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LIFTINGS(DEFORMATIONS) OF GRADED ALGEBRAS

by

Jan Kleppe
Oslo

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Contents

INTRODUCTION

In this paper we study formal deformations of graded algebras and corresponding problems in projective geometry. Given a graded algebra A, we may forget the graded structure and deform (lift) A as an algebra. Clearly we also have a deformation theory respecting the given graded structure of A . This deformation theory is closely related to the corresponding theory of $X = \mathrm{Proj}(A)$. One objective of this paper is to compare these three theories of deformation.

A basic tool is the cohomology groups of André and Quillen. Let $S \to A$ be a graded ringhomomorphism and let M be a graded A-module. We shall see that the groups

$$H^i(S,A,M)$$

are graded A-modules whenever S is noetherian and $S \to A$ is finitely generated. In fact, if we let

$$_{\nu}H^i(S,A,M)$$

correspond to S-derivations of degree ν, we shall prove that there are canonical isomorphisms

$$\coprod_{\nu=-\infty}^{\infty} {}_{\nu}H^i(S,A,M) \simeq H^i(S,A,M)$$

for every $i \geq 0$.

Deformations of A (forgetting the graded structure) are classified by the groups $H^i(S,A,A)$ for i=1,2. Restricting to graded deformations, we shall see that they are classified by the subgroups

$$_{0}H^i(S,A,A)$$

for i=1,2 . These generalities are proved or at least stated in chapter 1.

Let $\pi : R \to S$ be a graded surjection satisfying $(\ker \pi)^2=0$.

Since there is an injection

$$ {}_{o}H^2(S,\Lambda,\Lambda\otimes_S \ker\Pi) \to H^2(S,\Lambda,\Lambda\otimes_S \ker\Pi) $$

we deduce that Λ is liftable to R iff Λ is liftable as a graded algebra. We would like to generalize this result to arbritary surjections of complete local rings. This seems difficult. However if we assume

$$ {}_{\nu}H^1(S,\Lambda,\Lambda) = 0 $$

for $\nu > 0$ or $\nu < 0$ (called negative or positive grading respectively), then the statement above follows from 2.6 of chapter 2 when S is a field k. In fact, let $\underline{1}$ be the category of artinian local V-algebras with residue fields k, $V/m_V = k$, and let $\mathrm{Def}^{o}(\Lambda/k, -)$, resp $\mathrm{Def}(\Lambda/k, -)$, be the graded deformation functor, resp non-graded deformation functor on $\underline{1}$ with hulls $R^o(\Lambda)$ and $R(\Lambda)$ respectively. Consider the local V-morphism

$$ R(\Lambda) \to R^o(\Lambda) $$

Theorem 2.6 states that this morphism has a section whenever Λ has negative or positive grading. This follows from the existence of an isomorphisms.

$$ R(\Lambda) \simeq R^o(\Lambda[T]) $$

Here $\deg T = 1$ if we have negative grading.

In chapter 3 we enter into projective geometry assuming the graded algebras to be positively graded and generated by elements of degree 1. We compare the groups

$$ {}_{\nu}H^i(S,\Lambda,M) $$

with the corresponding groups $A^i(S,X,\widetilde{M}(\nu))$ in projective geometry, $X = \mathrm{Proj}(A)$. The groups $A^i(S,X,-)$ were introduced by Illusie in [I] and by Laudal [L1]. If X is S-smooth, then

$$A^i(S,X,\widetilde{M}) = H^i(X, \theta_X \otimes_S \widetilde{M})$$

where θ_X is the sheaf of S-derivations on X. If the depth of M with respect to the ideal

$$m = \coprod_{\nu=1}^{\infty} A_\nu$$

is sufficiently big, the groups

$${}_\nu H^i(S,A,M)$$

and

$$A^i(S,X,\widetilde{M}(\nu))$$

coincide. For instance, if $\mathrm{depth}_m A \geq 4$,

$${}_\nu H^1(S,A,A) \simeq A^1(S,X,O_X(\nu))$$

and

$${}_\nu H^2(S,A,A) \hookrightarrow A^2(S,X,O_X(\nu))$$

This implies that the deformations of A and X correspond uniquely to each other. When $\mathrm{depth}_m A \geq 3$ a rigidity theorem of Schlessinger, see (2.2.6) in [K,L], is generalized by the injection

$${}_\nu H^1(S,A,A) \to A^1(S,X,O_X(\nu))$$

Now these depth conditions are usually rather crude, and the exact sequences in which these groups fit are in many cases a better tool.

