Free resolutions, ghost terms and the Hilbert scheme

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Introduction

In this talk

- $R = k[x_0, x_1, x_2, x_3]$ polynom. ring, $k = \overline{k}$, $\mathfrak{m} = (x_0, x_1, x_2, x_3)$
- "$C \subseteq \mathbb{P}^3$ a curve" means "C locally CM of equidimension 1" with sheaf ideal $\mathcal{I}_C$, homogeneous ideal

$$I(C) := H^0_*(\mathcal{I}_C) := \oplus_v H^0(\mathcal{I}_C(v)) \ ,$$

and let $M := H^1_*(\mathcal{I}_C)$ be the (Hartshorne)-Rao module.

The **Hilbert scheme** $\mathrm{H}(d,g)$ is, as a set of closed points, equal to

$$\mathrm{H}(d,g) = \{\ (C) \mid C \subset \mathbb{P}^3 \textit{ curve of deg } C = d, \textit{ genus } C = g\}$$

Let $\mathrm{H}(d,g)_S$ be the open subscheme of smooth connected curves.

The main goal of this talk is to study $\mathrm{H}(d,g)$ **at** (C) **via a minimal** R**-free resolution of** I, by e. g.

- deforming C to a C' in various ways by making consecutive free summands in a minimal free resolution of $I(C)$ disappear in a free resolution of $I(C')$ (i.e. "killing" ghost terms).

For a diameter-1 curve C (i.e., $M_v \neq 0$ for only one v) we will

- show a one-to-one correspondence between the set of irred. components of $\mathrm{H}(d,g) \ni (C)$ and a set of min. 5-tuples specializing to a 5-tuple of graded Betti numbers of C
- see a specific description of the singular locus of the Hilbert scheme of diam ≤ 1 curves in terms of closures of Betti strata.

graded Betti numbers

Since $I(C) = H^0_*(\mathcal{I}_C)$ we have $\operatorname{depth}_{\mathfrak{m}} I \geq 2$, hence I has a minimal free resolution of the following form

$$0 \to \oplus_i R(-i)^{\beta_{3,i}} \to \oplus_i R(-i)^{\beta_{2,i}} \to \oplus_i R(-i)^{\beta_{1,i}} \to I \to 0\,. \qquad (1)$$

The numbers

$$\beta_{j,i} = \beta_{j,i}(C)$$

are the **graded Betti numbers** of $I(C)$. We say

- C is **arithmetically CM** or ACM if R/I is Cohen-Macaulay or equivalently, $\operatorname{depth}_{\mathfrak{m}} I = 3$ or all $\beta_{3,i} = 0$, i.e. the Rao module $M := H^1_*(\mathcal{I}_C) \cong H^2_{\mathfrak{m}}(I)$ vanishes.
- If ACM, the min.res. is given by the Hilbert-Burch matrix and H(d,g) is described by works of Peskine-Szpiro and Ellingsrud

Rao's theorem

Now we recall Rao's theorem concerning the form of a minimal free resolution of $I = I(C)$. Let

$$0 \to L_4 \xrightarrow{\sigma} L_3 \to L_2 \to L_1 \xrightarrow{\tau} L_0 \to M \to 0 \qquad (2)$$

be the minimal resolution of $M = H^1_*(\mathcal{I}_C)$. Then (1) and

$$0 \to L_4 \xrightarrow{\sigma \oplus 0} L_3 \oplus F_2 \longrightarrow F_1 \to I \to 0 \qquad (3)$$

are isomorphic [Rao, Thm. 2.5] ! Here the composition of $L_4 \to L_3 \oplus F_2$ with the natural projection $L_3 \oplus F_2 \to F_2$ is zero.

We may write (3) as a so-called *E-resolution* of I (cf. [MDP]):

$$0 \to E \oplus F_2 \to F_1 \to I \to 0\ , \qquad E := \operatorname{coker} \sigma\ . \qquad (4)$$

Rao's theorem

For a **diameter-1 curve** C with $r = \dim H^1_*(\mathcal{I}_C) = h^1(\mathcal{I}_C(c))$ ($M_v = 0$ for all $v \neq c$), we have the minimal free resolution:

$$0 \to R(-c-4)^r \xrightarrow{\sigma} R(-c-3)^{4r} \to R(-c-2)^{6r} \to R(-c-1)^{4r} \to R(-c)^r \to M \to 0$$

which is "r times" the Koszul resolution of

$$k(-c) \cong R/(x_0, x_1, x_2, x_3)(-c) =: M_{[c]}.$$

If $r = 1$ then the matrix of σ is the transpose of (x_0, x_1, x_2, x_3).

