The Hilbert Scheme of Buchsbaum space curves

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Abstract. We consider the Hilbert scheme $\mathrm{H}(d,g)$ of space curves C with homogeneous ideal $I(C) := H^0_*(\mathcal{I}_C)$ and Rao module $M := H^1_*(\mathcal{I}_C)$. By taking suitable generizations (deformations to a more general curve) C' of C, we simplify the minimal free resolution of $I(C)$ by e.g making consecutive free summands (ghost-terms) disappear in a free resolution of $I(C')$. Using this for Buchsbaum curves of diameter one ($M_v \neq 0$ for only one v), we establish a one-to-one correspondence between the set $\mathcal{S}$ of irreducible components of $\mathrm{H}(d,g)$ that contain (C) and a set of minimal 5-tuples that specializes in an explicit manner to a 5-tuple of certain graded Betti numbers of C related to ghost-terms. Moreover we almost completely (resp. completely) determine the graded Betti numbers of all generizations of C (resp. all generic curves of $\mathcal{S}$), and we give a specific description of the singular locus of the Hilbert scheme of curves of diameter at most one. We also prove some semi-continuity results for the graded Betti numbers of any space curve under some assumptions.

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1 Introduction

The goal of this paper is to give an explicit description of all irreducible components of the Hilbert scheme $\mathrm{H}(d,g)$ of space curves that contain a given Buchsbaum curve. Thus this paper completes the study we started in [20] where we only succeeded in some cases ([20], Prop. 4.6). Recall that a curve C (equidimensional and locally Cohen-Macaulay) with sheaf ideal $\mathcal{I}_C$ is called (arithmetically) Buchsbaum if the Rao module $M := H^1_*(\mathcal{I}_C)$ satisfies $(X_0,X_1,X_2,X_3)\cdot M = 0$ where $R = k[X_0,X_1,X_2,X_3]$ is the polynomial ring. Hence if $M_v = 0$ for all but one v, then C is certainly Buchsbaum; we call C a diameter-1 curve in this case.

In this paper we determine all components V of $\mathrm{H}(d,g)$ containing a diameter-1 curve C from the point of view of describing the graded Betti numbers of the generic curve of V in terms of the graded Betti numbers of C (Corollary 5.7, Theorem 5.10). There are 5 graded Betti numbers of C, related to ghost terms if they are non-zero, that play a very special role and determines for instance the number of components V containing (C) (Proposition 5.13). Moreover, if (C) is contained in the closure of a Betti stratum $H(\underline{\beta})$, necessarily irreducible by Proposition 3.5, then we determine the set of graded Betti numbers $\underline{\beta}$ almost completely (Theorem 5.3, Remark 5.6). As a consequence we describe the singular locus of the Hilbert scheme of curves of diameter at most one as an explicit union of certain Betti strata, up to closure (Theorem 6.1). To prove such results it is important to understand which graded Betti number are semi-continuous (Proposition 5.4). We also prove a semi-continuity result for the graded Betti numbers for any space curve under an assumption (Corollary 3.3). Moreover we need to find "all" generizations of C. In [20] we mainly found the generizations using some ideas appearing in [23]. In this work we describe the generization that does not preserve postulation in much more detail and with a new proof (Proposition 4.1). For the generization that preserves postulation and reduces $\dim M$ by one, we correct an inaccuracy in [20], Prop. 4.2 (a): the resolution may be non-minimal in one and only one degree, see Remark 2.11. All these results, together with those on the obstructedness and dimension of $\mathrm{H}(d,g)$ in [20], make us understand the Hilbert scheme of diameter-1 curves.

Thus this paper contributes to solving questions related to the number of components, irreducibility and smoothness of $\mathrm{H}(d,g)$, see [1], [9, 10], [12, 13], [23, 24] for some contributions which are relevant for this paper, and [4] for a thorough study of diameter-1 curves.

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1.1 Notations and terminology

Let $R = k[X_0, X_1, X_2, X_3]$ be a polynomial ring over an algebraically closed field $k = \overline{k}$ (of characteristic zero in the examples) and let $\mathbb{P}^3 := \mathrm{Proj}(R)$. A curve C in $\mathbb{P}^3$ is an *equidimensional, locally Cohen-Macaulay* (lCM) subscheme of $\mathbb{P} := \mathbb{P}^3$ of dimension one with sheaf ideal $\mathcal{I}_C$ and normal sheaf $\mathcal{N}_C := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_C, \mathcal{O}_C)$. If $\mathcal{F}$ is a coherent $\mathcal{O}_{\mathbb{P}}$-Module, we let $H_*^i(\mathcal{F}) := \oplus_v H^i(\mathcal{F}(v))$, $h^i(\mathcal{F}) := \dim H^i(\mathcal{F})$ and $\chi(\mathcal{F}) := \Sigma(-1)^i h^i(\mathcal{F})$. Moreover, $M = M(C)$ is the Hartshorne-Rao module $H_*^1(\mathcal{I}_C)$, or just the Rao module, and $I = I(C)$ is the homogeneous ideal $H_*^0(\mathcal{I}_C)$ of C. They are graded modules over R. Note that M is artinian since C is lCM. C is called ACM (arithmetically CM) if $M = 0$. The postulation $\gamma = \gamma_C$ (resp. deficiency $\rho = \rho_C$ and specialization $\sigma = \sigma_C$) of C is the function defined over the integers by $\gamma(v) = h^0(\mathcal{I}_C(v))$ (resp. $\rho(v) = h^1(\mathcal{I}_C(v))$ and $\sigma(v) = h^1(\mathcal{O}_C(v))$). If $M \neq 0$, let

$$c(C) = \max\{n | h^1(\mathcal{I}_C(n)) \neq 0\}\ , \quad b(C) = \min\{n | h^1(\mathcal{I}_C(n)) \neq 0\}\ ,$$

and let $\operatorname{diam} M := c(C) - b(C) + 1$ be the diameter of M (or of C). We say C has maximal rank if $H^0(\mathcal{I}_C(c)) = 0$ where $c = c(C)$. A curve C satisfying $\mathfrak{m} \cdot M = 0$, $\mathfrak{m} = (X_0, .., X_3)$, is an (arithmetically) Buchsbaum curve, thus *diameter-1 curves* are necessarily Buchsbaum.

We say C is *unobstructed* if the Hilbert scheme ([14]) of space curves of degree d and arithmetic genus g, $\mathrm{H}(d,g)$, is smooth at the corresponding point (C), otherwise C is *obstructed*. The open part of $\mathrm{H}(d,g)$ of *smooth connected* space curves is denoted by $\mathrm{H}(d,g)_S$, while $\mathrm{H}_{\gamma,\rho} = \mathrm{H}(d,g)_{\gamma,\rho}$ (resp. H_γ) denotes the subscheme of $\mathrm{H}(d,g)$ of curves with constant cohomology, i.e. γ_C and ρ_C do not vary with C (resp. constant postulation γ), cf. [23] for an introduction. Let V be an irreducible subset (resp. component) of $\mathrm{H}(d,g)$ containing (C). A curve in a sufficiently small open subset U of V (small enough so that any curve in U has all the openness properties that we want to require) is called a *generization* of $C \subseteq \mathbb{P}^3$ in $\mathrm{H}(d,g)$ (resp. a *generic* curve of $\mathrm{H}(d,g)$). We define generizations in H_γ and $\mathrm{H}_{\gamma,\rho}$ similarly.

2 Background

In this section we review techniques and results which we will need in this paper.

2.1 Minimal resolutions and graded Betti numbers

Let C be a curve in $\mathbb{P}^3$. Then the homogeneous ideal $I = I(C)$ has a minimal resolution of the following form

$$0 \to \oplus_i R(-i)^{\beta_{3,i}} \to \oplus_i R(-i)^{\beta_{2,i}} \to \oplus_i R(-i)^{\beta_{1,i}} \to I \to 0\,. \tag{1}$$

The numbers $\beta_{j,i} = \beta_{j,i}(C)$ are the graded Betti numbers of $I(C)$. We denote the set of all graded Betti numbers of $I(C)$ by $\underline{\beta}(C) := \{\beta_{j,i}(C)\}$. We define the *Betti stratum*, $\mathrm{H}(\underline{\beta})$, of $\mathrm{H}(d,g)_{\gamma,\rho}$ to consist of all curves C of $\mathrm{H}(d,g)_{\gamma,\rho}$ satisfying $\beta_{j,i}(C) = \beta_{j,i}$ for every i, j.

Now we recall Rao's theorem concerning the form of a minimal resolution of $I = I(C)$. Let

$$0 \to L_4 \xrightarrow{\sigma} L_3 \to L_2 \to L_1 \xrightarrow{\tau} L_0 \to M \to 0 \tag{2}$$

be the minimal resolution of $M = M(C) = H^1_*(\mathcal{I}_C)$ and let $L_j = \oplus_i R(-i)^{\beta_{j+1,i}(M)}$. Then (1) and

$$0 \to L_4 \xrightarrow{\sigma \oplus 0} L_3 \oplus F_2 \longrightarrow F_1 \to I \to 0 \tag{3}$$

are isomorphic ([30], Thm. 2.5)! Here the composition of $L_4 \to L_3 \oplus F_2$ with the natural projection $L_3 \oplus F_2 \to F_2$ is zero. We may write (3) as a so-called *E-resolution* of I (cf. [23]):

$$0 \to E \oplus F_2 \to F_1 \to I \to 0\ , \qquad E := \operatorname{coker} \sigma\ . \tag{4}$$

For a diameter-1 curve C with $r = \dim H^1_*(\mathcal{I}_C) = h^1(\mathcal{I}_C(c))$, we have the free resolution

$$0 \to R(-c-4)^r \xrightarrow{\sigma} R(-c-3)^{4r} \to R(-c-2)^{6r} \to R(-c-1)^{4r} \to R(-c)^r \to M \to 0 \tag{5}$$

which is "r times" the Koszul resolution of the R-module $k \cong R/\mathfrak{m}$ twisted by $-c$. Hence we may put $\oplus_i R(-i)^{\beta_{3,i}} = R(-c-4)^r$ in (1). If $r = 1$ then the matrix of σ is just the transpose of (X_0, X_1, X_2, X_3).

Example 2.1. *There is a curve in* $\mathrm{H}(33,117)_S$ *of* $\operatorname{diam} M = 1$ *with minimal resolution*

$$0 \to R(-9) \to R(-10)^2 \oplus R(-9) \oplus R(-8)^4 \to R(-9) \oplus R(-8) \oplus R(-7)^5 \to I \to 0\ ,$$

(see [3] or [32]). If we compare it to the Rao form (3), *we see that* $F_2 = R(-10)^2 \oplus R(-9)$ *and that* $0 \to L_4 = R(-9) \to L_3 = R(-8)^4$ *is the leftmost part in the minimal resolution of* M. *Note that* F_2 *and* L_4 *have the common free summand* $R(-9)$. *A repeated summand in two consecutive terms in the minimal resolution* (1) *will be called a* ghost term. *Also* F_1 *and* F_2 *have* $R(-9)$ *as a ghost term.*

Definition 2.2. *The Rao module* $M = M(C)$ *admits "a Buchsbaum component"* $M_{[t]}$ *if*

$$M \simeq M' \oplus M_{[t]}$$

as graded R*-modules where* $M_{[t]}$ *is the graded* R*-module* k *supported in degree* t $(M_{[t]} \cong k(-t))$.

Remark 2.3. *Suppose* $M = M(C)$ *admits a Buchsbaum component,* $M \simeq M' \oplus M_{[t]}$.

