for $\underset{\sim}{D}_{\underline{X}}$ defined on $\underline{l}^o$. Moreover to the projection morphisms

$$\underset{\sim}{\mathrm{pr}}_i : \underset{\sim}{D}_{\underline{X}} \to \underset{\sim}{\mathrm{Hilb}}_{X_i}$$

there are morphisms of prorepresenting objects

$$\hat{O}_{D,\underline{X}} \leftarrow \hat{O}_{\mathrm{Hilb}^{p_i},X_i}$$

which is just the completion of the local homomorphisms $O_{D,\underline{X}} \leftarrow O_{\mathrm{Hilb}^{p_i},X_i}$ deduced from morphisms $\mathrm{pr}_i : D = D(\underline{p}) \to \mathrm{Hilb}^{p_i}$ at $\underline{X}$.

Let $\underline{X} = (X_1,\dots,X_r) \in \underset{\sim}{D}(\underline{p})(\mathrm{Spec}(k))$ be a given k-point of $D(\underline{p})$. Our goal is to study $D(\underline{p})$ and $\mathrm{pr}_i : D(\underline{p}) \to \mathrm{Hilb}^{p_i}$ at $\underline{X}$ and moreover, Hilb^{p_i} at $\mathrm{pr}_i(\underline{X}) = X_i$. We shall use the local deformation theory as presented by Laudal in [L] and [L 1] to give an explicite description of the hull of the functor $\underset{\sim}{D}_{\underline{X}}$ in terms of its corresponding cohomology groups of algebras and of a certain total obstruction morphism. This hull is $\hat{O}_{D,\underline{X}}$ by (1.2.2). Moreover if D_{X_i} is the fiber of $pr_i : D = D(\underline{p}) \to \mathrm{Hilb}^{p_i}$ at X_i, we determine $\hat{O}_{D_{X_i},\underline{X}}$ correspondingly. Finally we shall examine when pr_i is smooth or unramified at $\underline{X}$.

In (1.2.3) and (1.2.6) we recall some of the basic notations and facts about the cohomology groups of algebras.

<u>(1.2.3).</u> To any closed embedding of k-schemes, say $f : X \hookrightarrow Y$, and to any quasi-coherent O_X-Module $\underline{F}$, we define $\underline{A}^p(f,\underline{F})$ by

$$\underline{A}^q(f,\underline{F})(U) = H^q(B,C,\underline{F}(f^{-1}(U))), \quad q \geq 0$$

where $U = \mathrm{Spec}(B) \subseteq Y$ is an open affine subscheme of Y, where $f^{-1}(U) = \mathrm{Spec}(C)$, and where $H^q(B,C,-)$ is the usual cohomology groups of algebras associated to the morphism $B \twoheadrightarrow C$. See the introduction of these groups by M. André in [An] or by L. Illusie in [I]. Note that if $\underline{I}_{X/Y} = \ker(O_Y \to O_X)$, then

$$\underline{A}^1(f,O_X) = \underline{\mathrm{Hom}}_{O_Y}(\underline{I}_{X/Y},O_X) = \underline{N}_{X/Y}$$

is the normal bundle of X in Y. Moreover by [L,(3.2.5) and (3.2.9)] there are cohomology groups of algebras

$$A^i(f,\underline{F}), \quad i \geq 0$$

which are the abutment of a spectral sequence given by

$$E_2^{p,q} = H^p(Y,\underline{A}^q(f,\underline{F})) .$$

It follows that

$$A^1(f,O_X) = H^0(X,\underline{N}_{X/Y})$$

and if $f: X \hookrightarrow Y$ is locally a complete intersection, we may prove that

$$A^i(f,\underline{F}) = H^{i-1}(X,\underline{N}_{X/Y} \underset{O_X}{\otimes} \underline{F}) \quad \text{for} \quad i \geq 0.$$

Let $X \overset{f}{\hookrightarrow} Y \overset{g}{\hookrightarrow} P$ be closed embeddings of k-schemes. Then there is a long exact sequence

$$\to A^i(f,\underline{F}) \to A^i(gf,\underline{F}) \to A^i(g,f_*\underline{F}) \to A^{i+1}(f,\underline{F}) \to$$

for all $i \geq 0$ (Use [L,(3.3.4)] combined with $A^i(gf,\underline{F}) = A^i(P,X,\underline{F})$).

<u>(1.2.4)</u>. Let $f: X \hookrightarrow Y$ and $\underline{F}$ be as in (1.2.3) and let $Z \subseteq X$ be locally closed. By [L; (3.2.10) and (3.2.11)] there are groups $A^i_Z(f,\underline{F})$ and a spectral sequence

$$E_2^{p,q} = A^p(f,\underline{H}^q_Z(\underline{F}))$$

converging to $A^{(\cdot)}_Z(f,\underline{F})$. If $f|_{X-Z}$ is the composition

$$X - Z \to X \xrightarrow{f} Y,$$

then there is a long exact sequence

$$\to A^i_Z(f,\underline{F}) \to A^i(f,\underline{F}) \to A^i(f|_{X-Z},\underline{F}) \to A^{i+1}_Z(f,\underline{F}) \to$$

for $i \geq 0$.

Let $(X_1,X_2,\ldots,X_r) \in \underset{\sim}{\underline{D}}(\underline{p})(\mathrm{Spec}(k))$ be given, let

$$g_{ij}: X_i \to X_j \quad \text{for} \quad i \leq i \leq j \leq r \quad \text{and}$$

$$f_i \;: X_i \to \mathbb{P} \quad \text{for} \quad 1 \leq i \leq r$$

be the obvious compositions and define a category $\underline{d} = \underline{d}^r$ by

$$\text{ob}\,\underline{d}^r = \{f_i \mid i = 1,\dots,r\}$$

$$\text{Mor}(f_i,f_j) = \begin{cases} \emptyset & \text{for } j<i \\ (g_{ij},1) & \text{for } j\geq i\,, \end{cases}$$

i.e. where $(g_{ij},1)$ is the morphism

$$\begin{array}{ccc} X_i & \xrightarrow{g_{ij}} & X_j \\ {\scriptstyle f_i}\downarrow & \circ & \downarrow{\scriptstyle f_j} \\ \mathbb{P} & \xrightarrow[=]{1} & \mathbb{P} \end{array}$$

and where 1 is the identity. Let $\underline{d}_i$ for $i = 1,\dots,r$ be the subcategory of $\underline{d} = \underline{d}^r$ consisting of the object f_i.

<u>(1.2.5).</u> The algebra cohomology associated to the categories $\underline{d}_i$ and $\underline{d}^r$ are given by

$$A^q(\underline{d}_i,O_{\underline{d}_i}) = A^q(f_i,O_{X_i}) \quad \text{for } i = 1,\dots,r$$

where $A^q(f_i,O_{X_i})$ are defined in (1.2.3), and by

$$A^{(\cdot)}(\underline{d},O_{\underline{d}})$$

which are the abutment of a spectral sequence $E_2^{p,q} =$

$$\varprojlim{}^{(p)} \left\{ \begin{array}{ccccccc} A^q(f_1,O_{X_1}) & & A^q(f_2,O_{X_2}) & & A^q(f_3,O_{X_3}) & \dots & A^q(f_r,O_{X_r}) \\ & {\scriptstyle l_1^q}\searrow \;\; \swarrow{\scriptstyle m_2^q} & & {\scriptstyle l_2^q}\searrow \;\; \swarrow{\scriptstyle m_3^q} & & \dots & \swarrow{\scriptstyle m_r^q} \\ & A^q(f_2,g_{12*}O_{X_1}) & & A^q(f_3,g_{23*}O_{X_2}) & \dots & A^q(f_r,g_{r-1,r*}O_{X_{r-1}}) & \end{array} \right\}$$

Since $g_{i,i+1}: X_i \hookrightarrow X_{i+1}$, there are morphisms $O_{X_{i+1}} \to g_{i,i+1*}O_{X_i}$ which define m^q_{i+1}, and since $f_i = f_{i+1} \circ g_{i,i+1}$, there are natural maps l^q_i: $A^q(f_i, O_{X_i}) \to A^q(f_{i+1}, g_{i,i+1*}O_{X_i})$ which correspond to one of the morphisms appearing in the long exact sequence of (1.2.3). See [L,(3.1.5) and (3.2.8)] for the definitions and [L1, § 2] for the spectral sequence.

<u>(1.2.6).</u> There are cohomology groups of algebras

$$A^{(\bullet)}_{\underline{d}_i}(\underline{d}, O_{\underline{d}})$$

which are the abutment of a spectral sequence ${}_iE^{p,q}_2$ given by the term $E^{p,q}$ of (1.2.5) <u>provided</u> we replace the group $A^q(f_i, O_{X_i})$ and the morphisms m^q_i and l^q_i in the expression of $E^{p,q}$ by the trivial group and the trivial morphisms. Moreover there is a long exact sequence

$$\to A^q_{\underline{d}_i}(\underline{d}, O_{\underline{d}}) \xrightarrow{t^q_i} A^q(\underline{d}, O_{\underline{d}}) \xrightarrow{p^q_i} A^q(\underline{d}_i, O_{\underline{d}_i}) \xrightarrow{\alpha^q_i} A^{q+1}_{\underline{d}_i}(\underline{d}, O_{\underline{d}}) \to$$

whose morphisms fit into a commutative diagram

$$\begin{tikzcd}
A^q_{\underline{d}_i}(\underline{d}, O_{\underline{d}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & A^q(\underline{d}, O_{\underline{d}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & A^q(\underline{d}_i, O_{\underline{d}_i}) \\
{}_iE^{0,q}_2 \arrow[r] & E^{0,q}_2 \arrow[ur] &
\end{tikzcd}$$

where the vertical morphisms are edge homomorphisms and where

$$E^{0,q}_2 \to A^q(\underline{d}_i, O_{\underline{d}_i}) = A^q(f_i, O_{X_i})$$

is the natural projection. See [L,(3.1.8)] and [L1, § 2]. Note that the vertical arrows are the identities if $q = 1$.

The main reason for being interested in the cohomology groups $A^q(\underline{d}^r, O_{\underline{d}^r})$ or in $A^q(\underline{d}_i, O_{\underline{d}_i})$ lies in the fact that these groups for $q = 1,2$ determine all deformations of the corresponding categories, i.e. they determine the functors $\underset{\sim}{D}_{\underline{X}}$ and $\underset{\sim}{\mathrm{Hilb}}_{X_i}$ respectively. To be precise, we have the following result [L;(4.1.14)].

<u>Theorem 1.2.7.</u> (Small deformation theorem). Let $\eta : (A',m') \to (A,m)$ be a surjective morphism of local k-algebras satisfying

$$m' \ker \eta = 0$$

and let $\varphi : S = \mathrm{Spec}(A) \to S' = \mathrm{Spec}(A')$ be the induced embedding. Suppose $\underline{X}_S = (X_{1S}, \ldots, X_{rS}) \in \underset{\sim}{D}_{\underline{X}}(S)$ is given. Then there is an element (called an obstruction)

$$o(\underline{X}_S) \in A^2(\underline{d}^r, O_{\underline{d}^r}) \underset{k}{\otimes} \ker \eta$$

which is zero if and only if there is an object $\underline{X}' = (X'_1, \ldots, X'_r) \in \underset{\sim}{D}_{\underline{X}}(S')$ satisfying

$$\underset{\sim}{D}_{\underline{X}}(\varphi)(\underline{X}') = \underline{X}_S.$$

Moreover if $o(\underline{X}_S) = 0$, then the set of deformations

$$\{\underline{X}' \in \underset{\sim}{D}_{\underline{X}}(S') \mid \underset{\sim}{D}_{\underline{X}}(\varphi)(\underline{X}') = \underline{X}_S\}$$

is a principal homogeneous space over

$$A^1(\underline{d}^r, O_{\underline{d}^r}) \underset{k}{\otimes} \ker \eta .$$

Applying (1.2.7) for $r = 1$ we find that $A^q(\underline{d}_i, O_{\underline{d}_i}) \otimes \ker \eta$ for $q = 1,2$ determine the deformations of $X_{iS} \in \underset{\sim}{\mathrm{Hilb}}_{X_i}(S)$ to S'.

Any $\eta : A' \twoheadrightarrow A$ or $\varphi : S \to S'$ as in (1.2.7) are called <u>small</u>, and any $\underline{X}' \in \underset{\sim}{D}_{\underline{X}}(S')$ satisfying $\underset{\sim}{D}_{\underline{X}}(\varphi)(\underline{X}') = \underline{X}_S$ is said to be a <u>deformation</u> or a <u>lifting</u> of $\underline{X}_S$ to S'.

Once having such a theorem as (1.2.7), it is possible to describe the hull of its local deformation functor [L;(4,2,4)].

<u>Theorem 1.2.8.</u> (Characterization of hulls).

Let $\underline{X} = (X_1,\ldots,X_r) \in \underset{\sim}{D}(\mathbb{P};\underline{p})(\mathrm{Spec}(k))$ define the category $\underline{d} = \underline{d}^r$, let

$$A^q(\underline{d},O_{\underline{d}})^{\vee} = \mathrm{Hom}_k(A^q(\underline{d},O_{\underline{d}}),k)$$

be the dual vector space and let

$$T^q(\underline{d}) = \mathrm{Sym}_k[A^q(\underline{d},O_{\underline{d}})^{\vee}]^{\wedge}$$

be the completion of $\mathrm{Sym}_k[A^q(\underline{d},O_{\underline{d}})^{\vee}]$ in its maximal ideal. Then there is a morphism of complete local k-algebras (called a total obstruction morphism)

$$o(\underline{d}) : (T^2(\underline{d}),m_{T^2}) \to (T^1(\underline{d}),m_{T^1})$$

satisfying

$$o(\underline{d})(m_{T^2}) \subseteq m^2_{T^1}$$

such that

$$T^1(\underline{d}) \mathop{\hat{\otimes}}_{T^2(\underline{d})} k$$

is a hull for $\underset{\sim}{D}_{\underline{X}}$ defined on $\underline{l}^{o}$.

So by (1.2.2) we have determined the completion of $O_{D,\underline{X}}$. And applying (1.2.8) for $r = 1$, we find a description of $\hat{O}_{\mathrm{Hilb}^{p_i},X_i}$.

Corollary 1.2.9. If $a^i = \dim_k A^i(\underline{d}, O_{\underline{d}})$, then

$$a^1 - a^2 \le \dim O_{D,\underline{X}} \le a^1.$$

Moreover

$$\dim O_{D,\underline{X}} = a^1$$

if and only if $D = D(\underline{p})$ is non-singular at $\underline{X}$ (by which we mean that the structure morphism $D \to \mathrm{Spec}(k)$ is smooth at the point $x \in D$ which corresponds to the k-point $\underline{X}$ of D).

Using (1.2.9) for $r = 1$ we have corresponding results for $O_{\mathrm{Hilb}^{p_i}, X_i}$ for each i.

The reason for being interested in the groups $A^q_{\underline{d}_i}(\underline{d}, O_{\underline{d}})$ lie in the fact that these groups are useful in the study of the projection morphisms

$$\underset{\sim}{pr}_i : \underset{\sim}{D}_{\underline{X}} \to \underset{\sim}{\mathrm{Hilb}}_{X_i}$$

which we now shall see.

Remark 1.2.10. If $k[\epsilon] \simeq k^{\oplus 2} \in \mathrm{ob}\,\underline{l}^o$ is the ring of dual numbers, then by (1.2.7) or (1.2.8) we find tangent spaces

$$\underset{\sim}{D}_{\underline{X}}(k[\epsilon]) = A^1(\underline{d}, O_{\underline{d}}),$$

$$\underset{\sim}{\mathrm{Hilb}}_{X_i}(k[\epsilon]) = A^1(\underline{d}_i, O_{\underline{d}_i}) = A^1(f_i, O_{X_i}).$$

Therefore the projection morphisms $\underset{\sim}{pr}_i : \underset{\sim}{D}_{\underline{X}} \to \underset{\sim}{\mathrm{Hilb}}_{X_i}$ give rise to tangent maps

$$\underset{\sim}{pr}_i(k[\epsilon]) : A^1(\underline{d}, O_{\underline{d}}) \to A^1(d_i, O_{\underline{d}_i})$$

and these maps coincide with the corresponding morphisms p_i^1 in the long exact sequence of (1.2.6) by [L,(4.1.15)].

In the situation of (1.2.7) the morphisms

$$p_i^2 \otimes 1_{\ker\eta} : A^2(\underline{d}, O_{\underline{d}}) \otimes \ker\eta \to A^2(\underline{d}_i, O_{\underline{d}_i}) \otimes \ker\eta$$

where $1_{\ker\eta}$ is the identity on $\ker\eta$, map the obstruction $o(\underline{X}_S)$ onto the obstructions $o(\underline{X}_{iS})$, again by [L,(4.1.15)]. We express this by saying that p_i^2 map "obstructions to obstructions".

Suppose $o(X_{iS}) = 0$ and pick a lifting $X_i' \in \underset{\sim}{\mathrm{Hilb}}_{X_i}(S')$ of X_{iS} to S'. Then there is an obstruction $o_{X_i}(\underline{X}_S)$ in $A^2_{\underline{d}_i}(\underline{d}, O_{\underline{d}}) \otimes \ker\eta$ which maps to $o(\underline{X}_S)$ via

$$t_i^2 \otimes 1_{\ker\eta} : A^2_{\underline{d}_i}(\underline{d}, O_{\underline{d}}) \otimes \ker\eta \to A^2(\underline{d}, O_{\underline{d}}) \otimes \ker\eta$$

where t_i^2 appears in the long exact sequence of (1.2.6). Therefore t_i^2 maps "obstructions to obstructions" as well. In the same way the morphisms

$$A^2(f, O_X) \to A^2(f|X-Z, O_X)$$

and

$$A^2_Z(f, O_X) \to A^2(f, O_X)$$

appearing in the long exact sequence of (1.2.4) map "obstructions to obstructions".

Recall that

$$\underset{\sim}{pr}_i : \underset{\sim}{D}_{\underline{X}} \to \underset{\sim}{\mathrm{Hilb}}_{X_i}$$

is <u>formally smooth</u> (resp. <u>formally unramified</u>) iff

$$(*) \qquad \underset{\sim}{D}_{\underline{X}}(S) \to \underset{\sim}{D}_{\underline{X}}(T) \underset{\underset{\sim}{\mathrm{Hilb}}_{X_i}(T)}{\times} \underset{\sim}{\mathrm{Hilb}}_{X_i}(S)$$

is surjective (resp. injective) whenever $\varphi : T \to S$ is small in $\underline{l}$. By [EGA,IV,(17.14.2)] $\underset{\sim}{pr}_i$ is formally smooth (resp. formally unramified) iff

$$pr_i : D(\underline{p}) \to Hilb^{p_i}$$

is smooth (resp. unramified) at the k-point $\underline{X}$ since pr_i is of finite presentation. If D_{X_i} is the fiber of $pr_i : D(\underline{p}) \to Hilb^{p_i}$ at the k-point $X_i = (X_i \subseteq \mathbb{P})$ of $Hilb^{p_i}$, we have the following result

Theorem 1.2.11. (Properties of pr_i).

i) There is a morphism of complete local k-algebras

$$T_i^2(\underline{d}) = Sym_k[A^2_{\underline{d}_i}(\underline{d},O_{\underline{d}})^{\vee}]^{\wedge} \to T_i^1(\underline{d}) = Sym_k[A^1_{\underline{d}_i}(\underline{d},O_{\underline{d}})^{\vee}]^{\wedge}$$

such that

$$\hat{O}_{D_{X_i},\underline{X}} \simeq T_i^1(\underline{d}) \hat{\otimes}_{T_i^2(\underline{d})} k$$

ii) If $A^1_{\underline{d}_i}(\underline{d},O_{\underline{d}}) = 0$, then

$$pr_i : D(\underline{p}) \to Hilb^{p_i}$$

is unramified at the point $\underline{X}$ of D.

iii) If $A^2_{\underline{d}_i}(\underline{d},O_{\underline{d}}) = 0$, then pr_i is smooth at $\underline{X}$.

Proof i) Using trivial liftings of $X_i \subseteq \mathbb{P}$ to S where $Spec(k) \to S$ is in $\underline{l}$, i.e. liftings of the form $X_i \times S \subseteq \mathbb{P} \times S$, and using (1.2.10) we prove a "small deformation theorem" (1.2.7) where the algebra cohomology involved are

$$A^q_{\underline{d}_i}(\underline{d},O_{\underline{d}}) \quad \text{for} \quad q = 1,2$$

(or use [L,(4.1.17)]). In the same way as we used (1.2.7) to give a description of $\hat{O}_{D,\underline{X}}$ via (1.2.8) and (1.2.2), we deduce the conclusion of i).

ii) Since $A^1_{\underline{d}_i}(\underline{d},O_{\underline{d}}) = 0$ it follows from i) that

$$\hat{O}_{D_{X_i},\underline{X}} \simeq k .$$

Unlike what happens to a smooth morphism, a morphism is unramified iff its fibers are unramified as [EGA, IV, (17,4,1)] says, and we are done.

iii) The surjectivity of (*) follows from [L,(4.1.17)].

Remark 1.2.12. The results of this section apply also to the case where $\underline{X} = (X_1,\dots,X_r)$ is a given K-point of $D(\underline{p})$. In this case $\underline{l}^o_K \subseteq (K/\underline{Sch}/K)^o$ is the subcategory of local artinian K-algebras with residue fields K, and we define the local Hilbert-flag functor $\underset{\sim}{D}_{\underline{X}}$ at $\underline{X}$ on $\underline{l}_K$ as in (1.2.1). Since D represents the functor $\underset{\sim}{D}(\mathbb{P};\underline{p})$: $\underline{Sch}/k \to \underline{Sets}$, it follows that

$$D \underset{k}{\times} \operatorname{Spec}(K)$$

represents the Hilbert-flag functor

$$\underset{\sim}{D}(\mathbb{P} \times \operatorname{Spec}(K);\underline{p}) : \underline{Sch}/K \to \underline{Sets},$$

and $\underline{X}$, which is a K-point of D, is also at K-point of $D \underset{k}{\times} \operatorname{Spec}(K)$. By (1.2.2) we deduce that

$$\hat{O}_{D\times\operatorname{Spec}(K),\underline{X}} = \hat{O}_{D,\underline{X}} \underset{k}{\otimes} K$$

is a hull of $\underset{\sim}{D}_X$ on $\underline{l}^o_K$. Now the algebra cohomology are

K-vector spaces, and the complete local rings $T^q(\underline{d})$ and $T_i^q(\underline{d})$ are K-algebras. (1.2.9) is still true, and in (1.2.11;i) we easily deduce

$$\hat{O}_{D_{X_i},\underline{X}} \underset{k}{\otimes} K \simeq T_i^1(\underline{d}) \underset{T_i^2(\underline{d})}{\hat{\otimes}} K .$$

Since the injectivity (resp. the surjectivity) of (*), where $\varphi : T \to S$ is small in $\underline{1}_K$, follows from the assumptions by [L,(4.1.17)], (1.2.11;ii,iii) holds.