In chapter 3 we also relate the groups corresponding to embeddings. Let $\varphi : B \to A$ be a surjective morphism of graded S-algebras such that $B_o = A_o = S$ and let

$$f : X = \mathrm{Proj}(A) \to X = \mathrm{Proj}(B)$$

be the induced embedding. We would like to compare the groups ${}_{\nu}H^i(B,A,M)$ and $A^i(S,f,\widetilde{M}(\nu))$. If f is locally a complete intersection, one knows that

$$A^i(S,f,O_X(\nu)) \cong H^{i-1}(X,N_f(\nu))$$

where N_f is the normal bundle of X in Y. Again putting depth conditions on M, we conclude that

$${}_{\nu}H^i(B,A,M)$$

and

$$A^i(S,f,\widetilde{M}(\nu))$$

coincide. If $\mathrm{depth}_m A \geq 2$, then

$${}_{\nu}H^1(B,A,A) \simeq A^1(S,f,\mathcal{O}_X(\nu))$$

and

$${}_{\nu}H^2(B,A,A) \hookrightarrow A^2(S,f,O_X(\nu))$$

From this follows that if B is $S = k$-free then

$$\mathrm{Def}^o(\varphi,-) \simeq \mathrm{Hilb}_X(-)$$

on $\underline{1}$ where $\mathrm{Def}^o(\varphi,-)$ is the graded deformation functor of φ and where $\mathrm{Hilb}_X(-)$ is the local Hilbert functor at X. From this and the isomorphism

$$R(A) \simeq R^o(A[T])$$

we generalize a theorem of Pinkham [P] as follows. If A has negative grading and $\operatorname{depth}_m A \geq 1$ and if $X = \operatorname{Proj}(A[T])$ is the projective cone of X in $\mathbb{P}_k^{n+1} = \operatorname{Proj}(B[T])$, then there is a smooth morphism of functors

$$\operatorname{Hilb}_X(-) \to \operatorname{Def}(A/k,-)$$

In chapter 4 we investigate the conditions of negative and positive grading. We shall assume A to be the minimal cone of a closed subscheme $X \subseteq \mathbb{P}_S^n$. By twisting the embedding we prove that the minimal cone B of $X \subseteq \mathbb{P}_S^N$ for large N very often has negative or positive grading. For instance, if X is S-smooth B will have negative grading. If X is of pure dimention ≥ 2 and locally Cohen-Macauley, then B will have positive grading. Combining these two results we deduce a theorem of Schlessinger [S3]. See also [M].

Using these results we find that the smooth unliftable projective variety of Serre [Se] gives rise to a graded k-algebra which is unliftable to characteristic zero. This is done in chapter 5. His example is of the form $X = Y/G$. Y is a complete intersection of dimention 3 and the order of G divides the characteristic.

The possibility of using this example to get an unliftable k-algebra may be looked upon as the beginning of this paper. The proof given here is due to O.A. Laudal and the author.

We end chapter 5 by proving that if $\operatorname{ord}(G)$ did not divide the characteristic and if Y was a complete intersection of dimention ≥ 3 then $X = Y/G$ would have been everywhere liftable. This paper contains all the results of [K]. I would like to thank O.A.Laudal for reading the manuscript.

CHAPTER 1

Cohomology groups of graded algebras.

Rings will be commutative with unit. Let S-alg be the category of S-algebras and

$$\underline{SF} \subset \underline{S\text{-}alg}$$

the full subcategory of free S-algebras. Given an S-algebra A and an A-module M, we define

$$H^i(S,A,M) = \varprojlim_{(\underline{SF}/A)^\circ}{}^{(i)} \ \mathrm{Der}_S\,(-,M)$$

where $\mathrm{Der}_S(-,M)$ is the functor on $(\underline{SF}/A)^\circ$ with values in Ab defined by

$$\mathrm{Der}_S(-,M)(F \xrightarrow{\varphi} A) = \mathrm{Der}_S(F,M)$$

M being an F-module via φ.