Putting this into the resolution of I, we get for a **diam. 1 curve**:

$$0 \to R(-c-4)^r \xrightarrow{\sigma \oplus 0} R(-c-3)^{4r} \oplus F_2 \longrightarrow F_1 \to I \to 0 \qquad (5)$$

Example. $\exists\, X$ of $\mathrm{H}(18, 39)_S$ with $c = 4$ and min. resolution:

$$0 \to R(-8) \to R(-7)^4 \oplus R(-8) \to R(-6)^4 \oplus R(-4) \to I(X) \to 0\,.$$

Compare it with the Rao form (5). Here $R(-8)$ is a **ghost term**.

I will explain the main results (from [K12]) of my talk by recalling

Theorem (K06; Ann. Inst. Fourier, 56 no. 5, 2006)

$C \subset \mathbb{P}^3$ a curve whose Rao module $M \cong M^r_{[c]} \neq 0$ (so $r = \beta_{3,c+4}$). Then C is obstructed (i.e $\mathrm{H}(d,g)$ is singular at (C)) if and only if

$$\beta_{1,c} \cdot \beta_{2,c+4} \neq 0 \quad \text{or} \quad \beta_{1,c+4} \cdot \beta_{2,c+4} \neq 0 \quad \text{or} \quad \beta_{1,c} \cdot \beta_{2,c} \neq 0 \ .$$

Moreover if C is **unobstructed**, *the dimension of $\mathrm{H}(d,g)$ at (C) is*

$$\dim_{(C)} \mathrm{H}(d,g) = 4d + \delta^2(0) + r(\beta_{1,c+4} + \beta_{2,c}), \text{ and}$$

$$\delta^2(0) = \sum_i \beta_{1,i} \cdot h^2(\mathcal{I}_C(i)) - \sum_i \beta_{2,i} \cdot h^2(\mathcal{I}_C(i)) + \sum_i \beta_{3,i} \cdot h^2(\mathcal{I}_C(i))$$

Example: Obstructed curve in $\mathrm{H}(33,117)_S$ with diam .1, $c = 5$:

$$R(-9) \hookrightarrow R(-10)^2 \oplus R(-9) \oplus R(-8)^4 \to R(-9) \oplus R(-8) \oplus R(-7)^5 \to I \to 0$$

5-tuples of graded Betti numbers

For a given diameter-1 curve $C \subseteq \mathbb{P}^3$, we consider the 5-tuple

$$\underline{\beta}(C)_5 := (\beta_{1,c+4}, \beta_{1,c}, \beta_{2,c+4}, \beta_{2,c}, \beta_{3,c+4}) .$$

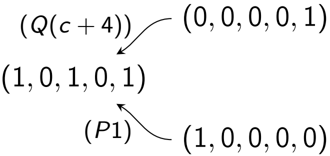
Remark The thm. says: C is unobstructed iff there are at least two consecutive zero's in the first 4 coordinates or $\beta_{3,c+4} = 0$.

Last Example: $C \in \mathrm{H}(33,117)_S$ satisfied: $\underline{\beta}(C)_5 = (1,0,1,0,1)$

On such 5-tuples we let **admissible** operations be possibly repeated use of vector-subtractions by: $\underline{q}_c := (0,1,0,1,0)$, $\underline{q}_{c+4} := (1,0,1,0,0)$, $\underline{p}_1 := (0,0,1,0,1)$, and $\underline{p}_2 = (0,1,0,0,1)$, provided the resulting 5-tuple is **non-negative** in every coordinate.

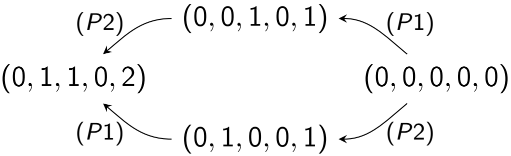
Example: For $\underline{\beta}(C)_5 = (1,0,1,0,1)$, $C \in \mathrm{H}(33,117)_S$, we have $\underline{\beta}(C)_5 - q_{c+4} = (0,0,0,0,1)$ and $\underline{\beta}(C)_5 - p_1 = (1,0,0,0,0)$; the resulting 5-tuples are called **minimal**.

a) The curve $C \in \mathrm{H}(33,117)_S$ with $\underline{\beta}(C)_5 = (1,0,1,0,1)$:



2 admissible 5-tuples, both minimal

b) $\exists C \in \mathrm{H}(32,109)_S$ with 5-tuple $\underline{\beta}(C)_5 = (0,1,1,0,2)$:



