ON THE NORMAL SHEAF OF DETERMINANTAL VARIETIES

JAN O. KLEPPE, ROSA M. MIRÓ-ROIG*

Abstract. Let X be a standard determinantal scheme $X \subset \mathbb{P}^n$ of codimension c, i.e. a scheme defined by the maximal minors of a $t \times (t+c-1)$ homogeneous polynomial matrix $\mathcal{A}$. In this paper, we study the main features of its normal sheaf $\mathcal{N}_X$. We prove that under some mild restrictions: (1) there exists a line bundle $\mathcal{L}$ on $X \setminus Sing(X)$ such that $\mathcal{N}_X \otimes \mathcal{L}$ is arithmetically Cohen-Macaulay and, even more, it is Ulrich whenever the entries of $\mathcal{A}$ are linear forms (2) $\mathcal{N}_X$ is simple (hence, indecomposable) and, finally, (3) $\mathcal{N}_X$ is μ-(semi)stable provided the entries of $\mathcal{A}$ are linear forms.

Contents

1. Introduction

The goal of this paper is to study the main properties of the normal sheaf of a determinantal scheme. Determinantal schemes have been a central topic in both Commutative Algebra and Algebraic Geometry. They are schemes defined by the vanishing of the $r \times r$ minors of a homogeneous polynomial matrix; and as a classical examples of determinantal schemes we have rational normal scrolls, Segre varieties and Veronese varieties. On the other hand, the normal sheaf $\mathcal{N}_X$ of a projective scheme $X \subset \mathbb{P}^n$ has been intensively studied since it reflects many properties of the embedding. For instance, if $X \subset \mathbb{P}^n$ is a smooth projective variety of dimension $d \geq n/2$, $n \geq 5$ and $\mathcal{N}_X$ splits into a sum of line bundles then X is a complete intersection (cf. [2]; Corollary 3.6). This paper is entirely devoted to study the main features of the normal sheaf $\mathcal{N}_X$ of a (linear) standard determinantal scheme $X \subset \mathbb{P}^n$, i.e. schemes defined by the maximal minors of

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a homogeneous polynomial matrix. In particular, we will study the Macaulayness, simplicity and μ-stability of $\mathcal{N}_X$.

Since the seminal work by Horrocks characterizing ACM sheaves on $\mathbb{P}^n$ as those that completely split into a sum of line bundles (cf. [19]), a big amount of papers devoted to study ACM sheaves on projective schemes has appeared. Recall that a sheaf $\mathcal{E}$ on a projective scheme $X \subset \mathbb{P}^n$ is arithmetically Cohen-Macaulay (ACM, for short) if it is locally Cohen-Macaulay and $\mathrm{H}^i_*(\mathcal{E}) = \sum_{t\in\mathbb{Z}} \mathrm{H}^i(X, \mathcal{E}(t)) = 0$ for $0 < i < \dim X$. It turns out that a natural way to measure the complexity of a projective scheme is to ask for the families of ACM sheaves that it supports. Mimicking an analogous trichotomy in Representation Theory, in [11] it was proposed a classification of ACM projective varieties as *finite, tame or wild* according to the complexity of their associated category of ACM vector bundles and it was proved that this trichotomy is exhaustive for the case of ACM curves: rational curves are finite, elliptic curves are tame and curves of higher genus are wild. Unfortunately very little is known for varieties of higher dimension.

Among ACM vector bundles $\mathcal{E}$ on a given variety X, it is interesting to spot a very important subclass for which its associated module $\oplus_t \mathrm{H}^0(X, \mathcal{E}(t))$ has the maximal number of generators, which turns out to be $\deg(X)\,\mathrm{rk}(\mathcal{E})$. The algebraic counterpart has also arisen a lot of interest. In fact, Ulrich proved that for a local ring R the minimal number $m(M)$ of generators of a Maximal Cohen-Macaulay (MCM) module M is bounded by $m(M) \le e(R)\,\mathrm{rk}(M)$ where $e(R)$ denotes de multiplicity of R (cf. [36]). MCM modules attaining this bound are called Ulrich modules. These algebraic considerations prompted to define, for a projective scheme $X \subset \mathbb{P}^n$, a sheaf $\mathcal{E}$ on X to be Ulrich if it is ACM and the associated graded module $\mathrm{H}^0_*(\mathcal{E})$ is Ulrich. When $\mathcal{E}$ is initialized (i.e. $\mathrm{H}^0(\mathcal{E}) \neq 0$ but $\mathrm{H}^0(\mathcal{E}(-1)) = 0$), the last condition is equivalent to $\dim \mathrm{H}^0(X, \mathcal{E}) = \deg(X)\,\mathrm{rk}(\mathcal{E})$. The search of Ulrich sheaves on a particular variety is a challenging problem. In fact, few examples of varieties supporting Ulrich sheaves are known, although in [14] it has been conjectured that any variety has an Ulrich sheaf. See [7]; Proposition 2.8, for the existence of Ulrich bundles of rank one on linear standard determinantal schemes. Moreover, the recent interest in the existence of Ulrich sheaves relies among other things on the fact that a d-dimensional variety $X \subset \mathbb{P}^n$ supports an Ulrich sheaf (bundle) if and only if the cone of cohomology tables of coherent sheaves (resp. vector bundles) on X coincides with the cone of cohomology tables of coherent sheaves (resp. vector bundles) on $\mathbb{P}^d$ ([15]; Theorem 4.2).

In this paper, we will address the following longstanding problems:

Problem 1.1. *Let $X \subset \mathbb{P}^n$ be a (linear) standard determinantal scheme of codimension c and let $\mathcal{N}_X$ be its normal sheaf:*

1. *Is there any invertible sheaf $\mathcal{L}$ on $X \setminus Sing(X)$ such that $\mathcal{N}_X \otimes \mathcal{L}$ is ACM (resp. Ulrich)? If so, how does the minimal free $\mathcal{O}_{\mathbb{P}^n}$-resolution of $\mathcal{N}_X \otimes \mathcal{L}$ looks like?*
2. *Is there any invertible sheaf $\mathcal{L}'$ on $X \setminus Sing(X)$ such that $\wedge^q \mathcal{N}_X \otimes \mathcal{L}'$ is ACM (resp. Ulrich)?*
3. *Is $\mathcal{N}_X$ simple? or, at least indecomposable?*
4. *Is $\mathcal{N}_X$ μ-(semi)stable?*

In our approach, we often use the well known fact that a locally free sheaf on the smooth locus of a reduced normal Cohen-Macaulay variety X extends uniquely to a reflexive sheaf on X, or more precisely that the morphism (2.2) of this paper is an isomorphism under appropriate depth conditions. We will start our work analyzing whether the normal sheaf $\mathcal{N}_X$ (and its exterior powers) to a standard determinantal scheme is ACM. It is well known that the normal sheaf $\mathcal{N}_X$ to a standard determinantal scheme $X \subset \mathbb{P}^n$ of codimension c is ACM if $1 \le c \le 2$ but it is no longer true for $c \ge 3$. Our first goal will be to prove that under some weak restriction there exists an invertible sheaf $\mathcal{L}$ on $X \setminus Sing(X)$ (i.e. a coherent $\mathcal{O}_X$-module that is invertible on $X \setminus Sing(X)$) such that $\mathcal{N}_X \otimes \mathcal{L}$ is ACM. Our next aim will be to prove that for linear standard determinantal schemes $X \subset \mathbb{P}^n$ satisfying some mild hypothesis, $\mathcal{N}_X \otimes \mathcal{L}$ is not only ACM but also Ulrich.

As Hartshorne and Casanellas pointed out in [9] there are few examples of indecomposable ACM bundles of arbitrarily high rank. So, once we know the Macaulayness of a suitable twist $\mathcal{N}_X \otimes \mathcal{L}$ of the normal sheaf $\mathcal{N}_X$ to a standard determinantal scheme $X \subset \mathbb{P}^n$ we are led to ask if it is indecomposable. Concerning Problem 1.1 (3), we are able to prove that under some weak conditions $\mathcal{N}_X$ is indecomposable and, even more, it is simple. As it is explained in section 4, this last property works in a much more general set up (cf. Theorem 4.7). Another challenging problem is the existence of μ-(semi)stable ACM bundles of high rank on projective schemes since μ-(semi)stable ACM bundles of higher ranks are essentially unknown due to the lack of criteria to check it when the rank is high. In the last part of the paper, we analyze the μ-(semi)stability of the normal sheaf $\mathcal{N}_X$ to a standard determinantal scheme X. We prove that it is μ-semistable provided the homogeneous matrix associated to X has linear entries and μ-stable if, in addition, we assume that X has codimension 2.

Next we outline the structure of the paper. Section 2 provides the background and basic results on (linear) standard determinantal schemes and the associated complexes needed in the sequel as well as the definition and main properties on ACM sheaves and Ulrich sheaves on a projective scheme. We refer to [6] and [13] for more details on standard determinantal schemes and to [10], [14], [31] and [33] for more details on Ulrich sheaves. In section 3, we address Problems 1.1 (1) and (2) and we prove our main results. Given a standard determinantal scheme $X \subset \mathbb{P}^n$ of codimension c associated to a homogeneous $t \times (t+c-1)$ polynomial matrix $\mathcal{A}$, we denote by M the cokernel of the graded morphism defined by $\mathcal{A}$ and we prove that, under some mild assumption, $\mathcal{N}_X \otimes \widetilde{S_{c-2}M}$ is ACM (cf. Theorem 3.7). As a by-product we obtain an $\mathcal{O}_{\mathbb{P}^n}$-resolution of $\mathcal{N}_X \otimes \mathcal{L}$ which turns out to be pure and linear in the case of linear standard determinantal schemes (cf. Theorem 3.7 and Corollary 3.8). To prove it we use a generalization of the mapping cone process together with a careful analysis of possible cancelations of repeated summands in the mapping cone construction. Parts of the results of this section are inspired by ideas developed in [27] leading to isomorphisms $\operatorname{Ext}^1(\widetilde{M}, \widetilde{S_iM}) \cong \operatorname{Hom}(\mathcal{I}_X/\mathcal{I}_X^2, \widetilde{S_{i-1}M})$ and to good estimates of the depth of $\operatorname{Ext}^1(M, S_iM)$. We also generalize these results to higher Ext-groups and to exterior powers of twisted normal sheaves (cf. Theorem 3.15). At the end of this section, we guess a result, analogous to Theorem 3.15 for the exterior power of the normal sheaf (cf. Conjectures 3.9 and 3.23) based on our computations with Macaulay2. Section 4 deals with Problem 1.1 (3) and we determine conditions under which the normal sheaf $\mathcal{N}_X$ of a standard determinantal scheme $X \subset \mathbb{P}^n$ is simple and, hence, indecomposable (cf. Theorem 4.3). We also determine the cohomology and hence the depth of the conormal sheaf of X (cf. Proposition 4.1 and Corollaries 3.21 and 4.2). Finally, in section 5, we face Problem 1.1 (4) and we prove that the normal sheaf $\mathcal{N}_X$ of a standard determinantal scheme $X \subset \mathbb{P}^n$ of codimension c associated to a $t \times (t+c-1)$ matrix with linear entries is always μ-semistable (cf. Theorem 5.3) and even more it is μ-stable when $c = 2$ and $n \geq 4$ (cf. Theorem 5.7).

Remark. This version on the arXiv makes a correction to Proposition 3.9 of previous versions on the arXiv, as well as to the published version in Crelle's journal, see Remark 3.18 for details.

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Notation. Throughout this paper K will be an algebraically closed field of characteristic zero, $R = K[x_0, x_1, \cdots, x_n]$, $\mathfrak{m} = (x_0, \ldots, x_n)$ and $\mathbb{P}^n = \operatorname{Proj}(R)$. Given a closed subscheme $X \subset \mathbb{P}^n$, we denote by $\mathcal{I}_X$ its ideal sheaf, $\mathcal{N}_X$ its normal sheaf and $I(X) = \mathrm{H}^0_*(\mathbb{P}^n, \mathcal{I}_X)$ its saturated homogeneous ideal unless $X = \emptyset$, in which case we let $I(X) = \mathfrak{m}$. If X is equidimensional and Cohen-Macaulay of codimension c, we set $\omega_X = \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n})(-n-1)$ to be its canonical sheaf. Given a coherent sheaf $\mathcal{E}$ on X we denote the twisted sheaf $\mathcal{E} \otimes \mathcal{O}_X(l)$ by $\mathcal{E}(l)$. As usual, $\mathrm{H}^i(X, \mathcal{E})$ stands for the cohomology groups, $\mathrm{h}^i(X, \mathcal{E})$ for their dimension and $\mathrm{H}^i_*(X, \mathcal{E}) = \oplus_{l \in \mathbb{Z}} \mathrm{H}^i(X, \mathcal{E}(l))$.

For any graded quotient A of R of codimension c, we let $I_A = \ker(R \twoheadrightarrow A)$ and we let $N_A = \operatorname{Hom}_R(I_A, A)$ be the normal module. If A is Cohen-Macaulay of codimension c, we let $K_A = \operatorname{Ext}^c_R(A, R)(-n-1)$ be its canonical module. When we write $X = \operatorname{Proj}(A)$, we let $A = R_X := R/I(X)$ and $K_X = K_A$. If M is a finitely generated graded A-module, let $\operatorname{depth}_J M$ denote the length of a maximal M-sequence in a homogeneous ideal J and let $\operatorname{depth} M = \operatorname{depth}_{\mathfrak{m}} M$.

2. Preliminaries

For convenience of the reader we include in this section the background and basic results on (linear) standard determinantal varieties as well as on arithmetically Cohen-Macaulay sheaves and Ulrich sheaves needed in the sequel.

2.1. **Determinantal varieties.** Let us start recalling the definition of arithmetically Cohen-Macaulay subschemes of $\mathbb{P}^n$.

Definition 2.1. A subscheme $X \subset \mathbb{P}^n$ is said to be arithmetically Cohen-Macaulay (briefly, ACM) if its homogeneous coordinate ring $R/I(X)$ is a Cohen-Macaulay ring, i.e. $\operatorname{depth} R/I(X) = \dim R/I(X)$.

Thanks to the graded version of the Auslander-Buchsbaum formula (for any finitely generated R-module M):

$$\operatorname{pd} M = n + 1 - \operatorname{depth} M,$$

we deduce that a subscheme $X \subset \mathbb{P}^n$ is ACM if and only if $\operatorname{pd} R/I(X) = \operatorname{codim} X$. Hence, if $X \subset \mathbb{P}^n$ is a codimension c ACM subscheme, a graded minimal free R-resolution of $I(X)$ is of the form:

$$0 \to F_c \xrightarrow{\varphi_c} F_{c-1} \xrightarrow{\varphi_{c-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} I(X) \to 0$$

where $F_i = \oplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}}$, $i = 1, \cdots, c$ (in this setting, minimal means that $\operatorname{im}(\varphi_i) \subset \mathfrak{m}F_{i-1}$).

We now collect the results on standard determinantal schemes and the associated complexes needed in the sequel and we refer to [6] and [13] for more details.

Definition 2.2. If $\mathcal{A}$ is a homogeneous matrix, we denote by $I(\mathcal{A})$ the ideal of R generated by the maximal minors of A and by $I_j(\mathcal{A})$ the ideal generated by the $j \times j$ minors of $\mathcal{A}$. A codimension c subscheme $X \subset \mathbb{P}^n$ is called a *standard determinantal* scheme if $I(X) = I(\mathcal{A})$ for some $t \times (t+c-1)$ homogeneous matrix $\mathcal{A}$. In addition, we will say that X is a *linear standard determinantal scheme* if all entries of $\mathcal{A}$ are linear forms.

As examples of standard determinantal schemes we have any ACM variety $X \subset \mathbb{P}^n$ of codimension 2 and as examples of linear determinantal schemes we have, for instance, rational normal curves, Segre varieties and rational normal scrolls. Now we are going to describe some complexes associated to a codimension c standard determinantal scheme X. To this end, we denote by $\varphi : F \longrightarrow G$ the morphism of free graded R-modules of rank $t + c - 1$ and t, defined by the homogeneous matrix $\mathcal{A}$ of X. We denote by $\mathcal{C}_i(\varphi)$ the (generalized) Koszul complex:

$$\mathcal{C}_i(\varphi): \ 0 \to \wedge^i F \otimes S_0(G) \to \wedge^{i-1} F \otimes S_1(G) \to \ldots \to \wedge^0 F \otimes S_i(G) \to 0$$

where $\wedge^i F$ stands for the i-th exterior power of F and $S_i(G)$ denotes the i-th symmetric power of G. Let $\mathcal{C}_i(\varphi)^*$ be the R-dual of $\mathcal{C}_i(\varphi)$. The dual map φ^* induces graded morphisms

$$\mu_i : \wedge^{t+i} F \otimes \wedge^t G^* \to \wedge^i F.$$

They can be used to splice the complexes $\mathcal{C}_{c-i-1}(\varphi)^* \otimes \wedge^{t+c-1} F \otimes \wedge^t G^*$ and $\mathcal{C}_i(\varphi)$ to a complex $\mathcal{D}_i(\varphi)$:

$$0 \to \wedge^{t+c-1} F \otimes S_{c-i-1}(G)^* \otimes \wedge^t G^* \to \wedge^{t+c-2} F \otimes S_{c-i-2}(G)^* \otimes \wedge^t G^* \to \ldots \to \tag{2.1}$$
$$\wedge^{t+i} F \otimes S_0(G)^* \otimes \wedge^t G^* \xrightarrow{\epsilon_i} \wedge^i F \otimes S_0(G) \to \wedge^{i-1} F \otimes S_1(G) \to \ldots \to \wedge^0 F \otimes S_i(G) \to 0.$$

The complex $\mathcal{D}_0(\varphi)$ is called the Eagon-Northcott complex and the complex $\mathcal{D}_1(\varphi)$ is called the Buchsbaum-Rim complex. Let us rename the complex $\mathcal{C}_c(\varphi)$ as $\mathcal{D}_c(\varphi)$. Denote by $I_m(\varphi)$ the ideal generated by the $m \times m$ minors of the matrix $\mathcal{A}$ representing φ. Then, we have the following well known result:

Proposition 2.3. *Let $X \subset \mathbb{P}^n$ be a standard determinantal subscheme of codimension c associated to a graded minimal (i.e. $\operatorname{im}(\varphi) \subset \mathfrak{m}G$) morphism $\varphi : F \to G$ of free R-modules of rank $t+c-1$ and t, respectively. Set $M = Coker(\varphi)$. Then, it holds:*

(i) $\mathcal{D}_i(\varphi)$ is acyclic for $-1 \le i \le c$.