If $S \to A$ is a graded S-algebra and if M is a graded A-module, we may consider the category of graded S-algebras Sg-alg and the corresponding category

$$\underline{SgF} \subset \underline{Sg\text{-}alg}$$

of free graded S-algebras. Let

$${}_k\mathrm{Der}_S(-,M) : \underline{SgF}/A \to Ab$$

be the functor defined by

$${}_k\mathrm{Der}_S(-,M)(F \xrightarrow{\varphi} A) = {}_k\mathrm{Der}_S(F,M) = \{D \in \mathrm{Der}_S(F,M) \mid D \text{ is graded of degree } k\}$$

Then we put

Definition 1.1

$${}_kH^i(S,A,M) = \varprojlim_{\underline{SgF}/A^\circ}{}^{(i)} \ {}_k\mathrm{Der}_S(-,M)$$

As mentioned in the introduction, the groups $H^i(S,A,-)$ and ${}_oH^i(S,A,-)$ classifies formal deformations. Recall that if

$$R \overset{\pi}{\to} S$$

is any surjection with nilpotent kernel, we say that an R-algebra A' is a lifting or deformation of A to R if there is given a cocartesian diagram

$$\begin{array}{ccc} R & \longrightarrow & A' \\ \pi\downarrow & & \downarrow \\ S & \longrightarrow & A \end{array}$$

such that

$$\mathrm{Tor}_1^R(A',S) = 0$$

Two liftings A' and A" are considered equivalent if there is an R-algebra isomorphism $A' \simeq A''$ reducing to the identity on A. If $\varphi : A \to B$ is a morphism of S-algebras and A' and B' are liftings of A and B respectively, we say that a morphism

$$\varphi' : A' \to B'$$

is a lifting or deformation of φ with respect to A' and B' if $\varphi' \underset{R}{\otimes} \mathrm{id}_S = \varphi$. We define graded liftings of graded algebras and graded liftings of graded morphisms in exactly the same way.

Assume that $R \overset{\pi}{\to} S$ satisfies $(\ker \pi)^2 = 0$

Then it is known that

<u>Theorem 1.2</u>

There is an element

$$\sigma(A) \in H^2(S,A,A \underset{S}{\otimes} \ker \pi)$$

which is zero if and only if A can be lifted to R. If $\sigma(A) = 0$, then the set of non-equivalent liftings is a principal homogeneous space over $H^1(S,A,A \otimes_S \ker \pi)$

Theorem 1.3

There is an element

$$\sigma(\varphi, A',B') \in H^1(S,A,B \otimes_S \ker \pi)$$

which is zero if and only if φ can be lifted to R with respect to A' and B'. If $\sigma(\varphi ; A',B') = 0$ then the set of liftings is a principal homogeneous space over

$$H^0(S,A,B \otimes_S \ker \pi) = \mathrm{Der}_S(A,B \otimes_S \ker \pi)$$

The elements $\sigma(A)$ and $\sigma(\varphi ; A',B')$ are called obstructions.

Then corresponding theorems in the graded case are

Theorem 1.4

There is an element

$$\sigma_0(A) \in {}_0H^2(S,A,A \otimes_S \ker \pi)$$

which is zero if and only if A can be lifted to a graded R-algebra. If $\sigma_0(A) = 0$, then the set of non-equivalent liftings is a principel homogeneous space over ${}_0H^1(S,A,A \otimes_S \ker \pi)$

Theorem 1.5

There is an element

$$\sigma_0(\varphi ; A',B') \in {}_0H^1(S,A,B \otimes_S \ker \pi)$$

which is zero if and only if φ can be lifted as a graded morphism to R with respect to A' and B'. Moreover, if $\sigma_0(\varphi ; A',B') = 0$, then the set of graded liftings is a principal homogeneous space over ${}_0H^0(S,A,B \otimes_S \ker \pi) =$ ${}_0\mathrm{Der}_S(A,B \otimes_S \ker \pi)$

In [L1]we find proofs of 1.2 and 1.3 and these can easily be carried over to the graded case.

If we want to compare the graded and non-graded theories of deformation, we need to know the relations between the groups ${}_{0}H^{i}(S,A,M)$ and $H^{i}(S,A,M)$. This is given by the following theorem. A proof of this can also be found in [I].