3 admissible 5-tuples, only the final one is minimal

Theorem (Main theorem 1)

Let $C \subseteq \mathbb{P}^3$ be a curve of diameter one. Then every admissible 5-tuple corresponds to a generization C' (i.e. a deformation to a more general curve) whose 5-tuple equals the admissible one. Moreover

$$\{\text{minimal } \underline{\beta}'_5 \,|\, \underline{\beta}'_5 \rightsquigarrow \underline{\beta}(C)_5\} \stackrel{1-1}{\longleftrightarrow} \{\text{irred. comp. } V \subset \mathrm{H}(d,g) |\, V \ni (C)\}\,.$$

Here V maps to the 5-tuple of its generic curve and all components V are generically smooth.

Generizations killing ghost terms in the Rao form

Consider the Rao form of a min. resolution:

$$0 \to L_4 \xrightarrow{\sigma \oplus 0} L_3 \oplus F_2 \longrightarrow F_1 \to I(C) \to 0$$

where $0 \to L_4 \xrightarrow{\sigma} L_3 \to .. \to M \to 0$ is a min. res. of $M = M(C)$.

Theorem

Let $C \subseteq \mathbb{P}^3$ be ***any curve*** *with minimal free resolution as above. If F_1 and F_2 have a common free summand;*

$$F_2 = F_2' \oplus R(-i)\,, \quad F_1 = F_1' \oplus R(-i),$$

then there is a generization C' (of type Qi) of C in $\mathrm{H}(d,g)$ with const. postulation and const. Rao module and with min. resolution

$$0 \to L_4 \xrightarrow{\sigma \oplus 0} L_3 \oplus F_2' \to F_1' \to I(C') \to 0\,.$$

Proof Let $M(\sigma)$ be the matrix of σ. As in [MDP], in the resolution

$$0 \to L_4 \xrightarrow{\begin{bmatrix} M(\sigma) \\ 0 \\ 0 \end{bmatrix}} L_3 \oplus F_2' \oplus R(-i) \xrightarrow{\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & 0 \end{bmatrix}} F_1' \oplus R(-i) \to I(C)$$

we replace the 0 in the rightmost corner with a parameter λ. Since

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & \lambda \end{bmatrix} \begin{bmatrix} M(\sigma) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ,$$

the cokernel of the deformed (i.e. changed) matrix defines a flat family of homogeneous ideals over an open in $\mathbb{A}^1_k \ni 0$. □

Remark If C' is a generic curve of an irred. comp. of $\mathrm{H}(d,g)$ containing C of diam. 1 with $c = c(C)$ and $\beta'_{i,j} = \beta_{i,j}(C')$, then

$$\beta'_{1,c+3} \cdot (\beta'_{2,c+3} - 4\beta'_{3,c+4}) = 0 \,, \quad \beta'_{1,i} \cdot \beta'_{2,i} = 0 \ \text{ for any } i \neq c+3 \,,$$

Example

Taking two general skew lines, we link twice via CI's of type $(5,2)$ and $(5,4)$, and we get a curve X, generic (by [K88], Prop. 3.8) in $\mathrm{H}(12,18)$, with min. res. and a ghost term $R(-5)$ in degree $c+3$:

$$0 \to R(-6) \to R(-7) \oplus R(-5)^4 \to R(-5) \oplus R(-4)^4 \to I(X) \to 0 \,.$$

Theorem (Main theorem 2)

Let C' be any generization of diam. 1 curve C in $\mathrm{H}(d,g)$ satisfying

$$\beta'_{1,c+3} \cdot (\beta'_{2,c+3} - 4\beta'_{3,c+4}) = 0 \,, \quad \beta'_{1,i} \cdot \beta'_{2,i} = 0 \text{ for any } i \in \{c+1, c+2\}$$

Then C' is a generization of C in $\mathrm{H}(d,g)$ generated by (PQ).

The generizations P1 and P2:

Suppose C admits a Buchsbaum component $M_{[t]}$, i.e.

$$M(C) \cong M' \oplus M_{[t]} \quad \text{as} \quad R-modules.$$

Remark If M' is a direct sum of other Buchsbaum components of possibly various degrees (resp. of the same degree t, i.e. $M \simeq M^r_{[t]}$), then C is a Buchsbaum curve (resp. of diameter one).