(ii) $\mathcal{D}_0(\varphi)$ is a minimal free graded R-resolution of $R/I(X)$ and $\mathcal{D}_i(\varphi)$ is a minimal free graded R-resolution of length c of $S_i(M)$, $1 \le i \le c$.

(ii) Up to twist, $K_X \cong S_{c-1}(M)$. So, up to twist, $\mathcal{D}_{c-1}(\varphi)$ is a minimal free graded R-module resolution of K_X.

(iv) $\mathcal{D}_i(\varphi)$ is a minimal free graded R-resolution of $S_i(M)$ for $c+1 \le i$ whenever $depth_{I_m(\varphi)} R \ge t+c-m$ for every m such that $t \ge m \ge max(1, t+c-i)$.

Proof. See, for instance [6]; Theorem 2.20 and [13]; Theorem A2.10 and Corollaries A2.12 and A2.13. □

It immediately follows from Proposition 2.3 (ii) that standard determinantal schemes are ACM. Let us also recall the following useful comparison of cohomology groups. If $Z \subset X$ is a closed subset such that $U = X \setminus Z$ is a local complete intersection, L and N are finitely generated $R/I(X)$-modules, $\widetilde{N}$ is locally free on U and $\operatorname{depth}_{I(Z)} L \ge r+1$, then the natural map

$$\operatorname{Ext}^i_{R/I(X)}(N, L) \longrightarrow \mathrm{H}^i_*(U, \mathcal{H}om_{\mathcal{O}_X}(\widetilde{N}, \widetilde{L})) \tag{2.2}$$

is an isomorphism, (resp. an injection) for $i < r$ (resp. $i = r$) cf. [21], exposé VI. Recall that we interpret $I(Z)$ as $\mathfrak{m}$ if $Z = \emptyset$.

We end this subsection describing the Picard group of a smooth standard determinantal scheme X. Assume that $X \subset \mathbb{P}^n$ is given by the maximal minors of a $t \times (t+c-1)$ homogeneous matrix $\mathcal{A}$ representing a homomorphism φ of free graded R-modules

$$\varphi : F = \bigoplus_{i=1}^{t+c-1} R(-a_i) \longrightarrow G = \bigoplus_{j=1}^{t} R(-b_j).$$

Without loss of generality, we may assume $a_1 \le a_2 \le \ldots \le a_{t+c-1}$ and $b_1 \le b_2 \le \ldots \le b_t$. Denote by H the general hyperplane section of X and by $Z \subset X$ the codimension 1 subscheme of X defined by the maximal minors of the $(t-1) \times (t+c-1)$ matrix obtained deleting the last row of $\mathcal{A}$. The following theorem computes the Picard group of X. Indeed, we have:

Theorem 2.4. *Let $X \subset \mathbb{P}^n$ be a smooth standard determinantal scheme of codimension $c \ge 2$. Set $\ell := \sum_{j=1}^{t+c-1} a_j - \sum_{i=1}^{t} b_i$. Assume $t > 1$. If $n - c > 2$ and $a_1 - b_t > 0$; or $n - c = 2$, $a_1 - b_t > 0$ and $\ell \ge n+1$, then $Pic(X) \cong \mathbb{Z}^2 \cong \langle H, Z \rangle$.*

Proof. See [12]; Corollary 2.4 for smooth standard determinantal varieties $X \subset \mathbb{P}^n$ of dimension $d \ge 3$ and [18]; Proposition 5.2 for the case $d = 2$ (see also [22]; Theorem II.4.2, for smooth surfaces $X \subset \mathbb{P}^4$). □

2.2. **ACM and Ulrich sheaves.** We set up here some preliminary notions mainly concerning the definitions and basic results on ACM and Ulrich sheaves needed later.

Definition 2.5. Let $X \subset \mathbb{P}^n$ be a projective scheme and let $\mathcal{E}$ be a coherent sheaf on X. $\mathcal{E}$ is said to be *Arithmetically Cohen Macaulay* (shortly, ACM) if it is locally Cohen-Macaulay (i.e., $\operatorname{depth} \mathcal{E}_x = \dim \mathcal{O}_{X,x}$ for every point $x \in X$) and has no intermediate cohomology, i.e.

$$\mathrm{H}^i(X, \mathcal{E}(t)) = 0 \qquad \text{for all } t \text{ and } i = 1, \dots, \dim X - 1.$$

Notice that when X is a non-singular variety, any coherent ACM sheaf on X is locally free. A seminal result due to Horrocks (cf. [19]) asserts that, up to twist, there is only one indecomposable ACM bundle on $\mathbb{P}^n$: $\mathcal{O}_{\mathbb{P}^n}$. Ever since this result was stated, the study of the category of indecomposable arithmetically Cohen-Macaulay bundles on a given projective scheme X has raised a lot of interest since it is a natural way to understand the complexity of the underlying variety X (for more information the reader can see [9], [10], [33], [31] and [32]). One of the goals of this paper is to study whether the normal sheaf $\mathcal{N}_X$ and the exterior power $\wedge^q \mathcal{N}_X$ of the normal sheaf (or suitable twists $\mathcal{N}_X \otimes \mathcal{L}$ and $\wedge^q \mathcal{N}_X \otimes \mathcal{L}'$ by invertible sheaves $\mathcal{L}$ and $\mathcal{L}'$) of a standard determinantal scheme $X \subset \mathbb{P}^n$ are ACM. ACM sheaves are closely related to their algebraic counterpart, the maximal Cohen-Macaulay modules.

Definition 2.6. A graded R_X-module E is a *maximal Cohen-Macaulay* module (MCM for short) if $\operatorname{depth} E = \dim E = \dim R_X$.

In fact, we have:

Proposition 2.7. *Let $X \subseteq \mathbb{P}^n$ be an ACM scheme. There exists a bijection between ACM sheaves $\mathcal{E}$ on X and MCM R_X-modules E given by the functors $E \to \widetilde{E}$ and $\mathcal{E} \to \mathrm{H}^0_*(X, \mathcal{E})$.*

Proof. See [8]; Proposition 2.1. □

Definition 2.8. Given a closed subscheme $X \subset \mathbb{P}^n$, a coherent sheaf $\mathcal{E}$ on X is said to be *initialized* if

$$\mathrm{H}^0(X, \mathcal{E}(-1)) = 0 \quad \text{but} \quad \mathrm{H}^0(X, \mathcal{E}) \neq 0.$$

Notice that when $\mathcal{E}$ is a locally Cohen-Macaulay sheaf, there always exists an integer k such that $\mathcal{E}_{init} := \mathcal{E}(k)$ is initialized.

Let us now introduce the notion of Ulrich sheaf.

Definition 2.9. *Given a projective scheme $X \subset \mathbb{P}^n$ and a coherent sheaf $\mathcal{E}$ on X, we say that $\mathcal{E}$ is an* Ulrich sheaf *if $\mathcal{E}$ is an ACM sheaf and $\mathrm{h}^0(\mathcal{E}_{init}) = \deg(X) \operatorname{rk}(\mathcal{E})$.*

We have the following result that justifies this definition:

Theorem 2.10. *Let $X \subseteq \mathbb{P}^n$ be an integral subscheme and let $\mathcal{E}$ be an ACM sheaf on X. Then the minimal number of generators $m(\mathcal{E})$ of the R_X-module $\mathrm{H}^0_*(\mathcal{E})$ is bounded by*

$$m(\mathcal{E}) \leq \deg(X) \operatorname{rk}(\mathcal{E}).$$

Therefore, since it is obvious that for an initialized sheaf $\mathcal{E}$, $\mathrm{h}^0(\mathcal{E}) \le m(\mathcal{E})$, the minimal number of generators of Ulrich sheaves is as large as possible. Modules attaining this upper bound were studied by Ulrich in [36]. A detailed account is provided in [14]. In particular we have:

Theorem 2.11. *Let $X \subseteq \mathbb{P}^N$ be an n-dimensional ACM variety and $\mathcal{E}$ be an initialized ACM coherent sheaf on X. The following conditions are equivalent:*

(i) *$\mathcal{E}$ is Ulrich.*

(ii) *$\mathcal{E}$ admits a linear $\mathcal{O}_{\mathbb{P}^N}$-resolution of the form:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-N+n)^{a_{N-n}} \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-1)^{a_1} \longrightarrow \mathcal{O}_{\mathbb{P}^N}^{a_0} \longrightarrow \mathcal{E} \longrightarrow 0.$$

(ii) $\mathrm{H}^i(\mathcal{E}(-i)) = 0$ *for* $i > 0$ *and* $\mathrm{H}^i(\mathcal{E}(-i-1)) = 0$ *for* $i < n$.

(iv) *For some (resp. all) finite linear projections $\pi : X \longrightarrow \mathbb{P}^n$, the sheaf $\pi_*\mathcal{E}$ is the trivial sheaf $\mathcal{O}_{\mathbb{P}^n}^t$ for some t.*

In particular, initialized Ulrich sheaves are 0*-regular and therefore they are globally generated.*

Proof. See [14] Proposition 2.1. □

In the next section, we will prove that under some weak restrictions the normal sheaf $\mathcal{N}_X$ of a linear standard determinantal scheme X twisted by a suitable invertible sheaf $\mathcal{L}$ on $X \setminus Sing(X)$ is Ulrich (cf. Theorem 3.7); and as an immediate consequence we will deduce the μ-semistability of the normal sheaf of a linear standard determinantal scheme (cf. Theorem 5.3).

3. The Cohen-Macaulayness of the normal sheaf of a determinantal variety

The main goal of this section is to answer Problem 1.1(1). More precisely, we prove that, under some mild assumptions, given a standard determinantal scheme $X \subset \mathbb{P}^n$ of codimension c there always exists an invertible sheaf $\mathcal{L}$ on $X \setminus Sing(X)$ such $\mathcal{N}_X \otimes \mathcal{L}$ is ACM (cf. Theorem 3.7) and as a by-product we obtain an $\mathcal{O}_{\mathbb{P}^n}$-resolution of $\mathcal{N}_X \otimes \mathcal{L}$ which turns out to be pure and linear in the case of linear standard determinantal schemes (cf. Theorem 3.7 and Corollary 3.8). At the end of this section, we conjecture an analogous result for the exterior power of the normal sheaf (cf. Conjecture 3.23).

In this section $X \subset \mathbb{P}^n$ will be a standard determinantal scheme of codimension c, $\mathcal{A}$ the $t \times (t+c-1)$ homogeneous matrix associated to X, $I = I_t(\mathcal{A})$, $A = R/I$, $N_A := \mathrm{Hom}_A(I/I^2, A)$, $\mathcal{N}_X := \widetilde{N_A}$,

$$\varphi : F := \bigoplus_{j=1}^{t+c-1} R(-a_j) \longrightarrow G := \bigoplus_{i=1}^{t} R(-b_i)$$

the morphism of free R-modules associated to $\mathcal{A}$ and $M := \mathrm{coker}(\varphi)$. We will assume $t > 1$, since the case $t = 1$ corresponds to a codimension c complete intersection $X \subset \mathbb{P}^n$ and its normal sheaf is well understood ($\mathcal{N}_X \cong \oplus_{i=1}^{c} \mathcal{O}_X(d_i)$). If $t \ge 2$, $\widetilde{M}$ is a locally free $\mathcal{O}_X$-module of rank 1 over $T := X \setminus V(J)$ where $J := I_{t-1}(\mathcal{A})$ and $T \hookrightarrow \mathbb{P}^n$ is a local complete intersection. Recall also that if $a_j > b_i$ for any i,j, then $V(J) = Sing(X)$, $\operatorname{codim}_X(Sing(X)) = c+2$ or $Sing(X) = \emptyset$ and $\operatorname{codim}_{\mathbb{P}^n} X = c$ for a general choice of $\varphi \in \operatorname{Hom}(F,G)$.

Recall that the normal sheaf $\mathcal{N}_X$ of a standard determinantal scheme is ACM if and only if $\mathrm{H}^i_*(\mathcal{N}_X) = 0$ for $0 < i < \dim X$. In particular, the normal sheaf $\mathcal{N}_X$ of a standard determinantal curve $X \subset \mathbb{P}^n$ ($c = n-1$) is always ACM as well as the normal sheaf $\mathcal{N}_X$ of a standard determinantal scheme $X \subset \mathbb{P}^n$ of codimension $1 \le c \le 2$. Indeed, if $c = 1$, then $\mathcal{N}_X \cong \mathcal{O}_X(\delta)$ where $\delta := \deg(X)$ and if $c = 2$, then there is an exact sequence (cf. [24], (26))

$$0 \to \widetilde{F} \otimes_R \widetilde{G}^* \to ((\widetilde{F}^* \otimes_R \widetilde{F}) \oplus (\widetilde{G}^* \otimes_R \widetilde{G}))/R \to \widetilde{G} \otimes_R \widetilde{F}^* \to \mathcal{N}_X \to 0. \tag{3.1}$$

Unfortunately, it is no longer true for higher codimension and we only have that under some mild conditions $\mathrm{H}^i_*(\mathcal{N}_X) = 0$ for $1 \le i \le n-c-2$ (see [25]; Lemma 35 for $c = 3$, [28]; Corollary 5.5 for $3 \le c \le 4$ and [27]; Theorem 5.11 for the general case). In this section we are going to prove that under some weak restrictions $\mathcal{N}_X \otimes \widetilde{S_{c-2}M}$ is ACM, i.e.

$$\mathrm{H}^i_*(X, \mathcal{N}_X \otimes \widetilde{S_{c-2}M}) = 0 \text{ for } 1 \le i \le n-c-1.$$

To prove it, we will use the following technical lemma which can be seen as a generalization of the mapping cone process and we include a proof for the sake of completeness. In this lemma the differentials of a complex, say $Q_\bullet$, are denoted by $d^i_Q : Q_i \to Q_{i-1}$.

Lemma 3.1. *Let $Q_\bullet \xrightarrow{\sigma_\bullet} P_\bullet \xrightarrow{\tau_\bullet} F_\bullet$ be morphisms of complexes satisfying $Q_j = P_j = F_j = 0$ for $j < 0$ and assume that all three complexes are acyclic (exact for $j \neq 0$) and that the sequence*

$$0 \longrightarrow \operatorname{coker} d^1_Q \longrightarrow \operatorname{coker} d^1_P \xrightarrow{\alpha} \operatorname{coker} d^1_F$$

is exact. Moreover assume that there exists a morphism $\ell_\bullet : Q_\bullet \longrightarrow F_\bullet[1]$ such that for any integer i:

$$d^{i+1}_F \ell_i + \ell_{i-1} d^i_Q = \tau_i \sigma_i. \tag{3.2}$$

Then, the complex $Q_\bullet \oplus P_\bullet[1] \oplus F_\bullet[2]$ given by

$$Q_i \oplus P_{i+1} \oplus F_{i+2} \xrightarrow{d^i_{Q,P,F}} Q_{i-1} \oplus P_i \oplus F_{i+1}$$

where

$$d^i_{Q,P,F} := \begin{pmatrix} d^i_Q & 0 & 0 \\ \sigma_i & -d^{i+1}_P & 0 \\ \ell_i & -\tau_{i+1} & d^{i+2}_F \end{pmatrix}$$

is acyclic (exact for $i \neq -2$) and $\operatorname{coker} d^{-1}_{Q,P,F} = \operatorname{coker} \alpha$.

Proof. It is straightforward to show that $d^{i-1}_{Q,P,F} \circ d^i_{Q,P,F} = 0$ by using (3.2) and that the differentials of the complexes commute with $\sigma_\bullet$ and $\tau_\bullet$. To see the acyclicity of $Q_\bullet \oplus P_\bullet[1] \oplus F_\bullet[2]$, let

$$\begin{pmatrix} d^{i-1}_Q & 0 & 0 \\ \sigma_{i-1} & -d^i_P & 0 \\ \ell_{i-1} & -\tau_i & d^{i+1}_F \end{pmatrix} \begin{pmatrix} q \\ p \\ f \end{pmatrix} = \begin{pmatrix} d^{i-1}_Q(q) \\ \sigma_{i-1}(q) - d^i_P(p) \\ \ell_{i-1}(q) - \tau_i(p) + d^{i+1}_F(f) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We must show the existence of $(q', p', f') \in Q_i \oplus P_{i+1} \oplus F_{i+2}$ whose transpose maps to $(q, p, f)^{\mathrm{tr}}$ via $d^i_{Q,P,F}$. Since $Q_\bullet$ is exact for $i \neq 0$ there exists a $q' \in Q_i$ such that $d^i_Q(q') = q$ provided $i > 1$. For $i = 1$ we use $\sigma_0(q) = d^1_P(p)$ and the injectivity of $\operatorname{coker} d^1_Q \to \operatorname{coker} d^1_P$ to see that q maps to zero in $\operatorname{coker} d^1_Q$. Hence there exists $q' \in Q_1$ such that $d^1_Q(q') = q$. We will treat the case $i = 0$ shortly.

Suppose $i > 0$. Since we have $\sigma_{i-1}(q) - d^i_P(p) = 0$, $q = d^i_Q(q')$ and $\sigma_{i-1} d^i_Q = d^i_P \sigma_i$ we get $d^i_P(\sigma_i(q') - p) = 0$. By the exactness of $P_\bullet$, there exists an element $p' \in P_{i+1}$ such that $d^{i+1}_P(p') = \sigma_i(q') - p$. Then we insert $p = \sigma_i(q') - d^{i+1}_P(p')$ and $q = d^i_Q(q')$ into $\ell_{i-1}(q) - \tau_i(p) + d^{i+1}_F(f) = 0$ and we use (3.2) and that d^{i+1}_P and d^{i+1}_F commute with $\tau_\bullet$ to conclude that $d^{i+1}_F(-\ell_i(q') + \tau_{i+1}(p') + f) = 0$. Hence there exists an element $f' \in F_{i+2}$ such that $f = \ell_i(q') - \tau_{i+1}(p') + d^{i+2}_F(f')$. The expressions for q, p and f above prove precisely what we needed to show if $i > 0$.

Now suppose $i = 0$. Since $d^{-1}_Q = d^0_Q = d^0_P = \sigma_{-1} = \ell_{-1} = 0$ and $q \in Q_{-1} = 0$ we have an element $(p, f) \in P_0 \oplus F_1$ that maps to $-\tau_0(p) + d^1_F(f) = 0$ in F_0. Since $\tau_0(p)$ maps to zero in $\operatorname{coker} d^1_F$, it follows that p is sent to an element in $\operatorname{coker} d^1_P$ that is contained in $\operatorname{coker} d^1_Q$ by the exactness assumption on the sequence of cokernels of Lemma 3.1. Hence there exists an element $q' \in Q_0$ such that $\sigma_0(q') - p$ maps to zero in $\operatorname{coker} d^1_P$, whence there is a $p' \in P_1$ such that $d^1_P(p') = \sigma_0(q') - p$. Since we have $d^1_F(-l_0(q') + \tau_1(p') + f) = 0$ there exists $f' \in F_2$ such that $d^2_F(f') = -l_0(q') + \tau_1(p') + f$, and we get the expressions for p and f which we wanted to show.