(a) *If* M' *is a direct sum of other Buchsbaum components of possibly various degrees (resp. of the same degree* t*, i.e.* $M \simeq M^r_{[t]}$*), then* C *is a Buchsbaum curve (resp. of diameter one).*

(b) *Buchsbaum curves are only a special class of curves having Buchsbaum components. Indeed every curve obtained from Liaison addition where one of the curves is Buchsbaum, has a Buchsbaum component up to a possible twist (see [25] for the notion of Liaison addition).*

If $M \simeq M' \oplus M_{[t]}$ and if we denote $(\sigma', \sigma_{[t]}) := \begin{pmatrix} \sigma' & 0 \\ 0 & \sigma_{[t]} \end{pmatrix}$, then M has the minimal resolution:

$$0 \to P_4 \oplus R(-t-4) \xrightarrow{(\sigma', \sigma_{[t]})} P_3 \oplus R(-t-3)^4 \to P_2 \oplus R(-t-2)^6 \to \ldots \to P_0 \oplus R(-t) \to M \to 0 \tag{6}$$

where $0 \to P_4 \xrightarrow{\sigma'} P_3 \to P_2 \xrightarrow{\tau_2} P_1 \xrightarrow{\tau_1} P_0 \to M' \to 0$ is a minimal resolution of M' and

$$0 \to R(-t-4) \xrightarrow{\sigma_{[t]}} R(-t-3)^4 \to R(-t-2)^6 \to R(-t-1)^4 \xrightarrow{\tau_{[t]}} R(-t) \to M_{[t]} \to 0 \qquad (7)$$

is the Koszul resolution of the R-module $R/\mathfrak{m}(-t)$. Note that $\sigma_{[t]} = (X_0, X_1, X_2, X_3)^{tr} = \tau_{[t]}^{tr}$. Combining with Rao's theorem concerning (3), we get the following minimal resolution of I:

$$0 \to P_4 \oplus R(-t-4) \xrightarrow{(\sigma', \sigma_{[t]}) \oplus 0} P_3 \oplus R(-t-3)^4 \oplus F_2 \to F_1 \to I \to 0\,. \qquad (8)$$

It was shown in [20] that certain Betti number were related to whether (C) sits in the intersection of different irreducible components of $\mathrm{H}(d,g)$, and hence to whether C is obstructed, or not. To define them, we write F_i as

$$F_2 \cong Q_2 \oplus R(-t-4)^{b_1} \oplus R(-t)^{b_2}\ , \qquad F_1 \cong Q_1 \oplus R(-t-4)^{a_1} \oplus R(-t)^{a_2} \qquad (9)$$

where Q_i, for $i = 1, 2$ are supposed to contain no free direct summand of degree t and $t + 4$.

Definition 2.4. *The 4-tuple associated to a curve C with Buchsbaum component $M_{[t]}$ is (a_1, a_2, b_1, b_2). Note that $(a_1, a_2) = (\beta_{1,t+4}, \beta_{1,t})$ are the 1^{st} graded Betti numbers of $I = I(C)$.*

Remark 2.5. *For a Buchsbaum curve of diameter one, we have $M(C) \simeq M_{[t]}^r$ and $t = c$. Then $(a_1, a_2, b_1, b_2) = (\beta_{1,c+4}, \beta_{1,c}, \beta_{2,c+4}, \beta_{2,c})$ and $r = \beta_{3,c+4}$ are the graded Betti numbers of $I(C)$ in degree $c+4$ and c. In this case, if we want to have r attached, we work with the 5-tuple (a_1, a_2, b_1, b_2, r). Note that this 5-tuple was denoted by (r, a_1, a_2, b_1, b_2) in [20].*

2.2 Linkage

We will need the notion of linkage and how we can find the minimal resolution of a linked curve (cf. [27] and see [25] for an introduction to linkage or liaison). Considering $\mathcal{I}_{C/Y} := \mathcal{I}_C/\mathcal{I}_Y$ as the sheaf ideal of C in Y, we define

Definition 2.6. *Two curves C and D in $\mathbb{P}^3$ are said to be (algebraically) CI-linked if there exists a complete intersection curve (a CI) Y such that*

$$\mathcal{I}_C/\mathcal{I}_Y \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_D, \mathcal{O}_Y) \quad \text{and} \quad \mathcal{I}_D/\mathcal{I}_Y \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_C, \mathcal{O}_Y)\,.$$

Suppose that Y is a complete intersection of two surfaces of degrees f and g (a CI of type (f,g)) containing C. Since the dualizing sheaf, ω_Y, of Y satisfies $\omega_Y \cong \mathcal{O}_Y(f+g-4)$, we get

$$\mathcal{I}_{C/Y} \cong \omega_D(4-f-g) \qquad \text{and} \qquad \mathcal{I}_{D/Y} \cong \omega_C(4-f-g) \qquad (10)$$

from the definition. By [30] the module $M(C)$ is a biliaison (linking twice several times) invariant, up to twist. Moreover, using (10) and the fact that $\omega_D \cong \mathcal{E}xt^2(\mathcal{O}_D, \mathcal{O}_{\mathbb{P}}(-4))$, hence that $I(C)/I(Y) \cong \mathrm{Ext}^1(I_D(f+g), R)$, one knows how to find a resolution of $I(D)$ in terms of the resolution of $I(C)$ and some part of the resolution of the dual of $M(C)$. Indeed using the E-resolution of $I(C)$, there exists vertical morphisms

$$\begin{array}{ccccccccc} 0 \longrightarrow & R(-f-g) & \to & R(-f) \oplus R(-g) & \to & I(Y) & \longrightarrow 0 \\ & \downarrow & \circ & \downarrow & \circ & \downarrow & \\ 0 \longrightarrow & E \oplus F_2 & \longrightarrow & F_1 & \longrightarrow & I(C) & \longrightarrow 0 \end{array}$$

The mapping cone construction yields a resolution of $I(C)/I(Y)$. Taking R-duals, $\mathrm{Hom}_R(-,R)$, of it, we get

$$0 \to F_1^\vee \to E^\vee \oplus F_2^\vee \oplus R(f) \oplus R(g) \to I(D)(f+g) \to 0\,. \tag{11}$$

Note that $0 \to L_0^\vee \xrightarrow{\tau^\vee} L_1^\vee \to L_2^\vee \to E^\vee \to 0$ is a free resolution of $E^\vee$ because the R-dual sequence of (2) is a resolution of $\mathrm{Ext}^4_R(M,R)$. Letting $G_1 := L_2^\vee(-f-g) \oplus F_2^\vee(-f-g) \oplus R(-g) \oplus R(-f)$, the mapping cone construction yields the following R-free resolution:

$$0 \to L_0^\vee(-f-g) \xrightarrow{\tau^\vee \oplus 0} L_1^\vee(-f-g) \oplus F_1^\vee(-f-g) \to G_1 \to I(D) \to 0 \tag{12}$$

If we need to find a free resolution of the homogeneous ideal of a curve X linked to D, using a CI Z of type (f',g') (so X and C are bilinked), we use (11) (and not (12)) and the mapping cone construction as in the big diagram above, to find a resolution of $I(D)/I(Z)(f'+g')$. Taking R-duals we get a free resolution of $R/I(X)$ (cf. [25]). We illustrate this by an example:

Example 2.7. *If C is a disjoint union of two lines, then it is easy to see that*

$$0 \to R(-4) \xrightarrow{\sigma} R(-3)^4 \to R(-2)^4 \to I(C) \to 0$$

is the minimal resolution, having $0 \to E \to R(-2)^4 \to I(C) \to 0$ *as its E-resolution (cf. (4)). We link twice, first via a CI of type $(4,2)$ to get a curve D with an exact sequence (cf. (11))*

$$0 \to R(2)^3 \to E^\vee \oplus R(4) \to I(D)(6) \to 0\,,$$

then we link via a CI Z of type $(4,6)$ to get a curve X in $\mathrm{H}(18,39)$ with E-resolution:

$$0 \to E(-4) \oplus R(-8) \to R(-6)^4 \oplus R(-4) \to I(X) \to 0\,,$$

which really is the R-dual sequence of the resolution of $I(D)/I(Z)(10)$ found by the mapping cone construction. Note that we use a common hypersurface of degree 4 in both linkages. The minimal resolution of $I(X)$ is

$$0 \to R(-8) \to R(-8) \oplus R(-7)^4 \to R(-6)^4 \oplus R(-4) \to I(X) \to 0\,.$$

One should compare the resolution with the Rao form (3). Note that $R(-8)$ is a ghost term.

2.3 Deformations

In [20] we proved that we can cancel repeated free consecutive summands (ghost terms) in (3) using deformations:

Theorem 2.8. *Let $C \subseteq \mathbb{P}^3$ be any curve with homogeneous ideal $I(C)$ and Rao module $M(C)$ and minimal free resolutions as in (2) and (3). If F_1 and F_2 have a common free summand; $F_2 = F_2' \oplus R(-i)$, $F_1 = F_1' \oplus R(-i)$, then there is a generization C' of C in $\mathrm{H}(d,g)$ with constant postulation and constant Rao module, and with minimal resolution*

$$0 \to L_4 \xrightarrow{\sigma \oplus 0} L_3 \oplus F_2' \to F_1' \to I(C') \to 0\,.$$

The proof is straightforward once we have proven a key lemma, and we refer to [20], Thm. 4.1 for the details. We remark that the proof of the case $M \cong k(-c)$ in [23] extends to get Theorem 2.8.

Corollary 2.9. *Let C be any curve and let $\{\beta_{j,i}\}$ (resp. $\{\beta_{j,i}(M)\}$) be the graded Betti numbers of $I(C)$ (resp. $M(C)$, whence $L_3 = \oplus_i R(-i)^{\beta_{4,i}(M)}$). If $\beta_{1,i} \cdot (\beta_{2,i} - \beta_{4,i}(M)) \neq 0$ for some i, then there is a generization C' of C in $\mathrm{H}(d,g)$ with constant postulation and Rao module whose graded Betti numbers $\{\beta'_{j,i}\}$ satisfy:*

$$\text{(Qi)} \qquad \begin{array}{lll} \beta'_{1,i} = \beta_{1,i} - 1\,, & \beta'_{1,j} = \beta_{1,j} \text{ for } j \neq i & \\ \beta'_{2,i} = \beta_{2,i} - 1\,, & \beta'_{2,j} = \beta_{2,j} \text{ for } j \neq i\,, & \text{and} \quad \beta'_{3,j} = \beta_{3,j} \text{ for every } j\,. \end{array}$$

In particular if C is a generic curve of $\mathrm{H}(d,g)$, then $\beta_{1,i} \cdot (\beta_{2,i} - \beta_{4,i}(M)) = 0$ for every i.

Proof. By the semi-continuity of the postulation, a generic curve belongs to some open irreducible subset U of $\mathrm{H}(d,g)$ with *constant postulation*. It follows that $\beta_{1,i}$ is semi-continuous in U, cf. the proof of Corollary 3.3 for a discussion. Hence also the final statement of the corollary is immediate. □

In [20], Prop. 4.2 (a) we also proved the following result.

Proposition 2.10. *Let $C \subseteq \mathbb{P}^3$ be a curve for which there is an isomorphism $M(C) \cong M' \oplus M_{[t]}$ as graded R-modules such that the minimal resolution (8) of $I(C)$ takes the form:*

$$0 \to P_4 \oplus R(-t-4) \xrightarrow{(\sigma',\sigma_{[t]})\oplus 0 \oplus 0} P_3 \oplus R(-t-3)^4 \oplus Q_2 \oplus R(-t-4) \xrightarrow{\beta} F_1 \to I(C) \to 0\,. \quad (13)$$

Then there is a generization $C' \subseteq \mathbb{P}^3$ of $C \subseteq \mathbb{P}^3$ in $\mathrm{H}(d,g)$ with constant postulation such that $I(C')$ has a free resolution of the following form:

$$0 \to P_4 \xrightarrow{\sigma' \oplus 0 \oplus 0} P_3 \oplus R(-t-3)^4 \oplus Q_2 \to F_1 \to I(C') \to 0\,, \quad (14)$$

and such that $M(C') \cong M'$ as graded R-modules. The resolution is minimal except possibly in degree $t+3$ in which case some of the summands of $R(-t-3)^4$ may be cancelled against free summands of F_1.

Proof (the main step). We replace the 0-coordinate in the matrix of $(\sigma', \sigma_{[t]}) \oplus 0 \oplus 0$ that corresponds to $R(-t-4) \to R(-t-4)$, by some indeterminate λ of degree zero (as in [23], page 189). To get a complex in (13), we change the four columns $\{h_{j,0}, h_{j,1}, h_{j,2}, h_{j,3}\}$ in the matrix of β, corresponding to the map $R(-t-3)^4 \to F_1$, as follows. Look at the column $\{y_j\}$ of the map $R(-t-4) \to F_1$ induced by β, and put $y_j = \sum_{i=0}^{3} a_{j,i} X_i$ (such $a_{j,i} \in k$ exist, and exactly here we use that the resolution is minimal because we need $y_j = 0$ when $y_j \in k$). If we replace the four columns above by $\{h_{j,0} - \lambda \cdot a_{j,0}, h_{j,1} - \lambda \cdot a_{j,1}, h_{j,2} - \lambda \cdot a_{j,2}, h_{j,3} - \lambda \cdot a_{j,3}\}$, leaving the rest of β unchanged, we get that the changed sequence (13) defines a complex, and we conclude by e.g. [20], Lem. 4.8. □

Remark 2.11. *In [20], Prop. 4.2 (a) the resolution* (14) *was claimed to be minimal. The proof of [20], Prop. 4.2 (a) only supports the minimality in degrees $\neq t+3$, leaving the possibility of some of the summands of $R(-t-3)^4$ to be cancelled against corresponding* *summands of F_1. This explains why we in Proposition 2.10 have to correct the conclusion of [20], Prop. 4.2 (a). Only the mentioned result of [20] needs a correction. Moreover if F_1 does not contain R-free summands of the form $R(-t-4)$, then it is not necessary to assume that* (13) *is minimal (cf. the proof above and note that one may show [20], Lem. 4.8 for possibly non-minimal resolutions). The final sentence of Proposition 2.10 requires, however, that* (13) *is minimal.*

Corollary 2.12. *Let $M(C) \cong M' \oplus M_{[t]}$ as graded R-modules and let (a_1, a_2, b_1, b_2) be the corresponding 4-tuple. If $b_1 \neq 0$, then there is a generization C' of C in $\mathrm{H}(d,g)$ with constant postulation and M' whose 4-tuple is*

$$(a_1, a_2, b_1 - 1, b_2)\,. \tag{P1}$$

Indeed for $i \in \{1,2\}$, $h^i(\mathcal{I}_{C'}(v)) = h^i(\mathcal{I}_C(v))$ for $v \neq t$ and $h^i(\mathcal{I}_{C'}(t)) = h^i(\mathcal{I}_C(t)) - 1$.