1.3. Local study of the Hilbert-flag scheme and of its projection morphisms.

In this section we shall concentrate on the second Hilbert-flag scheme $D = D(p,q)$ of $\mathbb{P}$ and on its corresponding projection morphisms pr_i appearing in

$$(*) \qquad \begin{array}{ccc} D(p,q) & \xrightarrow{pr_2} & \mathrm{Hilb}^q = H(q) \\ pr_1 \downarrow & & \\ \mathrm{Hilb}^p = H(p) & & \end{array}$$

We shall use the theory of Section 1.2 to study (*) at a point $x \in D$, corresponding to the k-point $(X \subseteq Y \subseteq \mathbb{P})$ of D. In particular we are interested in

(1) a diagram (**) which includes (*) on the tangent space level at x,

and also in an good description of the "obstruction spaces"

(2) $A^2(\underline{d},O_{\underline{d}})$ and $A^2_{\underline{d}_i}(\underline{d},O_{\underline{d}})$ for $i = 1,2$,

say in terms of other known groups. In some way the groups of (2), at least the obstructions given by these groups, tell us how the diagram (*) at x is determined by (**). Now it might happen that these "obstruction groups" are too big, i.e. that there might be subgroups

(3) $\quad A^2(\underline{d},O_{\underline{d}})_{res} \subseteq A^2(\underline{d},O_{\underline{d}}), \quad A^2_{\underline{d}_i}(\underline{d},O_{\underline{d}})_{res} \subseteq A^2_{\underline{d}_i}(\underline{d},O_{\underline{d}})$

which contain all obstructions involved.

Our main results (1.3.2) and (1.3.4) of this section define subgroups of $A^2(\underline{d},O_{\underline{d}})$ and of $A^1_{\underline{d}_1}(\underline{d},O_{\underline{d}})$ in case Hilb^q is non-singular at $pr_2(x)$ such that the conclusion of (3) holds. Finally we also study

(4) the fibers of the morphisms pr_i of (*).

Let $(X \subseteq Y \subseteq \mathbb{P})$ be a given k-point of D corresponding to $x \in D$, and let $f : X \hookrightarrow Y$ and $g : Y \hookrightarrow \mathbb{P}$ be the embeddings. By abusing the language, we let

$$x = (X \subseteq Y \subseteq \mathbb{P}) \in D.$$

Now there are categories $\underline{d}$ and $\underline{d}_i$ for $i = 1,2$ (1.2.5), ideals $\underline{I}_X$, $\underline{I}_Y$, $\underline{I}_{X/Y}$ and corresponding normal bundles $\underline{N}_X$, $\underline{N}_Y$ and $\underline{N}_{X/Y}$ (1.1.4). Combining (1.2.10) with (1.2.2) we find tangent maps p_i^1 for $i = 1,2$ and a diagram

$$\begin{array}{ccc}
A^1(\underline{d},O_{\underline{d}}) & \xrightarrow{p_2^1} & A^1(g,O_Y) = H^0(\underline{N}_Y) \\
{\scriptstyle p_1^1}\downarrow & & \\
A^1(gf,O_X) = H^0(\underline{N}_X) & &
\end{array}$$

corresponding to (*). In view of (1.2.5) there is a cartesian diagram

$$
(**) \qquad
\begin{tikzcd}
A^1(\underline{d},O_{\underline{d}}) \arrow[r, "p_2^1"] \arrow[d, dashed, "p_1^1"'] \arrow[dr, phantom, "\square"] & H^o(\underline{N}_Y) \arrow[d, "m^1"] \\
H^o(\underline{N}_X) \arrow[r, "l^1"'] & A^1(g,f_*O_X)
\end{tikzcd}
$$

including the diagram above where we in (1.2.5) let $m^q = m_2^q$ and $l^q = l_1^q$ for $q \geq 1$.

Remark 1.3.1. Consider the big diagram

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & & & A^1(g,\underline{I}_{X/Y}) \arrow[d] & & \\
 & & & A^1(g,O_Y) \arrow[d, "m^1"] & & \\
0 \arrow[r] & A^1(f,O_X) \arrow[r] & A^1(gf,O_X) \arrow[r, "l^1"] & A^1(g,f_*O_X) \arrow[r, "\delta_2^1"] \arrow[d, "\delta_1^1"] & A^2(f,O_X) \arrow[dddll, dashed, bend left] & \\
 & & & A^2(g,\underline{I}_{X/Y}) \arrow[d] & & \\
 & & & A^2(g,O_Y) \arrow[d, "m^2"] & & \\
 & & A^2(gf,O_X) \arrow[r, "l^2"] & A^2(g,f_*O_X) \arrow[r] \arrow[d] & A^3(f,O_X) \arrow[r] & {} \\
 & & & A^3(g,\underline{I}_{X/Y}) & &
\end{tikzcd}
$$

of exact sequences, the horizontal sequence is deduced from (1.2.3) and the vertical one is the long exact sequence associated to the short exact sequence

$$0 \to \underline{I}_{X/Y} \to O_Y \to f_*O_X \to 0.$$

It follows that

$$A^1(g,\underline{I}_{X/Y}) = \ker m^1,$$

and that there is an exact sequence

$$0 \to \operatorname{coker} m^1 \to A^2(g,\underline{I}_{X/Y}) \to \ker m^2 \to 0.$$

Using (1.2.6) we find that

$$A^1_{\underline{d}_1}(\underline{d},O_{\underline{d}}) = {}_1E_2^{0,1} = \ker m^1,$$

that

$${}_1E_2^{0,2} = \ker m^2 \quad \text{and} \quad {}_1E_2^{1,1} = \operatorname{coker} m^1,$$

and also that

$$0 \to \operatorname{coker} m^1 \to A^2_{\underline{d}_1}(\underline{d},O_{\underline{d}}) \to \ker m^2 \to 0$$

is exact. We deduce k-isomorphisms

$$A^i_{\underline{d}_1}(\underline{d},O_{\underline{d}}) \simeq A^i(g,\underline{I}_{X/Y})$$

for $i = 1,2$. In the same way we prove

$$A^i_{\underline{d}_1}(\underline{d},O_{\underline{d}}) \simeq A^i(g,\underline{I}_{X/Y}) \text{ for all } i \geq 0,$$

$$A^i_{\underline{d}_2}(\underline{d},O_{\underline{d}}) \simeq A^i(f,O_X) \quad \text{for } i \geq 0.$$

Now the long exact sequence of (1.2.6) induces long exact sequences

$$(A)\quad \begin{array}{l} 0 \to A^1(g,\underline{I}_{X/Y}) \xrightarrow{t_1^1} A^1(\underline{d},0_{\underline{d}}) \xrightarrow{p_1^1} H^0(\underline{N}_X) \xrightarrow{\alpha_1^1} \\ A^2(g,\underline{I}_{X/Y}) \xrightarrow{t_1^2} A^2(\underline{d},0_{\underline{d}}) \xrightarrow{p_1^2} A^2(gf,0_X) \xrightarrow{\alpha_1^2} A^3(g,\underline{I}_{X/Y}) \end{array}$$

$$(B)\quad \begin{array}{l} 0 \to A^1(f,0_X) \xrightarrow{t_2^1} A^1(\underline{d},0_{\underline{d}}) \xrightarrow{p_2^1} H^0(\underline{N}_Y) \xrightarrow{\alpha_2^1} \\ A^2(f,0_X) \xrightarrow{t_2^2} A^2(\underline{d},0_{\underline{d}}) \xrightarrow{p_2^2} A^2(g,0_Y) \xrightarrow{\alpha_2^2} A^3(f,0_X), \end{array}$$

and the spectral sequence of (1.2.6) implies that $\alpha_1^1 = \delta_1^1 \circ l^1$ and $\alpha_1^1 = \delta_2^1 \circ m^1$ where δ_i^1, l^1 and m^1 appear in the big diagram above. Therefore there is a morphism

$$\gamma = \gamma_{\underline{X \subseteq Y}} : H^0(\underline{N}_X) \to \operatorname{coker} m^1$$

such that the composition

$$H^0(\underline{N}_X) \xrightarrow{\gamma} \operatorname{coker} m^1 \subseteq A^2(g,\underline{I}_{X/Y})$$

is α_1^1. If we let

$$\alpha = \alpha_{\underline{X \subseteq Y}} : H^0(\underline{N}_Y) \to A^2(f,0_X)$$

be equal to α_2^1, we easily deduce a long exact sequence related to (A),

$$(C)\quad \begin{array}{l} 0 \longrightarrow A^1(g,\underline{I}_{X/Y}) \xrightarrow{t_1^1} A^1(\underline{d},0_{\underline{d}}) \xrightarrow{p_1^1} H^0(\underline{N}_X) \xrightarrow{\gamma} \operatorname{coker} m^1 \\ \xrightarrow{t^2} \operatorname{coker} \alpha \xrightarrow{p^2} A^2(gf,0_X) \xrightarrow{l^2} A^2(g,f_*0_X) \to A^3(f,0_X) \end{array}$$

and a diagram

$$\begin{tikzcd}
 & \operatorname{coker} m^1 \arrow[r, "t^2"] \arrow[d, hook] \arrow[dr, phantom, "\circ"] & \operatorname{coker} \alpha \arrow[d, hook] \arrow[dr, "p^2"] & \\
H^0(\underline{N}_X) \arrow[ur, "\gamma"] \arrow[r, "\alpha_1^1"'] & A^2(g,\underline{I}_{X/Y}) \arrow[r, "t_1^2"'] \arrow[d] \arrow[dr, phantom, "\circ"] & A^2(\underline{d},O_{\underline{d}}) \arrow[r, "p_1^2"'] \arrow[d, "p_2^2"] & A^2(gf,O_X) \\
 & A^2(g,O_Y) \arrow[r, "="'] & A^2(g,O_Y) &
\end{tikzcd}$$

where the sequences are exact. Consider the situation of (1.2.7) and tensorize this diagram by $\ker \eta$. By (1.2.10) one knows that $p_2^2 \otimes 1_{\ker \eta}$ maps the obstruction $o(X_S,Y_S)$ onto $o(Y_S)$. If Hilb^q is non-singular at $(Y \subseteq \mathbb{P})$, then $o(Y_S) = 0$, and $o(X_S,Y_S)$ which is an element of $A^2(\underline{d},O_{\underline{d}}) \otimes \ker \eta$, is contained in the subgroup

$$\operatorname{coker} \alpha \otimes \ker \eta .$$

In the same way the obstruction $o_{X'}(X_S,Y_S)$ (see 1.2.10)) of $A^2_{\underline{d}_1}(\underline{d},O_{\underline{d}}) \otimes \ker \eta$ sits in the subgroup

$$\operatorname{coker} m^1 \otimes \ker \eta ,$$

and we define

$$A^2(\underline{d},O_{\underline{d}})_{res} = \operatorname{coker} \alpha ,$$

$$A^2_{\underline{d}_1}(\underline{d},O_{\underline{d}})_{res} = \operatorname{coker} m^1 .$$

Using the discussion of (1.3.1;C) we prove a theorem similar to (1.2.7) where we replace $A^2(\underline{d},O_{\underline{d}})$ by $A^2(\underline{d},O_{\underline{d}})_{res} = \operatorname{coker} \alpha$. In the same way as (1.2.7) implies (1.2.8) we obtain

Theorem 1.3.2. Let $(X \subseteq Y \subseteq \mathbb{P}) \in \underset{\sim}{D}(p,q)(\mathrm{Spec}(k))$ correspond to $x \in D$, let

$$A^2(\underline{d},O_{\underline{d}})_{res} = \operatorname{coker}\alpha,$$

and let

$$T^2(\underline{d})_{res} = \mathrm{Sym}_k[A^2(\underline{d},O_{\underline{d}})^{\vee}_{res}]^{\wedge}$$

be the completion of $\mathrm{Sym}_k[A^2(\underline{d},O_{\underline{d}})^{\vee}_{res}]$ in its maximal ideal. If Hilb^q is non-singular at $pr_2(x) = (Y \subseteq \mathbb{P})$, then there is a morphism

$$o(\underline{d})_{res} : T^2(\underline{d})_{res} \to T^1(\underline{d}) ,$$

commuting with the corresponding total obstruction morphism $o(\underline{d})$ of (1.2.8), such that

$$\hat{O}_{D,x} \simeq T^1(\underline{d}) \mathop{\hat{\otimes}}_{T^2(\underline{d})_{res}} k$$

Let $a^2_{res} = \dim_k A^2(\underline{d},O_{\underline{d}})_{res}$. As in (1.2.9) we deduce that

$$a^1 - a^2_{res} \leq \dim O_{D,x} \leq a^1 ,$$

and that $D = D(p,q)$ is non-singular at $x = (X \subseteq Y \subseteq \mathbb{P})$ if

$$A^2(\underline{d},O_{\underline{d}})_{res} = \operatorname{coker}\alpha = 0$$

Corollary 1.3.3. Let $x = (X \subseteq Y \subseteq \mathbb{P}) \in D(p,q)$, let

$$T^2(\gamma) = \mathrm{Sym}_k[\operatorname{coker}\gamma^{\vee}]^{\wedge},$$

and suppose that Hilb^p and Hilb^q is non-singular at $pr_1(x) = (X \subseteq \mathbb{P})$ and $pr_2(x) = (Y \subseteq \mathbb{P})$ respectively. Then there is a morphism

$$o(\gamma) : T^2(\gamma) \to T^1(\underline{d}) ,$$

commuting with $o(\underline{d})_{res}$ of (1.3.2), such that

$$\hat{O}_{D,x} \simeq T^1(\underline{d}) \underset{T^2(\gamma)}{\hat{\otimes}} k .$$

Proof Consider the long exact sequence of (1.3.1;C) and recall that p^2 maps "obstructions to obstructions". It follows that we in (1.2.7) might replace $A^2(\underline{d},O_{\underline{d}})$ with $\ker p^2$ which by (1.3.1;C) is equal to $\operatorname{coker}\gamma$, and we are done.

In the same way, by using (1.2.11) and the discussion of (1.3.1;C), we deduce that pr_1 is smooth at x provided

$$A^2_{\underline{d}_1}(\underline{d},O_{\underline{d}})_{res} = \operatorname{coker} m^1 = 0.$$

Theorem 1.3.4. Let $(X \subseteq Y \subseteq \mathbb{P}) \in \underset{\sim}{D}(p,q)(\operatorname{Spec}(k))$ correspond to $x \in D = D(p,q)$ and suppose that Hilb^q is non-singular at $pr_2(x) = (Y \subseteq \mathbb{P})$. If $m^1 : H^0(\underline{N}_Y) \to A^1(g,f_*O_X)$ is surjective, then

$$pr_1 : D(p,q) \to \operatorname{Hilb}^p$$

is smooth at x. (The converse is true if γ is surjective).

Proof The converse is also rather easy since if pr_1 is smooth at x, the tangent map p^1_1 is surjective. Using (1.3.1;C) we deduce that

$$\operatorname{im}\gamma = 0 ,$$

and since γ is surjective, that

$$\operatorname{im}\gamma = \operatorname{coker} m^1$$

and we are done. (Compare (1.3.4) with [EGA, IV, (17.11.1)]).

Smooth morphisms are flat and flat morphisms take generic points onto generic points. Therefore

Corollary 1.3.5. Suppose there is element $x = (X \subseteq Y \subseteq \mathbb{P})$ satisfying the conditions of (1.3.4). If $W \subseteq D = D(p,q)$ is an irreducible non-embedded component, $x \in W$, then $pr_1(W)$ is an irreducible non-embedded component of Hilb^p.

In case $Y \subseteq \mathbb{P} = \mathbb{P}^n_k$ is a global complete intersection, we shall later prove (1.3.12) that the fiber D_X of $pr_1 : D(p,q) \to \mathrm{Hilb}^p$ at $pr_1(x) = (X \subseteq \mathbb{P})$ is non-singular at x. It follows that $O_{D_X,x}$ is Cohen Macaulay. If therefore $x = (X \subseteq Y \subseteq \mathbb{P}^n_k)$ satisfies the conditions of (1.3.4) and if $W \subseteq D$ is an embedded component containing x, $pr_1(W)$ is an embedded component of Hilb^p.

We shall include the follwing result concerning $H(p) = \mathrm{Hilb}^p$ at $pr_1(x) = (X \subseteq \mathbb{P})$.

Corollary 1.3.6. Let $x = (X \subseteq Y \subseteq \mathbb{P})$ satisfy the conditions of (1.3.4) and let

$$T^i(gf) = \mathrm{Sym}_k[A^i(gf, O_X)^\vee]^\wedge .$$

Then

$$A^2(\underline{d}, O_{\underline{d}})_{res} = \ker l^2 \subseteq A^2(gf, O_X) ,$$

and there is a morphism

$$o(gf)_{res} : T^2(\underline{d})_{res} \to T^1(gf)$$

commuting with the total obstruction morphism

$$o(gf) : T^2(gf) \to T^1(gf)$$

of (1.2.8) in which $X = (X \xrightarrow{gf} \mathbb{P})$, such that

$$O_{H(p),pr_1(x)} \simeq T^1(gf) \underset{T^2(\underline{d})_{res}}{\hat{\otimes}} k .$$

In particular if Hilb^q is non-singular at $pr_2(x) = (Y \subseteq \mathbb{P})$ and if m^1 is surjective, then

$$h^o(\underline{N}_X) - a^2_{res} \leq \dim O_{H(p),pr_1(x)} \leq h^o(\underline{N}_X) .$$

Moreover $H(p) = \mathrm{Hilb}^p$ is non-singular at $pr_1(x)$ if in addition $1^2 : A^2(gf, O_X) \to A^2(g, f_* O_X)$ is injective.

Proof Let $\eta : A' \twoheadrightarrow A$ be small in $\underline{1}^o$, let $S = \mathrm{Spec}(A)$ and pick $(X_S \subseteq \mathbb{P} \times S) \in \underset{\sim}{\mathrm{Hilb}}_X(S)$. By (1.3.4)

$$\underset{\sim}{pr}_1(S) : \underset{\sim}{D}_X(S) \to \underset{\sim}{\mathrm{Hilb}}_X(S)$$

is surjective, and it follows that the obstruction $o(X_S) \in A^2(gf, O_X) \otimes \ker \eta$ for the existence of a lifting of $X_S \subseteq \mathbb{P} \times S$ to S' is contained in

$$\ker 1^2 \otimes \ker \eta$$

since p^2 maps obstructions to obstructions". See (1.3.1;C) and its discussion. Moreover since $\operatorname{coker} m^1 = 0$ by assumption,

$$A^2(\underline{d}, O_{\underline{d}})_{res} = \ker 1^2 \subseteq A^2(gf, O_X)$$

by (1.3.1;C) and we are done.

Concerning the smoothness of the second projection $pr_2 : D(p,q) \to \mathrm{Hilb}^q$, we use (1.3.1) together with (1.2.11;iii) and we obtain (1.3.7;i). (1.3.7;ii) follows from [EGA, IV, (17.11.1)].

<u>Proposition 1.3.7.</u> i) If $A^2(f,O_X) = 0$, then

$$pr_2 : D(p,q) \to \mathrm{Hilb}^q$$

is smooth at $x = (X \subseteq Y \subseteq \mathbb{P})$.

ii) Suppose that Hilb^q is non-singular at $pr_2(x)$. Then pr_2 is smooth at x if and only if $D(p,q)$ is non-singular at x and the tangent map

$$p_2^1 : A^1(\underline{d},O_{\underline{d}}) \to H^o(\underline{N}_Y)$$

is surjective.

To use the theorems (1.3.2), (1.3.4) and also (1.3.7) we should like to know how to compute $A^2(\underline{d},O_{\underline{d}})_{res} = \mathrm{coker}\,\alpha$, $A^2_{\underline{d}_1}(\underline{d},O_{\underline{d}}) = \mathrm{coker}\,m^1$ and $A^2_{\underline{d}_2}(\underline{d},O_{\underline{d}}) \simeq A^2(f,O_X)$. We restrict to the case where $\mathbb{P} = \mathbb{P}^n_k$ and where $g : Y \hookrightarrow \mathbb{P}^n_k$ is a globale complete intersection, and in this case $A^2_{\underline{d}_1}(\underline{d},O_{\underline{d}})_{res}$ is easily found (1.3.8), and also $A^2_{\underline{d}_2}(\underline{d},O_{\underline{d}})$ in some special cases (1.3.9). It seems harder to give a good description of $A^2(\underline{d},O_{\underline{d}})_{res}$ which makes it possible to decide when it is trivial, or to find its k-dimension, and in Chapter 2 we shall deal with this problem and related topics.

<u>Remark 1.3.8.</u> ($g : Y \hookrightarrow \mathbb{P} = \mathbb{P}^n_k$ is a global complete intersection, $\dim Y \geq 1$). Suppose

$Y = V(F_1,\ldots,F_r) \subseteq \mathbb{P}^n_k$ is defined by a homogeneous $\bigoplus_\nu H^o(\mathbb{P}, O_{\mathbb{P}}(\nu))$-regular sequence $\{F_1,\ldots,F_r\}$, $f_i = \deg F_i$ for $1 \leq i \leq r$, and suppose $Y \supseteq X$. Since

$$m^1 : H^o(\underline{N}_Y) \simeq \bigoplus_{i=1}^r H^o(O_Y(f_i)) \to H^o(\underline{N}_Y \otimes_{O_Y} O_X) \simeq \bigoplus_{i=1}^r H^o(O_X(f_i))$$

it follows that

$$A^1_{\underline{d}_1}(\underline{d},O_{\underline{d}})_{res} = \operatorname{coker} m^1 \simeq \bigoplus_{i=1}^{r} H^1(\underline{I}_X(f_i)).$$

Notice that

$$A^2(g,O_Y) = H^1(\underline{N}_Y) \simeq \bigoplus_{i=1}^{r} H^1(O_Y(f_i))$$

and that this group is trivial if $\dim Y \geq 2$. We deduce by the diagram of (1.3.1;C) that

$$A^2_{\underline{d}_1}(\underline{d},O_{\underline{d}})_{res} = A^2_{\underline{d}_1}(\underline{d},O_{\underline{d}}),$$

$$A^2(\underline{d},O_{\underline{d}})_{res} = A^2(\underline{d},O_{\underline{d}}).$$

<u>Remark 1.3.9.</u> ($X \subseteq Y \subseteq \mathbb{P}^3_k$ where $Y = V(F)$ is a surface of degree s and where X is a divisor on Y).

In this case we shall compute

$$A^2(f,O_X) = H^1(\underline{N}_{X/Y})$$

and look closer to its consequences. One knows that

$$\underline{N}_{X/Y} = \underline{\operatorname{Ext}}^1_{O_Y}(O_X,O_X) \simeq \underline{\operatorname{Ext}}^1_{O_Y}(O_X,O_Y)$$

where the isomorphism to the right follows from the fact that X is a divisor on Y, i.e. from $\operatorname{pd}_{O_Y} O_X = 1$. So

$$\underline{N}_{X/Y} \simeq \underline{\operatorname{Ext}}^1_{O_Y}(O_X,\omega_Y)(4-s) = \omega_X(4-s).$$

It follows that

$$A^2(f,O_X) \simeq H^o(O_X(s-4))^{\vee}.$$

Therefore if $s \leq 3$ and if in addition X is reduced (or weaker if in addition $H^o(O_X(s-4)) = 0$), we deduce (A), (B), (C) and (D) below;

(A) $A^2(f,O_X) = 0$ and pr_2 is smooth at $x = (X \subseteq Y \subseteq \mathbb{P}^3_k)$.