Finally using that $(p, f) \in P_0 \oplus F_1$ maps to $-\tau_0(p) + d^1_F(f)$ in F_0 we easily get $\operatorname{coker} d^{-1}_{Q,P,F} = \operatorname{coker} \alpha$ and we are done. □

Theorem 3.2. *We keep the notation introduced above and, in addition, we assume* $\operatorname{depth}_J A \geq 2$. *Then, we have*

$$\operatorname{Ext}^1_R(M, S_{c-1}M)$$

is a maximal Cohen-Macaulay A-module of rank c. Moreover, if $a_j = 1$ for all j and $b_i = 0$ for all i, then $\widetilde{\operatorname{Ext}^1_R(M, S_{c-1}M)}$ *is an Ulrich sheaf of rank c.*

Proof. Our primary goal is to show that $\operatorname{Ext}^1_R(M, S_{c-1}M)$ is a MCM A-module by using the exact sequence

$$0 \longrightarrow S_{c-2}M \longrightarrow G^* \otimes S_{c-1}M \longrightarrow F^* \otimes S_{c-1}M \longrightarrow \operatorname{Ext}^1_R(M, S_{c-1}M) \longrightarrow 0. \tag{3.3}$$

and Lemma 3.1 to exhibit a minimal free resolution of $\operatorname{Ext}^1_R(M, S_{c-1}M)$ having length c. So let us start proving the existence of the exact sequence (3.3). Indeed, we look at the Buchsbaum-Rim complex (see Proposition 2.3(ii))

$$\cdots \longrightarrow \wedge^{t+1} F \otimes \wedge^t G^* \otimes S_0 G^* \xrightarrow{\epsilon_1^*} F \xrightarrow{\varphi} G \longrightarrow M \longrightarrow 0. \tag{3.4}$$

Since $I_t(\varphi) \cdot M = 0$ and $\operatorname{im}(\epsilon_1^*) \subseteq I_t(\varphi) \cdot F$, we get that the induced map $\operatorname{Hom}_R(\epsilon_1^*, S_{c-1}M) = 0$. Therefore, the exact sequence (3.3) comes from applying the functor $\operatorname{Hom}_R(-, S_{c-1}M)$ to the exact sequence (3.4), because under the assumption $\operatorname{depth}_J A \geq 2$, we have $\operatorname{depth}_J S_iM \geq 2$ for any $1 \le i \le c$ and hence

$$
\begin{array}{rcl}
\mathrm{Hom}_R(M, S_iM) & \cong & \mathrm{H}^0_*(X \setminus V(J), \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{M}^{\otimes(i)})) \\
& \cong & \mathrm{H}^0_*(X \setminus V(J), \widetilde{M}^{\otimes(i-1)}) \\
& \cong & S_{i-1}M.
\end{array} \tag{3.5}
$$

From the exact sequence (3.3) we deduce that $\mathrm{Ext}^1_R(M, S_{c-1}M)$ has rank c; let us prove that it is a maximal Cohen-Macaulay A-module. The idea will be be to apply Lemma 3.1 to the following diagram which we will define as an expansion of (3.3) (we set $\wedge^i := \wedge^i F$):

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \wedge^{t+c-1} \otimes S_1G^* \otimes \wedge^t G^* & \overset{\sigma_c}{\dashrightarrow} & G^* \otimes \wedge^{t+c-1} \otimes S_0G^* \otimes \wedge^t G^* & \overset{\varphi^* \otimes 1}{\longrightarrow} & F^* \otimes \wedge^{t+c-1} \otimes S_0G^* \otimes \wedge^t G^* \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \wedge^{t+c-2} \otimes S_0G^* \otimes \wedge^t G^* & \overset{\sigma_{c-1}}{\dashrightarrow} & G^* \otimes S_0G \otimes \wedge^{c-1} & \overset{\varphi^* \otimes 1}{\longrightarrow} & F^* \otimes S_0G \otimes \wedge^{c-1} \\
& & \downarrow & & \downarrow & & \downarrow \\
& & S_0G \otimes \wedge^{c-2} & \overset{\sigma_{c-2}}{\dashrightarrow} & G^* \otimes S_1G \otimes \wedge^{c-2} & \overset{\varphi^* \otimes 1}{\longrightarrow} & F^* \otimes S_1G \otimes \wedge^{c-2} \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \vdots & & \vdots & & \vdots \\
& & \downarrow & & \downarrow & & \downarrow \\
& & S_{c-4}G \otimes \wedge^2 & \overset{\sigma_2}{\dashrightarrow} & G^* \otimes S_{c-3}G \otimes \wedge^2 & \overset{\varphi^* \otimes 1}{\longrightarrow} & F^* \otimes S_{c-3}G \otimes \wedge^2 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & S_{c-3}G \otimes F & \overset{\sigma_1}{\dashrightarrow} & G^* \otimes S_{c-2}G \otimes F & \overset{\varphi^* \otimes 1}{\longrightarrow} & F^* \otimes S_{c-2}G \otimes F \\
& & \downarrow & & \downarrow & & \downarrow \\
& & S_{c-2}G & \overset{\sigma_0}{\dashrightarrow} & G^* \otimes S_{c-1}G & \overset{\varphi^* \otimes 1}{\longrightarrow} & F^* \otimes S_{c-1}G \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \to & S_{c-2}M & \to & G^* \otimes S_{c-1}M & \overset{\varphi^* \otimes 1}{\longrightarrow} & F^* \otimes S_{c-1}M \to \mathrm{Ext}^1_R(M, S_{c-1}M) \to 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}
$$

Diagram A

Let us call $Q_\bullet$, $P_\bullet$ and $F_\bullet$ the resolutions of $S_{c-2}M$, $G^* \otimes S_{c-1}M$ and $F^* \otimes S_{c-1}M$, respectively. We need to define morphisms of complexes: $\sigma_\bullet : Q_\bullet \to P_\bullet$ and $\ell_\bullet : Q_\bullet \to F_\bullet[1]$ satisfying all the hypothesis of Lemma 3.1. Let us first recall the definition of

$$\partial^p_q : S_pG \otimes \wedge^q F \longrightarrow S_{p+1}G \otimes \wedge^{q-1} F.$$

To this end, we take $\{x_i\}_{i=1}^t$ an R-free basis of G and let $\{x_i^*\}_{i=1}^t$ be the dual basis of G^*. According to [13]; pg. 592, ∂^p_q takes an element $m \otimes f \in S_pG \otimes \wedge^q F$ to the element $\sum_{i=1}^t x_i m \otimes \varphi^*(x_i^*)(f) \in S_{p+1}G \otimes \wedge^{q-1} F$. For any integer i, $0 \le i \le c-2$, we define:

$$\sigma_i : S_{c-2-i}G \otimes \wedge^i F \longrightarrow G^* \otimes S_{c-1-i}G \otimes \wedge^i F$$

sending an element $m\otimes f \in S_{c-2-i}G\otimes\wedge^i F$ to $\sigma_i(m\otimes f)=\sum_{j=1}^t x_j^*\otimes x_j m\otimes f \in G^*\otimes S_{c-1-i}G\otimes\wedge^i F$. It is easy to check that the following diagram commutes for any integer i, $0\le i\le c-3$:

$$\begin{array}{ccc} S_{c-3-i}G\otimes\wedge^{i+1}F & \xrightarrow{\sigma_{i+1}} & G^*\otimes S_{c-2-i}G\otimes\wedge^{i+1}F \\ \downarrow\partial_{i+1}^{c-3-i} & & \downarrow 1_{G^*}\otimes\partial_{i+1}^{c-2-i} \\ S_{c-2-i}G\otimes\wedge^{i}F & \xrightarrow{\sigma_{i}} & G^*\otimes S_{c-1-i}G\otimes\wedge^{i}F \end{array}.$$

In this setting, we point out that the definition of σ_{c-2} implies the commutativity of the diagram

$$\begin{array}{ccc} S_0G\otimes\wedge^{c-2}F & \xrightarrow{\sigma_{c-2}} & G^*\otimes S_1G\otimes\wedge^{c-2}F \\ \downarrow\simeq & & \downarrow\simeq \\ R\otimes\wedge^{c-2}F & \xrightarrow{\mathrm{tr}\otimes 1} & G^*\otimes G\otimes\wedge^{c-2}F \end{array}$$

i.e., σ_{c-2} is induced by the trace map $R\longrightarrow G^*\otimes G$ that is dual to the evaluation map $G^*\otimes G\longrightarrow R$ (i.e. $\sigma_{c-2}=\mathrm{tr}\otimes 1$). We will now define σ_c and σ_{c-1} in such a way that the two left upper squares of the Diagram A commute or anticommute. Dualizing (i.e. applying $\mathrm{Hom}_R(-,R)$) and using the isomorphism $\wedge^i F^*\cong\wedge^{t+c-1-i}F\otimes\wedge^{t+c-1}F^*\cong\wedge^{t+c-1-i}F$, it will be sufficient to prove the commutativity of the following diagram

$$\begin{array}{ccc} G & \xleftarrow{\sigma_c^*} & G\otimes R \\ \uparrow\varphi & & \uparrow\epsilon_0^* \\ F & \xleftarrow{\sigma_{c-1}^*} & G\otimes S_0G^*\otimes\wedge^tF\otimes\wedge^tG^* \\ \uparrow(-1)^{t+1}\epsilon_1^* & & \uparrow 1\otimes(\partial_{c-1}^0)^* \\ \wedge^{t+1}F\otimes\wedge^tG^*\otimes S_0G^* & \xleftarrow{\sigma_{c-2}^*} & G\otimes S_1G^*\otimes\wedge^{t+1}F\otimes\wedge^tG^* \\ \| & & \\ (S_0G\otimes\wedge^{c-2}F)^* & & \end{array}.$$

We take $\{y_i\}_{i=1}^{t+c-1}$ to be a free R-basis of F and $\{y_i^*\}_{i=1}^{t+c-1}$ to be the dual basis, and we let $\{i_1,i_2,\cdots,i_{t+1}\}$, $i_1<i_2<\cdots<i_{t+1}$ be a subset of $I:=\{1,2,\cdots,t+c-1\}$. According to [23] or [13]; pages 592 and 593 (see also the exact sequence (2.1)), ϵ_0^* and ϵ_1^* are defined by

$$\epsilon_0^*(g\otimes y_{i_1}\wedge y_{i_2}\wedge\cdots\wedge y_{i_t}) := s_{t+1}\cdot g\otimes\varphi(y_{i_1})\wedge\cdots\wedge\varphi(y_{i_t}),\ \text{and}$$

$$\epsilon_1^*(y_{i_1}\wedge y_{i_2}\cdots\wedge y_{i_{t+1}}) := \sum_{j=1}^{t+1} s_j\cdot(\varphi(y_{i_1})\wedge\cdots\wedge\varphi(y_{i_{j-1}})\wedge\varphi(y_{i_{j+1}})\wedge\cdots\varphi(y_{i_{t+1}}))y_{i_j}.$$

where s_j is the sign of the permutation of I that takes the elements of $\{i_1,i_2,\cdots,i_{j-1},i_{j+1},\cdots,i_{t+1}\}$ into the first t positions. Note that $\varphi(y_i)$ are the columns of the matrix $\mathcal{A}$ associated to φ, that $\varphi(y_{i_1})\wedge\cdots\wedge\varphi(y_{i_t})$ is the maximal minor corresponding to the columns $i_1,\cdots,i_t$ and that a replacement of ϵ_1^* by $(-1)^{t+1}\epsilon_1^*$, cf. the diagram above, still makes the leftmost column in Diagram A a free resolution of $S_{c-2}M$. We define:

$$\sigma_c^* = Id_G,\ \text{and}$$

$$\sigma_{c-1}^*(g\otimes y_{i_1}\wedge\cdots\wedge y_{i_t}) = \sum_{j=1}^{t}(\varphi(y_{i_1})\wedge\cdots\wedge\varphi(y_{i_{j-1}})\wedge g\wedge\varphi(y_{i_{j+1}})\wedge\cdots\varphi(y_{i_t}))y_{i_j}.$$

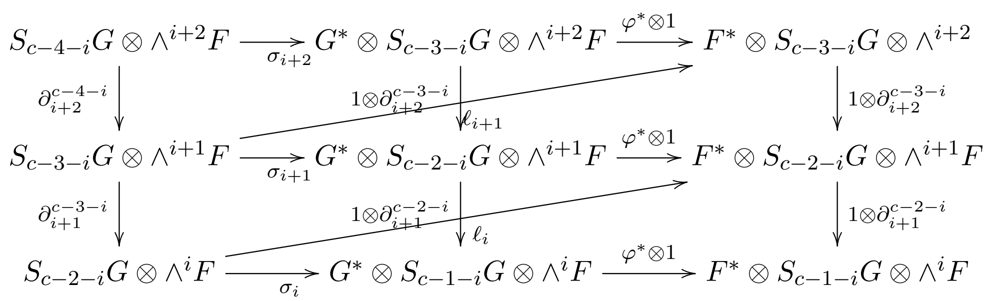
A straightforward computation gives us the desired commutativity, namely,

$$\sigma_c^* \cdot \epsilon_0^* = \varphi \cdot \sigma_{c-1}^* \quad \text{and} \quad (-1)^{t+1}\epsilon_1^* \cdot \sigma_{c-2}^* = \sigma_{c-1}^* \cdot (1 \otimes (\partial_{c-1}^0)^*).$$

We will now define the morphism $\ell_\bullet : Q_\bullet \longrightarrow F_\bullet[1]$. For any integer i, $0 \leq i \leq c-2$, we define:

$$\ell_i : S_{c-2-i}G \otimes \wedge^i F \longrightarrow F^* \otimes S_{c-2-i}G \otimes \wedge^{i+1}F$$

sending an element $m \otimes f \in S_{c-2-i}G \otimes \wedge^i F$ to $\ell_i(m \otimes f) = \sum_{j=1}^{t+c-1} y_j^* \otimes m \otimes (y_j \wedge f) \in F^* \otimes S_{c-2-i}G \otimes \wedge^{i+1}F$. Using the diagram



and [13]; Proposition A2.8 page 583 onto the derivation $\varphi^*(x_j^*)$, we check for any i, $0 \leq i \leq c-3$, that

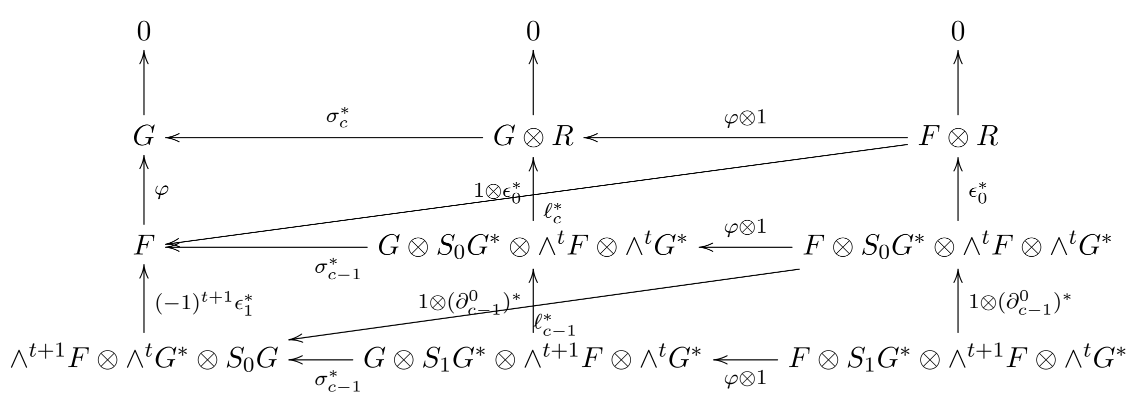
$$(\varphi^* \otimes 1) \cdot \sigma_{i+1} = (1_{F^*} \otimes \partial_{i+2}^{c-3-i}) \cdot \ell_{i+1} + \ell_i \cdot \partial_{i+1}^{c-3-i}.$$

We dualize the top part of Diagram A and we define ℓ_c^* and ℓ_{c-1}^* (obviously $\ell_i^* = 0$ for $i \geq c+1$) as follows:

$$\ell_c^* = Id_F, \text{ and}$$

$$\ell_{c-1}^*(f \otimes y_{i_1} \wedge \cdots \wedge y_{i_t}) = f \wedge y_{i_1} \wedge \cdots \wedge y_{i_t}.$$

A direct calculation using the following diagram



gives us (recall that $\sigma_c^* = 1$):

$$\varphi \cdot \ell_c^* = \sigma_c^* \cdot (\varphi \otimes 1), \text{ and}$$

$$(-1)^{t+1}\epsilon_1^* \cdot \ell_{c-1}^* + \ell_c^* \cdot \epsilon_0^* = \sigma_{c-1}^* \cdot (\varphi \otimes 1).$$

Now we are ready to apply Lemma 3.1. Since $\ell_c = Id$ and $\sigma_c = Id$, the corresponding summands split off and we deduce that $\operatorname{pd}\operatorname{Ext}^1_R(M, S_{c-1}M) = c$, i.e. $\operatorname{Ext}^1_R(M, S_{c-1}M)$ is a Maximal Cohen-Macaulay A-module.

Finally we prove the last assertion of the Theorem. We assume $a_j = 1$ for all j and $b_i = 0$ for all i. Recall that the degree of a linear standard determinantal scheme $X \subset \mathbb{P}^n$ of codimension c is $\binom{t+c-1}{c}$. Using the exact sequence (3.3), we get

$$_{-2}\operatorname{Ext}^1(M, S_{c-1}M) = 0, \text{ and}$$

$$\dim_K({}_{-1}\operatorname{Ext}^1(M, S_{c-1}M)) = (t+c-1)\cdot\binom{t+c-2}{c-1} = \operatorname{rk}(\operatorname{Ext}^1(M, S_{c-1}M))\cdot\deg(X).$$

Therefore, $\widetilde{\operatorname{Ext}^1(M, S_{c-1}M)}$ is a maximal Cohen-Macaulay module maximally generated or, equivalently, $\widetilde{\operatorname{Ext}^1_R(M, S_{c-1}M)}$ is an Ulrich sheaf on X of rank c. $\square$

Given an ACM scheme $X \subset \mathbb{P}^N$ with dualizing sheaf ω and a coherent sheaf $\mathcal{E}$ on X, we denote by $\mathcal{E}^\omega$ the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \omega)$. It is well known that $\mathcal{E}$ is ACM if and only if $\mathcal{E}^\omega$ is ACM. So, as a Corollary of Theorem 3.2, we have:

Corollary 3.3. *We keep the notation introduced above and we set* $N := \operatorname{Ext}^1_R(M, S_{c-1}M)$. *If* $\operatorname{depth}_J A \geq 2$ *then,* $\operatorname{Hom}_A(N, K_A)$ *is a maximal Cohen-Macaulay A-module of rank c. Moreover, if* $a_j = 1$ *for all* j *and* $b_i = 0$ *for all* i*, then* $\widetilde{\operatorname{Hom}_A(N, K_A)}$ *is an Ulrich sheaf of rank* c.