In [20], Cor. 3.3 and Thm. 3.4 we saw that the 4-tuple was important for discovering obstructedness:

Corollary 2.13. *Let C be a curve for which there is a graded R-module isomorphism $M(C) \cong M' \oplus M_{[t]}$, let (a_1, a_2, b_1, b_2) be the 4-tuple and suppose ${}_0\mathrm{Ext}^2_R(M, M) = 0$. Then C is obstructed if*

$$a_2 \cdot b_1 \neq 0 \quad \text{or} \quad a_1 \cdot b_1 \neq 0 \quad \text{or} \quad a_2 \cdot b_2 \neq 0\,.$$

Moreover if C is a diameter-1 curve (whence $t = c$), then C is obstructed if and only if

$$\beta_{1,c} \cdot \beta_{2,c+4} \neq 0 \quad \text{or} \quad \beta_{1,c+4} \cdot \beta_{2,c+4} \neq 0 \quad \text{or} \quad \beta_{1,c} \cdot \beta_{2,c} \neq 0\,.$$

Remark 2.14. *Let $M(C) \cong M' \oplus M_{[t]}$ as graded R-modules. For its 4-tuple (a_1, a_2, b_1, b_2), we have that*

$$a_2 \cdot b_1 = 0 \quad \text{and} \quad a_1 \cdot b_1 = 0 \quad \text{and} \quad a_2 \cdot b_2 = 0$$

is equivalent to requiring it to be of the form $(0, 0, b_1, b_2), (a_1, 0, 0, b_2)$ or *$(a_1, a_2, 0, 0)$. Hence by Corollary 2.13, if C is unobstructed, then there are "two consecutive 0's in the 4-tuple". This is equivalent to unobstructedness if* $\operatorname{diam} M = 1$. *Note that if* $\operatorname{diam} M = 0$ *(C is ACM), then C is always unobstructed by [10].*

Example 2.15. (a) *Start with the generic curve of $\mathrm{H}(8,5)_S$. It has 2-dimensional Rao module M and* $\operatorname{diam} M = 1$ *by [15]. We link with a CI of type $(4,6)$, then with a CI of type $(6,8)$, using the same degree-6 surface in both linkages. The minimal resolution of the bilinked curve is*

$$0 \to R(-10)^2 \to R(-10) \oplus R(-9)^8 \to R(-8)^7 \oplus R(-6) \to I \to 0\,,$$

whence $c = 6$ and $r = 2$. The corresponding 4-tuple is $(\beta_{1,c+4}, \beta_{1,c}, \beta_{2,c+4}, \beta_{2,c}) = (0,1,1,0)$, i.e. the curve C of $\mathrm{H}(32,109)_S$ is obstructed by Remark 2.14.

(b) *The curve C of $\mathrm{H}(33,117)_S$ of* $\operatorname{diam} M = 1$ *of Example 2.1 has 4-tuple $(1,0,1,0)$, i.e. C is obstructed by Remark 2.14. Since $c(C) = 5$, this curve has maximal rank.*

In the next section, we shall see that the curve of Example 2.15 (a) belongs to a unique irreducible component, while the curve of Example 2.15 (b) sits in the intersection of two irreducible components of $\mathrm{H}(d,g)_S$.

3 On the semi-continuity of graded Betti numbers

The goal of this section is to show a result on the semi-continuity of the graded Betti numbers of the homogeneous ideal $I(C)$ of a curve $C \subseteq \mathbb{P}^3$ considered as a point in $\mathrm{H}(d,g)$. We get the result as a consequence of the fact that the immersion $\mathrm{H}_\gamma \to \mathrm{H}(d,g)$ is an isomorphism in an open neighbourhood of (C) under a certain assumption. We also show a variation of a result of Bolondi, leading to the irreducibility of Betti strata with constant Rao modules. Letting ${}_0\mathrm{Ext}^i_R(-,-)$ be the degree-0 part of $\mathrm{Ext}^i_R(-,-)$, we have

Theorem 3.1. *Let C be any curve and let $I = H^0_*(\mathcal{I}_C)$ and $M = H^1_*(\mathcal{I}_C)$. Then*

$$ {}_0\mathrm{Hom}_R(I,M) = 0 \quad \Longrightarrow \quad \mathrm{H}_\gamma \cong \mathrm{H}(d,g) \ \text{ are isomorphic as schemes at } (C). $$

Proof. By mainly interpreting the exact sequence

$$ 0 \to {}_0\mathrm{Ext}^1_R(I,I) \to H^0(\mathcal{N}_C) \to {}_0\mathrm{Hom}_R(I,M) \to {}_0\mathrm{Ext}^2_R(I,I) \to H^1(\mathcal{N}_C) \to \tag{15} $$

in terms of deformation theories, as done in Prop. 2.10 of [20], we get the conclusion. □

Remark 3.2. *Theorem 3.1 holds in general for any closed subschemes C of $\mathbb{P}^n_k = \mathrm{Proj}(R)$, $k = \overline{k}$ under the sole assumption ${}_0\mathrm{Hom}_R(I,M) = 0$ by [21], Prop. 8 where the main ingredient in a proof (the isomorphism between the local graded deformation functor of $R \to R/I$ and the local Hilbert functor of $C \subset \mathbb{P}^n$) was proven already in 1979 ([18], Thm. 3.6 and Rem. 3.7). Note that if $H^1(\mathcal{I}_C(\deg F_i)) = 0$ for every minimal generator F_i of I, we get ${}_0\mathrm{Hom}_R(I,M) = 0$ and hence this result generalizes the comparison theorem of Piene-Schlessinger in [28].*

If C has maximal rank, then ${}_0\mathrm{Hom}_R(I,M) = 0$. In this case it is not so difficult to show $\mathrm{H}_\gamma \cong \mathrm{H}(d,g)$ at (C) by using the semi-continuity of $h^i(\mathcal{I}_C(v))$. The assumption ${}_0\mathrm{Hom}_R(I,M) = 0$ are, however, much weaker than requiring C to be of maximal rank, at least for generic unobstructed curves. In fact if ${}_0\mathrm{Ext}^2_R(M,M) = 0$ and C is unobstructed and generic in $\mathrm{H}(d,g)$, then it is shown in [20], Prop. 2.11 that ${}_0\mathrm{Hom}_R(I,M) = 0$.

As a surprising consequence of Theorem 3.1, we get the following result on the semi-continuity of the graded Betti numbers which we heavily use in the next section.

Corollary 3.3. *Inside H_γ and hence inside $\mathrm{H}_{\gamma,\rho}$ the graded Betti numbers are upper semi-continuous, i.e. if C' is a generization of C in H_γ, then*

$$ \beta_{i,j}(C') \le \beta_{i,j}(C) \quad \text{for any } i,j \ . $$

In particular if C is any curve satisfying ${}_0\mathrm{Hom}_R(I(C),M(C)) = 0$, then $\beta_{i,j}(C') \le \beta_{i,j}(C)$ for any i,j and every generization C' of C in $\mathrm{H}(d,g)$.

Proof. We apply Nakayama's lemma to the syzygy modules of (1) as explained in [21], Rem. 7 where we to a certain degree use [29], but our use of semi-continuity which takes place in a flat family with constant postulation is well known [6]. Then we combine with Theorem 3.1. □

Example 3.4. *It is known that the curve X of Example 2.7 sits in the intersection of two irreducible components of $\mathrm{H}(18,39)_S$ and that the generic curve $\tilde{X}$ of one of the components satisfies*

$$ 0 \to R(-8) \oplus R(-6)^2 \to R(-5)^4 \to I(\tilde{X}) \to 0 \, . $$

(Sernesi [31], cf. [8]). Looking at the minimal resolution of $I(X)$ in Example 2.7, we get $\beta_{1,5}(X) = \beta_{2,6}(X) = 0$ while $\beta_{1,5}(\tilde{X}) = 4$, $\beta_{2,6}(\tilde{X}) = 2$, i.e. we don't have semi-continuity for $\beta_{1,5}$ and $\beta_{2,6}$. In this example Corollary 3.3 does not apply because ${}_0\mathrm{Hom}_R(I(X), M(X)) \neq 0$!

Finally we consider the *Betti stratum* $\mathrm{H}(\underline{\beta}) := \{(D) \in \mathrm{H}_{\gamma,\rho} \mid \beta_{j,i}(D) = \beta_{j,i}$ for every $i, j\}$, see [17] and its references for papers on the Betti stratum. Thanks to Bolondi's proof of the irreducibility of $\mathrm{H}_{\gamma,\rho}$ in the Buchsbaum case ([2], Thm. 2.2, cf. [5], Prop. 4.3), we easily get

Proposition 3.5. *If $C \subseteq \mathbb{P}^3$ is a diameter-1 curve or C is ACM, then $\mathrm{H}(\underline{\beta}(C))$ is irreducible.*

Proof. Suppose $\mathrm{H}(\underline{\beta}(C))$ is not irreducible, containing at least two different irreducible components with generic curves D_1 and D_2. Then D_1 and D_2 have exactly the same R-free summands and the same morphism $\sigma \oplus 0$ in the minimal resolution (3), cf. (5), but the maps $L_3 \oplus F_2 \to F_1$ are different. In their E-resolutions the curves correspond to two maps φ_{D_1} and φ_{D_2} in $\mathrm{Hom}(E \oplus F_2, F_1)$, $E = \operatorname{coker} \sigma$. Consider the deformation induced by

$$\varphi_t := t\varphi_{D_1} + (1-t)\varphi_{D_2} \in \mathrm{Hom}(E \oplus F_2, F_1),\ \ t \in \mathbb{A}^1_k. \tag{16}$$

In some open subset $U \subset \mathbb{A}^1_k$ containing 0 and 1, φ_t defines a curve with the same graded Betti numbers as D_1 (and D_2) because in the minimal resolutions where 0-entries occur for the matrices of φ_{D_1} and φ_{D_2} due to repeated direct summands of F_2 and F_1, the same entry also vanishes for φ_t. Since U is irreducible we are done. □

Definition 3.6. *If $(D_1), (D_2) \in \mathrm{H}(\underline{\beta})$ are related as in (16), then the generic element $\tilde{D}$ of $\mathbb{A}^1_k$ is called a trivial generization of D_1 (or of D_2). Obviously, $(\tilde{D}) \in \mathrm{H}(\underline{\beta})$.*

Corollary 3.7 (of proof)**.** *Two arbitrary curves D_1 and D_2 of $\mathrm{H}(\underline{\beta})$ admit a trivial generization.*

Remark 3.8 (Bolondi, cf. [2], Cor. 2.3)**.** *Let $C \subseteq \mathbb{P}^3$ be any curve with Rao module M. By the same proof as above we get the irreducibility of:*

$$\{(D) \in \mathrm{H}_\gamma \mid M(D) \simeq M \text{ as graded } R\text{−modules, and } \beta_{j,i}(D) = \beta_{j,i}(C) \text{ for every } i, j\}\ .$$

4 Generizations not preserving postulation

In this section we study generizations of space curves, i.e. deformations to more general curves by "simplifying" their minimal resolutions. We start with the following generalization of [20], Prop. 4.2 (b) for which we give a new proof where we make ghost terms of a linked curve redundant under generization. Note that by *redundant* terms in a free resolution, we mean consecutive free summands that split off (disappear) when we make the free resolution minimal, while ghost terms don't split off! Recalling $M_{[t]} \cong R/\mathfrak{m}(-t)$, we have

Proposition 4.1. *Let C be a curve in $\mathbb{P}^3$ with Rao module $M(C)$, and suppose there is a graded R-module isomorphism $M(C) \cong M' \oplus M_{[t]}$. If $F_1 \cong Q_1 \oplus R(-t)$ in the minimal resolution (8) of the homogeneous ideal $I(C)$:*

$$0 \to P_4 \oplus R(-t-4) \xrightarrow{(\sigma', \sigma_{[t]}) \oplus 0} P_3 \oplus R(-t-3)^4 \oplus F_2 \to F_1 \to I(C) \to 0\,, \tag{17}$$

and if P_2 does not contain a direct summand $R(-t)$ (i.e. $\beta_{3,t}(M')=0$, cf. (6)), then there is a generization $C'\subseteq\mathbb{P}^3$ of $C\subseteq\mathbb{P}^3$ in $\mathrm{H}(d,g)$ with constant specialization and constant M' (up to a graded R-module isomorphism) such that $I(C')$ has the R-free resolution:

$$0\to P_4 \xrightarrow{\sigma'\oplus 0\oplus 0} P_3\oplus F_2\oplus R(-t-2)^6 \to Q_1\oplus R(-t-1)^4 \to I(C')\to 0\,. \qquad (18)$$

The resolution is minimal except possibly in degree $t+1$ and $t+2$ in which some of the summands of $R(-t-1)^4$ (resp. $R(-t-2)^6$) may be cancelled against corresponding free summands of F_2 (resp. Q_1). Moreover there exists a generization as above with a minimal resolution where all free common summands of $\{F_2,R(-t-1)^4\}$ and $\{R(-t-2)^6,Q_1\}$ are cancelled.