Theorem 1.6

Let $S \to A$ be a graded ringhomomorphism an let M be a graded A-module. If S is noetherian and $S \to A$ is finitely generated, then there is a canonical isomorphism

$$\coprod_{k=-\infty}^{\infty} {}_{k}H^{i}(S,A,M) \to H^{i}(S,A,M)$$

for every $i \geq 0$.

Remark In general, there is an injection

$$\coprod_{k=-\infty}^{\infty} {}_{k}H^{i}(S,A,M) \to H^{i}(S,A,M)$$

for every $i \geq 0$.

Proof

Let

$$(\underline{SgF}/A)_{fg} \subseteq \underline{SgF}/A$$

be the full subcategory defined by the objects $\varphi : F \to A$ where F is a finitely generated S-algebra.

Look at the diagram of categories

$$\begin{array}{ccc} (\underline{SF}/A)_{fg} & \longrightarrow & \underline{SF}/A \\ \uparrow & & \uparrow \\ (SgF/A)_{fg} & \longrightarrow & \underline{SgF}/A \end{array}$$

where all functors are forgetful. These induce morphisms

$$\begin{array}{ccc}
\varprojlim\limits_{(\underline{SF}/A)_{fg}}{}^{(i)} \mathrm{Der}_S(-,M) & \longleftarrow & \varprojlim\limits_{\underline{SF}/A}{}^{(i)} \mathrm{Der}_S(-,M) = H^i(S,A,M) \\
\downarrow & & \downarrow \\
\varprojlim\limits_{(\underline{SgF}/A)_{fg}}{}^{(i)} \mathrm{Der}_S(-,M) & \longleftarrow & \varprojlim\limits_{\underline{SgF}/A}{}^{(i)} \mathrm{Der}_S(-,M)
\end{array}$$

I claim that these maps are all isomorphisms for $i \geq o$. This will prove 1.6 since there is a canonical isomorphism of functors

$$*) \qquad \coprod_{k=-\infty}^{\infty} {}_k\mathrm{Der}_S(-,M) \longrightarrow \mathrm{Der}_S(-,M)$$

on $(\underline{SgF}/A)_{fg}$.

For $i = 0$, the contention of * is easily proved. For $i > 0$ let us prove that the right hand vertical morphisms are isomorphisms.

Let $F \to A$ be a graded S-algebrasurjection and let

$$F_i = F \underset{A}{x} F \underset{A}{x} \dots \underset{A}{x} F \qquad (i+1)\text{-times}$$

Consider the complex

$$\varprojlim_{\underline{SF}/F_o}{}^{(q)} \mathrm{Der}_S(-,M) \to \varprojlim_{\underline{SF}/F_1}{}^{(q)} \mathrm{Der}_S(-,M) \to \quad \to \varprojlim_{\underline{SF}/F_i}{}^{(q)} \mathrm{Der}_S(-,M) \to$$

where the differensials are the alterating sum of groupmorphisms

$$\varprojlim_{\underline{SF}/F_{i-1}}{}^{(q)} \mathrm{Der}_S(-,M) \to \varprojlim_{\underline{SF}/F_i}{}^{(q)} \mathrm{Der}_S(-,M)$$

induced by the projections $F_i \to F_{i-1}$. In this situation there is a Leray spectral sequence given by the term

$$E^{p,q} = H^p(\varprojlim_{\underline{SF}/F}{}^{(q)} \mathrm{Der}_S(-,M))$$

converging to

$$\varprojlim_{\underline{SF}/A}{}^{(\cdot)} \mathrm{Der}_S(-,M) = H^{(\cdot)}(S,A,M)$$

For a proof see (2.1.3) in [L1].

Similarely, there is a Leray spectral sequence with

$$'E_2^{p,q} = H^p(\varprojlim_{\underline{SgF}/F}{}^{(q)} \mathrm{Der}_S(-,M))$$

converging to

$$\varprojlim_{\underline{SgF}/A}{}^{(\cdot)} \mathrm{Der}_S(-,M)$$

To show that the morphisms

$$\varprojlim_{\underline{SF}/A}{}^{(i)} \mathrm{Der}_S(-,M) \to \varprojlim_{\underline{SgF}/A}{}^{(i)} \mathrm{Der}_S(-,M)$$

are isomorphisms, we use induction on i . If it is an isomorphism for $i \leq n$ and for every object A in $\underline{\text{Sg-alg}}$, we conclude that the morphism

$$E_2^{p,q} \to {}'E_2^{p,q}$$

is an isomorphism for $q \leq n$ and every p.