The generization P1:

Denoting $(\sigma', \sigma_{[t]}) := \begin{pmatrix} \sigma' & 0 \\ 0 & \sigma_{[t]} \end{pmatrix}$, then $M \cong M' \oplus M_{[t]}$ has the min. res.:

$$0 \to P_4 \oplus R(-t-4) \xrightarrow{(\sigma', \sigma_{[t]})} P_3 \oplus R(-t-3)^4 \to P_2 \oplus R(-t-2)^6 \to \ldots \to M \to 0$$

where $0 \to P_4 \xrightarrow{\sigma'} P_3 \to P_2 \xrightarrow{\tau_2} P_1 \xrightarrow{\tau_1} P_0 \to M' \to 0$ is a min. res.

$$0 \to R(-t-4) \xrightarrow{\sigma_{[t]}} R(-t-3)^4 \to R(-t-2)^6 \to R(-t-1)^4 \xrightarrow{\tau_{[t]}} R(-t) \to M_{[t]} \to 0$$

Combining with Rao's theorem, we get the min. resolution:

$$0 \to P_4 \oplus R(-t-4) \xrightarrow{(\sigma', \sigma_{[t]}) \oplus 0} P_3 \oplus R(-t-3)^4 \oplus F_2 \to F_1 \to I \to 0 .$$

Generizations preserving only postulation in the Rao form

Proposition (The generization P1)

Let $C \subseteq \mathbb{P}^3$ *a curve admitting an iso.* $M(C) \cong M' \oplus M_{[t]}$ *and*

$$0 \to P_4 \oplus R(-t-4) \xrightarrow{(\sigma', \sigma_{[t]}) \oplus 0 \oplus 0} P_3 \oplus R(-t-3)^4 \oplus Q_2 \oplus R(-t-4) \xrightarrow{\beta} F_1 \to I(C)$$

Then there is a generization C' *(of type P1) of* C *in* $\mathrm{H}(d,g)$ *with constant postulation such that* $I(C')$ *has a free resolution:*

$$0 \to P_4 \xrightarrow{\sigma' \oplus 0 \oplus 0} P_3 \oplus R(-t-3)^4 \oplus Q_2 \to F_1 \to I(C') \to 0\,,$$

and such that $M(C') \cong M'$ *as graded* R*-modules. The resolution is minimal except possibly in degree* $t+3$ *where some of the summands of* $R(-t-3)^4$ *may cancel against free summands of* F_1 *(and type* $Q(t+3)$ *generizations may apply).*

Proof Look (as in [MDP], page 189) at the min. res. of $I = I(C)$:

$$P_4 \oplus R(-t-4) \xrightarrow{(\sigma', \sigma_{[t]})\oplus 0 \oplus 0} P_3 \oplus R(-t-3)^4 \oplus Q_2 \oplus R(-t-4) \xrightarrow{\beta} F_1 \to I$$

where $\lambda = 0$ in

$$\begin{bmatrix} p_3 & h(\lambda) & q_2 & y \end{bmatrix} \begin{bmatrix} M(\sigma') & 0 \\ 0 & M(\sigma_{[t]}) \\ 0 & 0 \\ 0 & \lambda \end{bmatrix} = [0,0]\,, \text{ and } M(\sigma_{[t]}) = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Write the column $y = (y_j)$ as $y_j = \sum_{i=0}^{3} a_{j,i} x_i$ and the 4 columns $h(\lambda)$ as $h(\lambda) := (h'_0, h'_1, h'_2, h'_3)$ where $h'_i = (h_{j,i} - \lambda \cdot a_{j,i})$. Then coker β defines a flat family of ideals over an open in $\mathbb{A}^1_k \ni 0$. □

Remark (a generalization)
Suppose $M(C) \cong M' \oplus M_{CI}$, $M_{CI} = R/(f_0, f_1, f_2, f_3)$ a CI, and
$0 \to P_4 \oplus R(-d) \xrightarrow{(\sigma', \sigma_{CI})\oplus 0 \oplus 0} P_3 \oplus R(-d_i)^4 \oplus Q_2 \oplus R(-d) \to$
If each coordinate of y satisfies $y_j \in (f_0, f_1, f_2, f_3)$, then $R(-d)$ is removable through a deformation.

Generizations NOT preserving postulation

Example [Ser] We get a generization C' of type P1 of C:

$$0 \to R(-8) \to R(-8) \oplus R(-7)^4 \to R(-6)^4 \oplus R(-4) \to I(C) \to 0\,,$$

by deleting $R(-8)$. (C) sits in the intersection of two irred. comp. of $\mathrm{H}(18,39)_S$. The generic curve X of the other comp. satisfies

$$0 \to R(-8) \oplus R(-6)^2 \to R(-5)^4 \to I(X) \to 0\,.$$

Comparing, we get $\beta_{1,5}(C) = \beta_{2,6}(C) = 0$ while $\beta_{1,5}(X) = 4$, $\beta_{2,6}(X) = 2$, i.e. $\beta_{1,5}$ and $\beta_{2,6}$ are **not** (upper) **semi-continuous**.