The idea of a proof is to link C to a curve D by a CI of type (f,g) where $f\neq t$ and $g\neq t$, then to take a generization of D by using Proposition 2.10 because the degree-t generator of $I(C)$ leads to a ghost term for D exactly where it appears in Proposition 2.10. Finally we link back via a CI of the same type (f,g) as before. Since there are some technical challenges involved, we give an example which, to a certain extent, illustrate the proof.

Example 4.2. *Take the minimal resolution of a smooth Buchsbaum curve C of degree 6 and genus 3:*

$$0\to R(-6)\to R(-5)^4\to R(-4)^3\oplus R(-2)\to I(C)\to 0\,.$$

It has the form as in the resolution of $I(C)$ in Proposition 4.1 with $M'=0$ (and hence all $P_i=0$) and $t=2$. We claim there is a generization "cancelling the leftmost term $R(-6)$ (together with $R(-5)^4$) against $R(-2)$" at the cost of an increase in Betti numbers in degrees 3 and 4. To see it we link C to D via a CI of type (f,g) containing C. We take $f=g=4$ to simplify, but the argument works for any CI avoiding the quadric. Let

$$E_t := \operatorname{coker}\sigma_{[t]} \qquad \textit{where} \qquad \sigma_{[t]} := R(-t-4)\longrightarrow R(-t-3)^4 \qquad (19)$$

be given by the exact sequence (7). That sequence also give the exactness of

$$0\to R(t)\xrightarrow{\tau_{[t]}^{\vee}} R(t+1)^4\to R(t+2)^6\to E_t^{\vee}\to 0\,. \qquad (20)$$

The E-resolution of $I(C)$ is $0\to E_2\to R(-4)^3\oplus R(-2)\to I(C)\to 0$, which through (11) yields

$$0\to R(-6)\oplus R(-4)\to E_2^{\vee}(-8)\to I(D)\to 0 \qquad (21)$$

by removing 2 redundant terms. Using (20) and the mapping cone construction as in (12), we get:

$$0\to R(-6)\to R(-5)^4\oplus R(-6)\to R(-4)^5\to I(D)\to 0\,.$$

This resolution has the form as in Proposition 2.10 with $M'=0$ and $t=2$. By that Proposition there is a generization D' cancelling the ghost term $R(-6)$, and we get an ACM curve. Finally we link "back" via a general CI of type $(4,4)$, and we get a curve C' with minimal resolution,

$$0\to R(-4)^3\to R(-3)^4\to I(C')\to 0$$

which, thanks to [19], Prop. 3.7, is a generization of the original curve C.

Since we certainly do not want to have Proposition 4.1 only for curves whose Rao module $M(C)$ is a one-dimensional k-vector-space, we consider curves with a Buchsbaum component in Proposition 4.1, making any diameter-1 curve the special case $M(C) \cong M^r_{[t]}$.

Proof (of Proposition 4.1). First we find the E-resolution of $I(C)$. Using (3) and (4) and the notations from (6)–(8), we get the E-resolution

$$0 \to E \oplus E_t \oplus F_2 \to F_1 \to I(C) \to 0 \,, \qquad E := \operatorname{coker}(P_4 \xrightarrow{\sigma'} P_3) \tag{22}$$

where E_t is given by (19). Now linking C to D via a CI of type (f, g), $f, g \gg 0$, the resolution (11) of $I(D)$ is given by

$$0 \to F_1^\vee \to E^\vee \oplus E_t^\vee \oplus F_2^\vee \oplus R(f) \oplus R(g) \to I(D)(f+g) \to 0 \,. \tag{23}$$

The exact sequences (20) and $0 \to P_0^\vee \xrightarrow{\tau_1^\vee} P_1^\vee \xrightarrow{\tau_2^\vee} P_2^\vee \to E^\vee \to 0$ yield an R-free resolution of the middle term of (23), which through the mapping cone construction as in (12) implies an R-free resolution:

$$0 \to P_0^\vee \oplus R(t) \xrightarrow{(\tau_1^\vee, \tau_{[t]}^\vee) \oplus 0} P_1^\vee \oplus R(t+1)^4 \oplus F_1^\vee \xrightarrow{\beta} F_1' \to I(D)(f+g) \to 0 \tag{24}$$

where $F_1' := P_2^\vee \oplus R(t{+}2)^6 \oplus F_2^\vee \oplus R(f) \oplus R(g)$, noticing that the morphism $F_1^\vee \to P_2^\vee \oplus R(t{+}2)^6$ corresponding to a submatrix of β may be non-minimal because we in the mapping cone construction need to lift the morphism $F_1^\vee \to E^\vee \oplus E_t^\vee$ to $F_1^\vee \to P_2^\vee \oplus R(t+2)^6$. Note also that the mapping cone construction allows us to take the morphisms $P_1^\vee \to F_1'$ deduced from β (resp. the leftmost in (24)) as $\tau_2^\vee \oplus 0 \oplus 0 \oplus 0 \oplus 0$ (resp. $(\tau_1^\vee, \tau_{[t]}^\vee) \oplus 0$), see [34]. The resolution (24) has the form as in Proposition 2.10 because $F_1^\vee = Q_1^\vee \oplus R(t)$. Hence there is a generization D' cancelling the ghost term $R(t-f-g)$ from the resolution of $I(D')$ because Remark 2.11 allows to use Proposition 2.10 for non-minimal resolutions in the case $P_2^\vee$ does not contain $R(t)$. We get (where now the induced $Q_1^\vee \to P_2^\vee \oplus R(t{+}2)^6$ may be non-minimal):

$$0 \to P_0^\vee \xrightarrow{\tau_1^\vee \oplus 0 \oplus 0} P_1^\vee \oplus R(t+1)^4 \oplus Q_1^\vee \xrightarrow{\alpha} F_1' \to I(D')(f+g) \to 0 \,. \tag{25}$$

In addition the morphism $R(t+1)^4 \to F_1'$ corresponding to a submatrix of α may be non-minimal by Remark 2.11. Letting $E_{\tau_1} := \ker(P_1 \xrightarrow{\tau_1} P_0)$, then an E-resolution is

$$0 \to E_{\tau_1}^\vee \oplus R(t+1)^4 \oplus Q_1^\vee \to F_1' \to I(D')(f+g) \to 0 \,. \tag{26}$$

Since D' is a generization of D with constant postulation, there is a generization $Y' \supset D'$ of Y of type (f, g), such that the linked curve C' is a generization of C, cf. [19], Prop. 3.7 (the assumptions of Prop. 3.7 are weak, and they are at least satisfied if $H^1(\mathcal{I}_C(v)) = 0$ for $v = f, g, f-4$ and $g-4$, which we may assume by $f, g \gg 0$). Using (11), we get the resolution

$$0 \to F_1'^\vee \to E_{\tau_1} \oplus R(-t-1)^4 \oplus Q_1 \oplus R(-f) \oplus R(-g) \to I(C') \to 0 \,. \tag{27}$$

Noting that $0 \to P_4 \xrightarrow{\sigma'} P_3 \to P_2 \to E_{\tau_1} \to 0$ is exact and letting the lifting of $F_1'^\vee \to E_{\tau_1}$ to $F_1'^\vee \to P_2$ be the natural one (the form of $P_1^\vee \to F_1'$ above allows us to take the dual of $F_1'^\vee \to P_2$ as $id \oplus 0 \oplus 0 \oplus 0 \oplus 0$, id the identity), the mapping cone construction yields (cf. (12))

$$0 \to P_4 \xrightarrow{\sigma' \oplus 0} P_3 \oplus F_1'^\vee \to P_2 \oplus R(-t-1)^4 \oplus Q_1 \oplus R(-f) \oplus R(-g) \to I(C') \to 0 \,. \tag{28}$$

If we now replace F_1' with its defining expression, we get exactly the resolution of the proposition provided we can show that the repeated free summand $P_2 \oplus R(-f) \oplus R(-g)$ is redundant. This is obvious for P_2. Note that in the resolution where P_2 is deleted, the possibly non-minimality of $Q_1^\vee \to P_2^\vee \oplus R(t+2)^6$ reduces to a possibly non-minimality of $Q_1^\vee \to R(t+2)^6$ and moreover, ghost terms between Q_1 and F_2 remain ghost terms (easily seen from the form of $F_1'^\vee \to P_2$ above). Finally even though it is rather easy to see that $R(-f) \oplus R(-g)$ is redundant because $f, g \gg 0$, we choose instead to use the idea in the proof of Theorem 2.8 which imply that this free summand becomes at least redundant after a generization (and no ghost terms between Q_1 and F_2 become redundant), whence we get the desired R-free resolution. We also get the minimality of the resolution in degree $\neq t+1, t+2$ by observing that in this proof, there are eventually only two places where the resolution may be non-minimal, namely for the above mentioned morphisms $Q_1^\vee \to R(t+2)^6$ and $R(t+1)^4 \to F_2^\vee$. Since we get the final statement from Theorem 2.8, we are done. □

Corollary 4.3. *Let $M(C) \cong M' \oplus M_{[t]}$ as graded R-modules, let (a_1, a_2, b_1, b_2) be the corresponding 4-tuple and suppose $\beta_{3,t}(M') = 0$. If $a_2 \neq 0$ (recall $a_2 := \beta_{1,t}$), then there is a generization C' of C in $\mathrm{H}(d, g)$ with constant specialization and M' whose 4-tuple is*

$$(a_1, a_2 - 1, b_1, b_2)\ . \tag{P2}$$

Moreover for $i \in \{0, 1\}$, $h^i(\mathcal{I}_{C'}(v)) = h^i(\mathcal{I}_C(v))$ for $v \neq t$ and $h^i(\mathcal{I}_{C'}(t)) = h^i(\mathcal{I}_C(t)) - 1$.

Remark 4.4. *Strictly speaking we need an extension of the notion of a 4-tuple for the generization C' of C because $M_{[t]}$ disappear for C' (e.g. C' may be ACM). We have, however, the number t attached to C and so it is clear which Betti numbers decrease.*

5 The graded Betti numbers of diameter-1 curves

Since our results become quite complete for a diameter-1 (Buchsbaum) curve $C \subseteq \mathbb{P}^3$, we now consider such curves closely. The main result of this section describes "all" generizations of a diameter-1 curve C in $\mathrm{H}(d, g)$, from the point of view of describing their minimal resolutions. In other word, we give essentially all possible choices of the graded Betti numbers of a generization of a diameter-1 curve. In particular we determine the form of the minimal resolutions of all generic curves of the irreducible components of $\mathrm{H}(d, g)$ that contain (C) and we find how many such components exist. Note that these results somehow complete works of Chang ([7], Ex. 1, [32], Thm. 4.1, [33]) which, to a large degree, determine the set of graded Betti numbers for which there exists (even smooth connected) diameter-1 curves.