Recall that

$$E_2^{o,q} \subseteq \varprojlim_{\underline{SF}/F}{}^{(q)} \mathrm{Der}_S(-,M) = H^q(S,F,M)$$

Hence $E_2^{o,q} = 0$ for $q \geq 1$.

Since $F \in \mathrm{ob}\underline{SgF}$, we get

$$ {}^1E_2^{o,q} \subseteq \varprojlim_{\underline{SgF}/F}{}^{(q)} \mathrm{Der}_S(-,M) = 0 \qquad \text{for } q \geq 1$$

as well. Since for $r \geq 2$ the differensials of the spectral sequence are of bidegree $(r,1-r)$, and since for p and q given, $E_\infty^{p,q} = E_r^{p,q}$ for some r , we easily deduce isomorphisms

$$E_\infty^{p,q} \longrightarrow {}^1E_\infty^{p,q}$$

for every p and q with $p+q \leq n+1$. Hence there is an isomorphism

$$\varprojlim_{\underline{SF}/A}{}^{(n+1)} \mathrm{Der}_S(-,M) \to \varprojlim_{\underline{SgF}/A}{}^{(n+1)} \mathrm{Der}_S(-,M)$$

Q.E.D.

Let $R \xrightarrow{\pi} S$ be a graded surjection such that

$$(\ker \pi)^2 = 0$$

It is easy to see that the injection

$$ {}_oH^2(S,A,A \underset{S}{\otimes} \ker \pi) \longrightarrow H^2(S,A,A \underset{S}{\otimes} \ker \pi)$$

maps the obstruction $\sigma_o(A)$ onto $\sigma(A)$. For definitions of the obstructions see [L1]. This proves

<u>Corollary 1.7</u>

Let $R \xrightarrow{\pi} S$ be a graded surjection such that $(\ker \pi)^2 = 0$. If A is a graded S-algebra, then A can be lifted to R iff A can be lifted to R as a graded algebra

<u>Remark</u>

Let F_A be the set of non-equivalent liftings of A to

R and $F_A{}^o$ the corresponding set of graded liftings. If A' is a graded lifting of A to R , then there are isomorphisms and obvions vertical injections fitting into the diagram

$$\begin{array}{ccc} F_A & \simeq & H^1(S,A,A) \\ \uparrow & & \uparrow \\ F_A{}^o & \simeq & {}_oH^1(S,A,A) \end{array}$$

Hence there is a projection

$$p : F_A \rightarrow F_A{}^o$$

Now 1.7 can be generalized as follows. Let

$$\varphi : A \longrightarrow B$$

be a graded S-algebrahomomorphism Assume there are liftings A' and B' , not necessarily graded, of A and B such that φ is liftable to R with respect to A' and B'. Then φ admits a graded lifting to R with respect $p(A')$ and $p(B')$.
We omit the proof.

Similar results for graded S-modules and for graded module morphisms are valid.

CHAPTER 2

Defomation functors and formal moduli.

For the rest of this paper we shall deform only finitely generated algebras.

Let $\Pi : R \to R'$ be a surjective ringhomomorphism. If $(\ker \pi)^2 = 0$, then 1.7 say that A is liftable to R iff A is liftable to R as a graded algebra. We would like to drop the condition $(\ker \pi)^2 = 0$ in 1.7. To do this we shall introduce defomation functors.

Let V be a noetherian local ring with maximal ideal m_V and residue field $k = V/m_V$. Let $\underline{l}$ be the category whose objects are artinian local V-algebras with residue fields k and whose morphisms are local V-homomorphisms. Let S be a finitely generated k-algebra and assume that we can find graded liftings S_R of S to R for any $R \in \text{ob}\underline{l}$ such that for any morphism $\pi : R \to R'$ of $\underline{l}$ there is a morphism $S_R \to S_{R'}$ with

$$S_R \underset{R}{\otimes} R' \simeq S_{R'}$$

For each R, fix one S_R with this property and let

$$\varphi : S \to A$$

be a finitely generated graded S-algebra. Relative to the choice of liftings S_R we define

$$\mathrm{Def}^o(A/S,R) = \left\{ \begin{array}{ccc} S_R & \longrightarrow & A' \\ \downarrow & & \downarrow \\ S & \longrightarrow & A \end{array} \;\middle|\; A' \text{ is a graded lifting of } A \text{ to } S_R \right\} \Big/ \sim$$

It is easy to see that $\mathrm{Def}^o(A/S,-)$ is a covariant functor on $\underline{l}$ with values in Setz . This is the <u>graded deformation functor</u> or A/S. Correspondingly, we denote by $\mathrm{Def}(A/S,-)$ the non-graded deformation functor of A/S .