Since $\alpha : H^o(\underline{N}_Y) \to A^2(f,O_X)$, we get

(B) $A^2(\underline{d},O_{\underline{d}})_{res} = \operatorname{coker}\alpha = 0$ and $D(p,q)$ is non-singular at x

by (1.3.2) or directly by (A). Another consequence is this. Since $\operatorname{coker}\alpha = 0$ we deduce by (1.3.1;C) that γ is surjective and that 1^2 is injective, from which it follows that 1^2 is an isomorphism since

$$A^3(f,O_X) = H^2(\underline{N}_{X/Y}) = 0.$$

By (1.3.4) and (1.3.8) we find

(C) $H^1(\underline{I}_X(s)) = 0$ iff pr_1 is smooth at x,

and by (1.3.6) and the isomorphism of

$$1^2 : H^1(\underline{N}_X) \to H^1(O_X(s))$$

that

(D) if $H^1(\underline{I}_X(s)) = 0$ <u>or</u> $H^1(O_X(s)) = 0$, then Hilb^p is non-singular at $pr_1(x) = (X \subseteq \mathbb{P}^3_k)$ and

$$\dim O_{H(p),pr_1(x)} = 4d + h^1(O_X(s)).$$

If $Y \subseteq \mathbb{P} = \mathbb{P}^3_k$ is a surface of degree s containing X, and if there is an open dense subset $U \subseteq X$ such that U is a divisor on Y, we can by the proof of (2.3.11) deduce that 1^2 is surjective.

So if $\operatorname{coker}\alpha = 0$, we easily prove (B), (C) and (D) of (1.3.9).

<u>Example 1.3.10</u>. Suppose $X \subseteq \mathbb{P}^3_k$ is a divisor on a smooth conique Y, i.e. $X \subseteq Y = \mathbb{P}^1_k \times \mathbb{P}^1_k$ is defined by $\underline{I}_{X/Y} = O_Y(-q_1,-q_2)$, $q_1 \leq q_2$. Then $d = q_1 + q_2$ is the degree of X, $g = (q_1-1)(q_2-1)$ is its genus and $p(x) = dx + 1 - g$ is its Hilbert polynomial. We claim that $H(p) = \operatorname{Hilb}^p(\mathbb{P}^3)$ is non-singular at $\operatorname{pr}_1(x) = (X \subseteq \mathbb{P}^3)$ and that

$$\dim O_{H(p),\operatorname{pr}_1(x)} = \begin{cases} 4d & \text{if } q_1 \leq 3 \\ 4d + (q_1-3)(q_2-3) & \text{if } q_1 > 3. \end{cases}$$

In fact by (1.3.9;D) it suffices to prove (1), (2) and (3) below;

(1) $$H^0(O_X(-2)) = 0$$

which guarantees that $A^2(f,O_X) = 0$ without assuming that X is reduced,

(2) $$h^1(O_X(2)) = \begin{cases} (q_1-3)(q_2-3) & \text{if } q_1 > 3 \\ 0 & \text{if } q_1 \leq 3 \end{cases}$$

(3) $$h^1(\underline{I}_X(2)) = 0 \quad \text{for } q_1 > 3.$$

Every thing is easy. Since

$$0 \to O_Y(-q_1,-q_2) \to O_Y \to O_X \to 0$$

is exact, we deduce

$$H^0(O_X(-2)) = H^1(O_Y(-2-q_1,-2-q_2)),$$
$$H^1(O_X(2)) \subseteq H^2(O_Y(2-q_1,2-q_2)),$$
$$H^1(\underline{I}_X(2)) = H^1(\underline{I}_{X/Y}(2)) = H^1(O_Y(2-q_1,2-q_2)).$$

Since $Y = \mathbb{P}^1_k \times \mathbb{P}^1_k$ we apply the Künneth formula and we are done.

Finally we study the fibers of the projection morphisms pr_i. The fiber of

$$pr_2 : D(\mathbb{P}; p,q) \to \mathrm{Hilb}^q(\mathbb{P})$$

at a K-point $(Y \subseteq \mathbb{P} \times_k \mathrm{Spec}(K))$ is just the Hilbert scheme $\mathrm{Hilb}^p(Y)$ over Spec(K) which is thouroughly treated elsewhere (see [M1]). We shall here deal with

$$pr_1 : D(\mathbb{P}^n_k; p,q) \to \mathrm{Hilb}^p(\mathbb{P}^n_k).$$

<u>Terminology 1.3.11.</u> Let $k \hookrightarrow K$ be a field extension.

i) We shall say that

$$g : Y \hookrightarrow \mathbb{P}^n_K = \mathbb{P}_K$$

<u>is of type</u> $\underline{f} = (f_1,\ldots,f_r)$ or that g is a <u>global complete intersection of type</u> $\underline{f}$ if

$$g : Y = V(F_1,\ldots,F_r) \subseteq \mathbb{P}^n_K$$

is a closed subscheme defined by a homogeneous $\bigoplus_\nu H^0(O_{\mathbb{P}_K}(\nu))$-regular sequence $\{F_1,\ldots,F_r\}$ where $f_i = \deg F_i$ for $1 \leq i \leq r$.

ii) If $T \in \mathrm{ob}\,\underline{Sch}/k$ we say that

$$g : Y_T \to \mathbb{P}^n_T = \mathbb{P}^n_k \times T$$

<u>is of type</u> $\underline{f}$ if g is a closed embedding of flat T-scheme such that the fiber morphisms

$$g_t : Y_t \to \mathbb{P}^n_{k(t)}$$

are of type $\underline{f}$ for each $t = \mathrm{Spec}(k(t)) \in T$.

iii) We define

$$D(\mathbb{P}^n_k; p; \underline{f}) = D(p;\ f_1,\ldots,f_r)$$

to be the representing object of the open subfunctor of $\underset{\sim}{D}(\mathbb{P}^n_k;\, p,q)$ consisting of elements $(X \subseteq Y \subseteq \mathbb{P}^n_T)$ such that $Y \subseteq \mathbb{P}^n_T$ is of type $\underline{f}$. It is an open subscheme of $D(p,q)$ since morphisms of type $\underline{f}$ are stable under generization [EGA, IV, (1.10.4)]. Notice that if $r = 1$ and $\underline{f} = (s)$,

$$D(p;s) = D(p,q)$$

since in this case any point $(Y \subseteq \mathbb{P}^n_k)$ of Hilb^q is of type (s).

If Z is a scheme and $z \in Z$, we let

$$\dim_z Z = \dim O_{Z,z}.$$

<u>Proposition 1.3.12</u>. The fibers of the composition $\mathrm{pr}_1^{\underline{f}}$:

$$D(p;\underline{f}) \subseteq D(p,q) \xrightarrow{\mathrm{pr}_1} \mathrm{Hilb}^p$$

are smooth and geometrically connected provided the degree of q is strictly positive. If $(X \subseteq Y \subseteq \mathbb{P}^n_K)$ is a K-point of $D(p;\underline{f})$ corresponding to $x \in D(p;\underline{f})$, and if D_z is the fiber of pr_1 at $z = \mathrm{pr}_1(x)$, then

$$\dim_x D_z = \dim_K A^1(g, \underline{I}_{X/Y})$$

where $g : Y \hookrightarrow \mathbb{P}^n_K$ is the embedding

<u>Proof</u> Let $\mathbb{P} = \mathbb{P}^n_K$ and $Y = V(F_1,\dots,F_r) \subseteq \mathbb{P}$, let $\eta : A' \twoheadrightarrow A$ be small in $\underline{l}^o_K$ (1.2.12) and let $\varphi : S = \mathrm{Spec}(A) \hookrightarrow S' = \mathrm{Spec}(A')$ be the induced morphism. To prove that $D_z \to \mathrm{Spec}(K)$ is smooth at x, we consider the diagram

$$\begin{array}{ccccc} S & \longrightarrow & D_z & \longrightarrow & D(p,q) \\ \varphi\downarrow & \circ & \downarrow & \square & \downarrow \\ S' & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & \mathrm{Hilb}^p \end{array}$$

Therefore there is a diagram of deformations

$$\begin{array}{ccccc} S' & \longleftarrow & X \times_K S' & \hookrightarrow & \mathbb{P} \times_K S' \\ \uparrow & & \uparrow & & \uparrow \\ S & \longleftarrow & X \times_K S \subseteq Y_S & \subseteq & \mathbb{P} \times_K S \end{array}$$

where $(X \times S \subseteq Y_S \subseteq \mathbb{P} \times S) \in \underset{\sim}{D}_{X \subseteq Y}(S)$. Since $\deg(q) \geq 1$, we deduce that $\dim Y \geq 1$, and since $Y \subseteq \mathbb{P}$ is of type $\underline{f}$, that $H^1(\underline{I}_Y(f_i)) = 0$ for each i. If $\underline{I}_{Y_S} = \ker(O_{\mathbb{P} \times S} \to O_{Y_S})$, then by base change theorem

$$H^0(\mathbb{P} \times S, \underline{I}_{Y_S}(f_i)) \twoheadrightarrow H^0(\mathbb{P}, \underline{I}_Y(f_i))$$

is surjective for each $i \in [1,r]$. So there is a sequence $\{F_1', \dots, F_r'\}$, $F_i' \in H^0(\underline{I}_{Y_S}(f_i))$, of elements of $R \otimes_K A$ where $R = \bigoplus_\nu H^0(O_{\mathbb{P}}(\nu))$ which reduces to $\{F_1, \dots, F_r\}$ in R via the natural map $R \otimes_K A \twoheadrightarrow R \otimes_K K \simeq R$. Since $\{F_1, \dots, F_r\}$ is a R-regular sequence, it follows that

$$R \otimes_K A/(F_1', \dots, F_r')$$

is A-flat and moreover that

$$Y_S = V(F_1', \dots, F_r') \subseteq \mathbb{P} \times S.$$

Since

$$H^0(\underline{I}_{Y_S}(f_i)) \subseteq H^0(\underline{I}_{X \times S}(f_i)),$$

and since

$$H^0(\underline{I}_{X \times S'}(f_i)) = H^0(\underline{I}_X(f_i)) \otimes_K A' \twoheadrightarrow H^0(\underline{I}_{X \times S}(f_i)) = H^0(\underline{I}_X(f_i)) \otimes_K A$$

is surjective, we may lift each F_i' to $F_i'' \in H^0(\underline{I}_{X \times S'}(f_i))$ and thus define a scheme

$$Y_{S'} = V(F_1'', \dots, F_r'') \subseteq \mathbb{P} \times S'.$$

Clearly $X \times S' \subseteq Y_{S'}$, and $Y_{S'} \to S'$ is flat. Therefore $D_z \to \mathrm{Spec}(K)$ is smooth at x and moreover

$$\dim_x D_z = \dim_K A^1(g, \underline{I}_{X/Y})$$

by (1.2.11;i) and (1.2.12).

Suppose K is algebraically closed and let $(X \subseteq Y \subseteq \mathbb{P}^n_K)$ and $(X \subseteq Y' \subseteq \mathbb{P}^n_K)$ be K-points of $D(p;\underline{f})$, say

$$Y = V(F_1, \ldots, F_r) \subseteq \mathbb{P}^n_K \quad \text{and} \quad Y' = V(G_1, \ldots, G_r) \subseteq \mathbb{P}^n_K .$$

If $B = K[t]$ is a polynomial algebra in one variable, we let

$$H_i = F_i + t(G_i - F_i) \quad \text{for} \quad 1 \leq i \leq r,$$

and $H_i \in H^0(\underline{I}_X(f_i)) \otimes_K B = H^0(\underline{I}_{X \times \mathrm{Spec}(B)}(f_i))$. If we define

$$Y_B = V(H_1, \ldots, H_r) \subseteq \mathbb{P} \times \mathrm{Spec}(B) ,$$

then $X \times \mathrm{Spec}(B) \subseteq Y_B$, and its fibers at $t = 0$ and $t = 1$, say at $x_0, x_1 \in \mathrm{Spec}(B)$ respectively, are just $X \subseteq Y$ and $X \subseteq Y'$. The morphism $Y_B \to \mathrm{Spec}(B)$ is flat at x_0 and x_1 by the first part of this proof, and also over some open subset $U_1 \subseteq \mathrm{Spec}(B) = \mathbb{A}^1_K$ by [M1, Lect 8]. Therefore it is flat over the open subset $U_2 = U \cup \{x_0\} \cup \{x_1\}$ of $\mathrm{Spec}(B)$. In particular if $z \in \mathrm{Hilb}^p$ corresponds to the K-point $(X \subseteq \mathbb{P}^n_K)$ of Hilb^p, there is a morphism

$$\psi : U_2 \to D_z$$

which induces a morphism

$$\psi|_U : U = U_2 \cap \psi^{-1}(D(p;\underline{f})_z) \to D(p;\underline{f})_z$$

where $D(p;\underline{f})_z = D(p;\underline{f}) \cap D_z$. U is irreducible, and x_0 and x_1 are contained in U and we are done.

Recall that a morphism is <u>connected</u> in the sense of Hartshorne [H 3] iff

c_1) its fibers are geometrically connected and

c_2) it is universally submersive.

Moreover one knows that a morphism is <u>universally submersive</u> in the following two cases

us_1) it is surjective and proper,

us_2) it is surjective and flat and quasicompact.

Correspondingly we define a morphism to be <u>irreducible</u> iff

i_1) its fibers are geometrically irreducible and

i_2) it is surjective and universally open.

By [EGA,IV,(2.4.6)],if a morphism is locally of finite presentation, (us_2) implies (i_2). Notice that these definitions of a morphism to be connected or irreducible are stronger than those given in [EGA,IV,(4.5.5)]. However, [EGA,IV,(4.5.7)] implies the following result:

Let $f : X \to Y$ be connected (resp. irreducible) and let $T \to Y$ be any morphism. If T is connected (resp. irreducible), then so is $X \times_Y T$.

See also [H 3, (1.8)].

<u>Corollary 1.3.13.</u> Let $\psi : T \to H(p) = \mathrm{Hilb}^p(\mathbb{P}^n_k)$ be a morphism, let

$$(pr^{\underline{f}}_T, \psi) : D(p;\underline{f}) \times_{H(p)} T \to T$$

be the second projection, and suppose the degree of the Hilbert polynomial q, corresponding to $\underline{f} = (f_1,\dots,f_r)$, is strictly positive.

i) If $r = 1$ and if (pr_1^f, ψ) is surjective, then (pr_1^f, ψ) is connected and its fibers are smooth.

ii) If $r \geq 1$ and if for any K-point $(X \subseteq \mathbb{P}_K^n)$ of T,

$$H^1(\underline{I}_X(f_i)) = 0 \quad \text{for} \quad i \leq 1 \leq r,$$

then (pr_1^f, ψ) is smooth and irreducible.

Proof. i) follows from (1.3.12), (us_1) and (c_i) for $i = 1,2$. Using (1.3.4) we find that (pr_1^f, ψ) is a smooth morphism and in particular that (us_2) holds. Therefore (i_2) holds and (ii) follows from (i).

The following remark generalizes (1.3.4) in the case $(X \subseteq Y \subseteq \mathbb{P}_k^n)$ is a point of $D(p;\underline{f})$, and it indicates a direct proof of (1.3.4). It may also be used to generalize (1.3.13) in an obvious manner.

Remark 1.3.14. Let $x = (X \subseteq Y \subseteq \mathbb{P}_k^n)$ be a point of $D(p;\underline{f})$ where $\dim Y \geq 1$, let $\underline{I}_{X_H}$ be the sheaf of ideals which defines the universal object $X_H \subseteq \mathbb{P} \times H$ of $H = \mathrm{Hilb}^p$, and let $\pi : \mathbb{P} \times H \to H$ be the second projection. We claim that if

$$\text{(A)} \qquad \pi_* \underline{I}_{X_H}(f_i) \otimes k(x) \twoheadrightarrow H^0(\underline{I}_X(f_i))$$

is surjective for all $i = 1,2,\ldots,r$, pr_1 is smooth at x. In fact if $\varphi : S = \mathrm{Spec}(A) \hookrightarrow S' = \mathrm{Spec}(A')$ is small in $\underline{l}$ and if

$$((X_S \subseteq Y_S \subseteq \mathbb{P} \times S), (X_{S'} \subseteq \mathbb{P} \times S')) \in \underset{\sim}{D}(S) \times_{\underset{\sim}{\mathrm{Hilb}}_X(S)} \underset{\sim}{\mathrm{Hilb}}_X(S'),$$

it suffices to prove that

$$\text{(B)} \qquad H^0(\underline{I}_{X_{S'}}(f_i)) \twoheadrightarrow H^0(\underline{I}_{X_S}(f_i))$$

is surjective for each i

since we then might lift each $F_i' \in H^0(\underline{I}_{X_S}(f_i))$ where $Y_S = V(F_1',\ldots,F_r')$ to $F_i'' \in H^0(\underline{I}_{X_{S'}}(f_i))$ and thus obtain a lifting $Y_{S'} = V(F_1'',\ldots,F_r'') \subseteq \mathbb{P}\times S'$ of $Y_S \subset \mathbb{P}\times S$ to S' as in the proof of (1.3.12). Moreover

(B) follows from (A) since by base change theorem and by the surjectivity of $A' \to A$,

$$H^0(\underline{I}_{X_{S'}}(f_i)) \simeq \pi_*\underline{I}_{X_H}(f_i)\otimes A' \twoheadrightarrow \pi_*\underline{I}_{X_H}(f_i)\otimes A \simeq H^0(\underline{I}_{X_S}(f_i))$$

is surjective for each i. Notice that if $(X\subseteq Y\subseteq \mathbb{P}^n_K)$ is a K-point of $D(p;\underline{f})$ corresponding to $x\in D(p;\underline{f})$ and if $k(pr_1(x))$ is the residue field of $O_{H,pr_1(x)}$, then pr_1 is smooth at x provided

$$\pi_*\underline{I}_{X_H}(f_i)\otimes k(pr_1(x)) \to H^0(\underline{I}_X(f_i))$$

is surjective for $1\leq i\leq r$.

1.4. The relationship between the local Hilbert-flag functor and the corresponding graded deformation functor.

In Section 1.3 we characterized the completion of the local k-algebra $O_{D,x}$ of $D = D(p,q)$ at $x = (X\subseteq Y\subseteq \mathbb{P})$ via a total obstruction morphism

$$o(\underline{d})_{res}: T^2(\underline{d})_{res} \to T^1(\underline{d})$$

if $Y\subseteq \mathbb{P}$ was sufficiently nice (1.3.2). The main result of this section (1.4.6) shows that under some extra conditions, stated as the triviality of a certain group ${}_oH^2_m(B,A,A)$, there is a subspace

$$ {}_oH^2(R,A,A) \subseteq A^2(\underline{d},O_{\underline{d}})_{res} $$

and an obstruction morphism

$$ \underset{\sim}{o}(\underline{d})_{gr} : T^2(\underline{d})_{gr} = \mathrm{Sym}_k[{}_oH^2(R,A,A)^{\vee}]^{\wedge} \to T^1(\underline{d}) $$

such that the diagram

$$ \begin{tikzcd} T^2(\underline{d})_{res} \arrow[rr, "o(\underline{d})_{res}"] \arrow[dr] & \circ & T^1(\underline{d}) \\ & T^2(\underline{d})_{gr} \arrow[ur, "o(\underline{d})_{gr}"'] & \end{tikzcd} $$

commutes. This implies that

$$ \hat{O}_{D,x} \simeq T^1(\underline{d}) \hat{\otimes}_{T^2(\underline{d})_{gr}} k , $$

and we deduce in Section 2.2 some useful consequences since we there succeed in computing ${}_oH^2(R,A,A)$ provided X is sufficiently nice and of codimension 2 in $\mathbb{P}$.

To be precise we fix for the remainder of this section the following situation.

(1.4.1). If Z is a closed subscheme of $\mathbb{P}^n = \mathbb{P}^n_k$, defined by an ideal $\underline{I}$, the minimal cone of Z in $\mathbb{P}^n$ is the graded k-algebra

$$ \Gamma_*(O_{\mathbb{P}^n})/\Gamma_*(\underline{I}) $$

where $\Gamma_*(\underline{F}) = \bigoplus_{\nu=-\infty}^{\infty} \Gamma(\underline{F}(\nu))$ and where $\underline{F}$ is an $O_{\mathbb{P}^n}$-Module.

Correspondingly put

$$H^i_*(\underline{F}) = \underset{\nu}{\oplus} H^i(\underline{F}(\nu)).$$

Now let $X \xrightarrow{f} Y \xrightarrow{g} \mathbb{P}$ be closed embeddings of projective k-schemes where $\mathbb{P} \subseteq \mathbb{P}^n_k$, and let R, B and A be the minimal cones of $\mathbb{P} \subseteq \mathbb{P}^n_k$, $Y \subseteq \mathbb{P}^n_k$ and $X \subseteq \mathbb{P}^n_k$ respectively with corresponding irrelevant maximal ideals $m_R \subseteq R$, $m_B \subseteq B$ and $m = m_A \subseteq A$. Then there are canonical surjections

$$R \xrightarrow{\psi} B \xrightarrow{\varphi} A,$$

and we let

$$I_B = \ker\psi,\ I = I_A = \ker\varphi\psi \quad \text{and} \quad I_{B/A} = \ker\varphi,$$

and as always $\underline{I}_Y = \ker(O_{\mathbb{P}} \to O_Y)$ etc. Moreover we shall suppose that

$$\operatorname{depth}_{m_R} R \geq 2, \quad \operatorname{depth}_{m_B} B \geq 2$$

It follows that

$$H^i_{m_R}(R) = 0 \quad \text{and} \quad H^i_{m_B}(B) = 0$$

for $i = 0,1$ and that

$$H^o_m(A) = 0,\ H^1_m(A) \simeq H^1_*(\underline{I}_X) \simeq H^1_*(\underline{I}_{X/Y}) \quad \text{and} \quad H^{i+1}_m(A) \simeq H^i_*(O_X)$$

for $i \geq 1$.

Definition 1.4.2. i) The graded deformation functor

$$\operatorname{Def}^o_{\varphi\psi} : \underline{l}^o \to \underline{\text{Sets}}$$

is defined by

$$\operatorname{Def}^o_{\varphi\psi}(C) = \left\{ R \underset{k}{\otimes} C \to A_C \,\middle|\, \begin{array}{l} R\otimes C \to A_C \text{ is a homogeneous} \\ \text{lifting of } \varphi\psi \text{ to } C \end{array} \right\}$$

ii) Correspondingly we let

$$\mathrm{Def}^{o}_{(\psi,\varphi)} : \underline{1}^{o} \to \underline{\mathrm{Sets}}$$

by the functor defined by

$$\mathrm{Def}^{o}_{(\psi,\varphi)}(C) = \left\{ R \underset{k}{\otimes} C \xrightarrow{\psi_C} B_C \xrightarrow{\varphi_C} A_C \;\middle|\; \begin{array}{l} B_C \text{ and } A_C \text{ are graded and C-flat,} \\ \varphi_C \underset{C}{\otimes} 1_k = \varphi,\ \psi_C \underset{C}{\otimes} 1_k = \psi \end{array} \right\}$$

where $1_k : k \to k$ is the identity on k.