However, the triple $(h^0(\mathcal{I}_Z(4)), h^1(\mathcal{I}_Z(4)), h^1(\mathcal{O}_Z(4)))$ equals $(1,0,0)$ for $Z = C'$ and $(0,0,1)$ for $Z = X$, so (X) and (C') sits in different irred. comp. of $\mathrm{H}(18,39)_S$ by the semi-cont. of $h^i(\mathcal{I}_Z(v))$.

Generizations NOT preserving postulation in the Rao form

Proposition (The generization P2)

C a space curve admitting a graded R-module isomorphism $M(C) \cong M' \oplus M_{[t]}$. If $F_1 \cong Q_1 \oplus R(-t)$ in the min. res.:

$$0 \to P_4 \oplus R(-t-4) \xrightarrow{(\sigma', \sigma_{[t]}) \oplus 0} P_3 \oplus R(-t-3)^4 \oplus F_2 \to F_1 \to I(C) \to 0\,,$$

and if P_2 does not contain a summand $R(-t)$, then there is a generization C' (of type P2) of C in $\mathrm{H}(d,g)$ with constant specialization and constant M' such that $I(C')$ has the resolution:

$$0 \to P_4 \xrightarrow{\sigma' \oplus 0 \oplus 0} P_3 \oplus F_2 \oplus R(-t-2)^6 \to Q_1 \oplus R(-t-1)^4 \to I(C') \to 0\,.$$

The resolution is min. except possibly in degree $t+1$ and $t+2$ where type Qi generizations for $i \in \{t+1, t+2\}$ may apply

Proof. P2 for C is proved using P1 to the linked curve!

Liaison and a theorem of Peskine-Szpiro-Ferrand et al

How do we find the min. resolution of the linked curve. Considering $\mathcal{I}_{C/Y} := \mathcal{I}_C/\mathcal{I}_Y$ as the sheaf ideal of C in Y, we recall

Definition

Two curves C and D in $\mathbb{P}^3$ are said to be (algebraically) *CI-linked* if there exists a complete intersection curve (a CI) Y such that

$$\mathcal{I}_C/\mathcal{I}_Y \cong \mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\mathcal{O}_D, \mathcal{O}_Y) \quad \text{and} \quad \mathcal{I}_D/\mathcal{I}_Y \cong \mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\mathcal{O}_C, \mathcal{O}_Y).$$

The dualizing sheaf a CI $Y \supset C$ of type (f, g) satisfies $\omega_Y \cong \mathcal{O}_Y(f+g-4)$, so

$$\mathcal{I}_{C/Y} \cong \omega_D(4-f-g) \cong \mathcal{E}xt^2(\mathcal{O}_D, \mathcal{O}_\mathbb{P})(-f-g)$$

D lCM equidim codim 2 $\Rightarrow \mathcal{E}xt^2(\mathcal{E}xt^2(\mathcal{O}_D, \mathcal{O}_\mathbb{P}), \mathcal{O}_\mathbb{P}) \cong \mathcal{O}_D$, whence

$Ext^2_R(I(C)/I(Y)(f+g), R) \cong \mathrm{Ext}^2_{\mathcal{O}_\mathbb{P},*}(\mathcal{I}_{C/Y}(f+g), \mathcal{O}_\mathbb{P}) \cong H^0_*(\mathcal{O}_D)$.

We give the main ideas of the proof through an example:

Example Take the minimal resolution of a curve $C \in H(6,3)_S$:

$$0 \to R(-6) \xrightarrow{\sigma} R(-5)^4 \to R(-4)^3 \oplus R(-2) \to I(C) \to 0\,.$$

i.e. as $I(C)$ in last Prop. with $M' = 0$ (i.e all $P_i = 0$) and $t = 2$.

We **claim** there is a generization "cancelling $R(-6)$ (together with $R(-5)^4$) and $R(-2)$" at the cost of an increase in Betti numbers in deg. 3 and 4.