For a diameter-1 curve $C \subseteq \mathbb{P}^3$, we have $M(C) \cong M_{[t]}^r$ with $t = c$, and a 5-tuple $(a_1, a_2, b_1, b_2, r) = (\beta_{1,c+4}, \beta_{1,c}, \beta_{2,c+4}, \beta_{2,c}, \beta_{3,c+4})$. The minimal resolution (the Rao form) of $I(C)$ is

$$0 \to R(-c-4)^r \xrightarrow{\sigma_{[c]} \oplus 0} R(-c-3)^{4r} \oplus F_2 \to F_1 \to I(C) \to 0\,. \tag{29}$$

Remark 5.1. *Suppose* $\operatorname{diam} M = 1$, *i.e.* $M(C) \cong M_{[c]}^r$ *and let* $\beta_{j,i} := \beta_{j,i}(C)$.

(a) *By Remark 2.11 there is a generization given by* (P1), *see Corollary 2.12, whose graded Betti numbers do not change except for $\beta_{3,c+4}$ and $\beta_{2,c+4}$, which both decrease by* 1, *and $\beta_{1,c+3}$ and $\beta_{2,c+3}$, which may decrease by at most* 4, *keeping, however, $\beta_{1,c+3} - \beta_{2,c+3}$ unchanged.*

Moreover if we combine with Theorem 2.8, we may suppose that $\beta_{2,c+3}$ *decreases by exactly* $\min\{\beta_{1,c+3},4\}$ *after possibly further generizations (i.e. using* $(\mathrm{Q(c+3)})$ *of Corollary 2.9).*

(b) *By Proposition 4.1 we can describe the possible changes of the graded Betti numbers of a generization given by* (P2) *in detail. Indeed the graded Betti numbers of a generization as in Corollary 4.3 do not change except* $\beta_{3,c+4}$ *and* $\beta_{1,c}$*, which both decrease by* 1*, and* $\beta_{1,v}$ *and* $\beta_{2,v}$ *for* $v \in \{c+3,c+2,c+1\}$ *for which* $\beta_{1,c+1}-\beta_{2,c+1}$ *increases by* 4*,* $\beta_{2,c+2}-\beta_{1,c+2}$ *increases by* 6 *and* $\beta_{2,c+3}$ *decreases by* 4*. Moreover combining with Theorem 2.8, we may suppose that* $\beta_{1,c+1}$ *increases by* $4-\min\{\beta_{2,c+1},4\}$ *and* $\beta_{2,c+1}$ *decreases by* $\min\{\beta_{2,c+1},4\}$ *while* $\beta_{2,c+2}$ *increases by* $6-\min\{\beta_{1,c+2},6\}$ *and* $\beta_{1,c+2}$ *decreases by* $\min\{\beta_{1,c+2},6\}$ *after possibly further generizations.*

(c) *Combining* (a) *and* (b) *by mainly using* (Pi) p_i *times for* $i=1,2$*, we get the existence of a generization* C' *of* C *in* $\mathrm{H}(d,g)$ *whose Betti numbers* $\{\beta'_{j,i}\}$ *satisfy:*

$$\begin{array}{ll}
\beta'_{1,c+4}=\beta_{1,c+4}\,,\quad \beta'_{2,c+4}=\beta_{2,c+4}-p_1\,, & \beta'_{3,c+4}=\beta_{3,c+4}-p_1-p_2\,,\\
\beta'_{1,c+3}=\beta_{1,c+3}-\min\{4p_1,\beta_{1,c+3}\}\,, & \beta'_{2,c+3}=\beta_{2,c+3}-4p_2-\min\{4p_1,\beta_{1,c+3}\},\\
\beta'_{1,c+2}=\beta_{1,c+2}-\min\{6p_2,\beta_{1,c+2}\}\,, & \beta'_{2,c+2}=\beta_{2,c+2}+6p_2-\min\{6p_2,\beta_{1,c+2}\},\\
\beta'_{1,c+1}=\beta_{1,c+1}+4p_2-\min\{4p_2,\beta_{2,c+1}\}\,, & \beta'_{2,c+1}=\beta_{2,c+1}-\min\{4p_2,\beta_{2,c+1}\},\\
\beta'_{1,c}=\beta_{1,c}-p_2\,,\quad \beta'_{2,c}=\beta_{2,c}, & \beta'_{j,i}=\beta_{j,i}\quad \text{for } j=1,2 \text{ and every } i\notin B\,,
\end{array}$$

where $B=\{c,c+1,c+2,c+3,c+4\}$*. In particular the 5-tuple of* C' *is*

$$(\beta_{1,c+4},\beta_{1,c}-p_2,\beta_{2,c+4}-p_1,\beta_{2,c},\beta_{3,c+4}-p_1-p_2)\,.$$

Now we come to the main theorems of the paper. But first we need a definition.

Definition 5.2. *Let* C *be a diameter-*1 *curve in* $\mathbb{P}^3$*,* $(C)\in \mathrm{H}(d,g)$*, and let* J *be a subset of the natural numbers* $\mathbb{N}$*. Then a generization* C' *of* C *in* $\mathrm{H}(d,g)$ *that is given by repeatedly using some of the generizations furnished by* (P1)*,* (P2) *and* (Qj) *for* $j\in J$ *in some order, is called a generization in* $\mathrm{H}(d,g)$ *generated by* $(\mathrm{PQ_J})$*. If only* (Qj)*,* $j\in J$ *is used, we call it a generization generated by* $(\mathrm{Q_J})$*. We omit the index* J *in* $(\mathrm{PQ_J})$ *and* $(\mathrm{Q_J})$ *in the case* $J=\mathbb{N}$*. Moreover we allow* $J=\emptyset$ *in the definitions, in which case* C' *is a trivial generization of* C*.*

Since the generizations given by (Pi) and (Qj) composed with a trivial generization (Definition 3.6), is again a generization given by (Pi) and (Qj) respectively, we get that e.g. a generization C' of C generated by $(\mathrm{PQ_J})$ is, up to a trivial generization, independent of the order in which we use (Pi) and (Qj). Indeed if we change the order we still get a generization C'' of C in $\mathrm{H}(d,g)$ in which C'' and C' belong to the same Betti stratum, and we conclude from Corollary 3.7.

Now we can prove that any generization of C in $\mathrm{H}(d,g)$ is generated by (PQ), up to the removal of some ghost terms between F_2 and F_1 in the degrees $c+1,c+2,c+3$ of (29).

Theorem 5.3. *Let* $C\subseteq\mathbb{P}^3$ *be a Buchsbaum curve of diameter one and let* C' *be any generization of* C *in* $\mathrm{H}(d,g)$*. If* $A=\{c+1,c+2,c+3\}$ *then there exists a generization* C'' *of* C' *generated by* $(\mathrm{Q_A})$ *such that* C'' *is a generization of* C *in* $\mathrm{H}(d,g)$ *generated by* (PQ)*.*

The proof relies on the following semi-continuity result:

Proposition 5.4. *Let* C *be a Buchsbaum curve in* $\mathbb{P}^3$ *of diameter one. If* $v\notin\{c+1,c+2,c+3\}$*, then the Betti numbers* $\beta_{1,v}$ *and* $\beta_{2,v}$ *are upper semi-continuous. In particular the 5-tuple* $(\beta_{1,c+4},\beta_{1,c},\beta_{2,c+4},\beta_{2,c},\beta_{3,c+4})$ *is upper semi-continuous, i.e. each of these 5 numbers do not increase under generization.*

Remark 5.5. *If C is ACM, then the Betti numbers $\beta_{1,v}$ and $\beta_{2,v}$ are upper semi-continuous for any integer v. This is well known, but the result also follows from Corollary 3.3.*

Proof. We will prove the result by using so-called Ω-resolutions of a Buchsbaum curve ([7], it is really the dual of an E-resolution involving $M_{[t]}$ for $t=0$). Recall that Ω is by definition given by the exact sequences

$$0 \to \widetilde{\Omega} \to \mathcal{O}_{\mathbb{P}}(-1)^4 \to \mathcal{O}_{\mathbb{P}} \to 0 \ \text{ and } \ 0 \to R(-4) \to R(-3)^4 \to R(-2)^6 \to \Omega \to 0 \qquad (30)$$

which we deduce from the Koszul resolution of the regular sequence $\{X_0, X_1, X_2, X_3\}$, whence

$$H^2_*(\widetilde{\Omega}) = 0 \ , H^1(\widetilde{\Omega}(0)) \simeq k \ , \ H^1(\widetilde{\Omega}(v)) = 0 \text{ for } v \neq 0 \ . \qquad (31)$$

Note also that $\widetilde{\Omega}(2)$ is 0-regular and generated by global sections. It follows that if we tensor the 1^{st} exact sequence of (30) by $\widetilde{\Omega}(v)$ and take cohomology, we get

$$H^1(\widetilde{\Omega}^{\otimes 2}(v)) = 0 \quad \text{for } v \neq 1 \text{ and } 2 \ . \qquad (32)$$

Since $r = h^1(\mathcal{I}_C(c))$, the Ω-resolution of C of Proposition 5.4, twisted by c, is given by

$$0 \to G_2 \to \Omega^r \oplus G_1 \to I(C)(c) \to 0 \qquad (33)$$

where G_i for $i = 1, 2$ is free and the induced map $G_2 \to G_1$ is minimal. Using that a minimal resolution of Ω^r is just a direct sum of the resolution given in (30), we get by the mapping cone construction the following free resolution of $I(C)(c)$ (cf. the proof of Proposition 4.1)

$$0 \to R(-4)^r \xrightarrow{\sigma \oplus 0} R(-3)^{4r} \oplus G_2 \to R(-2)^{6r} \oplus G_1 \to I(C)(c) \to 0 \qquad (34)$$

that is minimal except possibly in degree 2 and 3. Comparing we see that $G_j(-c)$, $j = 1, 2$, contains exactly the free summand $R(-i)^{\beta_{j,i}}$ of degree i for $i \notin \{2, 3\}$. We *claim* that

$$h^1(\mathcal{I}_C \otimes \widetilde{\Omega}(v)) = \beta_{1,v} \ , \quad \text{for} \ \ v \notin \{c+1, c+2, c+3\}. \qquad (35)$$

To prove it we sheafify (33) and tensor with $\widetilde{\Omega}(v-c)$. Since $H^2_*(\widetilde{\Omega}) = 0$ and $H^1(\widetilde{\Omega}^{\otimes 2}(v-c)) = 0$, it follows that the sequence

$$H^1(\widetilde{G_2}(-c) \otimes \widetilde{\Omega}(v)) \to H^1(\widetilde{G_1}(-c) \otimes \widetilde{\Omega}(v)) \to H^1(\mathcal{I}_C \otimes \widetilde{\Omega}(v)) \to 0 \qquad (36)$$

is exact. Due to (31) the sequence (36) yields $H^1(\widetilde{\Omega}^{\beta_{2,v}}) \to H^1(\widetilde{\Omega}^{\beta_{1,v}}) \to H^1(\mathcal{I}_C \otimes \widetilde{\Omega}(v)) \to 0$. By the minimality of $G_2 \to G_1$, we deduce the equality in (35).

Using the proven claim, we get that each of the $\beta_{1,v}$ is semi-continuous since $h^1(\mathcal{I}_C \otimes \widetilde{\Omega}(v))$ is semi-continuous. To see the corresponding statement for $\beta_{2,v}$, we use again linkage. Note that if we link C to D via a CI of type (f, g), we get $c(D) = f + g - 4 - c$, and

$$\beta_{2,v}(C) = \beta_{1,c+c(D)+4-v}(D) \ , \quad \text{for} \ \ v \notin \{c+1, c+2, c+3\}.$$

by (12). By (35) we get that $\beta_{1,c+c(D)+4-v}(D)$ is semi-continuous, because $v \notin \{c+1, c+2, c+3\}$ is equivalent to $c+c(D)+4-v \notin \{c(D)+1, c(D)+2, c(D)+3\}$. Finally since $r = h^1(\mathcal{I}_C(c))$ is clearly semi-continuous, we get the semi-continuity for every $\beta_{i,v}$ of Proposition 5.4, as well as for the 5-tuple of graded Betti numbers, and we are done. □

Proof (of Theorem 5.3). We denote the 5-tuple $(\beta_{1,c+4}, \beta_{1,c}, \beta_{2,c+4}, \beta_{2,c}, \beta_{3,c+4})$ of C shortly by (a_1, a_2, b_1, b_2, r) and there is a corresponding 5-tuple, $(a'_1, a'_2, b'_1, b'_2, r')$ for the generization C'. We write the 5-tuple of the operations (P1) as $(a_1, a_2, b_1 - 1, b_2, r - 1)$ and (P2) as $(a_1, a_2-1, b_1, b_2, r-1)$. Repeated use of (Pq) for $q = 1, 2$ implies the existence of a generization of C with 5-tuple $(a_1, a_2-i, b_1-j, b_2, r-i-j)$ provided $a_2-i \geq 0$, $b_1-j \geq 0$ and $r-i-j \geq 0$. Recalling $\gamma_C(v) = h^0(\mathcal{I}_C(v))$ and $\sigma_C(v) = h^1(\mathcal{O}_C(v))$, we *claim* that

$$h^0(\mathcal{I}_{C'}(c)) - a'_2 \geq h^0(\mathcal{I}_C(c)) - a_2 \quad \text{and} \quad h^1(\mathcal{O}_{C'}(c)) - b'_1 \geq h^1(\mathcal{O}_C(c)) - b_1 \,. \qquad (37)$$

We only prove the first inequality since the latter is the "dual" result which one may get from the first inequality by linkage. To prove it we remark that $\gamma_{C'}(v) = \gamma_C(v)$ for $v < c$ by the semi-continuity of $\gamma_C(v)$ and $\sigma_C(v)$ because $\chi(\mathcal{I}_{C'}(v)) = \chi(\mathcal{I}_C(v))$ implies

$$\gamma_C(v) + \sigma_C(v) = \gamma_{C'}(v) + \sigma_{C'}(v) \quad \text{for } v \neq c \,. \qquad (38)$$

Using the exactness of the minimal resolutions of $\mathcal{I}_{C'}$ and $\mathcal{I}_C$ in degree $v = c$, we get that $h^0(\mathcal{I}_{C'}(c)) - a'_2 + b'_2 = h^0(\mathcal{I}_C(c)) - a_2 + b_2$ since the exactness of these resolutions in degree $v < c$ implies $\beta_{1,v}(C) - \beta_{2,v}(C) = \beta_{1,v}(C') - \beta_{2,v}(C')$ for every $v < c$. Since we know $b'_2 \leq b_2$ by the semi-continuity of Proposition 5.4, we get $\gamma_{C'}(c) - a'_2 \geq \gamma_C(c) - a_2$, i.e. the claim.