Then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Def}^{o}_{(\psi,\varphi)} & \hookrightarrow & \underset{\sim}{D}_{X\subseteq Y} \\ \downarrow & \circ & \downarrow \\ \mathrm{Def}^{o}_{\varphi\psi} & \hookrightarrow & \underset{\sim}{\mathrm{Hilb}}_X \end{array}$$

where the horizontal morphisms are defined by applying the functor Proj. The main ideas of the proof (1.4.6) are as follows. In (1.3.2) we described the hull of $\underset{\sim}{D}_{X\subseteq Y}$; its "obstruction space" was $A^2(\underline{d},O_{\underline{d}})_{res} = \mathrm{coker}\,\alpha$. By exactly the same technique we characterize in (1.4.5) the hull of $\mathrm{Def}^{o}_{(\psi,\varphi)}$, and its "obstruction space" is ${}_{o}H^2(R,A,A)$ provided $\psi : R \to B$ is a complete intersection. The main theorem (1.4.6) is obtained by examining when $\mathrm{Def}^{o}_{(\psi,\varphi)} \hookrightarrow \underset{\sim}{D}_{X\subseteq Y}$ is an isomorphism (true if ${}_{o}H^2_{\mathfrak{m}}(B,A,A) = 0$), and in this case we also find ${}_{o}H^2(R,A,A) \subseteq \mathrm{coker}\,\alpha$.

Recall that the cohomology groups of algebras $H^i(R,A,A)$, introduced in (1.2.3), are graded A-modules, i.e.

$$H^i(R,A,A) \simeq \bigoplus_{\nu=-\infty}^{\infty} {}_{\nu}H^i(R,A,A).$$

See [Kl,(1.7)]. Moreover there is a theorem similar to (1.2.7) for deforming the morphism $\varphi\psi : R \to A$ as a graded morphism where the cohomology groups involved are ${}_oH^i(R,A,A)$ for $i = 1,2$ [Kl,(1.5)]. It follows that the tangent space of $\mathrm{Def}^o_{\varphi\psi}$ is

$$\mathrm{Def}^o_{\varphi\psi}(k[\epsilon]) = {}_oH^1(R,A,A)\ ,$$

and that we may describe a hull of $\mathrm{Def}^o_{\varphi\psi}$ as in (1.2.8) with an obvious change in the notations.

Remark 1.4.3. Let $Z' = V(m) \subseteq X' = \mathrm{Spec}(A)$ and let $f' : X' \hookrightarrow Y' = \mathrm{Spec}(B)$ be the morphism induced by $\varphi : B \to A$. To any graded A-module M there are groups $A^i_{Z'}(f',\tilde{M})$ introduced in (1.2.4). Put

$$H^i_m(B,A,M) = A^i_{Z'}(f',\tilde{M})$$

Since f' is a morphism of affine schemes, we deduce by (1.2.3) that

$$A^i(f',\tilde{M}) \simeq H^i(B,A,M).$$

Therefore there is a spectral sequence

$$H^p(B,A,H^q_m(A))$$

converging to $H^{(\cdot)}_m(B,A,A)$ by (1.2.4). These groups are graded A-modules, and the spectral sequence preserves the grading. In particular

$${}_oH^2_m(B,A,A) \simeq {}_oH^1(B,A,H^1_m(A)) = {}_o\mathrm{Hom}_B(I_{B/A},H^1_m(A)),$$

and if $\bigoplus_{i=1}^{r} B(-n_i) \twoheadrightarrow I_{B/A}$ is a graded surjection and if $H^1(\underline{I}_X(n_i)) = 0$ for $1 \le i \le r$, then

$${}_oH^2_m(B,A,A) = 0.$$

Moreover, using the isomorphism

$$A^i(f'|_{X'-Z'},O_{X'}) \simeq \bigoplus_{\nu=-\infty}^{\infty} A^i(f,O_X(\nu))$$

which is easy to prove via spactral sequence of (1.2.3), and using (1.2.4), we find a long exact sequence

$$\to {}_{\nu}H^{i}_{m}(B,A,A) \to {}_{\nu}H^{i}(B,A,A) \to A^{i}(f,O_X(\nu)) \to {}_{\nu}H^{i+1}_{m}(B,A,A) \to .$$

For details, see [Kl]. Notice that

$${}_{o}H^{2}(B,A,A) \to A^{2}(f,O_X)$$

maps "obstructions to obstructions" since the morphism

$$A^{2}(f',O_{X'}) \to A^{2}(f'|_{X'-Z'},O_{X'})$$

does (1.2.10).

<u>Remark 1.4.4.</u> (The analogue of (1.3.1)). Consider the big diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & & & {}_{o}H^{1}(R,B,I_{B/A}) \arrow[d] & & \\
 & & & {}_{o}H^{1}(R,B,B) \arrow[d, "{}^{1}\varphi_{*}"] & & \\
0 \arrow[r] & {}_{o}H^{1}(B,A,A) \arrow[r] & {}_{o}H^{1}(R,A,A) \arrow[r, "{}^{1}\varphi^{*}"] & {}_{o}H^{1}(R,B,A) \arrow[r] \arrow[d] & {}_{o}H^{2}(B,A,A) \arrow[dddll, dashed, bend left] & \\
 & & & {}_{o}H^{2}(R,B,I_{B/A}) \arrow[d] & & \\
 & & & {}_{o}H^{2}(R,B,B) \arrow[d, "{}^{2}\varphi_{*}"] & & \\
 & & {}_{o}H^{2}(R,A,A) \arrow[r, "{}^{2}\varphi^{*}"] & {}_{o}H^{2}(R,B,A) \arrow[r] \arrow[d] & {}_{o}H^{3}(B,A,A) \arrow[r] & {} \\
 & & & {} & &
\end{tikzcd}$$

of exact sequences, and let

$$ {}_{gr}E_2^{p,q} = \varprojlim^{(p)} \left\{ \begin{tikzcd} {}_oH^q(R,A,A) \arrow[dr, "{}^q\varphi^*"'] & & {}_oH^q(R,B,B) \arrow[dl, "{}^q\varphi_*"] \\ & {}_oH^q(R,B,A) & \end{tikzcd} \right\} $$

Then we define $\alpha_{B\to A}$ and $\gamma_{B\to A}$ to be the compositions

$$ {}_oH^1(R,B,B) \xrightarrow{{}^1\varphi_*} {}_oH^1(R,B,A) \to {}_oH^2(B,A,A) $$

and

$$ {}_oH^1(R,A,A) \xrightarrow{{}^1\varphi^*} {}_oH^1(R,B,A) \twoheadrightarrow \operatorname{coker} {}^1\varphi_* $$

respectively. If $\psi : R \to B$ is a complete intersection, then ${}_oH^2(R,B,-)$, and this implies that

$$ \operatorname{coker} \alpha_{B\to A} \simeq {}_oH^2(R,A,A) $$

and that

$$ \operatorname{coker} {}^1\varphi_* = 0. $$

The analogue of (1.3.2) is

<u>Proposition 1.4.5.</u> If $\psi : R \to B$ is a complete intersection, then there is a morphism

$$ o(\underline{d})_{gr}: T^2(\underline{d})_{gr} = \mathrm{Sym}_k[{}_oH^2(R,A,A)^\vee]^\wedge \to T^1(\underline{d})_{gr} = \mathrm{Sym}_k[{}_{gr}E^{o,1\vee}]^\wedge $$

such that

$$ T^1(\underline{d})_{gr} \ \hat{\otimes}_{T^2(\underline{d})_{gr}} k $$

is a hull for the functor $\mathrm{Def}^o_{(\psi,\varphi)}$.

The proof is the same as for (1.3.2), only with a change in the notations.

<u>Theorem 1.4.6.</u> Let $R \xrightarrow{\psi} B \xrightarrow{\varphi} A$ be as in (1.4.1) and assume that $\psi : R \to B$ is a complete intersetion and that

$$ {}_oH^2_m(B,A,A) = 0. $$

Then there is an injection

$$ {}_oH^2(R,A,A) \hookrightarrow \operatorname{coker}\alpha $$

and an isomorphism on $\underline{1}^o$

$$ \underset{\sim}{\mathrm{Def}}^o_{(\psi,\varphi)} \simeq \underset{\sim}{D}_{\underline{X\subseteq Y}}. $$

In fact, under the conditions above, there is a commutative diagram

$$ \begin{array}{ccc} T^2(\underline{d})_{res} & \xrightarrow{o(\underline{d})_{res}} & T^1(\underline{d}) \\ \Downarrow & \circ & \downarrow\simeq \\ T^2(\underline{d})_{gr} & \xrightarrow{o(\underline{d})_{gr}} & T^1(\underline{d})_{gr} \end{array} $$

where the total obstruction ~~morphisms~~ $o(\underline{d})_{res}$ and $o(\underline{d})_{gr}$ are as in (1.3.2) and (1.4.5) respectively.

Proof It suffices to prove that the tangent spaces of $\mathrm{Def}^o_{(\psi,\varphi)}$ and of $\underset{\sim}{D}_{\underline{X\subseteq Y}}$ are isomorphic and that there is an injective map of "obstruction spaces"

$$ {}_oH^2(R,A,A) \hookrightarrow A^2(\underline{d},O_{\underline{d}})_{res} $$

which maps "obstructions to obstructions". The tangent space of $\mathrm{Def}^o_{(\psi,\varphi)}$ is ${}_{gr}E^{o,1}_2$, see (1.4.4), and the tangent space of $\underset{\sim}{D}_{\underline{X\subseteq Y}}$ is $A^1(\underline{d},O_{\underline{d}})$ which by the spectral sequence of (1.2.5) is precisely $E^{o,1}_2$. Combining one part of the diagram of (1.3.1) with (1.4.4) we obtain the following commutative diagram

(*)
$$\begin{tikzcd}
 & & {}_oH^1(R,B,B) \arrow[r] \arrow[d, "{}^1\varphi_*"'] & A^1(g,O_Y) \arrow[dd, dash] & \\
{}_oH^1(B,A,A) \arrow[r, hook] \arrow[d, dashed] & {}_oH^1(R,A,A) \arrow[r, "{}^1\varphi^*"] \arrow[d, dashed] & {}_oH^1(R,B,A) \arrow[rr] \arrow[dr] & & {}_oH^2(B,A,A) \arrow[d] \\
A^1(f,O_X) \arrow[r, hook] & A^1(gf,O_X) \arrow[rr] & & A^1(g,f_*O_X) \arrow[r] & A^2(f,O_X)
\end{tikzcd}$$

Therefore to prove that

$$_{gr}E_2^{o,1} \simeq E_2^{o,1} = A^1(\underline{d},O_{\underline{d}})$$

it suffices to prove that

(1) $$ {}_oH^1(R,B,B) \xrightarrow{\simeq} A^1(g,O_Y)$$

and that

(2) $$ {}_oH^i(B,A,A) \to A^i(f,O_X)$$

is an isomorphism (resp. an injection) for $i = 1$ (resp. $i = 2$). The isomorphism (1) follows from $\text{depth}_{m_B} B \geq 2$ since by (1.4.3) we know that there is an exact sequence

$$0 \to {}_oH^1(R,B,B) \to A^1(g,O_Y) \to {}_oH^2_{m_B}(R,B,B) \to$$

and that

$${}_oH^2_{m_B}(R,B,B) = {}_oH^1(R,B,H^1_{m_B}(B)) = 0.$$

Moreover (2) follows from ${}_oH^2_m(B,A,A) = 0$ by corresponding arguments. Now to prove that there is an injection

$${}_oH^2(R,A,A) = \operatorname{coker}\alpha_{B\to A} \hookrightarrow \operatorname{coker}\alpha\ ,$$

we consider the diagram (*) above and the definitions of $\alpha_{B\to A}$ (1.4.4) and α (1.3.1), and we easily get a well-defined map

$$\operatorname{coker}\alpha_{B\to A} \to \operatorname{coker}\alpha\ .$$

This map is injective by (1) and the injection of (2), and it maps "obstructions to obstructions" since the morphisms

$$ {}_{o}H^{2}(B,A,A) \to {}_{o}H^{2}(R,A,A) \quad \text{and} $$
$$ {}_{o}H^{2}(B,A,A) \to A^{2}(f,O_X) $$

do correspondingly, see (1.4.3) and (1.2.10), and we are done.

Usually we apply (1.4.6) in the following form.

Corollary 1.4.7. Let $\{F_1,\dots,F_r\}$ be a homogeneous R-regular sequence such that $I_B = (F_1,\dots,F_r)$ and let $I_A = (F_1,\dots,F_r, F_{r+1},\dots,F_t)$. Put $f_i = \deg F_i$ for all i. If

$$ H^1(\underset{\sim}{I}_X(f_i)) = 0 \quad \text{for} \quad r+1 \leq i \leq t $$

and if

$$ {}_{o}H^{2}(R,A,A) = 0 \ , $$

$D(p,q)$ is non-singular at $x = (X \subseteq Y \subseteq \mathbb{P})$.

This result follows from (1.4.6) and (1.4.3).

Remark 1.4.8. Consider the diagram

$$\begin{array}{ccc} \mathrm{Def}^{o}_{(\psi,\varphi)} & \hookrightarrow & \underset{\sim}{D}_{X\subseteq Y} \\ \downarrow & \circ & \downarrow \\ \mathrm{Def}^{o}_{\varphi\psi} & \hookrightarrow & \underset{\sim}{\mathrm{Hilb}}_X \end{array}$$

and suppose $\psi : R \to B$ is a complete intersection. Then

$$ \mathrm{Def}^{o}_{(\psi,\varphi)} \hookrightarrow \underset{\sim}{D}_{X\subseteq Y} $$

is an isomorphism if ${}_{o}H^{2}_{m}(B,A,A) = 0$. Correspondingly if ${}_{o}H^{2}_{m}(R,A,A) = 0$, then

$$ \mathrm{Def}^{o}_{\varphi\psi} \longrightarrow \underset{\sim}{\mathrm{Hilb}}_X $$

is an isomorphism. This follows mainly from

$$0 \to {}_oH^1(R,A,A) \to H^0(\underline{N}_X) \to {}_oH^2_m(R,A,A) \to$$

$$_oH^2(R,A,A) \to A^2(gf,O_X) \to {}_oH^3_m(R,A,A) \to$$

since $_oH^2(R,A,A) \to A^2(gf,O_X)$ maps "obstructions to obstructions" (1.4.3). Moreover

(A) $\quad {}_oH^2(R,A,A) = 0$ and $_oH^3_m(R,A,A) = 0$

or

(B) $\quad {}_oH^2(R,A,A) = 0$ and $_oH^2_m(R,A,A) = 0$

implies that Hilb^p is non-singular at $(X \subseteq \mathbb{P})$ since in the first case, $A^2(gf,O_X) = 0$, and in the second case, $\underset{\sim}{\mathrm{Hilb}}_X \simeq \mathrm{Def}^o_{\varphi\psi}$ is formally smooth. Correspondingly if $\psi : R \to B$ is a complete intersection, $D(p,q)$ is non-singular at $(X \subseteq Y \subseteq \mathbb{P})$ provided $_oH^2(R,A,A) = 0$ and $_oH^3_m(B,A,A) = 0$ or $_oH^2_m(B,A,A) = 0$. Notice that if $I_A = (F_1,\dots,F_t)$ and if

$$H^1(\underline{I}_X(f_i)) = 0$$

for $1 \leq i \leq t$, then

$$_oH^2_m(R,A,A) = {}_o\mathrm{Hom}_R(I_A, H^1_m(A)) = 0$$

by the spectral sequence of (1.4.3). This spectral sequence is also useful if we want to compute $_oH^3_m(R,A,A)$ or $_oH^3_m(B,A,A)$.

In Section 1.3 we considered the smoothness of

$$\underset{\sim}{pr}_1 : \underset{\sim}{D}_{X\subseteq Y} \to \underset{\sim}{\mathrm{Hilb}}_X .$$

Correspondingly

$$\mathrm{Def}^{o}_{(\psi,\varphi)} \to \mathrm{Def}^{o}_{\varphi\psi}$$

is formally smooth if $\psi : R \to B$ is a complete intersection. This follows from the surjectivity of ${}^{1}\varphi_{*}$.

Chapter 2.

2.1. Gorenstein duality.

Let R be any local Gorenstein ring of dimension $n+1$, and let m be its maximal ideal. If M is an R-module of finite type, then it is well known that the Yoneda pairing

$$(1) \qquad H^{i+1}_m(M) \times \mathrm{Ext}^{n-i}_R(M,R) \to H^{n+1}_m(R)$$

induces isomorphisms

$$\varphi_i : H^{i+1}_m(M) \longrightarrow \mathrm{Hom}_R(\mathrm{Ext}^{n-i}_R(M,R), H^{n+1}_m(R))$$

$$\psi_i : \mathrm{Ext}^{n-i}_R(M,R)^\wedge \to \mathrm{Hom}_R(H^{i+1}_m(M), H^{n+1}_m(R))$$

for any integer $i \in \mathbb{Z}$, where $(-)^\wedge$ means completion with respect to the m-adic topology. See [H 2].

Correspondingly, let k be a field, and let R be a graded k-algebra of dimension $n+1$ with irrelevant maximal ideal m. Usually we will denote by ${}_\nu M$ the component of degree ν of a graded R-module M. If R is a quotient of a finitely generated free k-algebra, and if R_m is Gorenstein, then there is an integer p such that the Yoneda pairing

$$(2) \qquad {}_\nu H^{i+1}_m(M) \times {}_{-\nu-p}\mathrm{Ext}^{n-i}_R(M,R) \to {}_{-p}H^{n+1}_m(R) \cong k$$

is non-singular for any $i \in Z$ and $\nu \in Z$ and any graded R-module M of finite type. Let $\mathbb{P} = \mathrm{Proj}(R)$. Using that

$$H^i(\mathbb{P}, \tilde{M}(\nu)) \simeq {}_\nu H^{i+1}_m(M), \qquad i \geq 1 \text{ and } \nu \in \mathbb{Z},$$

$$\mathrm{Ext}^{n-i}_{O_\mathbb{P}}(\tilde{M}, O_\mathbb{P}(\nu)) \simeq {}_\nu\mathrm{Ext}^{n-i}_R(M,R), \quad i \geq 1 \text{ and } \nu \in \mathbb{Z},$$

and some exact sequences involving the terms $i = 0,-1$, we find that (2) is equivalent to the fact that the Yoneda pairing

$$(3) \qquad H^i(\mathbb{P},\tilde{M}(\nu)) \times \mathrm{Ext}^{n-i}_{O_{\mathbb{P}}}(\tilde{M}(\nu),O_{\mathbb{P}}(-p)) \to H^n(\mathbb{P},O_{\mathbb{P}}(-p)) \cong k$$

is non-singular for any $i \in \mathbb{Z}$. The integer p above is therefore given by $\omega_{\mathbb{P}} = O_{\mathbb{P}}(-p)$ where $\omega_{\mathbb{P}}$ is the dualizing sheaf on $\mathbb{P}$. See [A.K., (1.3.)].

In particular (1) and (2) tells us how to relate $\mathrm{Ext}^i_R(M,R)$, resp. ${}_\nu\mathrm{Ext}^i_R(M,R)$ in the graded case, to other known cohomology groups. The main theorem of this section says that we may in a similar way relate the groups $\mathrm{Ext}^i_R(M,N)$, resp. ${}_\nu\mathrm{Ext}^i_R(M,N)$, to other groups provided M and N are R-modules of finite type and of finite projective dimension. If $M = N$ is an ideal I of R and $A = R/I$, this gives a nice description of $\mathrm{Ext}^i_R(I,I)$. Moreover since we in Section 2.2 will prove that

$$H^i(R,A,A) \simeq \mathrm{Ext}^i_R(I,I)$$

for $i = 1,2$ in most cases where $\dim R - \dim A = 2$, we will there deduce useful informations about the cohomology group $H^2(R,A,A)$. In particular we have a vanishing criterion for ${}_0H^2(R,A,A)$. In view of the results of Section 1.4, such as (1.4.7), it is important for the study of the Hilbert-flag scheme to have such vanishing theorems, and as corollaries we will prove that, under some conditions, $D(p,q)$ and Hilb^p are non-singular at $(X \subseteq Y \subseteq \mathbb{P})$ and $(X \subseteq \mathbb{P})$ respectively where $X = \mathrm{Proj}(A)$ and where Y is some global complete intersection containing X.

In the following let R be a noetherian ring, let $m \subseteq R$ be an ideal, and let M and N be R-modules. Recall that $\Gamma_m(M) = H^0_m(M)$

is the kernel of the restriction map $M \to \Gamma(P',\tilde{M})$ where $P' = \mathrm{Spec}(R) - V(m)$, and that $H^i_m(-)$ is the right derived functor of $\Gamma_m(-)$. See [H 2].

Definition 2.1.1. Put

$$\mathrm{Hom}_m(M,-) = \Gamma_m \cdot \mathrm{Hom}_R(M,-).$$

So $\mathrm{Hom}_m(M,-)$ is a left exact covariant A-linear functor, and we let $\mathrm{Ext}^i_m(M,-)$ be its right derived functor.

Proposition 2.1.2. Let M be of finite type.

i) There are two spectral sequences given by

$$E_2^{p,q} = H^p_m(\mathrm{Ext}^q_R(M,N))$$

$$'E_2^{p,q} = \mathrm{Ext}^p_R(M,H^q_m(N))$$

converging to $\mathrm{Ext}^{(\cdot)}_m(M,N)$.

ii) Moreover there is a long exact sequence

$$\to \mathrm{Ext}^i_m(M,N) \to \mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_{O_{P'}}(\tilde{M},\tilde{N}) \to \mathrm{Ext}^{i+1}_m(M,N) \to$$

which is functorial in both M and N.

See [SGA 2, exp. VI] for proofs and details.

Remark 2.1.3. If R, m, M and N are graded, then we may in a similar way define ${}_\nu H^i_m(-)$ and ${}_\nu \mathrm{Ext}^i_m(M,-)$. If M is of finite type, then

$$H^i_m(N) \simeq \bigoplus_{\nu \in \mathbf{Z}} {}_\nu H^i_m(N)$$

and

$$\mathrm{Ext}^i_m(M,N) \simeq \bigoplus_{\nu \in \mathbf{Z}} {}_\nu \mathrm{Ext}^i_m(M,N)$$

are isomorphisms. Moreover the spectral sequences of (2.1.2) preserve the grading, and if $\mathbb{P} = \mathrm{Proj}(R) - V(m)$, there is a long exact sequence

$$\to {}_{\nu}\mathrm{Ext}^i_m(M,N) \to {}_{\nu}\mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_{O_{\mathbb{P}}}(\tilde{M},\tilde{N}(\nu)) \to {}_{\nu}\mathrm{Ext}^{i+1}_m(M,N) \to$$

We can now state and prove the following duality theorem.