Remark 3.4. Keeping the notation introduced above and arguing as in the proof of Theorem 3.2, we can prove that for any i, $0 \leq i \leq c$, there exits an exact sequence

$$0 \longrightarrow S_{i-1}M \longrightarrow G^* \otimes S_iM \longrightarrow F^* \otimes S_iM \longrightarrow \operatorname{Ext}^1_R(M, S_iM) \longrightarrow 0 \tag{3.6}$$

provided $\operatorname{depth}_J A \geq 2$ where $J = I_{t-1}(\mathcal{A})$. Note that we interpret $S_{-1}M$ as $\operatorname{Hom}_A(M, A)$.

Hence we get a big diagram, similar to diagram A above, where we have replaced $c-1$ by i. The part of the proof of Theorem 3.2 where we show the existence of $\{\sigma_j\}_{j=0}^c$ between the two leftmost columns (we call this part of the big diagram by (*)), seems to hold for any i, $0 \leq i \leq c$. Since σ_i is injective (except for $i = c$), this would imply that the length of the projective resolution of $\operatorname{Ext}^1_R(M, S_iM)$ is at most $\operatorname{pd} A + 1$ for $0 \leq i \leq c-1$. The only problem to get this result will be to define σ_j more generally and to verify the commutativity of the first diagram in the proof of Theorem 3.2 where the dual of the "splice" maps ϵ_j, $j = 0, 1$ occur (we call this diagram (**) after having replaced $c-1$ by i and made the corresponding obvious changes).

The case $i = 1$ was treated in [27]; Proof of Theorem 3.1. That proof is almost the dual of the proof of this part of Theorem 3.2. Indeed the lower part of diagram (*) for $i = 1$ is exactly diagram (**) provided we move G^* from the second column to G in the first column in diagram (*). Thus the proof of Theorem 3.1 in [27] implies this part of the proof of Theorem 3.2 and vice versa.

The existence of the morphisms σ_j in diagram (*) in the case $i = c$ is very similar (and easier) to what we had to prove in Theorem 3.2 for $i = c-1$. Indeed, in this case we only need to check diagram (**) where now only one ϵ_j occur. The mapping cone construction leads to a resolution of $\operatorname{Ext}^1_R(M, S_cM)$ where, however, the leftmost free module (of rank one) clearly does not split off, whence the length of a minimal resolution must be $\operatorname{pd} A + 2$. As explained for $i = 1$ as almost the "dual" of $i = c - 1$ above, the case $i = 0$ is similarly "dual" to $i = c$. We deduce the existence of a morphism between the R-free resolutions of $S_{-1}M$ and $G^* \otimes A$. In this case it is easy to see that the leftmost free module in the resolution of $\operatorname{Ext}^1_R(M, A)$ split off. In particular, we get

$$\operatorname{pd} \operatorname{Ext}^1_R(M, S_iM) \le \operatorname{pd} A + 1 \quad \text{for} \quad i \in \{0, 1, c-1\} .$$

The importance of proving $\operatorname{pd} \operatorname{Ext}^1_R(M, S_iM)$ to be small follows from our next proposition because it leads to good depth of "twisted normal modules". Indeed, in [27] we use this to prove both conjectures appearing in [29] on the dimension and smoothness of the locus of determinantal schemes inside the Hilbert scheme.

Proposition 3.5. *With the above notation, set* $J = I_{t-1}(\mathcal{A})$. *We have*

(1) $\operatorname{Ext}^1_R(M, S_{c-1}M) \cong \operatorname{Hom}_A(I/I^2, S_{c-2}M)$ *provided* $\operatorname{depth}_J A \ge 2$; *and*
(2) $\operatorname{Ext}^1_R(M, S_iM) \cong \operatorname{Hom}_A(I/I^2, S_{i-1}M)$ *for* $0 \le i \le c$, $i \ne c-1$, *provided* $\operatorname{depth}_J A \ge 4$.

Proof. We prove (1) and (2) simultaneously. By Proposition 2.3 (ii), the canonical module $K_A(v) \cong S_{c-1}(M)$ for some integer v and by Proposition 2.3 (ii) M is a maximal Cohen-Macaulay A-module. Therefore, we have

$$\operatorname{Ext}^i_A(M, K_A) = 0 \text{ for } i \ge 1. \tag{3.7}$$

Using (2.2) as in the proof of Theorem 4.1 in [27], we get that

$$\operatorname{Ext}^j_A(M, S_iM) = 0 \text{ for } 1 \le j \le 2, \tag{3.8}$$

provided $\operatorname{depth}_J A \ge 4$.

Under the assumption $\operatorname{depth}_J A \ge 2$, we have seen in (3.5) that for any i, $1 \le i \le c$

$$\operatorname{Hom}_A(M, S_iM) \cong S_{i-1}M$$

(true also for $i = 0$). In particular, we have

$$\operatorname{Hom}_A(M, K_A) \cong S_{c-2}(M)(-v). \tag{3.9}$$

Notice that the isomorphism

$$\operatorname{Hom}_R(M, S_iM) \cong \operatorname{Hom}_A(M \otimes_R A, S_iM)$$

leads to a spectral sequence

$$\operatorname{Ext}^p_A(\operatorname{Tor}^R_q(M, A), S_iM) \Rightarrow \operatorname{Ext}^{p+q}_R(M, S_iM);$$

a part of the usual 5-term associated sequence is

$$0 \longrightarrow \operatorname{Ext}^1_A(M \otimes_R A, S_iM) \longrightarrow \operatorname{Ext}^1_R(M, S_iM) \longrightarrow \operatorname{Hom}_A(\operatorname{Tor}^R_1(M, A), S_iM) \longrightarrow \operatorname{Ext}^2_A(M \otimes_R A, S_iM),$$

which using (3.7), (3.8) and the exactness of the sequence

$$0 \longrightarrow \operatorname{Tor}^R_1(M, A) \longrightarrow M \otimes_R I \longrightarrow M \otimes_R R \xrightarrow{\simeq} M \otimes_R A \longrightarrow 0$$

allows us to conclude that

$$\operatorname{Ext}^1_R(M, S_iM) \cong \operatorname{Hom}_A(M \otimes_R I, S_iM).$$

Using (2.2) we get

$$\begin{aligned} \operatorname{Hom}_A(M \otimes_R I, S_iM) &\cong \operatorname{Hom}_A(M \otimes I/I^2, S_iM) \\ &\cong \operatorname{Hom}_A(I/I^2, \operatorname{Hom}_A(M, S_iM)) \\ &\cong \operatorname{Hom}_A(I/I^2, S_{i-1}M), \end{aligned}$$

and we are done. □

Remark 3.6. We can improve upon (2) of Proposition 3.5 in the case $i = 0$ and get

$$\operatorname{Ext}^1_R(M, A) \cong \operatorname{Hom}_A(I/I^2, \operatorname{Hom}_A(M, A))$$

only assuming $\operatorname{depth}_J A \geq 3$. Indeed this depth condition implies $\operatorname{Ext}^1_A(M, A) = 0$ by (2.2) and if we can show $\operatorname{Ext}^2_A(M, A) = 0$ of (3.8) by another argument, then the proof above applies to get the claim. To see $\operatorname{Ext}^2_A(M, A) = 0$ we remark that $S_{c-1}M$ is a twist of the canonical module. This implies $\operatorname{Ext}^2_A(M, A) \cong \operatorname{Ext}^2_A(M \otimes S_{c-1}M, S_{c-1}M)$ by a spectral sequence argument, while we get $\operatorname{Ext}^2_A(M \otimes S_{c-1}M, S_{c-1}M) \cong \operatorname{Ext}^2_A(S_cM, S_{c-1}M)$ by using that $\widetilde{M} \otimes \widetilde{S_{c-1}M} \cong \widetilde{S_cM}$ if we restrict to $\operatorname{Proj}(A) - V(J)$, cf. [27]; proof of Theorem 4.5 (the text after the diagram (4.3)) for details. Then we conclude by Gorenstein duality.

Theorem 3.7. *Let $X \subset \mathbb{P}^n$ be a standard determinantal scheme of codimension $c \geq 2$ associated to a $t \times (t + c - 1)$ matrix $\mathcal{A}$. Set $J = I_{t-1}(\mathcal{A})$ and assume $\operatorname{depth}_J R/I(X) \geq 2$. Then, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}^2_X, \widetilde{S_{c-2}M})$ is an ACM sheaf of rank c. In addition, if X is a linear standard determinantal scheme, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}^2_X, \widetilde{S_{c-2}M})(-H)$ is an initialized Ulrich sheaf of rank c and it has a pure linear $\mathcal{O}_{\mathbb{P}^n}$-resolution of the following type:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-c)^{a_c} \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{a_1} \longrightarrow \mathcal{O}^{a_0}_{\mathbb{P}^n} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}^2_X, \widetilde{S_{c-2}M})(-H) \longrightarrow 0$$

with $a_0 = c \cdot \deg(X) = c \cdot \binom{t+c-1}{c}$ and $a_i = \binom{c}{i} \cdot a_0$ for $1 \leq i \leq c$. Finally, if X is a local complete intersection then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}^2_X, \widetilde{S_{c-2}M}) \cong \mathcal{N}_X \otimes \widetilde{S_{c-2}M}$ and

$$\operatorname{H}^i_*(X, \mathcal{N}_X \otimes \widetilde{S_{c-2}M}) = 0 \quad \text{for } 1 \leq i \leq n - c - 1.$$

Proof. We can apply Proposition 3.5 and we get $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}^2_X, \widetilde{S_{c-2}M}) \cong \operatorname{Ext}^1_R(\widetilde{M, S_{c-1}M})$. Therefore, applying Theorem 3.2 we conclude that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}^2_X, \widetilde{S_{c-2}M})$ is an ACM sheaf of rank c on X and if X is a linear standard determinantal scheme, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}^2_X, \widetilde{S_{c-2}M})(-H)$ is an initialized Ulrich sheaf of rank c, cf. the Corollary below for the twist. In this case the minimal $\mathcal{O}_{\mathbb{P}^n}$-resolution of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}^2_X, \widetilde{S_{c-2}M})(-H)$ is given by [14], Proposition 2.1. □

As an immediate application of Theorem 3.2 and its proof we obtain a free R-resolution of $\operatorname{Ext}^1_R(M, S_{c-1}M)$, and more generally of $\operatorname{Ext}^1_R(M, S_iM)$ for $0 \leq i \leq c$, which we make explicit for later use. In fact, we have:

Corollary 3.8. *We keep the notation introduced above and we call $Q_\bullet$, $P_\bullet$ and $F_\bullet$ the minimal free R-resolutions of $S_{i-1}M$, $G^*\otimes S_iM$ and $F^*\otimes S_iM$, respectively. Then $\operatorname{Ext}^1_R(M,S_iM)$, for $0\le i\le c$, has a free R-resolution of the following type:*

$$0\longrightarrow Q_c\longrightarrow Q_{c-1}\oplus P_c\longrightarrow Q_{c-2}\oplus P_{c-1}\oplus F_c\longrightarrow Q_{c-3}\oplus P_{c-2}\oplus F_{c-1}\longrightarrow\cdots$$

$$\longrightarrow Q_0\oplus P_1\oplus F_2\longrightarrow P_0\oplus F_1\longrightarrow F_0\longrightarrow \operatorname{Ext}^1_R(M,S_iM)\longrightarrow 0$$

provided $\operatorname{depth}_J A\ge 2$ *where $J=I_{t-1}(\mathcal{A})$. In particular, we have*

$$\operatorname{depth}\operatorname{Ext}^1(M,S_iM)\ge \operatorname{depth}A-2.$$

Moreover if $i=c-1$ (resp. $i\in\{0,1\}$) we may delete Q_c,P_c and Q_{c-1},F_c (resp. Q_c and a subsummand of P_c) from this resolution, and we get

$$\operatorname{depth}\operatorname{Ext}^1(M,S_iM)\ge \operatorname{depth}A-1 \text{ for } i=0,1; \text{ and}$$

$$\operatorname{depth}\operatorname{Ext}^1(M,S_{c-1}M)=\operatorname{depth}A.$$

Proof. Using Lemma 3.1 and Remark 3.4 we get a free R-resolution of $\operatorname{Ext}^1_R(M,S_iM)$ of the form above. If $i=c-1$ (resp. $i\in\{0,1\}$) the mentioned summands split off by the proof of Theorem 3.2 (resp. Remark 3.4). □

Note that we do not claim that this resolution is minimal (not even for $i=c-1$) since we have not carefully analyzed all possible cancelation of repeated direct summands in the mapping cone construction. Nevertheless, there is a particular case where we can assure that *all* repeated summands split off. Indeed, if $i=c-1$, $a_j=1$ for all j and $b_s=0$ for all s then the module $\operatorname{Ext}^1_R(M,S_{c-1}M)$ is Ulrich. By Proposition 2.11 (ii), it has a pure linear resolution and therefore, we may delete not only Q_c,P_c and Q_{c-1},F_c but also any other repeated summand split off.

We would like to know whether the rank c ACM (resp. Ulrich) sheaves on X constructed in the Theorem 3.2 are indecomposable. We will see in the next section that under some weak conditions $\mathcal{N}_X$ and $\mathcal{H}om(\mathcal{I}_X/\mathcal{I}_X^2,\widetilde{S_iM})$, $0\le i\le c-1$ are in fact indecomposable (see Theorems 4.3 and 4.14).

Now we consider and discuss a Conjecture which generalizes the main result of this section. It is based on a series of examples computed with Macaulay2 ([20]):

Conjecture 3.9. *Let $X\subset\mathbb{P}^n$ be a standard determinantal scheme of codimension $c\ge 2$ associated to a $t\times(t+c-1)$ matrix $\mathcal{A}$. Set $I=I_t(\mathcal{A})$, $J=I_{t-1}(\mathcal{A})$ and assume* $\operatorname{depth}_J R/I\ge 2$. *With the above notation, we conjecture that for all integer i, $0\le i\le c$, $\operatorname{Ext}^i_R(S_iM,S_{c-i}M)$ is an (indecomposable) Maximal Cohen-Macaulay R/I-module of rank $\binom{c}{i}$. In addition, if X is a linear determinantal scheme then $\operatorname{Ext}^i_R(S_iM,S_{c-i}M)$ is an (indecomposable) Ulrich R/I-module of rank $\binom{c}{i}$.*

Our next goal will be to use the spectral sequence

$$E_2^{p,q}=\operatorname{Ext}^p_A(\operatorname{Tor}^R_q(S_rM,A),S_sM)\Rightarrow \operatorname{Ext}^{p+q}_R(S_rM,S_sM)\quad\text{for }\ -1\le s\le c, \tag{3.10}$$

to get some non-obvious isomorphic variations of the above Conjecture. Indeed under some natural depth_J-conditions we will see that the spectral sequence degenerates and that we get

$$\operatorname{Ext}^a_R(S_rM, S_sM) \cong \operatorname{Hom}_A(\wedge^a(I/I^2), S_{s-r}M) \quad \text{for} \quad 0 \le a \le r-s+c. \tag{3.11}$$

Firstly, we remark that since $T := \operatorname{Spec}(A) \setminus V(J)$ is locally a complete intersection, then the $\mathcal{O}_T$-modules

$$\widetilde{\operatorname{Tor}}^R_q(\widetilde{S_rM}, \mathcal{O}_T) \cong \widetilde{\operatorname{Tor}}^R_q(\mathcal{O}_T, \mathcal{O}_T) \otimes_{\mathcal{O}_T} \widetilde{S_rM} \quad \text{and}$$

$$\widetilde{\operatorname{Tor}}^R_q(\mathcal{O}_T, \mathcal{O}_T) \cong \wedge^q(\mathcal{I}/\mathcal{I}^2) \tag{3.12}$$

are locally free on T, whence if $\operatorname{depth}_J S_sM \ge i+2$ then (2.2) implies

$$\begin{aligned} E_2^{p,q} = \operatorname{Ext}^p_A(\operatorname{Tor}^R_q(S_rM, A), S_sM) &\cong \operatorname{H}^p(T, \mathcal{H}om(\wedge^q(\mathcal{I}/\mathcal{I}^2) \otimes \widetilde{S_rM}, \widetilde{S_sM})) \\ &\cong \operatorname{H}^p(T, \mathcal{H}om(\wedge^q(\mathcal{I}/\mathcal{I}^2), \widetilde{S_{s-r}M})), \ \text{for } p \le i. \end{aligned} \tag{3.13}$$

Lemma 3.10. *Let r, s and $a \ge 1$ be integers satisfying $-1 \le s \le c$ and assume*

$$\operatorname{depth}_J \operatorname{Hom}_A(\wedge^q(I/I^2), S_{s-r}M) \ge a+3-q \quad \textit{for all} \quad 0 \le q < a\,. \tag{3.14}$$

Then for $j = r$, and more generally for every integer j such that $r-s-1 \le j \le r-s+c$ we get

$$\operatorname{Ext}^a_R(S_jM, S_{s-r+j}M) \cong \operatorname{Hom}_A(\wedge^a(I/I^2), S_{s-r}M)\,.$$

Remark 3.11. The inequality (3.14) for $q = 0$ just means $\operatorname{depth}_J S_{s-r}M \ge a+3$, whence $\operatorname{depth}_J A \ge a+3$. It follows that (3.13) holds for $p \le a+1$ provided $-1 \le s \le c$.