Now let $\Delta\gamma(c) := \gamma_C(c) - \gamma_{C'}(c)$. Using Corollary 4.3 $\Delta\gamma(c)$ times, we get the existence of a generization C_{P2}, furnished by (P2), with constant specialization ($\sigma_{C_{P2}}(v) = \sigma_C(v)$) and with the same postulation as C'. Indeed this is possible because $a_2 \geq \Delta\gamma(c) \geq 0$ by (37) and $r \geq \Delta\gamma(c)$ by the semi-continuity of $h^1(\mathcal{O}_C(c))$ that implies

$$h^0(\mathcal{I}_{C'}(c)) - r' = \chi(\mathcal{I}_{C'}(c)) - h^1(\mathcal{O}_{C'}(c)) \geq \chi(\mathcal{I}_C(c)) - h^1(\mathcal{O}_C(c)) = h^0(\mathcal{I}_C(c)) - r.$$

Next we use Corollary 2.12 $\Delta\sigma(c) := \sigma_C(c) - \sigma_{C'}(c)$ times to get the existence of a generization C_P of C_{P2}, furnished by (P1), with constant postulation ($\gamma_{C_P}(c) = \gamma_{C_{P2}}(v)$) and with the same specialization as C'. This is possible because $b_1 \geq \Delta\sigma(c) \geq 0$ by (37) and $r - \Delta\gamma(c) \geq \Delta\sigma(c)$. Indeed the latter follows at once from the equality $\chi(\mathcal{I}_{C'}(c)) = \chi(\mathcal{I}_C(c))$ that implies $r - r' = \Delta\gamma(c) + \Delta\sigma(c)$.

So far we have two curves C_P and C' that by (38) and the construction of C_P have the same postulation and specialization functions, whence $h^1(\mathcal{I}_{C'}(c)) = h^1(\mathcal{I}_{C_P}(c))$. It follows that $\beta_{3,v}(C') = \beta_{3,v}(C_P)$ for $v = c + 4$ and hence for every v. Since $\gamma_{C'} = \gamma_{C_P}$, we get

$$\beta_{1,v}(C') - \beta_{2,v}(C') = \beta_{1,v}(C_P) - \beta_{2,v}(C_P) \qquad (39)$$

for every v by [26]. We *claim* that $\beta_{i,j}(C') \leq \beta_{i,j}(C_P)$ for $i = 1, 2$ and $j \notin A$. First take $j \notin \{c, c+4\} \cup A$. Then $\beta_{i,j}(C) = \beta_{i,j}(C_P)$ by the construction of C_P and $\beta_{i,j}(C') \leq \beta_{i,j}(C)$ by Proposition 5.4, and we get the claim. Next we consider $j = c$. Then $\beta_{1,c}(C_P) = \beta_{1,c}(C) - \Delta\gamma(c)$ and $\beta_{2,c}(C_P) = \beta_{2,c}(C)$ by the construction of C_P or by Remark 5.1 (c). Since $\beta_{1,c}(C') = a'_2 \leq \beta_{1,c}(C) - \Delta\gamma(c)$ by (37) and $\beta_{2,c}(C') \leq \beta_{2,c}(C)$ by Proposition 5.4, we get the claim for $j = c$. Finally for $j = c+4$ we use the other inequality of (37), Remark 5.1 (c) and Proposition 5.4 to see $\beta_{i,c+4}(C') \leq \beta_{i,c+4}(C_P)$ for $i = 1, 2$, and the claim is proved.

If the inequality of the claim is strict for some $j \notin A$ and some $i \in \{1, 2\}$, then both $\beta_{1,j}(C_P)$ and $\beta_{2,j}(C_P)$ are non-zero by their semi-continuity and (39), and $R(-j)$ is a common free summand of F_2 and F_1 in the minimal resolution of $I(C_P)$. Hence Theorem 2.8 applies to $R(-j)$ as well as to any other ghost term between F_2 and F_1 in the minimal resolution of $I(C_P)$ for which the inequality of the claim is strict. It follows that there is a generization D of C_P generated by $(Q_{\mathbb{N}-A})$ such that $\beta_{i,j}(C') = \beta_{i,j}(D)$ for $i = 1, 2$ and $j \notin A$.

Finally if $j \in A$, we still have (39). It follows that we either have $\beta_{i,j}(C') = \beta_{i,j}(C_P)$ for $i = 1, 2$, *or* $\beta_{i,j}(C') < \beta_{i,j}(C_P)$ for $i = 1, 2$, whose corresponding ghost term in the minimal resolution of $I(C_P)$ is removed by a generization of D, *or* $\beta_{i,j}(C') > \beta_{i,j}(C_P)$ for $i = 1, 2$, leading to a ghost term in the minimal resolution of $I(C')$ that is removed by a generization given by (Qj) of C'. Removing all such ghost terms corresponding to strict inequalities of the graded Betti numbers above, we get the existence of generizations C''_1 of C', and D' of D, generated by $(\mathrm{Q_A})$ such that $\beta_{i,j}(C''_1) = \beta_{i,j}(D')$ for every $i \in \{1, 2\}$ and $j \in \mathbb{N}$. Since (Qj) do not change $\beta_{3,c+4}$, then the generizations C''_1 and D' of C belong to the same Betti stratum. Using Corollary 3.7 and Definition 3.6, we get the theorem. □

Remark 5.6. *Let C' be a generic curve of the Betti stratum of a diameter-1 curve C. Then it follows from the last paragraph of the proof that if $\beta_{i,j}(C') \le \beta'_{i,j}$ for $i = 1, 2$ and $j \in A$ where $\beta'_{i,j}$ is given as in Remark 5.1 (c), we may take $C'' = C'$ in Theorem 5.3, i.e. C' is a generization of C in* $\mathrm{H}(d, g)$ *generated by* (PQ).

A main application of Theorem 5.3 is the first statement ("the hard part") of the following:

Corollary 5.7. *Let C' be a generic curve of an irreducible component of* $\mathrm{H}(d, g)$ *containing a diameter-1 curve C, and let $c = c(C)$ and $\beta'_{i,j} = \beta_{i,j}(C')$. Then C' is a generization of C in* $\mathrm{H}(d, g)$ *generated by* (PQ). *Moreover* $\beta'_{2,c+4} \cdot \beta'_{3,c+4} = \beta'_{1,c} \cdot \beta'_{3,c+4} = 0$,

$$\beta'_{1,c+3} \cdot (\beta'_{2,c+3} - 4\beta'_{3,c+4}) = 0 \ , \quad \beta'_{1,i} \cdot \beta'_{2,i} = 0 \ \text{ for any } i \neq c+3 \,,$$

and its 5-tuple is either $(\beta'_{1,c+4}, 0, 0, \beta'_{2,c}, \beta'_{3,c+4})$ *with* $\beta'_{3,c+4} \neq 0$ *or* $(\beta'_{1,c+4}, \beta'_{1,c}, \beta'_{2,c+4}, \beta'_{2,c}, 0)$.

Proof. The generic curve C' is a generization of C in $\mathrm{H}(d, g)$, whence is generated by (PQ) by Theorem 5.3 or Remark 5.6. Moreover the generic curve C' must satisfy $r' \cdot \beta'_{1,c} = 0$ and $r' \cdot \beta'_{2,c+4} = 0$ where $r' = \beta'_{3,c+4}$, because otherwise there exists by Corollaries 2.12 and 4.3 a generization C'' of C' such that $\beta_{3,c+4}(C'') = \beta'_{3,c+4} - 1$ contradicting the semi-continuity of $\beta'_{3,c+4}$ (Proposition 5.4). Similarly we get the conclusion for $\beta'_{1,i} \cdot \beta'_{2,i}$ by Corollary 2.9. □

So generic curves may have ghost terms in degree $c + 3$ (only). To find an example, recall that if we link C to a curve D using a general CI of type (f, g) such that $H^1(\mathcal{I}_C(v)) = 0$ for $v = f, g, f-4$ and $g-4$, then C is generic if and only if D is generic ([19], Prop. 3.8).

Example 5.8. *Using this we take two general skew lines as in Example 2.7 and we link twice, first via a CI of type* $(5, 2)$, *then via a CI of type* $(5, 4)$. *This gives us a curve X, generic in* $\mathrm{H}(12, 18)$, *with minimal resolution and a ghost term $R(-5)$ in degree $c + 3$:*

$$0 \to R(-6) \to R(-7) \oplus R(-5)^4 \to R(-5) \oplus R(-4)^4 \to I(X) \to 0\,.$$

Since our concern is about irreducible components of $\mathrm{H}(d, g)$ containing (C), it is only the graded Betti numbers in the 5-tuple and e.g. ghost terms there that play a role, as we now shall see.

Definition 5.9. *Let C be a diameter-1 curve and denote its 5-tuple by $\underline{\beta}(C)_5$. We say a 5-tuple $\underline{\beta}'_5$ specializes to $\underline{\beta}(C)_5$, and we write $\underline{\beta}'_5 \rightsquigarrow \underline{\beta}(C)_5$ if we obtain $\underline{\beta}'_5$ from $\underline{\beta}(C)_5$ by repeatedly using some of the operations* (Pi) *for $i = 1, 2$ and* (Qj) *for $j = c, c+4$ in some order. A 5-tuple $\underline{\beta}'_5$ is called minimal if it has the property that it does not allow further reductions by using the mentioned operations, i.e. $\underline{\beta}'_5$ is given as in Corollary 5.7.*

Theorem 5.10. *Let $C \subseteq \mathbb{P}^3$ be a Buchsbaum curve of diameter one. Then there is a one-to-one correspondence between the set of minimal 5-tuples that specialize to $\underline{\beta}(C)_5$ via the operations* $(\mathrm{PQ_J})$ *for $J = \{c, c+4\}$, and the set of irreducible (non-embedded) components of* $\mathrm{H}(d,g)$ *containing (C), i.e.*

$$\{\text{minimal } \underline{\beta}'_5 \,|\, \underline{\beta}'_5 \rightsquigarrow \underline{\beta}(C)_5\} \overset{1-1}{\longleftrightarrow} \{\text{irreducible components } V \subset \mathrm{H}(d,g) \,|\, V \ni (C)\}\,.$$

Here V maps to the 5-tuple of its generic curve and all components V are generically smooth.

Proof. Let $\underline{\beta}'_5$ be a minimal 5-tuple that specializes to $\underline{\beta}(C)_5$. We want to define the corresponding irreducible component $V(\underline{\beta}'_5)$ whose generic curve has $\underline{\beta}'_5$ as its 5-tuple. Since the operations (Pi) for $i = 1, 2$ and (Qj) for $j = c, c+4$ on 5-tuples correspond to the existence of generizations, there is a generization $\tilde{C}$ of C in $\mathrm{H}(d,g)$ such that $\underline{\beta}(\tilde{C})_5 = \underline{\beta}'_5$. Then $\tilde{C}$ is unobstructed by Corollary 2.13. Let $V(\tilde{C})$ be the unique irreducible component of $\mathrm{H}(d,g)$ containing $(\tilde{C})$ and let C' be the generic curve of $V(\tilde{C})$. Then $\underline{\beta}(C')_5$ is minimal by Corollary 5.7 and we have $\underline{\beta}(C')_5 \leq \underline{\beta}(\tilde{C})_5$ by the semi-continuity of 5-tuples, whence equality by the minimality of $\underline{\beta}(\tilde{C})_5$. Put $V(\underline{\beta}'_5) := V(\tilde{C})$.