<u>Theorem 2.1.4.</u> Let (R,m) be a local Gorenstein ring of dimension $n+1$, and let M and N be R-modules of finite type and of finite projective dimension. Then there is an R-linear map

$$\pi : \mathrm{Ext}^{n+1}_m(N,N) \to H^{n+1}_m(R)$$

which composed with the Yoneda pairing

$$\mathrm{Ext}^{i+1}_m(N,M) \times \mathrm{Ext}^{n-i}_R(M,N) \to \mathrm{Ext}^{n+1}_m(N,N)$$

induces a pairing which gives rise to isomorphisms

$$\varphi_i : \mathrm{Ext}^{i+1}_m(N,M) \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Ext}^{n-i}_R(M,N),H^{n+1}_m(R))$$

$$\psi_i : \mathrm{Ext}^{n-i}_R(M,N)^{\wedge} \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Ext}^{i+1}_m(N,M),H^{n+1}_m(R))$$

for all $i \in \mathbb{Z}$ where $(-)^{\wedge}$ means completion with respect to m.

<u>Proof.</u> We will use the notation $(-)^{\vee} = \mathrm{Hom}_R(-,H^{n+1}_m(R))$.

<u>Step 1.</u> Let

$$0 \to P_r \to P_{r-1} \to \dots \to P_o \to N \to 0$$

be a projective resolution of N by free R-modules, and let

$$N_i = \begin{cases} \mathrm{coker}(P_{i+1} \to P_i) & \text{for } 0 \le i < r \\ P_r & \text{for } i = r. \end{cases}$$

So $N_o = N$. We claim that there are R-linear maps

$$\mathrm{Tr}_i : \mathrm{Ext}_m^{n+1}(P_i,P_i) \to H_m^{n+1}(R) \quad \text{for} \quad 0 \leq i \leq r \quad \text{and}$$

$$T_i \ : \mathrm{Ext}_m^{n+1}(N_i,N_i) \to H_m^{n+1}(R) \quad \text{for} \quad 0 \leq i \leq r$$

and moreover obvious maps such that, for each $i \geq 0$, the diagram

$$(*)\quad \begin{tikzcd}
\mathrm{Ext}_m^{n+1}(P_i,P_{i+1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Ext}_m^{n+1}(P_i,N_{i+1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Ext}_m^{n+1}(P_i,P_i) \arrow[d, "\mathrm{Tr}_i"] \\
\mathrm{Ext}_m^{n+1}(N_{i+1},P_{i+1}) \arrow[r, "\alpha_{i+1}"] \arrow[d, "\beta_{i+1}"] \arrow[dr, phantom, "\circ"] & \mathrm{Ext}_m^{n+1}(N_{i+1},N_{i+1}) \arrow[r, "T_{i+1}"'] \arrow[d, "T_{i+1}"] & H_m^{n+1}(R) \\
\mathrm{Ext}_m^{n+1}(P_{i+1},P_{i+1}) \arrow[r, "\mathrm{Tr}_{i+1}"] & H_m^{n+1}(R) &
\end{tikzcd}$$

commutes, and such that the diagram

$$(**)\quad \begin{tikzcd}
\mathrm{Ext}_m^{n+1}(N,P_o) \arrow[r, "\alpha_o"] \arrow[d, "\beta_o"] \arrow[dr, phantom, "\circ"] & \mathrm{Ext}_m^{n+1}(N,N) \arrow[d, "T_o"] \\
\mathrm{Ext}_m^{n+1}(P_o,P_o) \arrow[r, "\mathrm{Tr}_o"'] & H_m^{n+1}(R)
\end{tikzcd}$$

also does. We then define

$$\pi : \mathrm{Ext}_m^{n+1}(N,N) \to H_m^{n+1}(R)$$

to be the map T_o.

First we define Tr_i as follows. If $\mathrm{Tr} : \mathrm{Hom}_R(P_i,P_i) \to R$ is the usual trace map, then let

$$\mathrm{Tr}_i : \mathrm{Ext}_m^{n+1}(P_i,P_i) = H_m^{n+1}(\mathrm{Hom}_R(P_i,P_i)) \xrightarrow{H_m^{n+1}(\mathrm{Tr})} H_m^{n+1}(R).$$

See the first spectral sequence of (2.1.2). Since the diagram of trace maps and obvious maps

$$\begin{tikzcd} \mathrm{Hom}_R(P_i,P_{i+1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Hom}_R(P_i,P_i) \arrow[d, dash, "Tr"] \\ \mathrm{Hom}_R(P_{i+1},P_{i+1}) \arrow[r, "Tr"'] & R \end{tikzcd}$$

commutes, the diagram

$$(***) \qquad \begin{tikzcd} \mathrm{Ext}_m^{n+1}(P_i,P_{i+1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Ext}_m^{n+1}(P_i,P_i) \arrow[d, "Tr_i"] \\ \mathrm{Ext}_m^{n+1}(P_{i+1},P_{i+1}) \arrow[r, "Tr_{i+1}"'] & H_m^{n+1}(R) \end{tikzcd}$$

also does. In particular, if we define $T_r = Tr_r$, then (*) commutes for $i = r-1$ because $N_r = P_r$.

Next, we define T_i for $0 \leq i < r$ inductively such that (*) and (**) commute. In fact assume that T_{i+2} is defined, $i \geq -1$, and that (*) holds for $i+2$ instead of $i+1$. It follows that the diagram

$$\begin{tikzcd} \mathrm{Ext}_m^{n+1}(N_{i+1},N_{i+2}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Ext}_m^{n+1}(N_{i+1},P_{i+1}) \arrow[r, "\alpha_{i+1}"] \arrow[d, "\beta_{i+1}"] & \mathrm{Ext}_m^{n+1}(N_{i+1},N_{i+1}) \\ \mathrm{Ext}_m^{n+1}(P_{i+1},N_{i+2}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Ext}_m^{n+1}(P_{i+1},P_{i+1}) \arrow[d, "Tr_{i+1}"] & \\ \mathrm{Ext}_m^{n+1}(N_{i+2},N_{i+2}) \arrow[r, "T_{i+2}"'] & H_m^{n+1}(R) & \end{tikzcd}$$

commutes. Since the upper horizontal sequence is exact and since the vertical sequence to the left is part of a complex, we find that the dottet arrow in the diagram

$$\begin{tikzcd}
\mathrm{Ext}^{n+1}_m(N_{i+1},N_{i+2}) \arrow[r] \arrow[dr, dashed, "\circ"] & \mathrm{Ext}^{n+1}_m(N_{i+1},P_{i+1}) \arrow[r] \arrow[d, "\mathrm{Tr}_{i+1}\circ\beta_{i+1}"] & \mathrm{Im}\,\alpha_{i+1} \arrow[r] & 0 \\
 & H^{n+1}_m(R) & &
\end{tikzcd}$$

is the zero map. So there is a map $T' : \mathrm{Im}\,\alpha_{i+1} \to H^{n+1}_m(R)$ which coincides with $\mathrm{Tr}_{i+1}\circ\beta_{i+1}$ on $\mathrm{Ext}^{n+1}_m(N_{i+1},P_{i+1})$. Now we have

$$\begin{tikzcd}
\mathrm{Im}\,\alpha_{i+1} \arrow[r, hook] \arrow[d, "T'"] & \mathrm{Ext}^{n+1}_m(N_{i+1},N_{i+1}) \\
H^{n+1}_m(R) &
\end{tikzcd}$$

and $H^{n+1}_m(R)$ is injective as an R-module, so there exists an R-linear map $T_{i+1} : \mathrm{Ext}^{n+1}_m(N_{i+1},N_{i+1}) \to H^{n+1}_m(R)$ extending T'. In particular, the diagram (**) commutes and also (*) if we prove that the diagram

$$\begin{tikzcd}
\mathrm{Ext}^{n+1}_m(P_i,N_{i+1}) \arrow[r] \arrow[d] & \mathrm{Ext}^{n+1}_m(P_i,P_i) \arrow[d, "\mathrm{Tr}_i"] \\
\mathrm{Ext}^{n+1}_m(N_{i+1},N_{i+1}) \arrow[r, "T_{i+1}"'] & H^{n+1}_m(R)
\end{tikzcd}$$

commutes. This, however, follows from (***) since

$$\mathrm{Ext}^{n+1}_m(P_i,P_{i+1}) \twoheadrightarrow \mathrm{Ext}^{n+1}_m(P_i,N_{i+1})$$

is surjective, and this surjectivity follows from the last spectral sequence of (2.1.2).

<u>Step 2</u>. We claim that the Yoneda pairings composed with Tr_j and π,

$$\mathrm{Ext}^{i+1}_m(P_j,M)\times\mathrm{Ext}^{n-i}_R(M,P_j) \to \mathrm{Ext}^{n+1}_m(P_j,P_j) \xrightarrow{\mathrm{Tr}_j} H^{n+1}_m(R)$$

$$\mathrm{Ext}^{i+1}_m(N,M)\times\mathrm{Ext}^{n-i}_R(M,N) \to \mathrm{Ext}^{n+1}_m(N,N) \xrightarrow{\pi} H^{n+1}_m(R)$$

1) This injection can be shown to be an isomorphism.

give rise to commutative diagrams

$$\begin{tikzcd}
\operatorname{Ext}^{i+1}_m(N,M) \arrow[r] \arrow[d, "\varphi_i"] \arrow[dr, phantom, "\circ"] & \operatorname{Ext}^{i+1}_m(P_o,M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \operatorname{Ext}^{i+1}_m(P_1,M) \arrow[d] \\
\operatorname{Ext}^{n-i}_R(M,N)^\vee \arrow[r] & \operatorname{Ext}^{n-i}_R(M,P_o)^\vee \arrow[r] & \operatorname{Ext}^{n-i}_R(M,P_1)^\vee
\end{tikzcd}$$

Indeed, to see that the diagram to the right is commutative, we observe that there is a commutative diagram of Yoneda pairings

$$\begin{tikzcd}
\operatorname{Ext}^{i+1}_m(P_1,M) & \times \operatorname{Ext}^{n-i}(M,P_1) \arrow[r] \arrow[dr, phantom, "\circ"] & \operatorname{Ext}^{n+1}_m(P_1,P_1) \\
\operatorname{Ext}^{i+1}_m(P_o,M) \arrow[u] \arrow[d, "1"] & \times \operatorname{Ext}^{n-i}_R(M,P_1) \arrow[u, "1"] \arrow[d] \arrow[r] \arrow[dr, phantom, "\circ"] & \operatorname{Ext}^{n+1}_m(P_o,P_1) \arrow[u] \arrow[d] \\
\operatorname{Ext}^{i+1}_m(P_o,M) & \times \operatorname{Ext}^{n-i}_R(M,P_o) \arrow[r] & \operatorname{Ext}^{n+1}_m(P_o,P_o)
\end{tikzcd}$$

where 1 means equality, the proof of which is straightforward, using for instance [A.K.,IV,(1.2)]. Now we use (***) of step 1 for $i = 0$, and we conclude as expected. The commutativity of the diagram to the left is similarly treated, we need to use (**) of step 1.

<u>Step 3</u>. First we observe that ψ_i is an isomorphism for any i if we can prove that

$$\varphi_i : \operatorname{Ext}^{i+1}_m(N,M) \to \operatorname{Ext}^{n-i}_R(M,N)^\vee$$

also is. To see that φ_i is an isomorphism, start with $M = R$. If $i > n$, then

$$\operatorname{Ext}^{i+1}_m(N,R) = \operatorname{Ext}^{i-n}_R(N,H^{n+1}_m(R)) = 0$$

by the injectivity of $H_m^{n+1}(R)$. If $i < n$, then both groups vanish trivially. For $i = n$, we need to prove that the pairing induces an isomorphism

$$\varphi_n = \varphi_n(N) : \mathrm{Ext}_m^{n+1}(N,R) = \mathrm{Hom}_R(N,H_m^{n+1}(R)) \to \mathrm{Hom}_R(N,H_m^{n+1}(R)).$$

However, by step 2, we have a commutative diagram and exact horizontal sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(N,H_m^{n+1}(R)) \arrow[r] \arrow[d, "\varphi_n(N)"] & \mathrm{Hom}_R(P_o,H_m^{n+1}(R)) \arrow[r] \arrow[d, "\varphi_n(P_o)"] & \mathrm{Hom}_R(P_1,H_m^{n+1}(R)) \arrow[d, "\varphi_n(P_1)"] \\
0 \arrow[r] & \mathrm{Hom}_R(N,H^{n+1}(R)) \arrow[r] & \mathrm{Hom}_R(P_o,H_m^{n+1}(R)) \arrow[r] & \mathrm{Hom}_R(P_1,H_m^{n+1}(R)).
\end{tikzcd}$$

It will therefore be sufficient to prove that $\varphi_n(P_i)$ are isomorphisms for $i = 0,1$. If $R \hookrightarrow P_i$ is a direct factor in P_i, then the arguments of step 2 show that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(P_i,H_m^{n+1}(R)) \arrow[r, twoheadrightarrow] \arrow[d, "\varphi(P_i)"] & \mathrm{Hom}(R,H_m^{n+1}(R)) \arrow[d, "\varphi_n(R)"] \\
\mathrm{Hom}_R(P_i,H_m^{n+1}(R)) \arrow[r, twoheadrightarrow] & \mathrm{Hom}(R,H_m^{n+1}(R))
\end{tikzcd}$$

is commutative. Moreover, the horizontal arrows are split, and so it will be sufficient to show that $\varphi_n(R)$ is an isomorphism. This, however, follows from the usual Gorenstein duality.

We have now proved that φ_i is an isomorphism in case $M = R$ for any $i \in \mathbb{Z}$, and it follows easily that φ_i is an isomorphism if M is R-free. For the general case, let

$$0 \to M_o \to F_o \to M \to 0$$

be exact where F_o is R-free, and suppose inductively that the

φ_i's concerning M_o and F_o are isomorphisms for all $i \in \mathbb{Z}$. Since the Yoneda pairing is δ-functorial, see [A.K.,IV,(1.1)], there are commutative diagrams

$$\begin{tikzcd}
\to \mathrm{Ext}^i_m(N,M_o) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Ext}^i_m(N,F_o) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Ext}^i_m(N,M) \arrow[r] \arrow[d, "\varphi_{i-1}"] \arrow[dr, phantom, "\circ"] & \mathrm{Ext}^{i+1}_m(N,M_o) \to \arrow[d] \\
\to \mathrm{Ext}^{n+1-i}_R(M_o,N)^\vee \arrow[r] & \mathrm{Ext}^{n+1-i}_R(F_o,N)^\vee \arrow[r] & \mathrm{Ext}^{n+1-i}_R(M,N)^\vee \arrow[r] & \mathrm{Ext}^{n-i}_R(M_o,N)^\vee \to
\end{tikzcd}$$

and exact horizontal sequences, and φ_i is an isomorphism for all $i \in \mathbb{Z}$ by the five-lemma, as required.

<u>Remark 2.1.5</u>. Clearly a similar theorem holds in the graded case too. With assumptions as in (2), i.e. that R is a quotient of a finitely generated free k-algebra and that R_m is Gorenstein, in which case we say that the cone of $\mathbb{P} = \mathrm{Proj}(R)$ is Gorenstein, we may prove that if M and N is of finite type and of finite projective dimension as graded R-modules, then there is a map

$$\pi : {}_{-p}\mathrm{Ext}^{n+1}_m(N,N) \to {}_{-p}H^{n+1}_m(R) \cong k$$

which composed with the Yoneda pairing

$${}_\nu\mathrm{Ext}^{i+1}_m(N,M) \times {}_{-\nu-p}\mathrm{Ext}^{n-i}_R(M,N) \to {}_{-p}\mathrm{Ext}^{n+1}_m(N,N)$$

induces a pairing which is non-singular for every integer i and ν. In particular

$$\varphi_i : {}_\nu\mathrm{Ext}^{i+1}_m(N,M) \xrightarrow{\sim} {}_{-\nu-p}\mathrm{Ext}^{n-i}_R(M,N)^\vee$$

$$\psi_i : \mathrm{Ext}^{n-i}_R(M,N) \xrightarrow{\sim} {}_\nu\mathrm{Ext}^{i+1}_m(N,M)^\vee$$

where now $(-)^\vee = \mathrm{Hom}_k(-,k)$.

For later applications we state and prove the following result.

Corollary 2.1.6. Let $R = k[X_0,X_1,X_2,X_3]$ be a polynomial ring over k, let $X \subseteq \mathbb{P}_k^3 = \mathrm{Proj}(R)$ be a closed subscheme with minimal cone $A = R/I$, and let

$$0 \to \bigoplus_{i=1}^{r_3} R(-n_{3i}) \xrightarrow{N} \bigoplus_{i=1}^{r_2} R(-n_{2i}) \xrightarrow{M} \bigoplus_{i=1}^{r_1} R(-n_{1i}) \to I \to 0$$

be a graded resolution of I. If we define

$$\varphi_A(\nu) : \bigoplus_{i=1}^{r_1} H^1(\underline{I}_X(n_{1i}+\nu)) \to \bigoplus_{i=1}^{r_2} H^1(\underline{I}_X(n_{2i}+\nu))$$

to be the map induced by the transpose of the matrix M appearing in the resolution of I, then

$${}_{\nu}\mathrm{Ext}_R^2(I,I)^{\vee} \simeq {}_{-\nu-4}\mathrm{Ext}_m^2(I,I) = \ker \varphi_A(-\nu-4).$$

In particular if

$$H^1(\underline{I}_X(n_{1i}-4)) = 0 \quad \text{for} \quad 1 \leq i \leq r_1,$$

then

$${}_0\mathrm{Ext}_R^2(I,I) = 0.$$

Note that since A is the minimal cone of $X \subseteq \mathbb{P}_k^3$, $\mathrm{depth}_m A \geq 1$, and since

$$\mathrm{pd}_R A + \mathrm{depth}_m A = \dim R = 4,$$

it follows that $\mathrm{pd}_R I \leq 2$. So there is always a resolution of I as in (2.1.6).

Proof. According to (2.1.5)

$${}_{\nu}\mathrm{Ext}_R^2(I,I)^{\vee} \simeq {}_{-\nu-4}\mathrm{Ext}_m^2(I,I),$$

and by (2.1.2 i)

$${}_{-\nu-4}\mathrm{Ext}_m^2(I,I) = {}_{-\nu-4}\mathrm{Hom}_R(I,H_m^2(I)).$$

Now applying ${}_{-\nu-4}\mathrm{Hom}_R(-,H^2_m(I))$ to the resolution of I, and using that

$$H^2_m(I) \simeq \bigoplus_{\nu=-\infty}^{\infty} H^1(\underline{I}_X(\nu))$$

we find

$$_{-\nu-4}\mathrm{Hom}_R(I,H^2_m(I)) = \ker \varphi_A(-\nu-4)$$

as required.

2.2. Relationship between cohomology groups of algebras and Ext-groups in codimension 2.

Let $R \twoheadrightarrow R/I = A$ be a surjective ring-homomorphism, and let M be an A-module. Recall that a common way of relating cohomology groups of algebras to Ext-groups goes via the spectral sequence

$$\mathrm{Ext}^p_A(H_q(R,A,A),M)$$

which converges to $H^{p+q}(R,A,M)$. See [An,(16.1)] for details. It follows that

$$H^1(R,A,M) \simeq \mathrm{Hom}_A(I/I^2,M) \simeq \mathrm{Hom}_R(I,M)$$

which is easy to see anyway. Moreover if $R \twoheadrightarrow A$ is locally a complete intersection outside an ideal m of A and if $\mathrm{depth}_m A \geq 1$, the spectral sequence above proves that

$$H^2(R,A,A) \simeq \mathrm{Ext}^1_A(I/I^2,A).$$

Clearly one may also try to compare $H^i(R,A,A)$ with $\mathrm{Ext}^{i-1}_R(I,A)$ and also with $\mathrm{Ext}^i_R(I,I)$ for $i = 1,2$. If we work in the codimension 2 case, i.e. if $\dim R - \dim A = 2$, it turns out that the comparison of $H^i(R,A,A)$ with $\mathrm{Ext}^i_R(I,I)$ is quite natural, and the main theorem of this section shows that they are isomorphic, i.e. we have

<u>Theorem 2.2.1.</u> Let $X \subseteq \mathbb{P}$ be generically a complete intersection of projective schemes over a field k, and suppose that $\mathbb{P}$ is non-singular along X and that there is an embedding $\mathbb{P} \subseteq \mathbb{P}_k^N$ whose minimal cone R is Cohen Macaulay. Moreover suppose that X is equidimensional and Cohen Macaulay, and that $\dim \mathbb{P} - \dim X = 2$ and $\dim \mathbb{P} \geq 3$. If A is the minimal cone of $X \subseteq \mathbb{P}_k^N$, then

$$_{\nu}H^i(R,A,A) \simeq {}_{\nu}\mathrm{Ext}_R^i(I,I)$$

for $i = 1,2$. Moreover $R \simeq \mathrm{Hom}_R(I,I)$.

As applications we use in the end of this section the main theorem above for $i = 2$ and $\nu = 0$ to study curves in projective 3-space. We deduce a vanishing criterion for $H^1(\underline{N}_X)$ and also another result implying the non-singularity of Hilb^p at $(X \subseteq \mathbb{P}_k^3)$. Even more interesting are, perhaps, the corresponding results for $D(p,q)$ since they usually apply to a larger class of examples. Finally to illustrate, several examples of curves of low degree and genus are considered.

To prove the theorem above, we need the following lemma.

<u>Lemma 2.2.2.</u> Let $R \to R/I = A$ be a morphism of noetherian rings, and let M be an A-module of finite type. Then there is a natural injective morphism

$$H^2(R,A,M) \hookrightarrow \mathrm{Ext}_R^1(I,M).$$

Moreover if there is an ideal $m \subseteq A$ such that $\mathrm{depth}_m M \geq 1$, then the morphism above induces an isomorphism

$$H_m^0(H^2(R,A,M)) \xrightarrow{\sim} H_m^0(\mathrm{Ext}_R^1(I,M)).$$

Proof. Choose a surjective R-linear map $\psi : P \to I$ where P is a free R-module, and let $E = \ker \psi$. If $f : \Lambda^2 P \to \Lambda P = P$ is given by $f(x\wedge y) = x\psi(y) - \psi(x)y$, then let $K = \operatorname{im} f \subseteq P$. Since $K \subseteq \ker \psi = E$, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & E & \to & P & \to & I & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & E/K & \to & P/K & \to & I & \to & 0 \end{array}$$

consisting of exact horizontal sequences. Recall that

$$H^2(R,A,M) = \operatorname{coker}[\operatorname{Hom}_R(P/K,M) \to \operatorname{Hom}_R(E/K,M)]$$

by [SGA 7, exp.VI]. Now since $\operatorname{Hom}_R(P/K,M) \xrightarrow{\sim} \operatorname{Hom}_R(P,M)$, we easily deduce an exact sequence

$$0 \to H^2(R,A,M) \to \operatorname{Ext}^1_R(I,M) \to \operatorname{Hom}_R(K,M).$$

Hence

$$0 \to H^0_m(H^2(R,A,M)) \to H^0_m(\operatorname{Ext}^1_R(I,M)) \to H^0_m(\operatorname{Hom}_R(K,M))$$

is exact and since $\operatorname{depth}_m M \geq 1$,

$$H^0_m(\operatorname{Hom}_R(K,M)) = \operatorname{Hom}_R(K,H^0_m(M)) = 0,$$

and we are done.

The key lemma of the proof of the theorem is this.