Indeed link C to D via a CI of type (f,g) containing C, taking $f = g = 4$ to simplify. The E-resolution is :

$$0 \to E \to R(-4)^3 \oplus R(-2) \to I(C) \to 0\;, \qquad E := \operatorname{coker} \sigma\;.$$

Using the mapping cone:

$$\begin{array}{ccccccccc} 0 \longrightarrow & R(-8) & \longrightarrow & R(-4)^2 & \longrightarrow & I(Y) \longrightarrow 0 \\ & \downarrow & \circ & \downarrow & \circ & \downarrow \\ 0 \longrightarrow & E & \rightarrow & R(-4)^3 \oplus R(-2) & \rightarrow & I(C) \longrightarrow 0 \end{array}$$

we get a resolution of $I(C)/I(Y)$. Taking R-duals, $\mathrm{Hom}_R(-, R)$, and using $I(D) = \ker(R \to H^0_*(\mathcal{O}_D))$, we get the exact

$$0 \to R(-6) \oplus R(-4) \to E^\vee(-8) \to I(D) \to 0 \qquad (6)$$

having removed 2 redundant terms (to make the res. min.). Using

$$0 \to R(2) \to R(3)^4 \to R(4)^6 \to E^\vee \to 0\,.$$

and the mapping cone construction, we get:

$$0 \to R(-6) \to R(-5)^4 \oplus R(-6) \to R(-4)^5 \to I(D) \to 0\,.$$

This resolution has the form as in Prop. P1 with $M' = 0$ and $t = 2$.

By that Prop. there is a generization D' cancelling $R(-6)$, so D' is ACM. Linking "back" via a general CI Y' of type $(4,4)$:

$$\begin{array}{ccccccc} 0 \longrightarrow & R(-8) & \longrightarrow & R(-4)^2 & \longrightarrow & I(Y') \longrightarrow 0 \\ & \downarrow & \circ & \downarrow & \circ & \downarrow \\ 0 \longrightarrow & R(-5)^4 & \longrightarrow & R(-4)^5 & \longrightarrow & I(D') \longrightarrow 0 \end{array}$$

we get a res. of $I(D')/I(Y')$ whose dual yields a curve C' with res.,

$$0 \to R(-4)^3 \to R(-3)^4 \to I(C') \to 0\,.$$

By [K88], Prop. 3.7, C' is a generization of the curve C. □

Proof. P2 for C is proved using P1 onto the linked curve! Indeed if we link C to D via a CI of type (f,g), then $c(D) = f + g - 4 - c$,

$$\beta_{j,v}(C) = \beta_{3-j,c+c(D)+4-v}(D)\ , \quad \text{for} \quad v \notin \{c+1, c+2, c+3\}$$

for $j = 1$ and 2. Thus

$$\underline{p}_2(C) \quad \text{correspond} \quad \text{to} \quad \underline{p}_1(D)$$

For this and the change for $v \in \{c+1, c+2, c+3\}$, see Example.

To prove main theorem 2 we need this semi-continuity result.

Proposition

C a diameter-1 curve. If $v \notin \{c+1, c+2, c+3\}$, then the Betti numbers $\beta_{1,v}$ and $\beta_{2,v}$ are upper semi-continuous. In particular the 5-tuple $(\beta_{1,c+4}, \beta_{1,c}, \beta_{2,c+4}, \beta_{2,c}, \beta_{3,c+4})$ is upper semi-continuous, i.e. each of these 5 numbers do not increase under generization.

Proof, main ideas We use the so-called Ω-resolutions of a Buchsbaum curve. Here Ω is defined by the exact sequences

$$0 \to R(-4) \to R(-3)^4 \to R(-2)^6 \to \Omega \to 0$$

deduced from the Koszul resolution of $M_{[0]}$. Then we prove

$$h^1(\mathcal{I}_C \otimes \widetilde{\Omega}(v)) = \beta_{1,v} \ , \quad \text{for} \quad v \notin \{c+1, c+2, c+3\}.$$

Hence $\beta_{1,v}$ is semi-continuous since $h^1(\mathcal{I}_C \otimes \widetilde{\Omega}(v))$ is. Moreover if we link C to D via a CI of type (f,g), we conclude the proof by

$$\beta_{2,v}(C) = \beta_{1,c+c(D)+4-v}(D) \ , \quad \text{for} \quad v \notin \{c+1, c+2, c+3\} \qquad \square$$

Definition

C a diam. 1 curve. A generization C' of C in $\mathrm{H}(d,g)$ that is given by repeatedly using some of the generizations of type $(\mathrm{P1})$, $(\mathrm{P2})$ and (Qj) for $j \in \mathbb{N}$ and trivial generizations in some order, is called a generization in $\mathrm{H}(d,g)$ generated by (PQ).