Proof. Using (3.13) and the hypothesis (3.14) which yields

$$\operatorname{H}^{a+1-q}(T, \mathcal{H}om(\wedge^q(\mathcal{I}/\mathcal{I}^2), \widetilde{S_{s-r}M})) = \operatorname{H}^{a+2-q}_J(\operatorname{Hom}(\wedge^q(I/I^2), S_{s-r}M)) = 0\,, \tag{3.15}$$

we get $E_2^{p,q} := \operatorname{Ext}^p_A(\operatorname{Tor}^R_q(S_rM, A), S_sM) = 0$ for $p+q = a+1, \ 0 \le q < a$. In the same way

$$\operatorname{H}^{a-q}(T, \mathcal{H}om(\wedge^q(\mathcal{I}/\mathcal{I}^2), \widetilde{S_{s-r}M})) = 0$$

implies that $E_2^{p,q} = 0$ for $p+q = a, \ 0 \le q < a$. Hence all terms $E_2^{p,q}$ with $p+q = a$ vanish except $E_2^{0,a}$and we get

$$\operatorname{Ext}^a_R(S_rM, S_sM) \cong E_\infty^{0,a}.$$

We claim that (3.15) implies $E_\infty^{0,a} \cong E_2^{0,a}$. Indeed, by (3.15), $E_\mu^{p,q} = 0$ for any $\mu \ge 2$ and (p,q) satisfying $p+q = a+1$, $0 \le q < a$. Therefore, the differentials of the spectral sequence

$$d_{\mu,1-\mu} : E_\mu^{0,a} \longrightarrow E_\mu^{\mu,a+1-\mu} \quad , \ \mu \ge 2$$

vanish for $\mu \ge 2$ because $E_\mu^{\mu,a+1-\mu} = 0$ for $\mu \ge 2$. It follows that

$$\operatorname{Ext}^a_R(S_rM, S_sM) \cong E_2^{0,a} \cong \operatorname{Hom}_A(\wedge^a(I/I^2), S_{s-r}M) \tag{3.16}$$

where the isomorphism to the right follows from $\operatorname{depth}_J S_{s-r}M \ge 2$. Finally since the arguments above apply similarly to the spectral sequence ${}'E_2^{p,q} := \operatorname{Ext}^p_A(\operatorname{Tor}^R_q(S_jM, A), S_{s-r+j}M)$ as they did for $E_2^{p,q}$ we are done. □

Example 3.12. Suppose $\operatorname{depth}_J A \geq 5$ and take $a = r = 2$. Then (3.14) is satisfied for $s = 2$; the case $q = 0$ by hypothesis and the case $q = 1$ follows from [27]; Theorem 5.11. Therefore, we get

$$\operatorname{Ext}^2_R(S_2M, S_2M) \cong \operatorname{Hom}_A(\wedge^2 I/I^2, A). \tag{3.17}$$

If $s = c$ then (3.14) is satisfied; the case $q = 1$ follows from Theorem 3.2 and Proposition 3.5, whence

$$\operatorname{Ext}^2_R(S_2M, S_cM) \cong \operatorname{Hom}_A(\wedge^2 I/I^2, S_{c-2}M).$$

Lemma 3.13. *Let r, s and k be integers such that $r-1 \leq s \leq r-1+c$ and $1 \leq k \leq \min\{r-s+c, c\}$, and suppose $\operatorname{depth}_J A \geq 2$. Moreover for every i, j satisfying $1 \leq i < k$, $i \leq j \leq k$ we assume that*

$$\operatorname{Ext}^i_R(S_kM, S_{s-r+k}M) \cong \operatorname{Ext}^i_R(S_jM, S_{s-r+j}M)\,.$$

Then we have

$$\operatorname{pd} \operatorname{Ext}^k_R(S_kM, S_{s-r+k}M) \leq \operatorname{pd} A + 2k\,. \tag{3.18}$$

Proof. We proceed by induction on k. The case $k = 1$ follows from Corollary 3.8. To prove it for $k > 1$, we will use that

$$\operatorname{Ext}^i_R(S_iM, S_{s-r+i}M) \cong \operatorname{Ext}^i_R(S_kM, S_\sigma M)\,, \quad \sigma := s - r + k\,,$$

for $i < k$ by assumption. By Proposition 2.3, we have a resolution of length c of S_kM of the following type:

$$\to L_{k+1} \xrightarrow{\epsilon} L_k = \wedge^k F \to L_{k-1} = G \otimes \wedge^{k-1} F \to \cdots \to L_1 = S_{k-1}G \otimes F \to L_0 = S_kG \to S_kM \to 0$$

where ϵ is the "splice map". Applying the contravariant functor $\operatorname{Hom}_R(-, S_\sigma M)$, we get a complex

$$0 \to S_\sigma M \otimes L_0^* \xrightarrow{\delta_0} S_\sigma M \otimes L_1^* \xrightarrow{\delta_1} \cdots \to S_\sigma M \otimes L_{k-1}^* \xrightarrow{\delta_{k-1}} S_\sigma M \otimes L_k^* \xrightarrow{\epsilon^*} S_\sigma M \otimes L_{k+1}^* \tag{3.19}$$

where $\ker(\delta_0) = \operatorname{Hom}(S_kM, S_\sigma M) \cong S_{\sigma-k}M$, $\epsilon^* = \operatorname{Hom}(\epsilon, S_\sigma M) = 0$, whence

$$\operatorname{Ext}^k_R(S_kM, S_\sigma M) = S_\sigma M \otimes L_k^* / \operatorname{im}(\delta_{k-1}).$$

Due to the horseshoe lemma and the exact sequence

$$0 \longrightarrow \operatorname{im}(\delta_{i-1}) \longrightarrow \ker(\delta_i) \longrightarrow \operatorname{Ext}^i_R(S_kM, S_\sigma M) \longrightarrow 0$$

we get the implication

$$\operatorname{pd} \operatorname{im}(\delta_{i-1}) \leq \operatorname{pd} A + 2i - 1 \;\Rightarrow\; \operatorname{pd} \ker(\delta_i) \leq \operatorname{pd} A + 2i$$

by the induction hypothesis. The exact sequence

$$0 \longrightarrow \ker(\delta_i) \longrightarrow S_\sigma M \otimes L_i^* \longrightarrow \operatorname{im}(\delta_i) \longrightarrow 0$$

shows the implication

$$\operatorname{pd} \ker(\delta_i) \leq \operatorname{pd} A + 2i \;\Rightarrow\; \operatorname{pd} \operatorname{im}(\delta_i) \leq \operatorname{pd} A + 2i + 1 \quad \text{for} \quad 1 \leq i < k.$$

Since by (3.19) and $\ker(\delta_0) \cong S_{\sigma-k}M$ we have

$$\operatorname{pd} \operatorname{im}(\delta_0) \leq pdA + 1\,,$$

we conclude that $\operatorname{pd}\operatorname{im}(\delta_{k-1}) \leq \operatorname{pd} A + 2k - 1$ and using the exact sequence

$$0 \longrightarrow \operatorname{im}(\delta_{k-1}) \longrightarrow S_k(M) \otimes L_k^* \longrightarrow \operatorname{Ext}^k_R(S_kM, S_\sigma M) \longrightarrow 0$$

we get

$$\operatorname{pd}\operatorname{Ext}^k_R(S_kM, S_\sigma M) \leq \operatorname{pd} A + 2k$$

which proves the lemma. □

Remark 3.14. If $s \in \{r-1, r, r+c-2\}$, $s \leq c$, we have $\operatorname{pd}\operatorname{Ext}^k_R(S_kM, S_{s-r+k}M) \leq \operatorname{pd} A + 1$ for $k = 1$ by Remark 3.4. Using the proof above, we show $\operatorname{pd}\ker(\delta_1) \leq pdA + 1$, whence we can improve upon the conclusion for such s and we get

$$\operatorname{pd}\operatorname{Ext}^k_R(S_kM, S_{s-r+k}M) \leq \operatorname{pd} A + 2k - 1 \quad \text{for } 1 \leq k \leq \min\{r-s+c, c\}.$$

Theorem 3.15. *Let $X \subset \mathbb{P}^n$ be a standard determinantal scheme of codimension $c \geq 2$ associated to a $t \times (t+c-1)$ matrix $\mathcal{A}$. Let $I = I_t(\mathcal{A})$, $J = I_{t-1}(\mathcal{A})$ and $A = R/I$. Fix integers $s \geq -1$, $r-1 \leq s \leq c$, and assume* $\operatorname{depth}_J A \geq 2a+2$ *(resp.* $\operatorname{depth}_J A \geq 2a+1$ *if $s \in \{r-1, r, r+c-2\}$ and $a > 1$). Then, for $0 \leq a \leq r-s+c$, we have*

$$\operatorname{pd}_R \operatorname{Ext}^a_R(S_rM, S_sM) \leq \operatorname{pd}_R A + 2a \ ,$$

$$\operatorname{Ext}^a_R(S_rM, S_sM) \cong \operatorname{Hom}_A(\wedge^a(I/I^2), S_{s-r}M) \ , \quad \textit{and}$$

$$\operatorname{Ext}^a_R(S_jM, S_{s-r+j}M) \cong \operatorname{Hom}_A(\wedge^a(I/I^2), S_{s-r}M) \text{ for } r-s-1 \leq j \leq \min\{r-s+c, c\} \, .$$

Remark 3.16. (i) Since we in general have $\operatorname{depth}_J A \leq c+2$ and the assumption $\operatorname{depth}_J A \geq 2a+2$ (resp. $\operatorname{depth}_J A \geq 2a+1$) in Theorem 3.15, we get that a satisfies $a \leq \frac{c}{2}$ (resp. $a \leq \frac{c+1}{2}$).

(ii) It is worthwhile to point out that Theorem 3.15 with $r = s = 0$ holds more generally in the licci case (see [24]; Proposition 13 and Remark 14).

Proof. Let q be an integer such that $0 \leq q \leq a$ where $a \leq r-s+c$ (whence $s-r \leq c$). Then we will prove

$$\operatorname{depth}\operatorname{Ext}^q_R(S_aM, S_{s-r+a}M) \geq \dim A - 2q \ , \quad \text{and} \tag{3.20}$$

$$\operatorname{Ext}^q_R(S_jM, S_{s-r+j}M) \cong \operatorname{Hom}_A(\wedge^q(I/I^2), S_{s-r}M) \text{ for } r-s-1 \leq j \leq \min\{r-s+c, c\} \tag{3.21}$$

by induction on q. Note that (3.20) is equivalent to $\operatorname{pd}\operatorname{Ext}^q_R(S_aM, S_{s-r+a}M) \leq \operatorname{pd} A + 2q$ by Auslander-Buchsbaum's formula and that it implies $\operatorname{depth}_J \operatorname{Ext}^q_R(S_aM, S_{s-r+a}M) \geq \operatorname{depth}_J A - 2q$.

Since $S_{s-r}M$ is maximally Cohen-Macaulay, (3.20) and (3.21) holds for $q = 0$. Suppose both formulas hold for every non-negative $q < k$ for some positive $k \leq a$. It follows that

$$\operatorname{depth}_J \operatorname{Hom}_A(\wedge^q(I/I^2), S_{s-r}M) \geq \operatorname{depth}_J A - 2q \geq 2k + 2 - 2q \geq k + 3 - q; \tag{3.22}$$

and applying Lemma 3.10 (with k instead of a) we get (3.21) for $q = k$. Since we now have (3.21) for $q = k$ and $j \in \{k, a\}$ we get (3.20) for $q = k$ by Lemma 3.13, which completes the induction. Moreover since we can take $j = r$ in (3.21), we get the theorem for $s \notin \{r-1, r, r+c-2\}$.

Finally if $s \in \{r-1, r, r+c-2\}$ we use Remark 3.14 to improve upon (3.20) and (3.21) for $q > 0$. Using $\operatorname{depth}_J A \geq 2a+1$ we still get (3.22), and we are done. □

Corollary 3.17. *Let $s \geq -1$ and $a \geq 0$ be integer and suppose $r-1 \leq s \leq c$ and* $\operatorname{depth}_J A \geq 2a+2$ *(resp.* $\operatorname{depth}_J A \geq 2a+1$ *if $s \in \{r-1, r, r+c-2\}$ and $a > 1$). Then we have*

$$\operatorname{Ext}^i_R(S_r M, S_s M) \cong \operatorname{Ext}^i_R(A, S_{s-r} M) \quad \text{for every } i \leq \min\{a, r-s+c\}\,.$$

In particular, for $i \leq \min\{a, \frac{c+1}{2}\}$, we have

$$\operatorname{Ext}^i_R(S_i M, S_{c-i} M) \cong \operatorname{Ext}^i_R(A, S_{c-2i} M) \cong \operatorname{Ext}^i_R(S_e M, S_{c-2i+e} M) \quad \textit{for} \quad 0 \leq e \leq 2i.$$

Proof. Straightforward to verify using the two last formulas of Theorem 3.15. □

Remark 3.18. If we take a general determinantal scheme $X = \operatorname{Proj}(A)$ one knows that $\operatorname{depth}_J A = c+2$. Comparing this with $\operatorname{depth}_J A \geq 2a+2$ of Corollary 3.17 we get the isomorphisms of the Ext^i_R-groups of the corollary under the assumption $i \leq \frac{c}{2}$ (if $s-r \leq \frac{c}{2}$). Thus Corollary 3.17 corrects with a complete proof the previous version on the arXiv, as well as the published version in Crelle's journal, whose corresponding result (Proposition 3.9) only assumes $i \leq c$ (after renaming letters in Proposition 3.9). The problem with the proof there is that a morphism between the spectral sequences in the proof of Proposition 3.9 is not established. The proof of the Lemma's 3.10, 3.13 and Theorem 3.15 are mainly the same as those in Crelle's journal, and their assumptions imply that the spectral sequences degenerate on the E_2 level (see (3.16)). Note that the assumptions in Lemma 3.10 (resp. Theorem 3.15) are exactly (resp. added $s \geq -1$) as in previous versions/Crelle'journal, while we in Lemma 3.13 have introduced an assumption on certain Ext^i_R-groups which we obviously can replace by the assumptions in Theorem 3.15 if $k \leq a$. We only know counterexamples to Proposition 3.9 when $i = c$, and in fact we expect Corollary 3.17 to be true for $i \leq c-1$ and X general without assuming $\operatorname{depth}_J A \geq 2a+2$ (resp. $\geq 2a+1$), see Remark 3.22 for a follow-up. The inaccuracy in Proposition 3.9 implies that we had to weaken Remark 3.6 (Remark 3.10 in previous versions) to show this result. Moreover an assumption of generality is included in Theorem 4.14.

Corollary 3.19. *Let $i \geq 1$, $s \geq -1$ and a be integers satisfying $0 \leq a \leq c-s$.*

(i) *Suppose* $\operatorname{depth}_J A \geq 2a+2+i$ *(resp.* $\operatorname{depth}_J A \geq 2a+1+i$ *if $s \in \{-1, 0, c-2\}$ and $a > 0$). Then, for $1 \leq j \leq i$, we have*

$$\operatorname{Ext}^j_A(\wedge^a(I/I^2), S_s M) = 0 \quad \textit{and} \quad \operatorname{H}^j_*(X \setminus V(J), (\wedge^a \mathcal{I}_X/\mathcal{I}_X^2)^\vee \otimes \widetilde{S_s M}) = 0.$$

(ii) *Suppose* $\dim A \geq 2a+2+i$ *(resp. $\geq 2a+1+i$ if $s \in \{-1, 0, c-2\}$ and $a > 0$). Then we have*

$$\operatorname{H}^j_*(X, \mathcal{H}om(\wedge^a(\mathcal{I}_X/\mathcal{I}_X^2), \widetilde{S_s M})) = 0 \quad \text{for} \quad 1 \leq j \leq i.$$

Proof. (i) Taking $r = 0$ in Theorem 3.15, we get

$$\operatorname{Ext}^a_R(A, S_s M) \cong \operatorname{Hom}_A(\wedge^a(I/I^2), S_s M)$$

and applying Lemma 3.13 together with Auslander-Buchsbaum's formula, as in (3.20), we have

$$\operatorname{depth}_J \operatorname{Hom}_A(\wedge^a(I/I^2), S_s M) \geq 2+i. \tag{3.23}$$

Since $\wedge^a(\mathcal{I}_X/\mathcal{I}_X^2)$ is locally free on $X \setminus V(J)$, we apply (2.2) and, for $1 \le j \le i$, we get

$$\operatorname{Ext}^j_A(\wedge^a(I/I^2), S_sM) = 0 \quad \text{and} \quad \operatorname{H}^j_*(X \setminus V(J), \mathcal{H}om(\wedge^a(\mathcal{I}_X/\mathcal{I}_X^2), \widetilde{S_sM})) = 0.$$

Suppose $s \in \{-1, 0, c-2\}$. Then, for $a = 1$, we may use Remark 3.14 to improve upon (3.20) by 1 and since we still get (3.23), we conclude by the arguments above. If $a \ge 2$ then Theorem 3.15 applies, and using Remark 3.14 to improve upon (3.20), we get (3.23) and we conclude as previously.

(ii) The argument using the projective dimension and Auslander-Buchsbaum's formula also imply (3.23) with $\mathfrak{m}$ instead of J, whence we get $\operatorname{H}^{j+1}_{\mathfrak{m}}(\operatorname{Hom}_A(\wedge^a(I/I^2), S_sM)) = 0$ and we are done. □

The case $a = 1$, $i = 1$ and $s = -1$ of Corollary 3.19 is of special interest because it shows the vanishing of a group that we tried very much to compute in [28]. Indeed, we have

Corollary 3.20. *Let $J = I_{t-1}(\mathcal{A})$ and $A = R/I$ and suppose* $\operatorname{depth}_J A \ge 4$. *Then*

$$\operatorname{Ext}^1_A(I/I^2, \operatorname{Hom}(M, A)) = 0.$$

In Theorem 5.1 of [28] we repeatedly need to apply Corollary 3.20 to standard determinantal schemes obtained by deleting at least one column. Indeed, Corollary 3.20 shows that all assumptions of Theorem 5.1 (ii) are satisfied in the case $\dim X \ge 2$, $a_j \ge b_i$ for any i, j. It follows that the closure of the locus of determinantal schemes W inside the Hilbert scheme, is a generically smooth component of the Hilbert scheme (if $W \neq \emptyset$)! Thus Corollary 3.20 partially reproves Corollary 5.9 of [27]. In particular, we also get Conjecture 4.2 of [29] from Theorem 5.1 of [28] and Corollary 3.20.