Lemma 2.2.3. Let $\varphi : R \to R/I = A$ be a surjective morphism of noetherian rings such that $\operatorname{depth}_I R = 2$, and let $m \subseteq A$ be an ideal. Assume that $\operatorname{depth}_{\varphi^{-1}(m)} R \geq 3$ and that $A_{\wp}$ is of finite projective dimension over $R_{\varphi^{-1}(\wp)}$ for all prime ideals $\wp \in \operatorname{Spec}(A) - V(m)$ such that $\operatorname{depth} R_{\varphi^{-1}(\wp)} = 2$. Then there is an isomorphism

$$H^1(R,A,A) \xrightarrow{\sim} \operatorname{Ext}^1_R(I,I)$$

and an injection

$$H^0_m(H^2(R,A,A)) \hookrightarrow H^0_m(\mathrm{Ext}^2_R(I,I)).$$

Furthermore if $\mathrm{depth}_m A \geq 1$ and $\mathrm{depth}_{\varphi^{-1}(m)} R \geq 4$,

$$H^0_m(H^2(R,A,A)) \xrightarrow{\simeq} H^0_m(\mathrm{Ext}^2_R(I,I))$$

is an isomorphism.

Proof. We proceed in several steps.

Step 1. We claim that

$$\mathrm{Hom}_R(I,I) \xrightarrow{\simeq} \mathrm{Hom}_R(I,R) \xleftarrow{\simeq} \mathrm{Hom}_R(R,R) = R$$

are isomorphisms. In fact the depth condition $\mathrm{depth}_I R \geq 2$ is equivalent to $\mathrm{Ext}^i_R(R/I,R) = 0$ for $i = 0,1$, so

$$R = \mathrm{Hom}_R(R,R) \to \mathrm{Hom}_R(I,R)$$

is an isomorphism. Since

$$\mathrm{Hom}_R(I,I) \subseteq \mathrm{Hom}_R(I,R) \xleftarrow{\simeq} R$$

and since $\mathrm{Hom}_R(I,I)$ contains the identity, we are done.

Step 2. We claim that

$$\mathrm{Ext}^1_R(I,R) \to \mathrm{Ext}^1_R(I,A)$$

is injective. Indeed let $P' = \mathrm{Spec}(R) - V(m')$ where $m' = \varphi^{-1}(m)$ and let $X' = \mathrm{Spec}(A) - V(m)$ and consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_R(I,R) & \longrightarrow & \mathrm{Ext}^1_R(I,A) \\ \downarrow & \circ & \downarrow \\ H^0(P',\underline{\mathrm{Ext}}^1_{O_{P'}}(\tilde{I},O_{P'})) & \to & H^0(P',\underline{\mathrm{Ext}}^1_{O_{P'}}(\tilde{I},O_{X'})). \end{array}$$

It will be sufficient to show that the vertical arrow to the left and the lower horizontal arrow are injections, i.e. we need only show

$$(*) \qquad H^0_{m'}(\operatorname{Ext}^1_R(I,R)) = 0$$

and

$$(**) \qquad \operatorname{Ext}^1_R(I,R)_{\wp} \to \operatorname{Ext}^1_R(I,A)_{\wp}$$

injective for all prime ideals $\wp \in P'$. To see (*), we use one of the spectral sequences of (2.1.2), and we deduce that

$$\operatorname{Ext}^1_{m'}(I,R) \to H^0_{m'}(\operatorname{Ext}^1_R(I,R)) \to H^2_{m'}(\operatorname{Hom}_R(I,R))$$

is exact. Since $\operatorname{depth}_{m'} R \geq 3$, it follows that

$$\operatorname{Ext}^1_{m'}(I,R) = 0$$

by the other spectral sequence of (2.1.2). Moreover by step 1

$$H^2_{m'}(\operatorname{Hom}_R(I,R)) = H^2_{m'}(R) = 0$$

and (*) follows. Now we concentrate on (**). If $\operatorname{depth} R_{\wp} \geq 3$, then $H^0_{\wp R_{\wp}}(\operatorname{Ext}^1_R(I,R)_{\wp}) = 0$ by the proof of (*). So essentially by the commutative diagram above, it suffices to show (**) for all prime ideals $\wp' \subsetneqq \wp$. It is therefore enough to show (**) when $\operatorname{depth} R_{\wp} \leq 2$. Now if $A_{\wp} = 0$, then $I_{\wp} = R_{\wp}$ and (**) follows. And if $A_{\wp} \neq 0$, then $\wp \supseteq I$ and using $\operatorname{depth}_I R = 2$, we have that $\operatorname{depth} R_{\wp} \geq 2$. Since

$$\operatorname{pd} A_{\wp} + \operatorname{depth} A_{\wp} = \operatorname{depth} R_{\wp} = 2$$

we deduce that $\operatorname{pd} I_{\wp} \leq 1$. If $\operatorname{pd} I_{\wp} = 0$, then $\operatorname{Ext}^1_R(I,R)_{\wp} = 0$ and if $\operatorname{pd} I_{\wp} = 1$, then (**) looks like

$$\operatorname{Ext}^1_{R_{\wp}}(I_{\wp},R_{\wp}) \to \operatorname{Ext}^1_{R_{\wp}}(I_{\wp},A_{\wp}) \simeq \operatorname{Ext}^1_{R_{\wp}}(I_{\wp},R_{\wp}) \otimes_{R_{\wp}} A_{\wp}$$

and (**) is an isomorphism since $\operatorname{Ext}^1_R(I,R) \simeq \operatorname{Ext}^2_R(A,R)$ is an

A-module. And the claim follows.

Step 3. We claim that

$$H^1(R,A,A) \xrightarrow{\sim} \mathrm{Ext}^1_R(I,I)$$

and moreover that

$$H^0_m(H^2(R,A,A)) \hookrightarrow H^0_m(\mathrm{Ext}^2_R(I,I))$$

is injective. Indeed there is a long exact sequence

$$\mathrm{Hom}_R(I,I) \xrightarrow{\sim} \mathrm{Hom}_R(I,R) \to \mathrm{Hom}_R(I,A) \to \mathrm{Ext}^1_R(I,I) \to \mathrm{Ext}^1_R(I,R) \hookrightarrow \mathrm{Ext}^1_R(I,A)$$

and it follows that

$$H^1(R,A,A) = \mathrm{Hom}_R(I,A) \xrightarrow{\sim} \mathrm{Ext}^1_R(I,I)$$

are isomorphic by step 1 and 2. Moreover the sequence

$$(***) \qquad 0 \to \mathrm{Ext}^1_R(I,R) \to \mathrm{Ext}^1_R(I,A) \to \mathrm{Ext}^2_R(I,I)$$

is exact, so by (*) and (2.2.2), we deduce injections

$$(****) \quad H^0_m(H^2(R,A,A)) \hookrightarrow H^0_m(\mathrm{Ext}^1_R(I,A)) \hookrightarrow H^0_m(\mathrm{Ext}^2_R(I,I))$$

as required.

Step 4. It will be sufficient to show that the injections of (****) are isomorphisms. In fact $\mathrm{depth}_m A \geq 1$, and the injection to the left is an isomorphism by (2.2.2). To see that the injection to the right is an isomorphism, we continue the long exact sequence of (***), and so we see that it suffices to prove that

$$H^1_{m'}(\mathrm{Ext}^1_R(I,R)) = 0$$

and that

$$H^0_{m'}(\mathrm{Ext}^2_R(I,I)) \to H^0_{m'}(\mathrm{Ext}^2_R(I,R))$$

is the zero-map. In fact $\mathrm{Ext}^2_{m'}(I,R) = 0$, and it follows easily that

$$H^1_{m'}(\mathrm{Ext}^1_R(I,R)) \to H^3_{m'}(\mathrm{Hom}_R(I,R)) = H^3_{m'}(R)$$

is injective. Hence $H^1_{m'}(\mathrm{Ext}^1_R(I,R)) = 0$ because $\mathrm{depth}_{m'}R \geq 4$. Moreover we also get that

$$H^0_{m'}(\mathrm{Ext}^2_R(I,R)) \hookrightarrow H^2_{m'}(\mathrm{Ext}^1_R(I,R))$$

is injective. Since the diagram of differensials of spectral sequences

$$\begin{array}{ccc} H^0_{m'}(\mathrm{Ext}^2_R(I,I)) & \to & H^0_{m'}(\mathrm{Ext}^2_R(I,R)) \\ \downarrow & \circ & \downarrow \\ H^2_{m'}(\mathrm{Ext}^1_R(I,I)) & \to & H^2_{m'}(\mathrm{Ext}^1_R(I,R)) \end{array}$$

commutes and since the morphism $\mathrm{Ext}^1_R(I,R) \to \mathrm{Ext}^1_R(I,A)$ is injective by step 2, i.e. $\mathrm{Ext}^1_R(I,I) \to \mathrm{Ext}^1_R(I,R)$ is the zero morphism, we are done.

<u>Proof of (2.2.1)</u>. We apply (2.2.3) to the graded homomorphism $\varphi : R \to A = R/I$ of the cones of $X \subseteq \mathbb{P}$, and let m be the irrelevant maximal ideal of A. Since R is Cohen Macaulay, $\dim \mathbb{P} - \dim X = \mathrm{depth}_I R$, and the conditions of (2.2.3) are easily verified. Therefore there are graded-preserving isomorphisms

$$H^1(R,A,A) \simeq \mathrm{Ext}^1_R(I,I)$$

$$H^0_m(H^2(R,A,A)) \simeq H^0_m(\mathrm{Ext}^2_R(I,I)).$$

Now X is Cohen Macaulay and equidimensional, $\mathbb{P}$ is non-singular along X, and the codimension of X in $\mathbb{P}$ is 2, so

$$\mathrm{pd}_{O_{\mathbb{P}}} \underline{I}_X = 1.$$

It follows that the sheaf on $\mathbb{P}$

$$\widetilde{\mathrm{Ext}^2_R(I,I)} = \underline{\mathrm{Ext}}^2_{\mathcal{O}_{\mathbb{P}}}(\underline{I}_X,\underline{I}_X) = 0$$

and therefore that

$$H^0_m(\mathrm{Ext}^2_R(I,I)) \simeq \mathrm{Ext}^2_R(I,I).$$

So it will be sufficient to show that the sheaf $\widetilde{H^2(R,A,A)} = 0$ on $\mathbb{P}$. Since A is generically a complete intersection in R, one knows that

$$H^2(R,A,A)_{\wp} = 0$$

for all graded prime ideals $\wp \subseteq A$ where $\dim A_{\wp} = 0$. Now if $H^2(R,A,A) \neq 0$, then there exists a graded prime ideal $\wp \subseteq A$ such that

$$H^2(R,A,A)_{\wp} \neq 0$$

and such that

$$H^2(R,A,A)_{\wp'} = 0$$

for all graded prime ideals $\wp' \subsetneqq \wp$. Since $\dim A_{\wp} \geq 1$ and since the conditions of the first part of (2.2.3) are clearly satisfied for the morphism $R_{\varphi^{-1}(\wp)} \to A_{\wp}$, we find that

$$0 \neq H^2(R,A,A) = H^0_{\wp A_{\wp}}(H^2(R,A,A)_{\wp}) \hookrightarrow \mathrm{Ext}^2_R(I,I)_{\wp} ,$$

and because $\mathrm{pd}\, I_{\varphi^{-1}(\wp)} = 1$, we have a contradiction.

<u>Corollary 2.2.4</u>. Let $X \subseteq \mathbb{P}$, R and A be as in the theorem (2.2.1). If the cone A is Cohen Macaulay and if A is of finite projective dimension over R, then

$${}_{\nu}H^2(R,A,A) = 0$$

for any $\nu \in \mathbb{Z}$.

This follows from

$$\operatorname{depth}_I R = \dim R - \dim A = \operatorname{pd}_R A,$$

so $\operatorname{pd} I = 1$, and from the theorem (2.2.1).

<u>Corollary 2.2.5.</u> Let $f: X \subseteq \mathbb{P}$ be as in (2.2.1).

Then the sheaf on X

$$\underline{A}^2(f, O_X) = 0,$$

and this sheaf, defined in (1.2.3), is isomorphic to $H^2(R, \widetilde{A}, A)$. Moreover

$$A^2(f, O_X(\nu)) = H^1(\underline{N}_X(\nu))$$

for all $\nu \in \mathbb{Z}$.

<u>Proof.</u> It follows from the last part of the proof of (2.2.1) that $H^2(R, \widetilde{A}, A) = 0$, and from the definition (1.2.3) that $\underline{A}^2(f, O_X) = H^2(R, \widetilde{A}, A)$. By the spectral sequence of (1.2.3), we find

$$A^2(f, O_X(\nu)) = H^1(X, \underline{N}_X(\nu))$$

as required.

<u>Remark 2.2.6.</u> Let $X \subseteq \mathbb{P} \subseteq \mathbb{P}_k^N$ and $R \to A = R/I$ satisfy the conditions of (2.2.1), except for $X \subseteq \mathbb{P}$ being generically a complete intersection. If $n = \dim \mathbb{P}$, then there are isomorphisms

$$\underline{N}_X \simeq \underline{\operatorname{Ext}}^1_{O_{\mathbb{P}}}(\underline{I}_X, \underline{I}_X),$$

$$H^{i-1}(\underline{N}_X(\nu)) \simeq \operatorname{Ext}^i_{O_{\mathbb{P}}}(\underline{I}_X, \underline{I}_X(\nu)) \quad \text{for} \quad 1 \leq i \leq n-1,$$

$$H^0(O_{\mathbb{P}}(\nu)) \simeq \operatorname{Hom}_{O_{\mathbb{P}}}(\underline{I}_X, \underline{I}_X(\nu)),$$

$$H^n(O_{\mathbb{P}}(\nu)) \simeq \operatorname{Ext}^n_{O_{\mathbb{P}}}(\underline{I}_X, \underline{I}_X(\nu))$$

for any $\nu \in \mathbb{Z}$. In fact since $\operatorname{depth}_{\tilde{I}_x} O_{\mathbb{P},x} = \operatorname{pd} O_{X,x} = 2$ for any $x \in X$, it follows that

$$\underline{\operatorname{Hom}}_{O_{\mathbb{P}}}(\tilde{I},\tilde{I}) \xrightarrow{\sim} \operatorname{Hom}_{O_{\mathbb{P}}}(\tilde{I},O_{\mathbb{P}}) \xleftarrow{\sim} O_{\mathbb{P}}$$

as in step 1 of the proof of (2.2.3), and that

$$\underline{\operatorname{Ext}}^1_{O_{\mathbb{P}}}(\tilde{I},O_{\mathbb{P}}) \simeq \underline{\operatorname{Ext}}^1_{O_{\mathbb{P}}}(\tilde{I},O_X)$$

by step 2 of (2.2.3). Moreover as in step 3,

$$\underline{N}_X \simeq \operatorname{Ext}^1_{O_{\mathbb{P}}}(\tilde{I},\tilde{I}(\nu)).$$

Since there is a spectral sequence

$$H^p(\mathbb{P}, \underline{\operatorname{Ext}}^q_{O_{\mathbb{P}}}(\tilde{I},\tilde{I}(\nu)))$$

which converges to $\operatorname{Ext}^{(\cdot)}_{O_{\mathbb{P}}}(\tilde{I},\tilde{I}(\nu))$, the claims above follow if we can prove that the morphism $d_{2,-1}$ appearing in the exact sequence

$$0 \to \operatorname{Ext}^{n-1}_{O_{\mathbb{P}}}(\tilde{I},\tilde{I}(\nu)) \to H^{n-2}(\mathbb{P},\underline{\operatorname{Ext}}^1_{O_{\mathbb{P}}}(\tilde{I},\tilde{I}(\nu))) \xrightarrow{d_{2,-1}}$$

$$H^n(\mathbb{P},\underline{\operatorname{Hom}}_{O_{\mathbb{P}}}(\tilde{I},\tilde{I}(\nu))) \to \operatorname{Ext}^n_{O_{\mathbb{P}}}(\tilde{I},\tilde{I}(\nu)) \to 0.$$

is zero. This is easy to see if $\mathbb{P} = \mathbb{P}^N_k$ and $\nu \geq -n$ because

$$H^n(\mathbb{P},\underline{\operatorname{Hom}}_{O_{\mathbb{P}}}(\tilde{I},\tilde{I}(\nu))) = H^n(\mathbb{P},O_{\mathbb{P}}(\nu)) = 0$$

and a little bit more complicated if otherwise, and we just indicate why. Indeed to prove that $H^n(\mathbb{P},\underline{\operatorname{Hom}}_{O_{\mathbb{P}}}(\tilde{I},\tilde{I}(\nu)))$ and $\operatorname{Ext}^n_{O_{\mathbb{P}}}(\tilde{I},\tilde{I}(\nu))$ have the same k-dimension, we deduce by the spectral sequence above that $\operatorname{Ext}^{n+1}_{O_{\mathbb{P}}}(\tilde{I},-) = 0$. Therefore $\operatorname{Ext}^n_{O_{\mathbb{P}}}(\tilde{I},-)^{\vee}$ is left-exact where $(-)^{\vee} = \operatorname{Hom}_k(-,k)$, and since $\mathrm{Hom}_{O_{\mathbb{P}}}(-,\tilde{I}\otimes\omega_{\mathbb{P}})$ is left exact too where $\omega_{\mathbb{P}}$ is the dualizing sheaf on $\mathbb{P}$, we find that

$$\mathrm{Hom}_{O_{\mathbb{P}}}(\tilde{I}(\nu),\tilde{I}\otimes\omega_{\mathbb{P}})\simeq \mathrm{Ext}^{n}_{O_{\mathbb{P}}}(\tilde{I},\tilde{I}(\nu))^{\vee},$$

and we conclude as expected.

Now we consider curves in $\mathbb{P}^3_k$ such that $X\subseteq\mathbb{P}=\mathbb{P}^3_k$ satisfies the conditions of the theorem (2.2.1), and the results below appear as corollaries of this and of the duality theorem of the preceding section, and of the theory of Section 1.4. If $X\subseteq\mathbb{P}=\mathbb{P}^3_k$ is a curve, we let $R=k[X_0,X_1,X_2,X_3]$ be a polynomial ring over k, and we let $A=R/I$ be the minimal cone of X in $\mathbb{P}$. Moreover there is a minimal resolution of I as in (2.1.6), and the n_{ji} are therefore unique. Then we have a morphism $\varphi_A(\nu)$ as defined in (2.1.6). For the rest of this paper we will use the following definition of a curve in $\mathbb{P}^3_k$.

Definition 2.2.7. A curve X of $\mathbb{P}^3_k$ is a closed subscheme of $\mathbb{P}^3_k$ which is Cohen Macaulay and equidimensional of dimension 1. Moreover to any such curve X we define the numbers $s(X)$, $e=e(X)$ and $c=c(X)$ by

$$s(X)=\min_{1\le i\le r_1} n_{1i},$$

$$H^1(O_X(e))\neq 0 \quad\text{and}\quad H^1(O_X(\nu))=0 \quad\text{for}\quad \nu>e,$$

$$H^1(\underline{I}_X(c))\neq 0 \quad\text{and}\quad H^1(\underline{I}_X(\nu))=0 \quad\text{for}\quad \nu>c$$

provided $H^1(\underline{I}_X(\nu))$ does not vanish for all $\nu\in\mathbb{Z}$. Otherwise $c=-\infty$. Furthermore let

$$\delta^j = \sum_{i=1}^{r_1} h^j(\underline{I}_X(n_{1i})) - \sum_{i=1}^{r_2} h^j(\underline{I}_X(n_{2i})) + \sum_{i=1}^{r_3} h^j(\underline{I}_X(n_{3i})), \text{ and}$$

$$\delta = \delta^2 - \delta^1.$$

Note that by the sequence

$$0 \to \underline{I}_X \to O_{\mathbb{P}} \to O_X \to 0$$

we easily deduce $H^2(\underline{I}_X(\nu)) \simeq H^1(O_X(\nu))$, so

$$\delta^2 = \Sigma\, h^1(O_X(n_{1i})) - \Sigma\, h^1(O_X(n_{2i})) + \Sigma\, h^1(O_X(n_{3i})).$$

Moreover splitting the minimal resolution of $\underline{I}_X$

$$0 \to \bigoplus_1^{r_3} O_{\mathbb{P}}(-n_{3i}) \to \bigoplus_1^{r_2} O_{\mathbb{P}}(-n_{2i}) \to \bigoplus_1^{r_1} O_{\mathbb{P}}(-n_{1i}) \to \underline{I}_X \to 0$$

into two short exact sequences and taking cohomology after twisting by ν, we have by duality on $\mathbb{P}_k^3$ that

$$c(X) = \max n_{3i} - 4$$

and by $\max n_{1i} < \max n_{2i}$ that

$$e(X) \leq \max n_{2i} - 4.$$

Furthermore in case $\max n_{3i} \leq \max n_{2i}$ we have

$$e(X) = \max n_{2i} - 4$$

which implies $c(X) \leq e(X)$. Note also that

$$\min n_{1i} < \min n_{2i} < \min n_{3i}$$

since the resolution is minimal.

With these preparations in mind we now turn to the corollaries. First by using the duality theorem of the preceding section, we have in view of (2.1.6) the following description of the second cohomology groups of algebras $H^2(R,A,A)$.