The notion of **trivial generization** is needed to move around inside a *Betti stratum* $\mathrm{H}(\underline{\beta})$. We easily get it through the proof of the **irreducibility** of $\mathrm{H}(\underline{\beta})$. Indeed (cf. [Bo], Thm. 2.2)

Proof of irred. (in diam. 1 case) Two curves $D_1, D_2 \in \mathrm{H}(\underline{\beta})$ have exactly the same summands in their E-resolutions, but the maps $\varphi_{D_i} : E \oplus F_2 \to F_1$ are different. The **irred.** family given by

$$\varphi_t := t\varphi_{D_1} + (1-t)\varphi_{D_2} \in \mathrm{Hom}(E \oplus F_2, F_1), \ t \in \mathbb{A}^1_k.$$

is flat in open set $U \subset \mathbb{A}^1_k$ containing 0 and 1, and $U \subset \mathrm{H}(\underline{\beta})$. □

Definition The generic element $\tilde{D}$ of $\mathbb{A}^1_k$ is called a trivial generization of D_1 (or of D_2). Obviously, $(\tilde{D}) \in \mathrm{H}(\underline{\beta})$.

Theorem (Main theorem 2)

Let $C \subseteq \mathbb{P}^3$ be a Buchsbaum curve of diameter one and let C' be any generization of C in $\mathrm{H}(d,g)$. Then C', after possibly removing ghost terms from $I(C')$ of type Q_v for $v \in \{c+1, c+2, c+3\}$, is a generization of C in $\mathrm{H}(d,g)$ generated by (PQ).

Proof Let $\gamma_C(v) := h^0(\mathcal{I}_C(v))$ and $\Delta\gamma(c) := \gamma_C(c) - \gamma_{C'}(c)$. Let

$$\chi(\mathcal{I}_C(v)) = h^0(\mathcal{I}_C(v)) - h^1(\mathcal{I}_C(v)) + h^2(\mathcal{I}_C(v))$$

Due to $\chi(\mathcal{I}_{C'}(v)) = \chi(\mathcal{I}_C(v))$ and the semi-cont. of the 5-tuple, we prove $\beta_{1,c} \geq \Delta\gamma(c) \geq 0$ and $\beta_{3,c+4} \geq \Delta\gamma(c)$, whence we can use the operation (P2) $\Delta\gamma(c)$ times to get a generization C_{P2}, such that $\gamma_{C_{P2}}(c) = \gamma_{C'}(c)$ and $h^1(\mathcal{O}_{C_{P2}}(c)) = h^1(\mathcal{O}_C(c))$.

Next we use (P1) $\Delta\sigma(c) := h^1(\mathcal{O}_C(c)) - h^1(\mathcal{O}_{C'}(c))$ times to get the existence of a generization C_P of C_{P2}, furnished by (P1), without changing the postulation $\gamma_{C'}$ and such that $h^1(\mathcal{O}_{C_P}(c)) = h^1(\mathcal{O}_{C'}(c))$. This is possible because $\beta_{2,c+4} \geq \Delta\sigma(c) \geq 0$ and $\beta_{3,c+4} - \Delta\gamma(c) \geq \Delta\sigma(c)$.

Thus we have two curves C_P and C' such that $h^i(\mathcal{I}_{C_P}(v) = h^i(\mathcal{I}_{C'}(v))$ for $i = 0, 1, 2$ and $\forall v$. Then we use (Qi) to get curves in the same Betti stratum, and then a trivial generization □

Our final main result determines **the singular locus** of the open subscheme, $\mathrm{H}(d,g;c)$, of $\mathrm{H}(d,g)$ whose k-points are given by

$$\{(C) \in \mathrm{H}(d,g) | \; H^1(\mathcal{I}_C(v)) = 0 \text{ for every } v \neq c\} , \quad c \in \mathbb{Z}$$

Note that the main Theorem 1 really deals with $\mathrm{H}(d,g;c(C))$.

We define the **Betti stratum**, $\mathrm{H}(\underline{\beta})$, of $\mathrm{H}(d,g,c)$ to consist of all C satisfying $\beta_{j,i}(C) = \beta_{j,i} \; \forall i,j$. We write $\mathrm{H}(\underline{\beta})$ as $\mathrm{H}(\underline{\beta}_5)$ if

$$\beta_{1,c+3} \cdot (\beta_{2,c+3} - 4\beta_{3,c+4}) = 0, \; \beta_{1,i} \cdot \beta_{2,i} = 0 \; \text{ for } i \notin \{c, c+3, c+4\} .$$

Note that the closure $V(\underline{\beta}_5)_B := \overline{\mathrm{H}}(\underline{\beta}_5) \cap \mathrm{H}(d,g;c)$ is **irreducible**, cf. [B]. Let C be a generic curve of a Betti stratum $V(\underline{\beta}_5)_B$.