To see that the application $\underline{\beta}'_5 \rightsquigarrow V(\underline{\beta}'_5)$ is injective, we suppose $V(\underline{\beta}'_{1_5}) = V(\underline{\beta}'_{2_5})$. Then we can assume that their generic curves C'_1 and C'_2 coincide and we conclude the injectivity by

$$\underline{\beta}'_{1_5} = \underline{\beta}(C'_1)_5 = \underline{\beta}(C'_2)_5 = \underline{\beta}'_{2_5}\,.$$

The surjectivity of the application follows from Corollary 5.7 which implies that a generic curve C' is obtained by taking generizations in $\mathrm{H}(d,g)$ (starting with C) using (Pi) and (Qj) in some order. The corresponding operations (Pi) and (Qj) on the 5-tuples imply that $\underline{\beta}(C')_5$, which is minimal, specializes to $\underline{\beta}(C)_5$ using only (Pi) and (Qj) for $j = c, c+4$. ☐

Remark 5.11. *Theorem 5.10 significantly generalizes Prop. 4.6 of [20]. It also allows us to interpret geometrically the obstructedness result of [20], Thm. 1.3, see Corollary 2.13. Indeed given $(\beta_{1,c+4}, \beta_{1,c}, \beta_{2,c+4}, \beta_{2,c}, \beta_{3,c+4})$ with $\beta_{3,c+4} \neq 0$, then the obstructedness condition*

$$\beta_{1,c} \cdot \beta_{2,c+4} \neq 0 \quad \textbf{or} \quad \beta_{1,c+4} \cdot \beta_{2,c+4} \neq 0 \quad \textbf{or} \quad \beta_{1,c} \cdot \beta_{2,c} \neq 0$$

is equivalent to the following statement: there exist generizations given by (P1) *and* (P2)*,* **or** (P1) *and* (Q(c + 4))*,* **or** (P2) *and* (Qc) *respectively, where each of the three "and"-expressions correspond to two different ("directions for the") generizations, removing at least one ghost term in a minimal resolution of $I(C)$. Moreover each of the three expressions may correspond to two different irreducible components of* $\mathrm{H}(d,g)$*, but not necessarily, as we may see from:*

Example 5.12. (a) *The obstructed curve C of Example 2.15 (a) has 5-tuple* $(0,1,1,0,2)$*. It admits two generizations to two curves with 5-tuples* $(0,1,0,0,1)$ and $(0,0,1,0,1)$*. These*

5-tuples are not minimal. Indeed both curves admit generizations to curves with the same 5-tuple $(0,0,0,0,0)$. *By Theorem 5.10 C belongs to a unique irreducible components of* $\mathrm{H}(32,109)_S$*!*

(b) *The 5-tuple of the obstructed curve C in Example 2.15 (b) is* $(1,0,1,0,1)$, *i.e. the curve C admits two generizations to two curves with minimal 5-tuples* $(1,0,0,0,0)$ and $(0,0,0,0,1)$, *where one of the generizations is ACM and the other is Buchsbaum of diameter one. By Theorem 5.10 C belongs to exactly two irreducible components of* $\mathrm{H}(33,117)_S$, *cf. [3]. Note that both generizations correspond to the removal of ghost terms, cf. [32], Ex. 4.2. Hence we can not separate the two components by the usual semi-continuity of* $h^i(\mathcal{I}_C(v))$*!*

(c) *The 5-tuple of the curve X of Example 2.7 is* $(0,1,1,0,1)$, *having two generizations with 5-tuples* $(0,1,0,0,0)$ and $(0,0,1,0,0)$. *These 5-tuples are minimal and the corresponding curves are ACM. By Theorem 5.10 there are precisely two irreducible components* V_1, V_2 *of* $\mathrm{H}(18,39)_S$ *such that* $(X) \in V_1 \cup V_2$, *cf. Example 3.4. Note that we in this case may separate the two components by the semi-continuity of* $h^i(\mathcal{I}_Z(v))$ *because* $(h^0(\mathcal{I}_Z(4)), h^1(\mathcal{I}_Z(4)), h^1(\mathcal{O}_Z(4)))$ *is equal to* $(1,1,1)$ *for* $Z = X$, *while it is* $(1,0,0)$ *and* $(0,0,1)$ *for the two generizations.*

Our next proposition and remark, which was communicated to us by Johannes Kleppe together with a full proof and Example 5.14, determine explicitly how many irreducible components of $\mathrm{H}(d,g)$ that we have in the correspondence given in Theorem 5.10. Below $(a_1,a_2,b_1,b_2,r) = (\beta_{1,c+4}, \beta_{1,c}, \beta_{2,c+4}, \beta_{2,c}, \beta_{3,c+4})$ and we let $\binom{m}{n} = 0$ if $m < n$.

Proposition 5.13. *Let* (a_1,a_2,b_1,b_2,r) *be the 5-tuple of a Buchsbaum curve of diameter one, and let* $\hat{a}_2 = \max\{0, a_2 - b_2\}$ *and* $\hat{b}_1 = \max\{0, b_1 - a_1\}$. *The number of minimal 5-tuples that specialize to* (a_1,a_2,b_1,b_2,r) *is*

$$N_B + N_{CM}.$$

Here,

$$N_B = \binom{r - \hat{b}_1 - \hat{a}_2 + 1}{2} - \binom{r - b_1 - \hat{a}_2}{2} - \binom{r - \hat{b}_1 - a_2}{2} + \binom{r - b_1 - a_2 - 1}{2} \tag{40}$$

is the number of minimal 5-tuples that correspond to generic diameter-1 curves, and

$$N_{CM} = \begin{cases} \min\{b_1, a_2, r\} + 1, & \textit{if } r \le \max\{b_1, a_2\} \\ b_1 + a_2 - r + 1, & \textit{if } \max\{b_1, a_2\} \le r \le b_1 + a_2 \\ 0, & \textit{if } r > b_1 + a_2 \end{cases} \tag{41}$$

is the number of minimal 5-tuples that correspond to generic ACM curves.

Proof. The four basic reductions of a 5-tuple (a_1,a_2,b_1,b_2,r) are given by the vectors $\underline{\alpha}_1 = (1,0,1,0,0)$, $\underline{\alpha}_2 = (0,0,1,0,1)$, $\underline{\alpha}_3 = (0,1,0,0,1)$ and $\underline{\alpha}_4 = (0,1,0,1,0)$. Any reduction of the 5-tuple can be written as

$$(a_1,a_2,b_1,b_2,r) - \sum_{i=1}^{4} k_i \underline{\alpha}_i = (a_1 - k_1, a_2 - k_3 - k_4, b_1 - k_1 - k_2, b_2 - k_4, r - k_2 - k_3)$$

with each $k_i \ge 0$. These numbers cannot be negative, giving us the following five inequalities:

$$k_1 \le a_1 \qquad k_1 + k_2 \le b_1 \qquad k_2 + k_3 \le r \qquad k_3 + k_4 \le a_2 \qquad k_4 \le b_2\,.$$

Clearly, we have arrived at a minimal 5-tuple if and only if no k_i can be increased, implying that among each pair of neighbouring inequalities in the above, one must be an equality. To count the number of minimal 5-tuples, we will divide into two cases, depending on whether r is reduced to zero or not.

Case 1. If r is non-zero in the minimal 5-tuple, then the reductions of b_1 and a_2 must both be zero. Hence the minimal 5-tuple is of the form $(*, 0, 0, *, +)$, giving the following:

$$k_1 \le a_1 \qquad k_1 + k_2 = b_1 \qquad k_2 + k_3 < r \qquad k_3 + k_4 = a_2 \qquad k_4 \le b_2 \,.$$

This requires that $k_1 \le \min\{a_1, b_1\}$, and therefore $k_2 \ge b_1 - \min\{a_1, b_1\} = \max\{0, b_1 - a_1\} = \hat{b}_1$. Conversely, $\hat{b}_1 \le k_2 \le b_1$ implies $0 \le k_1 \le \min\{a_1, b_1\}$. Hence the minimal 5-tuples in *Case* 1 are in one-to-one correspondence with all pairs (k_2, k_3) within the square $\hat{b}_1 \le k_2 \le b_1$ and $\hat{a}_2 \le k_3 \le a_2$ that satisfy $k_2 + k_3 < r$. The number of such pairs can be expressed using triangular numbers as

$$N_B = \binom{r - \hat{b}_1 - \hat{a}_2 + 1}{2} - \binom{r - b_1 - \hat{a}_2}{2} - \binom{r - \hat{b}_1 - a_2}{2} + \binom{r - b_1 - a_2 - 1}{2}.$$

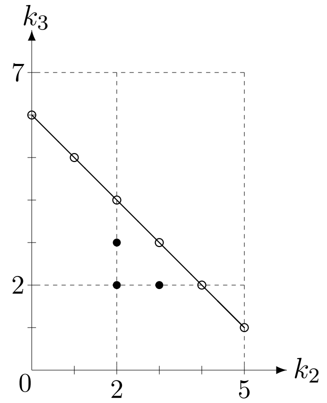
Note that $N_B \le (\min\{a_1, b_1\} + 1)(\min\{a_2, b_2\} + 1)$, with equality if and only if $r > b_1 + a_2$.

Case 2. If r is reduced to zero, we get a 5-tuple of the form $(*, *, *, *, 0)$. This form is a specialization of a unique minimal 5-tuple, found by reducing the pairs (a_1, b_1) and (a_2, b_2), i.e. increasing k_1 and k_4, until one of the integers in each pair reach zero. Therefore, we only have to count in how many ways r can be reduced to zero, and the constraints for these minimal 5-tuples are as follows:

$$k_1 = \min\{a_1, b_1 - k_2\} \qquad k_2 \le b_1 \qquad k_2 + k_3 = r \qquad k_3 \le a_2 \qquad k_4 = \min\{b_2, a_2 - k_3\} \,.$$

In other words, the minimal 5-tuples in *Case* 2 correspond to those pairs (k_2, k_3) on the line $k_2 + k_3 = r$ that satisfy $k_2 \le b_1$ and $k_3 \le a_2$, implying formula (41). □

Example 5.14. *Let us count the number of minimal 5-tuples that specialize to* $(3, 7, 5, 5, 6)$ *(disregarding if this is a 5-tuple of a diameter-1 curve that exists). In this case* $\hat{b}_1 = b_1 - a_1 = 2$ *and* $\hat{a}_2 = a_2 - b_2 = 2$. *The minimal 5-tuples are easily visualized in the* k_2k_3*-plane:*



The minimal 5-tuples counted by N_B *are determined by the points inside the rectangle* $2 \le k_2 \le 5$ *and* $2 \le k_3 \le 7$ *below the line* $k_2 + k_3 = 6$. *These are marked as filled dots. We see that* $N_B = 3$.

The minimal 5-tuples counted by N_{CM} *are given by the points on the line* $k_2 + k_3 = 6$ *inside the larger rectangle* $0 \le k_2 \le 5$ *and* $0 \le k_3 \le 7$. *These are marked as open dots. We easily count that* $N_{CM} = 6$.