Since the case $a = 1$ and $s = c-1$ is related to the dual of the conormal module of A, we remark

Corollary 3.21. *Let $J = I_{t-1}(\mathcal{A})$, $A = R/I$, $X = \operatorname{Proj}(A)$ and let K_A be the canonical module of A. If* $\operatorname{depth}_J A \ge 4 + i$, *then*

$$\operatorname{H}^k_{\mathfrak{m}}(I/I^2) = 0 \quad \text{for} \quad \dim X - i < k \le \dim X,$$

or equivalently ;

$$\operatorname{Ext}^j_A(I/I^2, K_A) = 0 \quad \text{for} \quad 1 \le j \le i.$$

Proof. This is immediate from Corollary 3.19 (i) since $K_A(v) \cong S_{c-1}(M)$ for some integer v by Proposition 2.3 (ii). For the equivalent statement, we use Gorenstein duality. □

Remark 3.22. Here we try to extend Corollary 3.17 to the range $\frac{c}{2} \le i \le c-1$, as well as to show

$$\operatorname{Ext}^i_R(S_rM, S_sM) \cong \operatorname{Hom}_A(\wedge^i(I/I^2) \otimes S_rM, S_sM) \quad \text{when} \quad -1 \le s \le c \tag{3.24}$$

for such i. Note that since S_sM is maximally Cohen-Macaulay as an A-module, the Hom-group has $\operatorname{depth}_J \ge 2$, and it is further isomorphic to $\operatorname{Hom}_A(\wedge^i(I/I^2), S_{s-r}M)$ if $-1 \le s - r \le c$.

(i) Conversely if $N := \operatorname{Ext}^i_R(S_rM, S_sM)$ satisfies $\operatorname{depth}_J N \ge 2$, then (3.24) holds. In fact if $T := \operatorname{Spec}(A) \setminus V(J)$, the depth assumption implies $N \cong H^0(T, \widetilde{N})$. Then (3.24) follows easily from the local version of the spectral sequence (3.10) which degenerates due to (3.12).

(ii) One may express (3.24) using $\operatorname{Tor}^R_{c-i}$-groups. Indeed note that the resolutions of S_kM and $S_{c-1-k}M$ given in Proposition 2.3 are R-dual to each other (up to twist by $\ell := \sum_{j=1}^{t+c-1} a_j - \sum_{i=1}^{t} b_i$). Combining with the fact that the homology groups of (3.19) may be interpreted as $\operatorname{Ext}^i_R(S_kM, S_\sigma M)$ as well as $\operatorname{Tor}^R_{c-i}(L, S_\sigma M)$ where L is determined by the R-dual resolution of S_kM, we get:

$$\operatorname{Ext}^i_R(S_kM, S_\sigma M) \cong \operatorname{Tor}^R_{c-i}(S_{c-1-k}M, S_\sigma M)(\ell) \ \text{ for } \ -1 \le k, \sigma \le c\ .$$

Indeed letting $j = c - i$ and $h = c - 1 - r$ we have

$$\operatorname{Ext}^i_R(S_rM, S_sM) \cong \operatorname{Ext}^i_R(S_{r+1}M, S_{s+1}M) \Longleftrightarrow \operatorname{Tor}^R_j(S_hM, S_sM) \cong \operatorname{Tor}^R_j(S_{h-1}M, S_{s+1}M)\,.$$

In particular Corollary 3.17 leads to a corresponding result for Tor^R_j-groups where $\frac{c}{2} \le j \le c$.

(iii) Using (ii) and $\operatorname{Tor}^R_j(L_1, L_2) \cong \operatorname{Tor}^R_j(L_2, L_1)$ we get for every i, $0 \le i \le c$, the isomorphism

$$\operatorname{Ext}^i_R(S_rM, S_sM) \cong \operatorname{Ext}^i_R(S_{c-1-s}M, S_{c-1-r}M) \quad \text{for} \quad -1 \le r, s \le c. \tag{3.25}$$

(iv) Exactly at the spot where the splice map in the complex (3.19) occurs, we do not only have $\operatorname{Ext}^k_R(S_kM, S_\sigma M)$ as a certain cokernel, but also $\operatorname{Ext}^{k+1}_R(S_kM, S_\sigma M)$ as the kernel of $S_\sigma M \otimes L^*_{k+1} \to S_\sigma M \otimes L^*_{k+2}$. Since $H^0_J(-)$ is left exact and $\operatorname{depth}_J S_\sigma M \ge 2$, we get $\operatorname{depth}_J \operatorname{Ext}^{k+1}_R(S_kM, S_\sigma M) \ge 2$ for $k < c$ (for $k = c$, $\operatorname{Ext}^{k+1}_R(S_kM, S_\sigma M) = 0$), thus (3.24) holds for $i = r + 1$, $-1 \le r, s \le c$ by (i). Using also (ii) it follows that both groups of (3.25) for $i = r + 1$ are further isomorphic to

$$\operatorname{Tor}^R_j(S_jM, S_sM)(\ell) \cong \operatorname{Hom}_A(\wedge^{r+1}(I/I^2) \otimes S_rM, S_sM)\,,\ j = c - 1 - r \text{ for } -1 \le r, s \le c\,.$$

(v) So the bottom line for (3.24) to hold is to show that $\operatorname{depth}_J \operatorname{Ext}^i_R(S_rM, S_sM) \ge 2$. Let us point out one more interesting case where this holds, namely the case $i = c - 1 = r - s$ (i.e. $(r, s) = (c, 1)$, $(c-1, 0)$ or $(c-2, -1)$). Using (ii) we have $\operatorname{Ext}^{c-1}_R(S_rM, S_sM) \cong \operatorname{Tor}^R_1(S_hM, S_sM)(\ell)$ with $h + s = 0$. Letting $(h, s) = (1, -1)$ or $(-1, 1)$ we get two Tor_1-groups that obviously are isomorphic and (iv) applies to get $\operatorname{Tor}^R_1(S_hM, S_sM)(\ell) \cong \operatorname{Hom}_A(\wedge^{c-1}(I/I^2) \otimes S_{c-1}M, A)$. If $(h, s) = (0, 0)$, the Tor_1-group is isomorphic to I/I^2, the conormal module. Its depth is found in the next section (Proposition 4.1), and the assumptions $n \ge 2c$, A general and $a_j > b_i$ suffice for having $\operatorname{depth}_J I/I^2 \ge 2$. Thus for $r - s = c - 1$, $n \ge 2c$, A general and $a_j > b_i$ for all i, j, we conclude that

$$\operatorname{Ext}^{c-1}_R(S_rM, S_sM) \cong \operatorname{Hom}_A(\wedge^{c-1}(I/I^2) \otimes S_{c-1}M, A) \cong I/I^2(\ell)\ .$$

As another application of Theorem 3.15, we can partially restate Conjecture 3.9. Indeed,

Conjecture 3.23. *Let $X \subset \mathbb{P}^n$ be a linear standard determinantal scheme of codimension $c \ge 2$ associated to a $t \times (t + c - 1)$ matrix $\mathcal{A}$. Set $I = I_t(\mathcal{A})$ and $J = I_{t-1}(\mathcal{A})$. For any integer a, $0 \le a \le \frac{c+1}{2}$ we conjecture that* $\operatorname{Hom}(\wedge^a(I/I^2), S_{c-2a}M)$ *is an (indecomposable) Ulrich R/I-module of rank $\binom{c}{a}$ provided* $\operatorname{depth}_J A \ge 2a + 2$.

Notice that under the above hypothesis Conjectures 3.9 and 3.23 are equivalent since, by Theorem 3.15, we have

$$\operatorname{Ext}^a_R(S_aM, S_{c-a}M) \cong \operatorname{Hom}_A(\wedge^a(I/I^2), S_{c-2a}M) \ \text{ for } 1 \le a \le \frac{c+1}{2}.$$

We want to point out that for $a = 0$, the conjecture was proved in [7]; Proposition 2.8 and in this section, we prove it for $a = 1$ (and, indecomposability for $n \geq 2c + 1$ in the next section).

4. The indecomposability of the normal sheaf of a determinantal variety

In this section we address Problem 1.1(3) and we determine conditions under which the normal sheaf $\mathcal{N}_X$, and more generally the "twisted" normal sheaves $\mathcal{N}_X(\mathcal{M}^i) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \widetilde{S_iM})$, $-1 \leq i \leq c-1$, of a standard determinantal scheme $X \subset \mathbb{P}^n$ of codimension c are simple (i.e. $\operatorname{Hom}(\mathcal{N}_X(\mathcal{M}^i), \mathcal{N}_X(\mathcal{M}^i)) \cong K$) and, hence, indecomposable (cf. Theorems 4.3 and 4.14). Obviously $\mathcal{N}_X(\mathcal{M}^0) = \mathcal{N}_X$. We keep the notation of previous sections. So, $\mathcal{A}$ will be the $t \times (t+c-1)$ matrix associated to the standard determinantal scheme $X \subset \mathbb{P}^n$, $I = I_t(\mathcal{A})$, $A = R/I$ and

$$\cdots \longrightarrow \oplus_j R(-n_{2j}) \longrightarrow \oplus_i R(-n_{1i}) \longrightarrow R \longrightarrow A \longrightarrow 0 \tag{4.1}$$

the Eagon-Northcott resolution of A. In this section we often have $d_{i,j} := a_j - b_i > 0$ for all i, j.

Again we will assume $c > 1$, since the case $c = 1$ corresponds to a hypersurface $X \subset \mathbb{P}^n$ and $\mathcal{N}_X \cong \mathcal{O}_X(\delta)$, $\delta := \deg(X)$, is simple. We will also assume $t > 1$, since the case $t = 1$ corresponds to a codimension c complete intersection $X \subset \mathbb{P}^n$ and $\mathcal{N}_X(\mathcal{M}^i)$ and $\mathcal{N}_X = \oplus_{i=1}^c \mathcal{O}_X(d_i)$, $d_i \in \mathbb{Z}$, are neither simple nor indecomposable. Let us start computing the depth of the conormal bundle of a standard determinantal scheme. We have:

Proposition 4.1. *Let $\mathcal{A}$ be a $t \times (t+c-1)$ homogeneous matrix with entries that are general forms of degree $d_{ij} > 0$, let $I = I_t(\mathcal{A})$, $J = I_{t-1}(\mathcal{A})$ and $X \subset \mathbb{P}^n$ be the standard determinantal scheme of codimension c associated to $\mathcal{A}$. Assume that $n \geq 2c - 2$. Then, it holds:*

1. $\operatorname{depth}_{\mathfrak{m}} I/I^2 = n - 2c + 2$.
2. $\operatorname{depth}_J I/I^2 \geq 3$ *(resp. $= n - 2c + 2$) provided $n \geq 2c + 1$ (resp. $n \leq 2c$).*

Therefore, if $n \geq 2c$ there is a closed subset $Z \subset X$ such that $X \setminus Z \hookrightarrow \mathbb{P}^n$ is a local complete intersection and $\operatorname{depth}_{I(Z)} I/I^2 \geq 2$.

Proof. (1) We will first prove Proposition 4.1 for an ideal generated by the maximal minors of a matrix with entries that are indeterminates and we will deduce that it also works for a homogeneous matrix with entries of general homogeneous polynomials.

So, we first assume that $n = t(t+c-1) - 1$ and $\mathcal{A}$ is a matrix with entries $x_0, x_1, \cdots, x_n$ of indeterminates. Set $R = k[x_0, \cdots, x_n]$, $I = I_t(\mathcal{A})$ and call Ω the module of differentials of R/I over k. By [6]; Theorem 14.12

$$\operatorname{depth}_{\mathfrak{m}} \Omega = \operatorname{depth}_{\mathfrak{m}} R/I - \operatorname{depth}_J R/I + 2.$$

and note that $\operatorname{depth}_J R/I = \operatorname{codim}(I_{t-1}(\mathcal{A}), R/I)$. Since,

$$\operatorname{depth}_{\mathfrak{m}} R/I = \dim R/I = n - c + 1, \text{ and}$$

$$\operatorname{codim}(I_{t-1}(\mathcal{A}), R/I) = \operatorname{codim}(I_{t-1}(\mathcal{A})) - \operatorname{codim}(I_t(\mathcal{A})) = 2(c+1) - c = c + 2,$$

we get

$$\operatorname{depth}_{\mathfrak{m}} \Omega = (n - c + 1) - (c + 2) + 2 = t(t + c - 1) - 2c.$$

Therefore, using the exact sequence

$$0 \longrightarrow I/I^{(2)} \longrightarrow (R/I)^{t(t+c-1)} \longrightarrow \Omega \longrightarrow 0,$$

we deduce that

$$\begin{aligned} \operatorname{depth}_{\mathfrak{m}} I/I^{(2)} &= \operatorname{depth}_{\mathfrak{m}} \Omega + 1 \\ &= t(t+c-1) - 2c + 1 \\ &= n - 2c + 2. \end{aligned}$$

By [6]; Corollary 9.18, we have $I^2 = I^{(2)}$. Therefore, we conclude that

$$\operatorname{depth}_{\mathfrak{m}} I/I^2 = n - 2c + 2.$$

Let us now assume $n < t(t+c-1) - 1$. We distinguish two cases:

Case 1. Assume $d_{ij} = 1$ for all i, j (i.e. the entries of the matrix $\mathcal{A}$ are linear forms). We have a $t \times (t+c-1)$ matrix $\overline{\mathcal{A}} = (x_{i,j})$ of indeterminates and the ideal $I_t(\overline{\mathcal{A}}) \subset S := k[x_{i,j}]$ which verifies

$$\operatorname{depth}_{\mathfrak{m}_S} I_t(\overline{\mathcal{A}})/I_t(\overline{\mathcal{A}})^2 = t(t+c-1) - 2c + 1.$$

We choose $t(t+c-1) - n - 1$ general linear forms $\ell_1, \cdots, \ell_{t(t+c-1)-n-1} \in S = k[x_{i,j}]$ and we set $S/(\ell_1, \cdots, \ell_{t(t+c-1)-n-1}) \cong k[x_0, x_1, \cdots, x_n] =: R$. Let us call $I \subset R$ the ideal of R isomorphic to the ideal $I_t(\overline{\mathcal{A}})/(\ell_1, \cdots, \ell_{t(t+c-1)-n-1})$ of $S/(\ell_1, \cdots, \ell_{t(t+c-1)-n-1})$. I is nothing but the ideal $I_t(\mathcal{A})$ where $\mathcal{A} = (m_{i,j})$ is a $t \times (t+c-1)$ homogeneous matrix with entries that are linear forms in $k[x_0, x_1, \cdots, x_n]$ obtained from $\overline{\mathcal{A}} = (x_{i,j})$ by substituting using the equations $\ell_1, \cdots, \ell_{t(t+c-1)-n-1}$. Since $\operatorname{depth}_{\mathfrak{m}_S} I_t(\overline{\mathcal{A}})/I_t(\overline{\mathcal{A}})^2 = t(t+c-1) - 2c + 1$ and $t(t+c-1) - n - 1 \leq t(t+c-1) - 2c + 1$, we can assume that $\ell_1, \cdots, \ell_{t(t+c-1)-n-1}$ is a regular sequence on both $I_t(\overline{\mathcal{A}})/I_t(\overline{\mathcal{A}})^2$ and $S/I_t(\overline{\mathcal{A}})$; and we conclude that

$$\operatorname{depth}_{\mathfrak{m}_R} I/I^2 = \operatorname{depth}_{\mathfrak{m}_S} I_t(\overline{\mathcal{A}})/I_t(\overline{\mathcal{A}})^2 - (t(t+c-1) - n - 1) = n - 2c + 2.$$

Case 2. Since $d_{i,j} > 0$ for all i, j, it is enough to raise the entry $m_{i,j}$ of the above matrix $\mathcal{A}$ to the power $d_{i,j}$.

(2) It follows from (1) and $\operatorname{depth}_J I/I^2 \geq \operatorname{depth}_J A - (\dim A - \operatorname{depth}_{\mathfrak{m}} I/I^2)$, cf. [26]; Lemma 7 because $\operatorname{depth}_J A = c + 2$ (resp. $n + 1 - c$) for $n \geq 2c + 1$ (resp. $n \leq 2c$). □

As an immediate and nice consequence of the above result, we get for $n \geq 2c + 1$ that the cohomology of the conormal bundle $\operatorname{H}^j_{\mathfrak{m}}(I/I^2)$ is non-zero for only one value of $j \neq \dim R/I$, cf. [1] for $c = 2$. Analogous result for the normal bundle was proved by Kleppe in [27]; Theorem 5.11.

Corollary 4.2. *Let $\mathcal{A}$ be a $t \times (t+c-1)$ homogeneous matrix with entries that are general forms of positive degree. Set $J = I_{t-1}(\mathcal{A})$, $A = R/I$ and let K_A be the canonical module of A. We have:*

(i) $\operatorname{H}^k_{\mathfrak{m}}(I/I^2) = 0$ *for* $k < n - 2c + 2$ *or, equivalently,* $\operatorname{Ext}^j_A(I/I^2, K_A) = 0$ *for* $c \leq j \leq n - c + 1$.

(ii) $\operatorname{H}^k_{\mathfrak{m}}(I/I^2) = 0$ *for* $max(3, n - 2c + 2) < k < n - c + 1$.

Proof. (i) It follows from Proposition 4.1 and Gorenstein duality.

(ii) Since $\operatorname{depth}_J A = min(c+2, n-c+1)$, we can apply Corollary 3.21 and we get $\operatorname{H}^k_{\mathfrak{m}}(I/I^2) = 0$ for $n - 2c + 2 < k \leq \dim X = n - c$. Combining with (i) we get what we want. □

Theorem 4.3. *Let $X \subset \mathbb{P}^n$ be a standard determinantal scheme of codimension $c \geq 2$ associated to a $t \times (t+c-1)$ matrix $\mathcal{A}$ with entries that are general forms of positive degree. Let $\mathcal{N}_X(\mathcal{M}^i) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \widetilde{S_iM})$ for $-1 \leq i \leq c-1$ and assume $n \geq 2c$. Then,*

$$\operatorname{Hom}(\mathcal{N}_X(\mathcal{M}^i), \mathcal{N}_X(\mathcal{M}^i)) \cong {}_0\operatorname{Hom}_A(I/I^2, I/I^2) \cong K$$

provided $max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}$. In particular, $\mathcal{N}_X$ and $\mathcal{N}_X(\mathcal{M}^i)$ are simple, and thus indecomposable.

Proof. First of all we observe that $\operatorname{Hom}_R(I, I) \cong R$. Indeed, we have the following diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \operatorname{Hom}_R(R,R) = R & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \operatorname{Hom}_R(I,I) & \longrightarrow & \operatorname{Hom}_R(I,R) & \longrightarrow & \operatorname{Hom}_R(I,A) & \longrightarrow & \operatorname{Ext}^1_R(I,I) \\ & & & & \downarrow & & & & \\ & & & & \operatorname{Ext}^1_R(A,R) = 0 & & & & \end{array}$$

Since $c \geq 2$, $\operatorname{Ext}^1_R(A, R) = 0$ and $R \cong \operatorname{Hom}_R(R, R) \cong \operatorname{Hom}_R(I, R)$. Hence, it follows that $\operatorname{Hom}_R(I, I)$ is an ideal of $\operatorname{Hom}_R(I, R) = R$ containing the identity; and so $\operatorname{Hom}_R(I, I) \cong R$.