Proposition 2.2.8. If $X \subseteq \mathbb{P} = \mathbb{P}^3_k$ is a curve which is generically a complete intersection in $\mathbb{P}$, then

$$_{\nu}H^2(R,A,A)^{\vee} \simeq \ker \varphi_A(-\nu-4) \simeq {}_{-\nu-4}H^2_m(R,A,A).$$

In particular if $H^1(\underline{I}_X(n_{1i}-4)) = 0$ for all i, $1 \leq i \leq r_1$, then

$$_{o}H^2(R,A,A) = 0.$$

Using spectral sequences we have that

$$\mathrm{Ext}^2_m(I,I) \simeq H^2_m(R,A,A)$$

because $H^2_m(I) \simeq H^1_m(A)$, and now (2.2.8) follows from (2.1.6) and (2.2.1). See also (1.4.3)

We may use (2.2.8) to obtain a vanishing criterion for $H^1(\underline{N}_X)$, or more generally to compute its k-dimension. If $Y \subseteq \mathbb{P}$ is a global complete intersection containing X, we may describe the cohomology group $A^2(\underline{d},O_{\underline{d}})_{res} = \operatorname{coker}\alpha$ of (1.3.2) correspondingly. Note that if we apply $_{o}\mathrm{Hom}_R(-,H^2_m(I))$ to the minimal resolution of I appearing in (2.1.6), we find that

$$_{o}\mathrm{Ext}^i_R(I,H^2_m(I)) = 0 \quad \text{for} \quad i \geq 1$$

provided $c(X) < \min n_{2i}$, and in this case

$$_{o}\mathrm{Hom}_R(I,H^2_m(I)) \simeq \bigoplus_{i=1}^{r_1} H^1(\underline{I}_X(n_{1i})).$$

Then we have

Corollary 2.2.9. 1) Let $f : X \subseteq \mathbb{P} = \mathbb{P}^3_k$ be a curve which is generically a complete intersection, and suppose

$$c(X) < \min_{1 \leq i \leq r_2} n_{2i} \,, \quad \text{and}$$

$$H^1(\underline{I}_X(n_{1i}-4)) = 0 \quad \text{for} \quad 1 \leq i \leq r_1.$$

Then

$$H^1(\underline{N}_X) \simeq A^2(f,O_X) \xrightarrow{\sim} {}_o\mathrm{Hom}_R(I,H^3_m(I)),$$

and its k-dimension is

$$h^1(\underline{N}_X) = \delta^2.$$

ii) Moreover let $Y = V(F_1,\dots,F_r) \subseteq \mathbb{P}$ for $r \leq 2$ be a global complete intersection whose minimal cone is B, let $I_{B/A} = \ker(B \to R/I)$, and suppose that each F_i is either a minimal generator for I or that $H^1(\underline{I}_X(f_i)) = 0$ for $f_i = \deg F_i$. Then

$$A^2(\underline{d},O_{\underline{d}})_{res} \xrightarrow{\sim} {}_o\mathrm{Hom}_R(I_{B/A},H^3_m(I))$$

and the morphism

$$\gamma : H^0(\underline{N}_X) \longrightarrow \bigoplus_{i=1}^{r} H^1(\underline{I}_X(f_i))$$

of (1.3.10) is surjective. Moreover

$$a^2_{res} - \dim_k \mathrm{coker}\, 1^2 = \delta^2 - \sum_{i=1}^{r} h^1(O_X(f_i)),$$

and if $e(X) < \min n_{2i}$ and if those F_i which are not minimal generators of I satisfy $f_i > e(X)$, then

$$\mathrm{coker}\, 1^2 = 0.$$

<u>Proof.</u> Indeed by (2.1.6), ${}_o\mathrm{Ext}^2_R(I,I) = 0$, and since $\mathrm{pd}\, I \leq 2$, we deduce by the exact sequence of (2.1.3) that

$$\mathrm{Ext}^2_{O_{\mathbb{P}}}(\underline{I}_X,\underline{I}_X) \simeq {}_o\mathrm{Ext}^3_m(I,I).$$

Moreover using (2.2.5) and (2.2.6)

$$A^2(f,O_X) \simeq H^1(\underline{N}_X) \simeq \mathrm{Ext}^2_{O_{\mathbb{P}}}(\underline{I}_X,\underline{I}_X),$$

and by the second spectral sequences of (2.1.2i)

$$ {}_0\mathrm{Ext}^3_m(I,I) = {}_0\mathrm{Hom}_R(I,H^3_m(I)) $$

because $c(X) < \min n_{2i}$. Thus

$$ H^1(\underline{N}_X) \simeq A^2(f,O_X) \simeq {}_0\mathrm{Hom}_R(I,H^3_m(I)). $$

By the next lemma (2.2.11) and by the exact sequence (**) of its proof we get that

$$ \dim {}_0\mathrm{Ext}^3_m(I,I) - \dim {}_0\mathrm{Ext}^2_m(I,I) = \delta. $$

Thus

$$ \dim {}_0\mathrm{Hom}_R(I,H^3_m(I)) = \delta + \dim {}_0\mathrm{Hom}_R(I,H^2_m(I)), $$

and since

$$ {}_0\mathrm{Hom}_R(I,H^2_m(I)) \simeq \bigoplus_{i=1}^{r_1} H^1(\underline{I}_X(n_{1i})) $$

and $c(X) < \min n_{2i}$, we deduce by the definition (2.2.7) of δ^1 that

$$ \dim {}_0\mathrm{Hom}_R(I,H^2_m(I)) = \delta^1. $$

So

$$ h^1(\underline{N}_X) = \delta + \delta^1 = \delta^2 $$

as required.

Next we prove that γ is surjective. For this we consider

$$ (*) \quad \begin{array}{ccccccc} 0 \to & {}_0\mathrm{Hom}_R(I_{B/A},H^2_m(I)) & \to & {}_0\mathrm{Hom}_R(I,H^2_m(I)) & \to & {}_0\mathrm{Hom}(I_B,H^2_m(I)) \\ & & & \wr\vert & \circ & \updownarrow \\ & & & \bigoplus_{i=1}^{r_1} H^1(\underline{I}_X(n_{1i})) & \to & \bigoplus_{i=1}^{r} H^1(\underline{I}_X(f_i)) \end{array} $$

where $I_B = \ker(R \to B)$, and by the assumption on the generators of I_B, we deduce that

$$\bigoplus_{i=1}^{r_1} H^1(\underline{I}_X(n_{1i})) \to \bigoplus_{i=1}^{r} H^1(\underline{I}_X(f_i))$$

is surjective, and so the morphism

$$_o\mathrm{Ext}^2_m(I,I) = {}_o\mathrm{Hom}_R(I,H^2_m(I)) \to {}_o\mathrm{Hom}_R(I_B,H^2_m(I)) \simeq \bigoplus_{i=1}^{r} H^1(\underline{I}_X(f_i))$$

also is. Moreover by $_o\mathrm{Ext}^2_R(I,I) = 0$ and by the exact sequence of (2.1.3)

$$H^o(\underline{N}_X) = \mathrm{Ext}^1_{O_{\mathbb{P}}}(\underline{I}_X,\underline{I}_X) \to {}_o\mathrm{Ext}^2_m(I,I)$$

is surjective, and since the morphism γ is the composition

$$H^o(\underline{N}_X) \to {}_o\mathrm{Ext}^2_m(I,I) \to \bigoplus_{i=1}^{r} H^1(\underline{I}_X(f_i))$$

of the morphisms we have just considered, γ is surjective. It follows by (1.3.10) that

$$0 \to A^2(\underline{d},O_{\underline{d}})_{res} \to A^2(f,O_X) \xrightarrow{1^2} \bigoplus_{i=1}^{r} H^1(O_X(f_i))$$

is exact. Now we have a commutative diagram

$$\begin{array}{ccc} A^2(f,O_X) & \xrightarrow{1^2} & \bigoplus_{i=1}^{r} H^1(O_X(f_i)) \\ \downarrow\cong & \circ & \uparrow\cong \\ {}_o\mathrm{Hom}_R(I,H^3_m(I)) & \to & \mathrm{Hom}_R(I_B,H^3_m(I)) \end{array}$$

and the kernels are therefore isomorphic, i.e.

$$A^2(\underline{d},O_{\underline{d}})_{res} \simeq {}_o\mathrm{Hom}_R(I_{B/A},H^3_m(I)).$$

Moreover the dimension formula for $a^2_{res} - \dim\operatorname{coker} 1^2$ follows easily. Finally to prove that 1^2 is surjective, we use the same arguments as in (*), and we are done.

Note that under the condition $c(X) < \min n_{3i}$, we may by the proof above see that

$$h^1(N_X) \geq \delta^2$$

and that equality holds iff

$${}_0\mathrm{Ext}^2_m(I,I) \twoheadrightarrow {}_0\mathrm{Ext}^2_R(I,I)$$

is surjective and

$${}_0\mathrm{Ext}^1_R(I,H^2_m(I)) = 0.$$

Moreover, again by the proof above, we may see that 1^2 is surjective if

$$c(X) < \min n_{3i},$$

$$e(X) < \min n_{2i}$$

and those F_i which are not minimal generators of I satisfy $f_i > e(X)$.

Now we give some examples of curves over an algebraically closed field k where we use (2.2.9) to prove that $H^1(\underline{N}_X) = 0$. Since this is always true for smooth curves (reduced is enough) having $e(X) \leq 0$, we consider curves where $e(X) \geq 1$.

Examples 2.2.10. i) We consider the Hilbert scheme $H(9,8)$ of curves of degree $d = 9$ and arithmetic genus $g = 8$. Without going into the details, we just state that there are smooth connected curves having $h^1(\underline{I}_X(2)) = 1$ and $h^1(\underline{I}_X(\nu)) = 0$ otherwise. Thus $c(X) = 2$ and so $\max n_{3i} = c(X) + 4 = 6$. Since $2d > 2g-2$, $H^1(O_X(2)) = 0$ by [M1, Lect 11], and it follows that $e(X) \leq 1$. We have already seen that $c(X) > e(X)$ implies $\max n_{3i} > \max n_{2i}$, and since $\max n_{2i} > \max n_{1i}$,

$$\max n_{1i} \leq 4.$$

Thus

$$H^1(\underline{I}_X(n_{1i}-4)) = 0 \quad \text{for all} \quad 1 \leq i \leq r_1.$$

Moreover it is trivial to see that

$$c(X) < \min n_{2i},$$

and so by (2.2.9),

$$H^1(\underline{N}_X) = 0.$$

Note that, in this case, the minimal resolution of I is easily found. Indeed by Riemann-Roch

$$\chi(\underline{I}_X(\nu)) = \chi(O_{\mathbb{P}}(\nu)) - \chi(O_X(\nu)) = \binom{\nu+3}{3} - (9\nu - 7).$$

Therefore $\chi(\underline{I}_X(3)) = h^0(\underline{I}_X(3)) = 0$, i.e.

$$\min n_{1i} \geq 4$$

and since we already have seen $\max n_{1i} \leq 4$,

$$n_{1i} = 4 \quad \text{for all} \quad 1 \leq i \leq r_1.$$

In the same way

$$n_{2i} = 5 \quad \text{for all} \quad 1 \leq i \leq r_2,$$

$$n_{3i} = 6 \quad \text{for all} \quad 1 \leq i \leq r_3.$$

Moreover $\chi(\underline{I}_X(4)) = h^0(\underline{I}_X(4)) = 6$, so $r_1 = 6$, and by $h^1(\underline{I}_X(2)) = 1$, $r_3 = 1$. The minimal resolution of I will therefore be of the form

$$0 \to R(-6) \to R(-5)^{\oplus 6} \to R(-4)^{\oplus 6} \to I \to 0.$$

Since $\chi(\underline{I}_X(1)) = 2$, $e(X)$ is in fact 1.

ii) There are smooth connected curves of degree $d = 9$ and genus $g = 8$ having $h^1(\underline{I}_X(2)) = h^1(\underline{I}_X(3)) = 1$

and $h^1(\underline{I}_X(\nu)) = 0$ otherwise. Now $c(X) = 3$, so $\max n_{3i} = 7$, and since $c(X) > e(X)$, it follows that $\max n_{1i} \leq 5$. Therefore

$$H^1(\underline{I}_X(n_{1i}-4)) = 0 \quad \text{for all} \quad i \in [1,r_1].$$

Moreover $\chi(\underline{I}_X(3)) = h^0(\underline{I}_X(3)) - h^1(\underline{I}_X(3)) = 0$, and we conclude $\min n_{1i} = 3$, so $c(X) < \min n_{2i}$. It follows that

$$H^1(\underline{N}_X) = 0$$

for such curves by (2.2.9).

iii) If we denote by $H(d,g)_S$ the open subscheme of $H(d,g)$ of smooth connected curves of degree d and genus g, we claim that $H(8,6)_S$ is smooth of dimension $4d = 32$. Indeed for any curve $X \subseteq \mathbb{P}$ of $H(8,6)_S$ it will be sufficient to show that $H^1(\underline{N}_X) = 0$. Now

$$\chi(\underline{I}_X(\nu)) = \binom{\nu+3}{3} - (8\nu - 5),$$

so $\chi(\underline{I}_X(1)) = 1$, $\chi(\underline{I}_X(2)) = -1$ and $\chi(\underline{I}_X(3)) = 1$. Moreover by $2d > 2g-2$, $e(X) \leq 1$, and in fact equal to 1 by $\chi(\underline{I}_X(1)) = 1$.

First note that there is no such curves lying on a smooth conique, nor on a singular one since the equations

$$q_1 + q_2 = 8$$
$$(q_1-1)(q_2-1) = 6$$

have no solutions among the positive integers. See (1.3.10) and [H 1, IV,(6.4.1cd)]. Thus

$$h^0(\underline{I}_X(2)) = 0, \quad h^1(\underline{I}_X(2)) = 1.$$

Next

$$\chi(\underline{I}_X(3)) = h^0(\underline{I}_X(3)) - h^1(\underline{I}_X(3)) = 1$$

and we claim that $h^0(\underline{I}_X(3)) = 1$. Indeed if $h^0(\underline{I}_X(3)) \geq 2$, we link X to another curve X' by a complete intersection Y of two surfaces of degree 3. Since the curve X' is of degree 1 and arithmetic genus -1 by (2.3.3), X' does not exist. So

$$h^0(\underline{I}_X(3)) = 1, \quad h^1(\underline{I}_X(3)) = 0.$$

Now using a result of Castelnuovo, see (2,ii) of **Section 3.3**, we get that

$$H^1(\underline{I}_X(\nu)) \otimes H^0(O_{\mathbb{P}}(1)) \twoheadrightarrow H^1(\underline{I}_X(\nu+1))$$

is surjective for all $\nu \geq e(X) + 2 = 3$. Thus any curve X of $H(8,6)_S$ satisfies $c(X) = 2$, and it follows that $\max n_{3i} = 6$, and by $c(X) > e(X)$ that $\max n_{1i} \leq 4$. Therefore

$$H^1(\underline{I}_X(n_{1i} - 4)) = 0$$

for all i, and if we use $\min n_{2i} > \min n_{1i} = 3$, then

$$c(X) < \min n_{2i}.$$

We conclude by (2.2.9) that

$$H^1(\underline{N}_X) = 0$$

as required.

Furthermore the resolution of I for any $X \subseteq \mathbb{P}$ of $H(8,6)_S$ is easily seen to be of the form

$$0 \to R(-6) \to R(-5)^{\oplus 5} \to R(-4)^{\oplus 4} \oplus R(-3) \to I \to 0$$

by corresponding arguments as in (i).

Next we apply (2.2.8) together with (1.4.6) or (1.4.8B) to prove another result implying that Hilb^p or $D(p,q)$ is non-singular at the points involved. We then seek for conditions for the vanishing of ${}_oH^2_m(R,A,A)$ and ${}_oH^2(R,A,A)$, resp. ${}_oH^2_m(B,A,A)$ and ${}_oH^2(R,A,A)$ where B is as in (1.4.6), simultaneously. We will need

Lemma 2.2.11. If $X \subseteq \mathbb{P} = \mathbb{P}^3_k$ is a curve of degree d, then

$$\chi(\underline{N}_X(\nu)) = 2d\nu + 4d.$$ 1)

Moreover if $X \subseteq \mathbb{P}$ is generically a complete intersection and if we let

$$h^i(\nu) = \dim_\nu H^i(R,A,A), \text{ and}$$

$$\delta^j(\nu) = \sum_{i=1}^{r_1} h^j(\underline{I}_X(n_{1i}+\nu)) - \sum_{i=1}^{r_2} h^j(\underline{I}_X(n_{2i}+\nu)) + \sum_{i=1}^{r_3} h^j(\underline{I}_X(n_{3i}+\nu)),$$

then

$$h^1(\nu) - h^2(\nu) = \chi(\underline{N}_X(\nu)) + \delta^2(\nu) - \delta^1(\nu)$$

for $\nu \geq -3$. In particular

$$h^1(0) - h^2(0) = 4d + \delta$$

with δ as in (2.2.7).

Proof. First we prove that

$$h^1(\nu) - h^2(\nu) = 2d\nu + 4d + \delta^2(\nu) - \delta^1(\nu)$$

for $\nu \geq -3$. We apply ${}_\nu\mathrm{Hom}_R(-,I)$ to the resolution of I, and we find

$$(*)\quad \begin{aligned} &\dim_\nu \mathrm{Ext}^0_R(I,I) - \dim_\nu \mathrm{Ext}^1_R(I,I) + \dim_\nu \mathrm{Ext}^2_R(I,I) = \\ &\sum h^0(\underline{I}_X(n_{1i}+\nu)) - \sum h^0(\underline{I}_X(n_{2i}+\nu)) + \sum h^0(\underline{I}_X(n_{3i}+\nu)). \end{aligned}$$

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1) Simpler proofs are available under some conditions on X.

Therefore by (2.2.1),

$$\chi(O_{\mathbb{P}}(\nu)) - (h^1(\nu) - h^2(\nu)) =$$

$$\Sigma\, \chi(\underline{I}_X(n_{1i}+\nu)) - \Sigma\, \chi(\underline{I}_X(n_{2i}+\nu)) + \Sigma\, \chi(\underline{I}_X(n_{3i}+\nu)) + \delta^1(\nu) - \delta^2(\nu)$$

for $\nu \geq -3$. If we use

$$\sum_1^{r_1} n_{1i} - \sum_1^{r_2} n_{2i} + \sum_1^{r_3} n_{3i} = 0$$

which follows easily from the fact that $\chi(O_X(\nu)) =$

$$\chi(O_{\mathbb{P}}(\nu)) - \Sigma\chi(O_{\mathbb{P}}(-n_{1i}+\nu)) + \Sigma\chi(O_{\mathbb{P}}(-n_{2i}+\nu)) - \Sigma\,\chi(O_{\mathbb{P}}(-n_{3i}+\nu))$$

is a polynomial in ν of degree 1, and if we use

$$\chi(\underline{I}_X(\nu)) = \chi(O_{\mathbb{P}}(\nu)) - \chi(O_X(\nu)) = \chi(O_{\mathbb{P}}(\nu)) - (d\nu + 1 - g)$$

then

$$\chi(O_{\mathbb{P}}(\nu)) - (h^1(\nu) - h^2(\nu)) =$$

$$\Sigma\chi(O_{\mathbb{P}}(n_{1i}+\nu)) - \Sigma\chi(O_{\mathbb{P}}(n_{2i}+\nu)) + \Sigma\chi(O_{\mathbb{P}}(n_{3i}+\nu)) - d\nu + g - 1 + \delta^1(\nu) - \delta^2(\nu)$$

Now by

$$\chi(\underline{I}_X(\nu')) = \Sigma\chi(O_{\mathbb{P}}(-n_{1i}+\nu')) - \Sigma\chi(O_{\mathbb{P}}(-n_{2i}+\nu')) + \Sigma\chi(O_{\mathbb{P}}(-n_{3i}+\nu'))$$

for $\nu' = -\nu - 4$ and by duality on $\mathbb{P} = \mathbb{P}^3_k$, we get

$$\chi(\underline{I}_X(-\nu-4)) = -\Sigma\chi(O_{\mathbb{P}}(n_{1i}+\nu)) + \Sigma\chi(O_{\mathbb{P}}(n_{2i}+\nu)) - \Sigma\chi(O_{\mathbb{P}}(n_{3i}+\nu)).$$

On the other hand

$$\chi(\underline{I}_X(-\nu-4)) = \chi(O_{\mathbb{P}}(-\nu-4)) - \chi(O_X(-\nu-4)) = -\chi(O_{\mathbb{P}}(\nu)) + (\nu+4)d - 1 + g.$$

Combining, we have the required formula for $h^1(\nu) - h^2(\nu)$.

By (2.1.3) there is an exact sequence

$$(**)\quad \begin{aligned} 0 \to {}_{\nu}\mathrm{Ext}^1_R(I,I) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}}}(\underline{I}_X,\underline{I}_X(\nu)) \to {}_{\nu}\mathrm{Ext}^2_{\mathfrak{m}}(I,I) \to \\ {}_{\nu}\mathrm{Ext}^2_R(I,I) \to \mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}}}(\underline{I}_X,\underline{I}_X(\nu)) \to {}_{\nu}\mathrm{Ext}^3_{\mathfrak{m}}(I,I) \to 0, \end{aligned}$$

and ${}_{\nu}\mathrm{Ext}^i_{\mathfrak{m}}(I,I)$ vanish for large ν by (2.1.2i). Using this and (*) for large ν and also (2.2.6), we obtain

$$\chi(\mathcal{O}_{\mathbb{P}}(\nu)) - \chi(\underline{N}_X(\nu)) =$$

$$\Sigma\chi(\underline{I}_X(n_{1i}+\nu)) - \Sigma\chi(\underline{I}_X(n_{2i}+\nu)) + \Sigma\chi(\underline{I}_X(n_{3i}+\nu)).$$

We may now proceed as in the first part of the proof, and we find the required formula for $\chi(\underline{N}_X(\nu))$, and the proof is complete.

If we consider (**) of the proof above for $\nu = 0$, it follows that

$$\delta = \dim {}_{o}\mathrm{Ext}^3_{\mathfrak{m}}(I,I) - \dim {}_{o}\mathrm{Ext}^2_{\mathfrak{m}}(I,I), \text{ and}$$
$$\dim {}_{o}\mathrm{Ext}^3_{\mathfrak{m}}(I,I) \le h^1(\underline{N}_X).$$

In particular

$$\delta \le h^1(\underline{N}_X),$$

and we have equality iff

$${}_{o}\mathrm{Ext}^2_{\mathfrak{m}}(I,I) = 0 \quad \text{and} \quad H^1(\underline{N}_X) \simeq {}_{o}\mathrm{Ext}^3_{\mathfrak{m}}(I,I)$$

which is equivalent to

$${}_{o}\mathrm{Ext}^2_{\mathfrak{m}}(I,I) = 0 \quad \text{and} \quad {}_{o}\mathrm{Ext}^2_R(I,I) = 0,$$

again by (**). However using spectral sequences

$${}_{o}\mathrm{Ext}^2_{\mathfrak{m}}(I,I) = {}_{o}H^2_{\mathfrak{m}}(R,A,A) = \ker\varphi_A(0),$$

and this together with (2.2.1), (2.2.8) and (1.4.8B) leads to

Proposition 2.2.12. Let $X \subseteq \mathbb{P}^3_k$ be a curve which is generically a complete intersection, let $A = R/I$ be the minimal cone of $X \subseteq \mathbb{P}_k^3$ and let $\varphi : R \to A$ be the natural surjection. Then $h^1(\underline{N}_X) \geq \delta$ and the following conditions are equivalent

i) $$h^1(\underline{N}_X) = \delta$$

ii) ${}_oH^2_m(R,A,A) = 0$ and ${}_oH^2(R,A,A) = 0$

iii) $\varphi_A(0)$ and $\varphi_A(-4)$ are injective.

Furthermore they are satisfied if

iv) $H^1(\underline{I}_X(n_{1i})) = 0$ and $H^1(\underline{I}_X(n_{1i}-4)) = 0$

for $1 \leq i \leq r_1$. Now if one of the equivalent conditions above holds, then

$$\mathrm{Def}_\varphi^o \simeq \underset{\sim}{\mathrm{Hilb}}_X$$

and $H = \mathrm{Hilb}^p(\mathbb{P}^3)$ is non-singular at $(X \subseteq \mathbb{P}^3)$ and $\dim O_{H,X} = 4d + \delta$.

Reviewing the examples of (2.2.10) we see that the curves of (i) and (iii) satisfy the conditions of (2.2.12i) because, in both cases, $\delta = 0$ and $h^1(\underline{N}_X) = 0$, while the curves of (ii) have $\delta = -1$. Note that the curves of (ii) do not satisfy the isomorphism $\mathrm{Def}_\varphi^o \simeq \underset{\sim}{\mathrm{Hilb}}_X$ of the conclusion either. Indeed under the assumption ${}_oH^2(R,A,A) = 0$,

$$\mathrm{Def}_\varphi^o \simeq \mathrm{Hilb}_X$$

is equivalent to $h^1(\underline{N}_X) = \delta$. We have one way by (2.2.12), and for the converse, we deduce by the isomorphism $\mathrm{Def}_\varphi^o \simeq \underset{\sim}{\mathrm{Hilb}}_X$ an isomorphism of tangent spaces

$${}_oH^1(R,A,A) \xrightarrow{\sim} H^o(N_X).$$

Since there is an exact sequence

$$0 \to H^1(R,A,A) \to H^o(\underline{N}_X) \to {}_oH^2_m(R,A,A) \to {}_oH^2(R,A,A)$$

by (1.4.8), we get that ${}_{o}H^2_m(R,A,A) = 0$, and so we have the conditions of (2.2.12ii) which is equivalent to $h^1(\underline{N}_X) = \delta$ as required. In view of the result which now follows, the curves of (ii) do not form an open subset of the Hilbert scheme.