In total we have $N_B + N_{CM} = 9$ *different minimal 5-tuples.*

Remark 5.15. *In some cases there is only one minimal 5-tuple that specializes to a given 5-tuple* (a_1, a_2, b_1, b_2, r). *This happens if and only if the original 5-tuple has the following* *property: if (x,y,z) is any of the triplets (a_1,b_1,r), (b_1,r,a_2) or (r,a_2,b_2), then either $xyz=0$ or $y \ge x+z$. Indeed, each of these triplets have two possible basic reductions, given by the vectors $(1,1,0)$ and $(0,1,1)$. If there is a unique minimal 5-tuple, then also these triplets must have a unique reduced version, and this is equivalent to the stated property. Note that this implies that the sequence (a_1,b_1,r,a_2,b_2) cannot have 4 neighbouring positive integers.*

In addition to four obvious cases (namely $r=0$, $a_1=a_2=0$, $a_2=b_1=0$ and $b_1=b_2=0$), this gives us the following three cases: $a_2=0$ and $b_1 \ge r+a_1$, $b_1=0$ and $a_2 \ge r+b_2$, or $a_1=b_2=0$ and $r \ge b_1+a_2$. Example 5.12 (a) *belongs to the last case. An example of each of the other two main cases is given below.*

Example 5.16. (a) *There is an obstructed curve C in $\mathrm{H}(42,177)_S$ with minimal resolution*

$$0 \to R(-10) \to R(-11)^2 \oplus R(-10)^2 \oplus R(-9)^4 \to R(-10) \oplus R(-9)^2 \oplus R(-8)^5 \to I \to 0$$

([32], Ex. 4.2). Since the 5-tuple of C is $(1,0,2,0,1)$, it admits two generizations to curves with 5-tuples; $(1,0,1,0,0)$ and $(0,0,1,0,1)$. *These 5-tuples are not minimal. Indeed both curves admit generizations to curves with the same 5-tuple $(0,0,0,0,0)$. By Theorem 5.10, C belongs to a unique component of $\mathrm{H}(42,177)$. Moreover since all generizations above correspond to the removal of ghost terms, they preserve postulation. It follows that (C), which is a singular point of $\mathrm{H}_\gamma = \mathrm{H}(42,177)_\gamma$, belongs to a unique component of H_γ (or one may use that ${}_0\mathrm{Hom}_R(I(C),M(C))=0$ implies $\mathrm{H}_\gamma \cong \mathrm{H}(42,177)$ at (C), cf. Theorem 3.1, to see it).*

(b) *If we link the curve of* (a) *via a CI of type $(8,8)$ we get an obstructed curve D with 5-tuple $(0,2,0,1,1)$. The curve D admits two generizations to two curves with 5-tuples* $(0,1,0,0,1)$ and $(0,1,0,1,0)$, *and two further generizations to curves with the same 5-tuple $(0,0,0,0,0)$. By Theorem 5.10, D belongs to a unique irreducible components of $\mathrm{H}(22,57)$.*

6 The Hilbert scheme of curves of diameter at most one

In this section we study the open subscheme, $\mathrm{H}(d,g;c)$, of $\mathrm{H}(d,g)$ whose k-points are given by

$$\{(C) \in \mathrm{H}(d,g) \mid H^1(\mathcal{I}_C(v)) = 0 \text{ for every } v \ne c\},$$

c an integer. Our main concern is to determine its singular locus. To do so, Theorem 5.3, which describe "all" generizations of curves in $\mathrm{H}(d,g;c)$, together with the characterization of obstructed curves in Corollary 2.13, will be the main ingredient. Note that Theorem 5.10, whose proof strongly needed Theorem 5.3, directly transfers to a theorem for $\mathrm{H}(d,g;c)$ with similar statements because all components of Theorem 5.10 properly intersect $\mathrm{H}(d,g;c(C))$.

In the following let C, $(C) \in \mathrm{H}(d,g;c)$, be a generic curve of a Betti stratum $\mathrm{H}(\underline{\beta})$, and let $\underline{\beta}_5$ be the 5-tuple of C. We write $\mathrm{H}(\underline{\beta})$ as $\mathrm{H}(\underline{\beta}_5)$ if the graded Betti numbers that do not belong to $\underline{\beta}_5$ are chosen as small as possible (cf. Corollary 2.9), i.e. so that they satisfy

$$\beta_{1,c+3} \cdot (\beta_{2,c+3} - 4\beta_{3,c+4}) = 0,\ \beta_{1,i} \cdot \beta_{2,i} = 0 \ \text{ for } i \notin \{c, c+3, c+4\}. \tag{42}$$

Note that if $\overline{\mathrm{H}}(\underline{\beta})$, $\overline{\mathrm{H}}(-)$ the closure of $\mathrm{H}(-)$ in $\mathrm{H}(d,g)$, is an irreducible component of $\mathrm{H}(d,g)$, then $\underline{\beta}$ satisfies (42) by Corollary 5.7. Suppose $\mathrm{H}(\underline{\beta}) = \mathrm{H}(\underline{\beta}_5)$, i.e. that C satisfies (42), and let $V(\underline{\beta}_5)_B := \overline{\mathrm{H}}(\underline{\beta}_5) \cap \mathrm{H}(d,g;c)$. If $(C') \in V(\underline{\beta}_5)_B$ then C is a generization of C' in $\mathrm{H}(d,g)$ generated by (PQ) by Theorem 5.3, see also [16], Ch. II, Ex. 3.17. Now we denote by

$$\underline{p}_1 := (0,0,1,0,1),\ \underline{p}_2 := (0,1,0,0,1),\ \underline{q}_c := (0,1,0,1,0),\ \underline{q}_{c+4} := (1,0,1,0,0)$$

the vectors that correspond to the operations (P1), (P2) and (Qj) for $j = c, c+4$ respectively. We define

$$V(\underline{\beta}_5 + \underline{q}_J)_B := \begin{cases} V(\underline{\beta}_5 + \underline{q}_c)_B \cup V(\underline{\beta}_5 + \underline{q}_{c+4})_B\,, & \text{if diam } M(C) = 1 \\ \emptyset & \text{if } C \text{ is ACM}\,. \end{cases} \tag{43}$$

Below $+$, resp. $*$ in an entry of a 5-tuple means a positive, resp. non-negative integer. Moreover if $V(\underline{\beta}_5)_B$ is an irreducible component of $\mathrm{H}(d, g; c)$, then we denote by *Sing* $V(\underline{\beta}_5)_B$ the part of the singular locus of $\mathrm{H}(d, g; c)$ that are contained in $V(\underline{\beta}_5)_B$. We get

Theorem 6.1. *With the above notations, suppose* $V(\underline{\beta}_5)_B$ *is an irreducible component of* $\mathrm{H}(d, g; c)$*. Then* $\underline{\beta}_5$ *is given as in* (i)-(v)*, and*

(i) *if* $\underline{\beta}_5$ *is equal to* $(+, 0, 0, +, *)$ or $(0, +, +, 0, 0)$, *then*

$$Sing\ V(\underline{\beta}_5)_B = V(\underline{\beta}_5 + \underline{p}_1)_B \cup V(\underline{\beta}_5 + \underline{p}_2)_B \cup V(\underline{\beta}_5 + \underline{q}_J)_B,$$

(ii) *if* $\underline{\beta}_5 = (0, 0, 0, +, *)$ or $(0, 0, +, *, 0)$, *then* $Sing\ V(\underline{\beta}_5)_B = V(\underline{\beta}_5 + \underline{p}_2)_B \cup V(\underline{\beta}_5 + \underline{q}_J)_B$,

(iii) *if* $\underline{\beta}_5 = (+, 0, 0, 0, *)$ or $(*, +, 0, 0, 0)$, *then* $Sing\ V(\underline{\beta}_5)_B = V(\underline{\beta}_5 + \underline{p}_1)_B \cup V(\underline{\beta}_5 + \underline{q}_J)_B$,

(iv) *if* $\underline{\beta}_5 = (0, 0, 0, 0, +)$, *then* $Sing\ V(\underline{\beta}_5)_B = V(\underline{\beta}_5 + \underline{p}_1 + \underline{p}_2)_B \cup V(\underline{\beta}_5 + \underline{q}_J)_B$,

(v) *if* $\underline{\beta}_5 = (0, 0, 0, 0, 0)$, *then* $Sing\ V(\underline{\beta}_5)_B = V(\underline{\beta}_5 + \underline{p}_1 + \underline{p}_2)_B \cup$

$$V(\underline{\beta}_5 + \underline{p}_1 + \underline{q}_c)_B \cup V(\underline{\beta}_5 + \underline{p}_1 + \underline{q}_{c+4})_B \cup V(\underline{\beta}_5 + \underline{p}_2 + \underline{q}_c)_B \cup V(\underline{\beta}_5 + \underline{p}_2 + \underline{q}_{c+4})_B\,.$$

Proof. It is easily checked that the minimal 5-tuples are of the form (i)-(v). Now let C be a generic curve of $V(\underline{\beta}_5)_B$.

(i) A generic curve $\tilde{C}$ of a non-empty $V(\underline{\beta}_5 + \underline{p}_1)_B$ has 5-tuple without consecutive 0's in its first 4 entries, whence $\tilde{C}$ is obstructed by Remark 2.14. The same argument, using Remark 2.14, holds for $V(\underline{\beta}_5 + \underline{p}_2)_B$. If C is not ACM, the argument also holds for the generic curve $\tilde{C}$ of $V(\underline{\beta}_5 + \underline{q}_i)_B$, $i = c$ and $c + 4$. Since C is a generization of $\tilde{C}$, it follows that $(\tilde{C})$ belongs to the closure of $\mathrm{H}(\underline{\beta}_5)$ in $\mathrm{H}(d, g)$, i.e. that $(\tilde{C}) \in V(\underline{\beta}_5)_B$ and we get

$$Sing\ V(\underline{\beta}_5)_B \supseteq V(\underline{\beta}_5 + \underline{p}_1)_B \cup V(\underline{\beta}_5 + \underline{p}_2)_B \cup V(\underline{\beta}_5 + \underline{q}_J)_B.$$

(also in the case some of the Betti strata to the right are empty).

Conversely suppose a curve C' of $V(\underline{\beta}_5)_B$ is not in the union of the V-sets above. If the generic curve C of $V(\underline{\beta}_5)_B$ is not ACM, then C is by Theorem 5.3 a generization of C' in $\mathrm{H}(d, g)$ generated by (PQ) without using (P1), (P2), nor (Qi) for $i = c$ and $c+4$. This follows from the fact that we can change the order in which we use (Pj) and (Qi). Indeed if e.g. (P2) is used, then $\underline{\beta}_5 + \underline{p}_2$ must specialize to the 5-tuple of C' which implies that (C') belongs to the closure of $\mathrm{H}(\underline{\beta}_5 + \underline{p}_2)$ and we get a contradiction. Thus C is a trivial generization of C', which implies that C' has exactly the same 5-tuple as C. It follows that C' is unobstructed.

If C is ACM, then C is a generization of C' in $\mathrm{H}(d, g)$ generated by (PQ) without using (P1) nor (P2), i.e. only generizations given by (Qi) are used. Then C' is ACM and hence unobstructed. This proves (i).

The other cases (ii)-(v) are proven similarly, and we get the theorem. □

Finally we remark that we can find the dimension of the singularities given in Theorem 6.1 in some cases. Indeed let $\mathrm{H}(\underline{\beta}_5) \subseteq \mathrm{H}_{\gamma,\rho}$ be a Betti stratum with generic curve C, $(C) \in \mathrm{H}(d,g;c)$, and let C' be a generic curve of $\mathrm{H}_{\gamma,\rho}$ satisfying (42) by Theorem 2.8. Then C' is a generization of C in $\mathrm{H}(d,g)$ without using (P1) and (P2). Indeed (P1) and (P2) change ρ. It follows that C' is a generization of C generated by ($\mathrm{Q_J}$), $J = \{c, c+4\}$. Suppose $\underline{\beta}_5 = \underline{\beta}_5(C)$ is of the form

$$\underline{\beta}_5 = (0, \beta_{1,c}, \beta_{2,c+4}, 0, \beta_{3,c+4}). \tag{44}$$

Then neither (Qc) nor (Q(c + 4)) are used, i.e. C' is a trivial generization of C and $(C') \in \mathrm{H}(\underline{\beta}_5)$. It follows that $V(\underline{\beta}_5)_B = \overline{\mathrm{H}}_{\gamma,\rho} \cap \mathrm{H}(d,g;c)$. Since $\dim \mathrm{H}_{\gamma,\rho}$ is known ([20], Rem. 2.3, first proved in [23], Thm. 3.8, p. 171), we can compute the dimension of the singularities $V(\underline{\beta}_5 + a\underline{p}_1 + b\underline{p}_2)_B$ for $a, b \in \{0,1\}$, of Theorem 6.1 because their generic curves satisfy (44):

Example 6.2. (a) *The singularity "$(0,1,1,0,2)$" of Example 5.12 (a) belongs to a unique irreducible component of* $\mathrm{H}(32,109)_S$ *with 5-tuple* $(0,0,0,0,0)$. *The codimension of the singularity, i.e.* $\dim V(0,0,0,0,0)_B - \dim V(0,1,1,0,2)_B$, *is* 3.

(b) *By [20], Ex. 3.12, there exists a singularity "$(0,1,1,0,r)$" belonging to a unique irreducible component of* $\mathrm{H}(d,g)_S$ *for any* $r \geq 2$, *and the codimension of the singularity is* $2r-1$.

(c) *The singularity of Example 5.12 (c) sits in the intersection of two irreducible components of* $\mathrm{H}(18,39)_S$, *and the codimension of the singularity in each of its components is* 1 *(cf. [31] and [8]).*

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