Claim: If $max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}$, then $K \cong {}_0\operatorname{Hom}_R(I, I) \cong {}_0\operatorname{Hom}_A(I/I^2, I/I^2)$.

Proof of the Claim: We apply $\operatorname{Hom}_R(-, I)$ and $\operatorname{Hom}_R(-, I/I^2)$ to the minimal resolution of I deduced from (4.1) and we get the following commutative diagram with exact horizontal sequences

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \operatorname{Hom}_R(I, I/I^2) & \longrightarrow & \oplus_i I/I^2(n_{1i}) & \longrightarrow & \oplus_j I/I^2(n_{2j}) & \longrightarrow & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \operatorname{Hom}_R(I, I) & \longrightarrow & \oplus_i I(n_{1i}) & \longrightarrow & \oplus_j I(n_{2j}) & \longrightarrow & \\ & & & & \uparrow & & \uparrow & & \\ & & & & \oplus_i I^2(n_{1i}) & \longrightarrow & \oplus_j I^2(n_{2j}) & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Since $\operatorname{Hom}_R(I, I/I^2) \cong \operatorname{Hom}_A(I/I^2, I/I^2)$ and ${}_0\operatorname{Hom}_R(I, I^2) = 0$, it suffices to show that $(I^2(n_{2j}))_0 = 0$ for all j. Using the natural surjective map $S_2I \twoheadrightarrow I^2$, it suffices to show that $(S_2I)_\mu = 0$ for $\mu := max\{n_{2j}\}$. But $(S_2I)_\mu = 0$ because we have a surjective map $\oplus_{i \leq j} R(-n_{1i} - n_{1j})_\mu \cong S_2(\oplus_i R(-n_{1i}))_\mu \twoheadrightarrow (S_2I)_\mu$ and $\oplus_{i \leq j} R(-n_{1i} - n_{1j})_\mu = 0$ by the assumption $max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}$. Hence, the claim is proved.

Let us now prove that $\mathcal{N}_X(\mathcal{M}^i)$ is simple, i.e. $\operatorname{Hom}(\mathcal{N}_X(\mathcal{M}^i), \mathcal{N}_X(\mathcal{M}^i)) \cong {}_0\operatorname{Hom}_A(I/I^2, I/I^2) \cong K$. Set $J = I_{t-1}(\mathcal{A})$. Since $\operatorname{depth}_J R/I = min(c+2, n+1-c) \geq 2$, we get that $\operatorname{depth}_J N_M \geq 2$

where $N_M := \operatorname{Hom}_A(I/I^2, S_iM)$. This also implies $\operatorname{depth}_J \operatorname{Hom}_A(N_M, N_M) \geq 2$, whence

$\operatorname{Hom}_A(N_M, N_M) \cong \mathrm{H}^0_*(X \setminus Z, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}_X(\mathcal{M}^i), \mathcal{N}_X(\mathcal{M}^i))) \cong \mathrm{H}^0_*(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}_X(\mathcal{M}^i), \mathcal{N}_X(\mathcal{M}^i)))$

where $Z := V(J)$. Using that $\mathcal{N}_X(\mathcal{M}^i) \cong (\mathcal{I}_X/\mathcal{I}_X^2)^\vee \otimes \widetilde{S_iM}$ is locally free on $X \setminus Z$, we have

$$\mathrm{H}^0_*(X \setminus Z, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}_X(\mathcal{M}^i), \mathcal{N}_X(\mathcal{M}^i))) = \mathrm{H}^0_*(X \setminus Z, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{I}_X/\mathcal{I}_X^2)).$$

Since $n \geq 2c$, we can apply Proposition 4.1 and we get $\operatorname{depth}_{I(Z)} I/I^2 \geq 2$ which implies that $\operatorname{depth}_{I(Z)} \operatorname{Hom}_A(I/I^2, I/I^2) \geq 2$, whence

$$\operatorname{Hom}_A(I/I^2, I/I^2) \cong \mathrm{H}^0_*(X \setminus Z, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{I}_X/\mathcal{I}_X^2)).$$

Putting altogether we obtain

$$\mathrm{H}^0_*(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}_X(\mathcal{M}^i), \mathcal{N}_X(\mathcal{M}^i))) \cong \operatorname{Hom}_A(I/I^2, I/I^2);$$

and taking the degree zero piece of these graded modules we get what we want. □

Remark 4.4. In Theorem 4.3, the hypothesis $n \geq 2c$ together with the generality of the entries can be replaced by

$$\mathrm{H}^1_{I(Z)}(I/I^2)_{n_{1i}} = \mathrm{H}^0_{I(Z)}(I/I^2)_{n_{1i}} = \mathrm{H}^0_{I(Z)}(I/I^2)_{n_{2j}} = 0 \quad \text{for any } i \text{ and } j \tag{4.2}$$

where $Z \subset X$ is a closed subset such that $X \setminus Z \hookrightarrow \mathbb{P}^n$ is a local complete intersection. Note that $I(Z) = \mathfrak{m}$ if we can take $Z = \emptyset$, e.g. if $n \leq 2c+1$ and the entries are general forms. To show it, observe that the exact cohomology sequence associated to $0 \to I^2 \to I \to I/I^2 \to 0$ gives us $(Z := V(J))$

$$\mathrm{H}^1_{I(Z)}(I/I^2)_\mu \cong \mathrm{H}^2_{I(Z)}(I^2)_\mu \cong \mathrm{H}^1(X \setminus Z, \mathcal{I}_X^2(\mu)) \quad \text{and}$$
$$\mathrm{H}^0_{I(Z)}(I/I^2)_{n_{2i}} \cong \mathrm{H}^1_{I(Z)}(I^2)_{n_{2i}} \cong \mathrm{H}^0(X \setminus Z, \mathcal{I}_X^2(n_{2i})) \ \text{ since } \ I^2(n_{2i})_0 = 0.$$

In the proof of Theorem 4.3, we show

$$\operatorname{Hom}_A(N_M, N_M) \cong \mathrm{H}^0_*(X \setminus Z, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{I}_X/\mathcal{I}_X^2))$$

and we use the hypothesis $n \geq 2c$ to get $\operatorname{depth}_{I(Z)} I/I^2 \geq 2$ and, hence,

$$\mathrm{H}^0_*(X \setminus Z, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{I}_X/\mathcal{I}_X^2)) \cong \operatorname{Hom}_A(I/I^2, I/I^2). \tag{4.3}$$

However, to get the Theorem 4.3 we only need the isomorphism (4.3) in degree 0. Letting $\operatorname{Ext}^j_{I(Z)}(N, -)$ be the right derived functor of the composed functor $\mathrm{H}^0_{I(Z)} \circ \operatorname{Hom}_R(N, -)$, cf. [21], Exposé VI for details, we have in degree 0 an exact sequence

$$0 \longrightarrow {}_0\operatorname{Hom}_{I(Z)}(I/I^2, I/I^2) \longrightarrow {}_0\operatorname{Hom}_A(I/I^2, I/I^2) \longrightarrow$$
$$\mathrm{H}^0(X \setminus Z, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{I}_X/\mathcal{I}_X^2)) \longrightarrow {}_0\operatorname{Ext}^1_{I(Z)}(I/I^2, I/I^2) \longrightarrow$$

where ${}_0\operatorname{Hom}_{I(Z)}(I/I^2, I/I^2) \cong {}_0\operatorname{Hom}(I/I^2, \mathrm{H}^0_{I(Z)}(I/I^2))$ and the terms in

$$0 \longrightarrow {}_0\operatorname{Ext}^1(I/I^2, \mathrm{H}^0_{I(Z)}(I/I^2)) \longrightarrow {}_0\operatorname{Ext}^1_{I(Z)}(I/I^2, I/I^2) \longrightarrow {}_0\operatorname{Hom}(I/I^2, \mathrm{H}^1_{I(Z)}(I/I^2))$$

vanish by the assumptions (4.2). Hence, we conclude

$$\mathrm{H}^0(X \setminus Z, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{I}_X/\mathcal{I}_X^2)) \cong {}_0\operatorname{Hom}_A(I/I^2, I/I^2)$$

which proves that in Theorem 4.3 we can replace the hypothesis $n \geq 2c$ by the assumptions (4.2).

Remark 4.5. If X is a linear standard determinantal scheme and $t > 1$ then $n_{1i} = t$ for all i, $n_{2j} = t+1$ for all j, and the hypothesis $max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}$ is satisfied.

Remark 4.6. It is worthwhile to point out that in Theorem 4.3 the hypothesis $max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}$ cannot be dropped when $c = 2$. To prove it we will compute the cokernel of the morphism $K = R_0 \cong {}_0\mathrm{Hom}_R(I,I) \xrightarrow{\phi_0} {}_0\mathrm{Hom}_A(I/I^2, I/I^2)$. To this end, we consider the diagram

$$\begin{array}{ccccccccc}
\mathrm{Hom}_R(I,I) & \xrightarrow{\phi} & \mathrm{Hom}_R(I,I/I^2) & \to & \mathrm{Ext}^1_R(I,I^2) & \to & \mathrm{Ext}^1_R(I,I) & \longrightarrow & \mathrm{Ext}^1_R(I,I/I^2) \\
 & & & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
 & & & & \mathrm{Ext}^1_R(I,R)\otimes I^2 & \to & K_A(n+1)\otimes_R I & \xrightarrow{\cong} & K_A(n+1)\otimes_R I/I^2. \\
 & & & & \downarrow\cong & & & & \\
 & & & & K_A(n+1)\otimes_R I^2 & & & &
\end{array}$$

where we have used $\mathrm{Ext}^2_R(I,-) = 0$ to get the isomorphisms. Therefore,

$$\mathrm{coker}(\phi) = K_A(n+1)\otimes_R I^2 \cong K_A(n+1)\otimes_A I^2/I^3.$$

On the other hand for a standard determinantal scheme $X \subset \mathbb{P}^n$ of codimension 2, we have the following well known exact sequences:

$$0 \longrightarrow G^* = \bigoplus_{j=1}^{t} R(-n_{2j}) \longrightarrow F^* = \bigoplus_{i=1}^{t+1} R(-n_{1i}) \longrightarrow I \longrightarrow 0, \text{ and}$$

$$0 \longrightarrow \wedge^2 G^* \longrightarrow G^*\otimes F^* \longrightarrow S_2F^* \longrightarrow S_2 I \longrightarrow 0. \tag{4.4}$$

Since a generic complete intersection of codimension 2 is syzygetic, i.e. $S_2 I \cong I^2$, the exact sequence

$$\cdots \longrightarrow F \longrightarrow G \longrightarrow K_A(n+1) \longrightarrow 0$$

leads to a commutative diagram

$$\begin{array}{ccccccc}
 & & G\otimes_R G^*\otimes_R F^* & & & & \\
 & & \downarrow & & & & \\
F\otimes_R S_2F^* & \longrightarrow & G\otimes_R S_2F^* & & & & . \\
\downarrow & & \downarrow & & & & \\
F\otimes_R I^2 & \longrightarrow & G\otimes_R I^2 & \longrightarrow & K_A(n+1)\otimes_R I^2 & \longrightarrow & 0.
\end{array}$$

It follows that

$$(F\otimes S_2F^*)\oplus(G\otimes G^*\otimes F^*) \longrightarrow G\otimes S_2F^* = \bigoplus_{\substack{1\le i\le j\le t+1\\ 1\le k\le t}} R(-n_{1i}-n_{1j}+n_{2k}) \to K_A(n+1)\otimes_R I^2 \to 0$$

is exact and ϕ_0 is not surjective e.g. in the case ($n_{21} = n_{22} = 4$ and $n_{11} = 2 < n_{12} = n_{13} = 3$):

$$0 \longrightarrow R(-4)^2 \longrightarrow R(-3)^2 \oplus R(-2) \longrightarrow I \longrightarrow 0.$$

We have stated Theorem 4.3 for standard determinantal schemes because the paper concerns the main features of the normal sheaf of a standard determinantal scheme $X \subset \mathbb{P}^n$. Nevertheless, the result works in a much more general set up. Indeed, the assumption that X is a standard determinantal scheme is not necessary and the result and its proof hold for any graded quotient of the polynomial ring provided $\operatorname{depth}_{I(Z)} I/I^2 \geq 2$. In fact, it holds

Theorem 4.7. *Let $X \subset \mathbb{P}^n$ be a closed subscheme of codimension $c \geq 2$ (not necessarily ACM) with a minimal free R-resolution*

$$\cdots \longrightarrow \oplus_j^{b_2} R(-n_{2j}) \longrightarrow \oplus_i^{b_1} R(-n_{1i}) \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

where $I := I(X)$. Let $Z \subset X$ be a closed subset such that $X \setminus Z \hookrightarrow \mathbb{P}^n$ is a local complete intersection. Let L be a finitely generated R/I-module that is invertible over $X \setminus Z$, put $\mathcal{N}_X(\mathcal{L}) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \widetilde{L})$ and assume $\operatorname{depth}_{I(Z)} L \geq 2$, $\operatorname{depth}_{I(Z)} I/I^2 \geq 2$ and $max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}$. Then,

$$\operatorname{Hom}(\mathcal{N}_X(\mathcal{L}), \mathcal{N}_X(\mathcal{L})) \cong {}_0\operatorname{Hom}_A(I/I^2, I/I^2) \cong K.$$

Remark 4.8. For a complete intersection of codimension $c \geq 2$ and dimension $n - c \geq 1$, the conclusion is false while all assumptions, except for $max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}$ are obviously satisfied.

Remark 4.9. As explained in Remark 4.4, the hypothesis $\operatorname{depth}_{I(Z)} I/I^2 \geq 2$ in Theorem 4.7 can be replaced by

$$\mathrm{H}^1_{I(Z)}(I/I^2)_{n_{1i}} = \mathrm{H}^0_{I(Z)}(I/I^2)_{n_{1i}} = \mathrm{H}^0_{I(Z)}(I/I^2)_{n_{2j}} = 0 \quad \text{for any } i \text{ and } j. \tag{4.5}$$

As an application we have:

Corollary 4.10. *Let $X \subset \mathbb{P}^n$ be either a codimension 2 ACM subscheme with a minimal free R-resolution*

$$0 \longrightarrow \oplus_j^{\nu} R(-n_{2j}) \longrightarrow \oplus_i^{\nu+1} R(-n_{1i}) \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

or a codimension 3 arithmetically Gorenstein subscheme with a minimal free R-resolution

$$0 \longrightarrow R(-e) \longrightarrow \oplus_j^{\nu} R(-n_{2j}) \longrightarrow \oplus_i^{\nu} R(-n_{1i}) \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Let $Z \subset X$ be a closed subset such that $X \setminus Z \hookrightarrow \mathbb{P}^n$ is a local complete intersection. Assume $\operatorname{depth}_{I(Z)} R/I \geq \begin{cases} 3 & \text{if } c = 2 \\ 2 & \text{if } c = 3 \end{cases}$ and $max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}$. Then, $\mathcal{N}_X$ is simple.

Proof. By Theorem 4.7 it suffices to show that $\operatorname{depth}_{I(Z)} I/I^2 \geq 2$. For $c = 2$, we get $\operatorname{depth}_{I(Z)} I/I^2 \geq \operatorname{depth}_{I(Z)} R/I - 1$ by [1], cf. (4.4). For $c = 3$ we know that I/I^2 is MCM by [4] because $I/I^2 \otimes K_A$ is MCM in the licci case and the canonical module K_A is trivial in the Gorenstein case. So, we are done □

Let us give some examples of ACM schemes $X \subset \mathbb{P}^N$ with simple normal sheaf.

Example 4.11. Let $X \subset \mathbb{P}^3$ be a (smooth) ACM curve. Since depth $I/I^2 \geq \dim R/I - 1$, we have $\mathrm{H}^0_{\mathfrak{m}}(I/I^2)_\mu = 0$ for any μ. So, according to Theorem 4.3 and Remark 4.4, we only need to check

$$\mathrm{H}^1_{\mathfrak{m}}(I/I^2)_{n_{1i}} \cong \mathrm{H}^1(X, \mathcal{I}^2_X(n_{1i})) = 0 \text{ for any } i; \text{ and} \tag{4.6}$$
$$max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}$$

to conclude that $\mathcal{N}_X$ is simple. Using Macaulay2, we get

(i) (4.6) does not hold in the linear case.

(ii) If $\deg \mathcal{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$, then (4.6) holds. However, since the condition $4 = max\{n_{2j}\} < 2 \cdot min\{n_{1i}\} = 4$ is not true, we cannot conclude that $\mathcal{N}_X$ is simple. In fact, we have seen in Remark 4.6 that $\mathcal{N}_X$ is not simple.

(ii) If $\deg \mathcal{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$ or $\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ or $\begin{pmatrix} 3 & 2 & 2 \\ 3 & 2 & 2 \end{pmatrix}$ or $\begin{pmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \end{pmatrix}$, then (4.6) holds as well as the inequality $max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}$; and we get that $\mathcal{N}_X$ is simple.

In conclusion, the assumptions (4.6) and $max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}$ seem weak for ACM curves in $\mathbb{P}^3$.

Example 4.12. Let $X \subset \mathbb{P}^4$ be a smooth standard determinantal curve. By Theorem 4.3 and Remark 4.4, to prove that $\mathcal{N}_X$ is simple, we only need to check that the following hypothesis are satisfied:

$$\mathrm{H}^1_{\mathfrak{m}}(I/I^2)_{n_{1i}} = \mathrm{H}^0_{\mathfrak{m}}(I/I^2)_{n_{1i}} = \mathrm{H}^0_{\mathfrak{m}}(I/I^2)_{n_{2j}} = 0 \text{ for any } i \text{ and } j; \text{ and} \tag{4.7}$$
$$max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}.$$

Using Macaulay2, we get

(i) (4.7) does not hold in the linear case.

(ii) If $\deg \mathcal{A} = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{pmatrix}$, then (4.7) holds while $max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}$ is not true and we cannot conclude that $\mathcal{N}_X$ is simple .

(ii) If $\deg \mathcal{A} = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}$, then (4.7) holds as well as the inequality $max\{n_{2j}\} < 2 \cdot min\{n_{1i}\}$. Hence, $\mathcal{N}_X$ is simple.

We will end this section with a result about the indecomposability of the normal sheaf of a standard determinantal scheme $X \subset \mathbb{P}^n$ which does not involve the degrees of the generators (resp. first syzygies) of $I(X)$. To achieve our goal we need the following preliminary lemma.