Example a) [W, BKM] $C \in \mathrm{H}(33,117)_S$ with $\underline{\beta}(C)_5 = (1,0,1,0,1)$:

$(Q(c+4))$ ← $(0,0,0,0,1)$

$(1,0,1,0,1)$

$(P1)$ ← $(1,0,0,0,0)$

2 minimal 5-tuples and their corresponding curves are unobstructed,

while the "subminimal" $\underline{\beta}(C)_5$ correspond to C which is obstructed.

b) $\exists C \in \mathrm{H}(32,109)_S$ with 5-tuple $\underline{\beta}(C)_5 = (0,1,1,0,2)$:

$(P2)$ ← $(0,0,1,0,1)$ ← $(P1)$

$(0,1,1,0,2)$ $(0,0,0,0,0)$

$(P1)$ ← $(0,1,0,0,1)$ ← $(P2)$

The final one is minimal, but also the "subminimal" ones correspond to unobstructed curves. C, however, is obstructed.

Definition If $Y := V(\underline{\beta}_5)_B$ is an irred. comp. of $\mathrm{H}(d,g;c)$, we let

$$V(\underline{\beta}_5+\underline{q}_J)_B := \begin{cases} V(\underline{\beta}_5+\underline{q}_c)_B \cup V(\underline{\beta}_5+\underline{q}_{c+4})_B\,, & \text{if diam } M(C)=1 \\ \varnothing & \text{if } C \text{ is ACM}\,. \end{cases}$$

Moreover for $i = 1$ and 2,

$$V(\underline{\beta}_5+\underline{p}_i+\underline{q}_J)_B := V(\underline{\beta}_5+\underline{p}_i+\underline{q}_c)_B \cup V(\underline{\beta}_5+\underline{p}_i+\underline{q}_{c+4})_B\,.$$

Below $+$, resp. $*$ means a positive, resp. non-neg. integer, and $SingY$ is the part of the sing. locus of $\mathrm{H}(d,g;c)$ contained in Y.

Theorem (Main theorem 3: The singular locus)

If $V(\underline{\beta}_5)_B$ *is an irred. comp. of* $\mathrm{H}(d,g;c)$, *then* $\underline{\beta}_5$ *is as in* (i)-(v);

(i) *if* $\underline{\beta}_5$ *is equal to* $(+,0,0,+,*)$ or $(0,+,+,0,0)$, *then*

$$Sing\ V(\underline{\beta}_5)_B = V(\underline{\beta}_5+\underline{p}_1)_B \cup V(\underline{\beta}_5+\underline{p}_2)_B \cup V(\underline{\beta}_5+\underline{q}_J)_B,$$

(ii) *if* $\underline{\beta}_5 = (0,0,0,+,*)$ or $(0,0,+,*,0)$, *then*

$$Sing\ V(\underline{\beta}_5)_B = V(\underline{\beta}_5+\underline{p}_2)_B \cup V(\underline{\beta}_5+\underline{q}_J)_B,$$

(iii) *if* $\underline{\beta}_5 = (+,0,0,0,*)$ or $(*,+,0,0,0)$, *then*

$$Sing\ V(\underline{\beta}_5)_B = V(\underline{\beta}_5+\underline{p}_1)_B \cup V(\underline{\beta}_5+\underline{q}_J)_B,$$

(iv) *if* $\underline{\beta}_5 = (0,0,0,0,+)$, *then*

$$Sing\ V(\underline{\beta}_5)_B = V(\underline{\beta}_5+\underline{p}_1+\underline{p}_2)_B \cup V(\underline{\beta}_5+\underline{q}_J)_B.$$

(v) *if* $\underline{\beta}_5 = (0,0,0,0,0)$, *then* $Sing\ V(\underline{\beta}_5)_B =$

$$V(\underline{\beta}_5+\underline{p}_1+\underline{p}_2)_B \cup V(\underline{\beta}_5+\underline{p}_1+\underline{q}_J)_B \cup V(\underline{\beta}_5+\underline{p}_2+\underline{q}_J)_B.$$

Proof Main thm 2 and the unobstr. thm for diam.1 curves □

For the **existence of diam. 1 curves**, see [C] and [W].

Thanks for listening !

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