<u>Proposition 2.2.13</u>. Let V be a reduced irreducible component of $\mathrm{Hilb}^p(\mathbb{P}^3_k)$ over an algebraically closed field k. Then there is a non-empty open smooth subscheme U of Hilb^p, contained in V, such that for any closed point $(X \subseteq \mathbb{P}) \in U$,

$$\mathrm{Def}^o_\varphi \simeq \underset{\sim}{\mathrm{Hilb}}_X.$$

Moreover for any $X \subseteq \mathbb{P}$ of U satisfying the conditions of (2.2.8) we have

$$\dim V = 4d + \delta + \dim_k \varphi_A(-4).$$

<u>Proof</u>. Let U be the open subscheme of V whose closed points $(X \subseteq \mathbb{P})$ satisfy

$$h^o(\underline{I}_X(\nu)) \leq h^o(\underline{I}_{X'}(\nu))$$

for all closed points $(X' \subseteq \mathbb{P}) \in V$ and all $\nu \in \mathbb{Z}$. Indeed U is open by the semi-continuity theorem [H1, III,(2.8)], and by shrinking U we may also suppose that U is open in Hilb^p and furthermore smooth since V is integral over an algebraically closed field. Now fix $X \subseteq \mathbb{P}$ of U, and let $S = \mathrm{Spec}(C) \in \mathrm{ob}\,\underline{l}$. We will define an inverse of

$$\mathrm{Def}^o_\varphi(S) \to \underset{\sim}{\mathrm{Hilb}}_X(S),$$

so let $(X_S \subseteq \mathbb{P}\times S) \in \underset{\sim}{\mathrm{Hilb}}_X(S)$ be given. By the exact sequence

$$0 \to \underline{I}_{X_S} \to O_{\mathbb{P}\times S} \to O_{X_S} \to 0,$$

and the corresponding sequence deduced from $X \subseteq \mathbb{P}$, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & H^0(\underline{I}_{X_S}(\nu)) & \to & H^0(O_{\mathbb{P}\times S}(\nu)) & \to & A_S(\nu) & \to & 0 \\ & & \downarrow & \circ & \downarrow & \circ & \downarrow & & \\ 0 & \to & H^0(\underline{I}_X(\nu)) & \to & H^0(O_{\mathbb{P}}(\nu)) & \to & A(\nu) & \to & 0 \end{array}$$

where by definition $A_S(\nu)$ and $A(\nu)$ are the cokernels. So $A = \oplus A(\nu)$ is the minimal cone of $X \subseteq \mathbb{P}$. Now

$$A_S(\nu) \underset{C}{\otimes} k \simeq A(\nu)$$

if the map

$$H^0(I_{X_S}(\nu)) \underset{C}{\otimes} k \to H^0(\underline{I}_X(\nu))$$

is surjective, and $A_S(\nu)$ is C-free if this map is injective. Thus

$$\mathrm{Def}^0_\varphi \simeq \underset{\sim}{\mathrm{Hilb}}_X$$

follows from the isomorphism

$$H^0(\underline{I}_{X_S}(\nu)) \underset{C}{\otimes} k \simeq H^0(\underline{I}_X(\nu)).$$

This, however, follows from a theorem of Grauert [H1, III,(12.9)] since U is integral, and from base change theorem [M1, Lect 7]. See also (1.3.14).

Finally to prove the dimension formula for V, we have by the isomorphism $\mathrm{Def}^0_\varphi \simeq \underset{\sim}{\mathrm{Hilb}}_X$ an isomorphism of tangent spaces which by (2.2.1) and (2.2.6) is

$${}_0\mathrm{Ext}^1_R(I,I) \simeq \mathrm{Ext}^1_{O_{\mathbb{P}}}(\underline{I}_X,\underline{I}_X).$$

Using the exact sequence (**) of the proof of (2.2.11) and the discussion after the proof,

$$h^1(\underline{N}_X) = \dim_o \mathrm{Ext}^3_m(I,I) - \dim_o \mathrm{Ext}^2_m(I,I) + \dim_o \mathrm{Ext}^2_R(I,I)$$
$$= \delta + \dim_k \ker \varphi_A(-4)$$

as required.

Roughly speaking (2.2.12) is therefore a criterion for proving that a sufficiently general curve of a component is a non-singular point of Hilb^p, i.e. for proving that components are reduced. Now we will use the theorem (1.4.6) to obtain a criterion which gives

$$\mathrm{Def}^o_{(\psi,\varphi)} \simeq \underset{\sim}{D}_{X\subseteq Y}$$

and $D(p,q)$ non-singular at $(X \subseteq Y \subseteq \mathbb{P})$. This should apply to components of $D(p,q)$ for proving that they are reduced, and the conditions we obtain should not be that special if $Y \subseteq \mathbb{P}$ is nicely chosen. Indeed it is natural to think about closed families of the Hilbert scheme such as those defined by, say $h^o(\underline{I}_X(s)) \geq 1$ for a given s, as the image of some component of the Hilbert-flag scheme $D(p;s)$[1] via the first projection

$$pr_1 : D(p;s) \to \mathrm{Hilb}^p.$$

Now let $X \subseteq \mathbb{P} = \mathbb{P}^3_k$ be a curve which is generically a complete intersection, let $g : Y = V(F_1,\dots,F_r) \hookrightarrow \mathbb{P}$ for $1 \leq r \leq 2$ be a global complete intersection containing X, and let $f : X \hookrightarrow Y$ be the inclusion. We consider the exact sequence of (1.3.10). Since g is a global complete intersection, it follows by (1.2.3) that

$$A^1(g,\underline{I}_{X/Y}) \simeq \bigoplus_{i=1}^{r} H^o(\underline{I}_{X/Y}(f_i)),$$

$$A^1(g,f_*O_X) \simeq \bigoplus_{i=1}^{r} H^1(O_X(f_i)),$$

1) For the notation in general, see (1.3.11).

and by (1.3.8) that

$$\operatorname{coker} m^1 \simeq \bigoplus_{i=1}^{r} H^1(\underline{I}_X(f_i)).$$

In view of (2.2.5)

$$A^2(gf, O_X) \simeq H^1(\underline{N}_X),$$

and the exact sequence of (1.3.10) looks like

$$0 \to \bigoplus_{i=1}^{r} H^o(\underline{I}_{X/Y}(f_i)) \to A^1(\underline{d}, O_{\underline{d}}) \to H^o(\underline{N}_X) \xrightarrow{\gamma}$$

$$\bigoplus_{i=1}^{r} H^1(\underline{I}_X(f_i)) \quad \to A^2(\underline{d}, O_{\underline{d}})_{res} \to H^1(\underline{N}_X) \xrightarrow{l^2} \bigoplus_{i=1}^{r} H^1(O_X(f_i)).$$

If we let

$$\chi(\underline{d}) = a^1 - a^2_{res} + \dim \operatorname{coker} l^2 ,$$

then

$$\chi(\underline{d}) = \chi(\underline{N}_X) + \sum_{i=1}^{r} (h^o(\underline{I}_{X/Y}(f_i)) - h^1(I_X(f_i)) + h^1(O_X(f_i))).$$

<u>Lemma 2.2.14</u>. Let X be a curve which is generically a complete intersection in $\mathbb{P} = \mathbb{P}^3_k$, and let $Y = V(F_1, \ldots, F_r)$ for $1 \le r \le 2$ be a global complete intersection containing X with $f_i = \deg F_i$. Then $\chi(\underline{d}) = a^1 - a^2_{res} + \dim \operatorname{coker} l^2$ is given by

$$\chi(\underline{d}) = 4d + \sum_{i=1}^{r} h^o(\underline{I}_{X/Y}(f_i)) + \sum_{i=1}^{r} \gamma(f_i)$$

where $\gamma(\nu) = h^1(O_X(\nu)) - h^1(\underline{I}_X(\nu))$. Moreover if $r = 1$ then

$$\chi(\underline{d}) = (4-f_1)d + \binom{f_i+3}{3} + g - 2 ,$$

and if $r = 2$ and $f_1 \le f_2$, then

$$\chi(\underline{d}) = \begin{cases} (4-f_1-f_2)d + \binom{f_1+3}{3} + \binom{f_2+3}{3} - \binom{f_2-f_1+3}{3} + 2g - 4 & \text{for } f_1 < f_2, \\ (4-2f_1)d + 2\binom{f_1+3}{3} + 2g - 6 & \text{for } f_1 = f_2. \end{cases}$$

Proof. Since Y is a global complete intersection, $H^1(\underline{I}_Y(f_i)) = 0$, and the sequence

$$0 \to H^o(\underline{I}_Y(f_i)) \to H^o(\underline{I}_X(f_i)) \to H^o(\underline{I}_{X/Y}(f_i)) \to 0$$

is therefore exact. Thus for any i,

$$h^o(\underline{I}_{X/Y}(f_i)) - h^1(\underline{I}_X(f_i)) + h^1(O_X(f_i)) = \chi(\underline{I}_X(f_i)) - h^o(\underline{I}_Y(f_i)),$$

and by Riemann-Roch

$$\chi(\underline{I}_X(f_i)) = \chi(O_{\mathbb{P}}(f_i)) - \chi(O_X(f_i)) = \binom{f_i+3}{3} - (df_i+1-g).$$

Now if $r = 1$, then $h^o(\underline{I}_Y(f_1)) = 1$, and if $r = 2$, there is an exact sequence

$$0 \to O_{\mathbb{P}}(-f_1-f_2) \to O_{\mathbb{P}}(-f_1) \oplus O_{\mathbb{P}}(-f_2) \to \underline{I}_Y \to 0.$$

We find $\Sigma\, h^o(\underline{I}_Y(f_i)) = 4$ in the case $f_1 = f_2$, and otherwise

$$\sum_{i=1}^{2} h^o(\underline{I}_Y(f_i)) = h^o(O_{\mathbb{P}}(f_2-f_1)) + 2 = \binom{f_2-f_1+3}{3} + 2.$$

Combining, we find the formulas, as required.

The preceding lemma allows us to prove

Proposition 2.2.15. Let $X \subseteq Y \subseteq \mathbb{P}^3_k$ correspond to surjections $R \overset{\psi}{\to} B \overset{\varphi}{\to} A$ of minimal cones. Then with assumptions as in (2.2.14)

$$a^2_{res} \geq \delta - \sum_{i=1}^{r} \gamma(f_i),$$

and we have equality iff

$${}_oH^2_m(B,A,A) = 0, \quad {}_oH^2(R,A,A) = 0 \quad \text{and } \operatorname{coker} 1^2 = 0.$$

Moreover if equality holds, then

$$\operatorname{Def}^o_{(\psi,\varphi)} \simeq \underset{\sim}{D}_{X \subseteq Y}$$

and $D = D(p,q)$ is non-singular at $x = (X \subseteq Y \subseteq \mathbb{P})$. Furthermore in this case

$$\dim O_{D,x} = 4d + \delta + \sum_{i=1}^{r} h^o(\underline{I}_{X/Y}(f_i)) = \chi(\underline{d}) + \delta - \sum_{i=1}^{r} \gamma(f_i),$$

and if $pr_1(D)$ is the scheme-theoretic image of the projection $pr_1 : D(p,q) \to \mathrm{Hilb}^p$, then

$$\dim O_{pr_1(D),pr_1(x)} = 4d + \delta .$$

<u>Proof.</u> To begin with we establish some inequalities. Indeed if $H_{(\psi,\varphi)}$ is the hull of $\mathrm{Def}^o_{(\psi,\varphi)}$, then

$$\dim H_{(\psi,\varphi)} \leq \dim O_{D,x} \leq a^1$$

because $\mathrm{Def}^o_{(\psi,\varphi)} \hookrightarrow \underset{\sim}{D}_{X \subseteq Y}$ is injective. And if $E_2^{o,1} = {}_{gr}E_2^{o,1}$ is defined by the spectral sequence of (1.4.4), we easily deduce an exact sequence

$$0 \to {}_oH^1(R,B,I_{B/A}) \to E_2^{o,1} \to {}_oH^1(R,A,A) \to 0$$

where $I_{B/A} = \ker(B \to A)$. Since $R \to B$ is a complete intersection,

$${}_oH^1(R,B,I_{B/A}) \simeq \bigoplus_{i=1}^{r} H^o(\underline{I}_{X/Y}(f_i)).$$

Now by (1.4.5), $E_2^{o,1}$ is the tangent space of $\mathrm{Def}^o_{(\psi,\varphi)}$ and

$$\dim E_2^{o,1} - h^2(0) \leq \dim H_{(\psi,\varphi)}.$$

Recall that $h^2(0) = \dim {}_oH^2(R,A,A)$. Combining, we find

$$(*)\qquad \begin{aligned} &h^1(0) - h^2(0) + \Sigma h^o(\underline{I}_{X/Y}(f_i)) \leq \dim H_{(\psi,\varphi)} \leq \dim O_{D,x} \leq \\ &a^1 \leq a^2_{res} + 4d + \Sigma h^o(\underline{I}_{X/Y}(f_i)) + \Sigma \gamma(f_i), \end{aligned}$$

where the last inequality follows from (2.2.14).

Since $h^1(O) - h^2(O) = 4d + \delta$ by (2.2.11), we deduce

$$a^2_{res} \geq \delta - \Sigma\gamma(f_i).$$

Now using (1.4.5) and (1.4.6) we find that if

$${}_oH^2_m(B,A,A) = 0, \quad {}_oH^2(R,A,A) = 0 \quad \text{and} \quad \operatorname{coker} l^2 = 0,$$

then all the inequalities of (*) are equalities. Thus

$$a^2_{res} = \delta - \Sigma\gamma(f_i).$$

Moreover by (1.4.6), $\mathrm{Def}^o_{(\psi,\varphi)} \simeq \underset{\sim}{D}_{X\subseteq Y}$ and $D(p,q)$ is non-singular at $(X \subseteq Y \subseteq \mathbb{P})$. Finally $\dim O_{D,x}$ is easily found, and also $\dim O_{pr_1(D),pr_1(x)}$ because $\hat{O}_{pr_1(D),pr_1(x)}$ is the hull of $\mathrm{Def}^o_{\varphi\psi}$, see (1.4.8) and recall that $\underset{\sim}{D}_{X\subseteq Y} \xleftarrow{\simeq} \mathrm{Def}^o_{(\psi,\varphi)} \to \mathrm{Def}^o_{\varphi\psi}$ is formally smooth and that $\mathrm{Def}^o_{\varphi\psi} \hookrightarrow \underset{\sim}{\mathrm{Hilb}}_X$ is injective.

Conversely suppose $a^2_{res} = \delta - \Sigma\gamma(f_i)$. It follows that all the inequalities of (*) are equalities. In particular $\operatorname{coker} l^2 = 0$, and $\mathrm{Def}^o_{(\psi,\varphi)} \simeq \underset{\sim}{D}_{X\subseteq Y}$ is an isomorphism of formally smooth functors, so

$$\dim H_{(\psi,\varphi)} = \dim E^{o,1}_2 = h^1(O) - \Sigma h^o(\underline{I}_{X/Y}(f_i)) .$$

Now one of the equalities of (*) implies that $h^2(O) = 0$, and it will therefore be sufficient to show

$${}_oH^2_m(B,A,A) = 0 .$$

If we consider the proof of (1.4.6), we find by using the isomorphism $\mathrm{Def}^o_{(\psi,\varphi)} \simeq \underset{\sim}{D}_{X\subseteq Y}$ which implies $E^{o,1}_2 \simeq A^1(\underline{d},O_{\underline{d}})$, and by using ${}_oH^2(R,A,A) = 0$ that

$${}_oH^1(B,A,A) \twoheadrightarrow A^1(f,O_X)$$
$${}_oH^2(B,A,A) \hookrightarrow A^2(f,O_X)$$

is surjective, resp. injective. Recall that $f : X \hookrightarrow Y$ is the inclusion. Then by (1.4.3) ${}_oH^2_m(B,A,A) = 0$ as required.

Finally we illustrate by considering examples over an algebraically closed field, and in these examples we let $Y = V(F) \supseteq X$ be a surface of the least possible degree, i.e. $\deg F = s(X)$. Now reviewing the examples of (2.2.10), we easily find that ${}_oH^2_m(B,A,A) = 0$ and ${}_oH^2(R,A,A) = 0$ in all three cases. Another example already treated in the literature is this, see [G.P.,§4].

<u>Example 2.2.16</u>. Following [G.P.] the Hilbert scheme $H(8,5)_S$ of smooth connected curves of degree 8 and genus 5 is normal and integral of dimension $4d = 32$, and there are five classes of curves to consider.

(A) $0 \to R(-6)^{\oplus 2} \to R(-5)^{\oplus 8} \to R(-4)^{\oplus 7} \to I \to 0$

which is the "generic" one, so $h^1(O_X(1)) = 0$,

(B) $0 \to R(-7) \to R(-6)^{\oplus 2} \oplus R(-5)^{\oplus 2} \to R(-4)^{\oplus 3} \oplus R(-3) \to I \to 0$

where $h^1(O_X(1)) = 1$,

(C) $0 \to R(-7) \to R(-6)^{\oplus 2} \oplus R(-5)^{\oplus 3} \to R(-5) \oplus R(-4)^{\oplus 3} \oplus R(-3) \to I \to 0$

where $h^1(O_X(1)) = 0$,

(D) Same resolution as in (C), but $h^1(O_X(1)) = 1$,

(E) $0 \to R(-8)^{\oplus 3} \to R(-7)^{\oplus 8} \to R(-6)^{\oplus 5} \oplus R(-2) \to I \to 0$

where $h^1(O_X(1)) = 0$ and where Y is non-singular.

Now by Riemann-Roch

$$\chi(\underline{I}_X(\nu)) = \binom{\nu+3}{3} - (8\nu-4)$$

so $\chi(\underline{I}_X(1)) = 0$, i.e. $h^1(\underline{I}_X(1)) = h^1(O_X(1))$. To use (2.2.15) we will first show that

$${}_oH^2_m(B,A,A) = 0$$

in all five cases. To see this, let $s = s(X) = n_{11}$, so by (1.4.3) it will be sufficient to show that $H^1(\underline{I}_X(n_{1i})) = 0$ for $2 \leq i \leq r_1$. By $c(X) = \max n_{3i} - 4$ and by considering the resolutions, we conclude easily.

Next we prove that ${}_oH^2(R,A,A) = 0$ except for the case (D). Indeed using (2.2.8), we find that ${}_oH^2(R,A,A)$ vanishes for the families (A) and (B), and also for (C) because $h^1(\underline{I}_X(1)) = 0$. For (D) we find

$$\dim {}_oH^2(R,A,A) \leq 1.$$

Moreover in the last case (E) it is not easy to use (2.2.8). However by (1.3.10), $A^2(\underline{d}, O_{\underline{d}})_{res} = \operatorname{coker} \alpha = 0$, and by (1.4.6), ${}_oH^2(R,A,A) \hookrightarrow \operatorname{coker} \alpha$ is injective because ${}_oH^2_m(B,A,A) = 0$. So ${}_oH^2(R,A,A) = 0$ for (E) as required. Thus (2.2.15) applies to all five families, except for the family (D), with $a^2_{res} = 0$, and by (2.2.15) and (2.2.14)

$$\dim O_{D,x} = \chi(\underline{d}) = \begin{cases} g + 33 = 38 & \text{for (A)} \\ d + g + 18 = 31 & \text{for (B)} \\ d + g + 18 = 31 & \text{for (C)} \\ 2d + g + 8 = 29 & \text{for (E)} \end{cases}$$

Moreover it can be shown that the families (A), (B), (C) and (E) form open subsets of $D(8,5;s)$.[1] Note also that $\dim O_{pr_1(D), pr_1(x)} = \dim O_{D,x}$ except for the family (A),

1) Let $D(d,g;s) = D(p,s)$ where p is the polynomial $p(\nu) = d\nu + 1 - g$.

in which case pr_1 is smooth by (1.3.4) because $H^1(\underline{I}_X(4)) = 0$. Since $h^0(\underline{I}_{X/Y}(4)) = 6$ in this case, the family (A) is open in $H(8,5)_S$ of dimension 32.

This coincides with [G,P]. Indeed they also say that the family (D) is the intersection of the closures of the families (B) and (C) in $H(8,5)_S$, and so also in $D(8,5;3)_S$ where $D(8,5;3)_S \subseteq D(8,5;3)$ consists of points $(X \subseteq Y \subseteq \mathbb{P})$ with X smooth and connected. This implies that all curves belonging to (D) are singular points of $D(8,5;3)$, and since $\mathrm{Def}^0_{(\psi,\varphi)} \simeq \underset{\sim}{D}_{X \subseteq Y}$, $\mathrm{Def}^0_{(\psi,\varphi)}$ can not be formally smooth. So

$$\dim_0 H^2(R,A,A) = 1,$$

and we shall soon see that

$$a^2_{res} = \dim \operatorname{coker} \alpha = 1$$

as well. Indeed we first prove that $H(8,5)_S$ is smooth by proving that smooth non-plane curves having $H^1(O_X(1)) = 0$ or $\omega_X \simeq O_X(1)$ satisfy $H^1(\underline{N}_X) = 0$. To see this, let θ_X, resp $\theta_{\mathbb{P}/X}$ be the sheaf of derivations on X, resp derivations on $\mathbb{P}$ restricted to X, and consider the exact sequences

$$0 \to \theta_X \to \theta_{\mathbb{P}/X} \to \underline{N}_X \to 0 ,$$

$$0 \to O_X \to O_X(1)^{\oplus 4} \to \theta_{\mathbb{P}/X} \to 0.$$

By taking cohomology we need only show

$$H^1(O_X) \to H^1(O_X(1))^{\oplus 4}$$

surjective. If $H^1(O_X(1)) = 0$, we are done, and if $\omega_X \simeq O_X(1)$, then by duality we must prove that

$$H^{o}(O_X) \underset{k}{\otimes} H^{o}(O_{\mathbb{P}}(1)) \to H^{o}(O_X(1))$$

is injective. This is, however, obvious because X is non-plane. Finally we have already seen that (1.3.1C) yields an exact sequence

$$H^{o}(\underline{N}_X) \xrightarrow{\gamma} H^{1}(\underline{I}_X(s)) \to A^{2}(d,O_{\underline{d}})_{res} \to H^{1}(\underline{N}_X) \xrightarrow{1^{2}} H^{1}(O_X(s))$$

Thus for (D)

$$a^{2}_{res} \leq h^{1}(\underline{I}_X(3)) = 1,$$

and since $A^{2}(\underline{d},O_{\underline{d}})_{res}$ can not vanish,

$$a^{2}_{res} = 1.$$