Lemma 4.13. *Let $X \subset \mathbb{P}^n$ be a codimension c standard determinantal subscheme associated to a graded morphism $\varphi : F \longrightarrow G$. Set $M = \mathrm{coker}(\varphi)$ and $J = I_{t-1}(\varphi)$. Let $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ be a hyperplane defined by a general form $h \in R_1$. Consider $X' = X \cap H \subset H$ the hyperplane section of X and $R' = R/(h) \cong K[z_0, \cdots, z_{n-1}]$. Then X' is a codimension c standard determinantal subscheme* *of* $\mathbb{P}^{n-1}$ *associated to the graded morphism* $\varphi' = \varphi \otimes 1 : F' := F \otimes_R R/(h) \longrightarrow G' := G \otimes_R R/(h)$ *and* $M' := \operatorname{coker}(\varphi') \cong M \otimes_R R/(h)$*. Assume* $\operatorname{depth}_J A \geq 3$*. Then, we have*

$$\operatorname{Ext}^1_R(M, S_iM) \otimes_R R/(h) \cong \operatorname{Ext}^1_{R'}(M', S_iM') \text{ for } 0 \leq i \leq c.$$

Proof. Let us consider the free R-resolution $W_\bullet$ of $\operatorname{Ext}^1_R(M, S_iM)$ of Corollary 3.8:

$$0 \longrightarrow Q_c \longrightarrow Q_{c-1} \oplus P_c \longrightarrow Q_{c-2} \oplus P_{c-1} \oplus F_c \longrightarrow Q_{c-3} \oplus P_{c-2} \oplus F_{c-1} \longrightarrow \cdots$$

$$\longrightarrow Q_0 \oplus P_1 \oplus F_2 \longrightarrow P_0 \oplus F_1 \longrightarrow F_0 \longrightarrow \operatorname{Ext}^1_R(M, S_iM) \longrightarrow 0\,;$$

and the free R'-resolution $W'_\bullet$ of $\operatorname{Ext}^1_{R'}(M', S_iM')$:

$$0 \longrightarrow Q'_c \longrightarrow Q'_{c-1} \oplus P'_c \longrightarrow Q'_{c-2} \oplus P'_{c-1} \oplus F'_c \longrightarrow Q'_{c-3} \oplus P'_{c-2} \oplus F'_{c-1} \longrightarrow \cdots$$

$$\longrightarrow Q'_0 \oplus P'_1 \oplus F'_2 \longrightarrow P'_0 \oplus F'_1 \longrightarrow F'_0 \longrightarrow \operatorname{Ext}^1_{R'}(M', S_iM') \longrightarrow 0$$

also given by Corollary 3.8 because $\operatorname{depth}_J A \geq 3$. Since $h \in R_1$ is a general linear form and $\operatorname{depth}_{\mathfrak{m}} \operatorname{Ext}^1_R(M, S_iM) \geq 1$ by Corollary 3.8, we have

$$\operatorname{Ext}^1_R(M, S_iM) : h = \operatorname{Ext}^1_R(M, S_iM)$$

and by [5]; Lemma 1.3.5, that $W_\bullet \otimes_R R/(h) = W'_\bullet$ is a free $R' = R/(h)$-resolution of $\operatorname{Ext}^1_R(M, S_iM) \otimes_R R/(h)$. Therefore, we conclude that

$$\operatorname{Ext}^1_R(M, S_iM) \otimes_R R/(h) \cong \operatorname{Ext}^1_{R'}(M', S_iM').$$

□

Now, we are ready to prove the indecomposability of the normal sheaf of a standard determinantal scheme $X \subset \mathbb{P}^n$ under some mild hypothesis which does not involve the degrees of the generators (resp. first syzygies) of $I(X)$. Recalling $\mathcal{N}_X(\mathcal{M}^k) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \widetilde{S_kM})$, we have

Theorem 4.14. *Let* $X \subset \mathbb{P}^n$ *be a standard determinantal scheme of codimension* $c \geq 2$ *defined by a matrix* $\mathcal{A}$ *with entries that are general forms. We keep the notation introduced above and we assume* $a_j - b_i > 0$ *for all* i, j*. It holds:*

If $c = 2$ *and* $n = 4$*, or* $c \geq 2$ *and* $n \geq 2c+1$*, then the normal sheaf* $\mathcal{N}_X$ *and more generally, the "twisted" normal sheaves* $\mathcal{N}_X(\mathcal{M}^k)$*,* $-1 \leq k \leq c-1$*, are indecomposable.*

Proof. The idea is to fix c and use induction on n. In fact in the special case $(c, n) = (2, 4)$, X is smooth because $\mathcal{A}$ is general and $\mathcal{N}_X$ is indecomposable by [3]; Théorème A. If $(c, n) = (c, 2c+1)$, $c \geq 2$, then X is again smooth, $\operatorname{Pic}(X) \cong \mathbb{Z}^2$ (cf. Theorem 2.4) and $\mathcal{N}_X$ is indecomposable by [2]; Theorem 3.2. It follows that $\mathcal{N}_X(\mathcal{M}^k) \cong (\mathcal{I}_X/\mathcal{I}_X^2)^\vee \otimes \widetilde{S_kM}$ is indecomposable. The result now follows from induction using Lemma 4.13 and taking into account that $\operatorname{Ext}^1_R(M, S_{k+1}M) \cong \operatorname{Hom}_A(I/I^2, S_kM)$ (see Proposition 3.5). □

Example 4.15. (1) We consider $X \subset \mathbb{P}^n$, $n \geq 4$, the standard determinantal subscheme of codimension 2 associated to the matrix $\begin{pmatrix} x_0 & x_1 & x_2^2 \\ x_3 & x_4 & f \end{pmatrix}$ where f is a general form of degree 2. We have seen in Remark 4.6 that $\mathcal{N}_X$ is not simple but it follows from Theorem 4.14 that $\mathcal{N}_X$ is indecomposable.

(2) We consider a *rational normal scroll* $S(a_0, \dots, a_k)$; i.e. the image of the map

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^k \longrightarrow \mathbb{P}^N$$

given by

$$\sigma(x, y; t_0, t_1 \cdots, t_k) := (x^{a_0}t_0, x^{a_0-1}yt_0, \cdots, y^{a_0}t_0, , \cdots, x^{a_k}t_k, x^{a_k-1}yt_k, \cdots, y^{a_k}t_k)$$

where $N = k + \sum_{i=0}^{k} a_i$. If we choose coordinates $X_0^0, \cdots, X_{a_0}^0, \cdots, X_0^k, \cdots, X_{a_k}^k$ in $\mathbb{P}^N$, the ideal of $S(a_0, \dots, a_k)$ is generated by the maximal minors of the $2 \times c$ matrix with two rows and $k+1$ catalecticant blocks:

$$M_{a_0,\cdots,a_k} := \begin{pmatrix} X_0^0 & \cdots & X_{a_0-1}^0 & \cdots & X_0^k & \cdots & X_{a_k-1}^k \\ X_1^0 & \cdots & X_{a_0}^0 & \cdots & X_1^k & \cdots & X_{a_k}^k \end{pmatrix}.$$

By Theorem 4.14, the normal bundle of $S(1, \cdots, 1)$ and of $S(2, 1, \cdots 1)$ are indecomposable. (Notice that $S(1, \cdots, 1)$ corresponds to the Segre variety $\mathbb{P}^1 \times \mathbb{P}^k \hookrightarrow \mathbb{P}^{2k+1}$ already discussed in [2]; Corollary 3.3).

Arguing as in the proof of Theorem 4.14, we get

Corollary 4.16. *Let $X \subset \mathbb{P}^n$ be a smooth standard determinantal scheme of codimension c and let $\mathbb{P}^{n-1} \cong H \subset \mathbb{P}^n$ a general hyperplane and set $X' = X \cap H$. Assume $n - c \geq 2$ and* $\operatorname{depth}_J A \geq 3$, $J = I_{t-1}(\mathcal{A})$. *Then,* $(\mathcal{N}_{X/\mathbb{P}^n})_{|\mathbb{P}^{n-1}} \cong \mathcal{N}_{X'/\mathbb{P}^{n-1}}$.

5. The μ-semistability of the normal sheaf of a determinantal varieties

The normal bundle of a smooth variety $X \subset \mathbb{P}^n$ has been intensively studied since it reflects many properties of the embedding; so far few examples of smooth varieties having μ-(semi)stable normal bundle are known. The first example of a curve $C \subset \mathbb{P}^3$ with normal bundle $\mathcal{N}_C$ μ-stable was given by Sacchiero ([35]). In [16]; Proposition 2, Ellia proved that the normal bundle of a linear determinantal curve $C \subset \mathbb{P}^3$ is linear μ-semistable and the normal bundle $\mathcal{N}_C$ of a general ACM curve $C \subset \mathbb{P}^3$ of degree 6 and genus 3 is μ-stable (see [17] and [34] for more information about the μ-stability of the normal sheaf of a curve in $\mathbb{P}^3$). The first goal of this last section is to generalize Ellia's result for linear determinantal curves C in $\mathbb{P}^3$ to linear determinantal schemes $X \subset \mathbb{P}^n$ of arbitrary dimension.

As in previous sections we will assume $c > 1$, since the case $c = 1$ corresponds to a hypersurface $X \subset \mathbb{P}^n$ and $\mathcal{N}_X \cong \mathcal{O}_X(\delta)$, $\delta := \deg(X)$, is μ-stable. We will also assume $t > 1$, since the case $t = 1$ corresponds to a codimension c complete intersection $X \subset \mathbb{P}^n$ and $\mathcal{N}_X = \oplus_{i=1}^{c} \mathcal{O}_X(d_i)$, $d_i \in \mathbb{Z}$, is not μ-stable.

Let us start recalling the definition of μ-(semi)stability

Definition 5.1. *Let $X \subset \mathbb{P}^n$ be a smooth projective scheme of dimension d and let $\mathcal{E}$ be a coherent sheaf on X. $\mathcal{E}$ is said to be μ-semistable if for any non-zero coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ we have the inequality*

$$\mu(\mathcal{F}) := \frac{deg(c_1(\mathcal{F}))}{\mathrm{rk}(\mathcal{F})} \leq \mu(\mathcal{E}) := \frac{deg(c_1(\mathcal{E}))}{\mathrm{rk}(\mathcal{E})}$$

where as usual $deg(c_1(\mathcal{F})) = c_1(F).H^{d-1}$. We say that $\mathcal{E}$ is μ-stable if strict inequality $<$ always holds.

Remark 5.2. Recall that μ-stable sheaves are simple and hence indecomposable but not vice versa.

Theorem 5.3. [1]*Let $X \subset \mathbb{P}^n$ be a smooth linear determinantal scheme of codimension $c \geq 2$ associated to a $t \times (t+c-1)$ matrix $\mathcal{A}$. Assume $n-c \geq 1$. Then, $\mathcal{N}_X$ is μ-semistable.*

Proof. Since the notion of μ-semistability is preserved when we twist by an invertible sheaf, it will be enough to prove that $\mathcal{N}_X(-H) \otimes \widetilde{S_{c-2}M}$ is μ-semistable. It follows from Theorem 3.7 that $\mathcal{N}_X(-H) \otimes \widetilde{S_{c-2}M}$ is a rank c Ulrich sheaf on X and by [9]; Theorem 2.9 any Ulrich sheaf is μ-semistable which proves what we want. □

Remark 5.4. Since Ulrich sheaves are Gieseker semistable, the above proof also shows that the normal sheaf $\mathcal{N}_X$ to a smooth linear determinantal scheme of codimension $c \geq 2$ is Gieseker semistable.

Remark 5.5. Without extra hypothesis the above result cannot be improved and the μ-stability of the normal sheaf $\mathcal{N}_X$ of a linear standard determinantal scheme $X \subset \mathbb{P}^n$ cannot be guaranteed. In fact, if we consider a rational normal curve $C \subset \mathbb{P}^n$ defined by the 2×2 minors of a $2 \times n$ matrix with general linear entries, it is well known that $\mathcal{N}_{X/\mathbb{P}^n} \cong \mathcal{O}_X(2)^{n-1}$. Therefore, $\mathcal{N}_X$ is μ-semistable but not μ-stable.

In the codimension 2 case, Theorem 5.3 can be improved using the following lemma:

Lemma 5.6. *Let $X \subset \mathbb{P}^n$ be a smooth linear determinantal scheme of codimension $c \geq 2$ and dimension $n-c \geq 2$ defined by the maximal minors of a $t \times (t+c-1)$ matrix $\mathcal{A}$. Assume $t \geq n$ when $n-c=2$. Let H be a general hyperplane section of X and let $Y \subset \mathbb{P}^n$ be the codimension 1 subscheme of X defined by the maximal minors of the $t \times (t+c)$ matrix $\mathcal{B}$ obtained adding to $\mathcal{A}$ a column of general linear forms. Let $\mathcal{L}$ be a line bundle on X. It holds:*

(i) *$\mathcal{L}$ is an ACM line bundle on X if and only if $\mathcal{L} \cong \mathcal{O}_X(aY+bH)$ with $-1 \leq a \leq c$ and $b \in \mathbb{Z}$;*
(ii) *$\mathcal{L}$ is an initialized Ulrich line bundle if and only if $\mathcal{L} \cong \mathcal{O}_X(-Y+tH)$ or $\mathcal{O}_X(cY-cH)$.*

[1]Note added in proof: After this paper appeared online in Crelle's journal the authors were informed that Ph. Ellia in his paper "Double structures and normal bundles of spaces curves" J. London Math. Soc. 58, 18-26 (1998), Remarks and Examples 20 (v) proved that the normal bundle of a smooth standard determinantal curve C in $\mathbb{P}^3$ defined by a matrix with either linear or quadratic entries is μ-semistable. Therefore, Theorem 5.3 can be seen as a generalization of his result in the linear case. The mentioned paper also contains interesting examples and results on the stability of normal bundles of space curves.

Proof. See [30]. □

In fact, we have:

Theorem 5.7. *Let X be a smooth linear determinantal scheme of codimension 2 in $\mathbb{P}^n$ defined by the maximal minors of a $t \times (t+1)$ matrix with linear entries. Assume that $n \geq 4$. Then, $\mathcal{N}_X$ is μ-stable. In particular, $\mathcal{N}_X$ is simple and indecomposable.*

Proof. For the case $(n,t) = (4,3)$ the reader can see [31]; Proposition 4.10. Assume $(n,t) \neq (4,3)$. Since μ-(semi)stability is preserved when we twist by line bundles, we know that $\mathcal{N}_X(-H)$ is μ-semistable (Theorem 5.3) and we want to prove that it is μ-stable, i.e we must rule out the existence of a coherent subsheaf $\mathcal{F} \subset \mathcal{N}_X(-H)$ with $\mu(\mathcal{F}) = \mu(\mathcal{N}_X(-H))$. By pulling-back torsion, if necessary, we may assume that $\mathcal{N}_X(-H)/\mathcal{F}$ is torsion free in which case $\mathcal{F}$ is locally free and we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{N}_X(-H) \longrightarrow \mathcal{N}_X(-H)/\mathcal{F} \longrightarrow 0$$

of coherent sheaves with $\mathcal{N}_X(-H)/\mathcal{F}$ torsion free and $\mu(\mathcal{F}) = \mu(\mathcal{N}_X(-H))$. By [9]; Theorem 2.9(b) $\mathcal{F}$ and $\mathcal{N}_X(-H)/\mathcal{F}$ are both Ulrich line bundles. So, according to our Lemma 5.6 we have 4 possibilities:

$$\begin{array}{ll}
(1) & 0 \longrightarrow \mathcal{O}_X(-Y+tH) \longrightarrow \mathcal{N}_X(-H) \longrightarrow \mathcal{O}_X(-Y+tH) \longrightarrow 0, \\
(2) & 0 \longrightarrow \mathcal{O}_X(2Y-2H) \longrightarrow \mathcal{N}_X(-H) \longrightarrow \mathcal{O}_X(-Y+tH) \longrightarrow 0, \\
(3) & 0 \longrightarrow \mathcal{O}_X(-Y+tH) \longrightarrow \mathcal{N}_X(-H) \longrightarrow \mathcal{O}_X(2Y-2H) \longrightarrow 0, \text{ or} \\
(4) & 0 \longrightarrow \mathcal{O}_X(2Y-2H) \longrightarrow \mathcal{N}_X(-H) \longrightarrow \mathcal{O}_X(2Y-2H) \longrightarrow 0.
\end{array}$$

Let us check that none of them is allowed. To this end, we will start computing the Chern classes of $\mathcal{N}_X(-H)$. We sheafify the exact sequence (3.3) and we get the exact sequence:

$$(5.1) \qquad 0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X(Y-2H)^t \longrightarrow \mathcal{O}_X(Y-H)^{t+1} \longrightarrow \mathcal{N}_X(-H) \longrightarrow 0.$$

Therefore, the Chern polynomial $c_u(\mathcal{N}_X(-H))$ of $\mathcal{N}_X(-H)$ is given by

$$\begin{array}{rcl}
c_u(\mathcal{N}_X(-H)) & = & \sum_u c_i(\mathcal{N}_X(-H))u^i \\
& = & \frac{(1-Hu)(1+(Y-H)u)^{t+1}}{(1+(Y-2H)u)^t} \\
& = & \frac{(1-Hu)(\sum_{k=0}^{t+1}\binom{t+1}{k}(Y-H)^k u^k)}{\sum_{k=0}^{t}\binom{t}{k}(Y-2H)^k u^k}
\end{array}$$

and a straightforward computation gives us

$$c_1(\mathcal{N}_X(-H)) = Y + (t-2)H, \text{ and}$$

$$c_2(\mathcal{N}_X(-H)) = -YH + \frac{t^2-t+2}{2}H^2.$$

Comparing the first Chern class we eliminate the possibility (1) and (4) because in case (1) we would get $c_1(\mathcal{N}_X(-H)) = -2Y + 2tH$ and in case (4) we would get $c_1(\mathcal{N}_X(-H)) = 4Y - 4H$. To rule out the two remaining cases, we compare the second Chern class; in both cases we would get $c_2(\mathcal{N}_X(-H)) = 2(1+t)YH - 2Y^2 - 2tH^2$ which is impossible and this concludes the proof of the Theorem. □

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Oslo and Akershus University College, Faculty of Technology, Art and Design, PB. 4, St. Olavs Plass, N-0130 Oslo, Norway

E-mail address: JanOddvar.Kleppe@hioa.no

Facultat de Matemàtiques, Departament d'Algebra i Geometria, Gran Via de les Corts Catalanes 585, 08007 Barcelona, SPAIN

E-mail address: miro@ub.edu