Recall that a morphism of covariant functors

$$F \longrightarrow G$$

on $\underline{l}$ is smooth iff the map

$$F(R) \longrightarrow F(R') \times_{G(R')} G(R)$$

is surjective whenever $R \to R'$ is surjective. The tangent space t_F of F is defined to be

$$t_F = F(k[\varepsilon])$$

when $k[\varepsilon] \in \mathrm{ob}\ \underline{l}$ is the dual ring of numbers.

<u>Definition 2.1</u>

A pro-$\underline{l}$ object $R(A/S)$, or just $R(A)$ is called a hull for $\mathrm{Def}(A/S,-)$ if there is a smooth morphism of functors

$$\mathrm{Hom}^C_{\underline{l}}(R(A),-) \longrightarrow \mathrm{Def}(A/S,-)$$

on $\underline{l}$ which induces an isomorphism on their tangent spaces. $R^o(A)$ is similarely defined as the hull of $\mathrm{Def}^o(A/S,-)$.

By 1.2 and 1.4 we see that

$$\mathrm{Def}(A/S,k[\varepsilon]) = H^1(S,A,A)$$

$$\mathrm{Def}^o(A/S,k[\varepsilon]) = {}_oH^1(S,A,A)$$

Look at the canonical morphism of functors

$$\mathrm{Def}^o(A/S,-) \longrightarrow \mathrm{Def}(A/S,-)$$

and the corresponding V-morphism

$$R(A) \longrightarrow R^o(A)$$

If this morphism splits we have solved the problem mentioned at the beginning of this paragraph.

In [L1] we find a very general theorem decribing these hulls. Following [L1] we notice that since A is a finitely generated S-algebra, the group $H^i(S,A,A)$ for a given i is finite as an A-module. We pick a countabel basis $\{v_j\}$ for $H^i(S,A,A)$ as a k-vectorspace and define a topology on H^i in which a basis for the neighbourhoods of zero are those subspaces containing all but a finite number of these v_j. Let

$$H^{i*} = \mathrm{Hom}_k^c(H^i,k) \qquad \text{for } i = 1,2$$

and let

$$T_A^i, \text{ or just } T^i \qquad i = 1,2$$

be the completion of $\mathrm{Sym}_V(H^{i*})$ in the topology induced by the topology on H^{i*}, i.e. the topology in which a basis for the neighbourhoods of zero are those ideals containing some power of the maximal ideal and intersecting H^{i*} in an open subspace. If H^i is a finite k-vectorspace then T^i is a convergent power series algebra on $\hat{V}$. The result we need is the following. See (4.2.4.) in [L1].

<u>Theorem 2.2</u>

There is a morphism of complete local rings

$$\sigma = \sigma(A) : T^2 \longrightarrow T^1$$

such that

$$R(A) \cong T^1 \hat{\otimes}_{T^2} \hat{V}$$

Short remark on the proof.

To simplify ideas, assume $V = k$ and $H^1(S,\Lambda,\Lambda)$ finite as a k-vectorspace. Let $\underline{l}_n \subseteq \underline{l}$ be the full subcategory of $\underline{l}$ consisting of objects R statisfying $m_R^n = 0$. Put

$$T_n^1 = T^1/m_{T^1}^n$$

and $R_2 = T_2^1$. If $R \in \mathrm{ob}\,\underline{l}_2$, then by 1.2

$$\mathrm{Def}(\Lambda/S,R) = H^1(S,\Lambda,\Lambda) \underset{k}{\otimes} m_R = \mathrm{Hom}_k^c\,(H^{1*},m_R) = \mathrm{Hom}_{\underline{l}}^c(T_2^1,R) = \mathrm{Hom}_{\underline{l}}^c(R_2,R)$$

Hence R_2 represents the functor $\mathrm{Def}(\Lambda/S,-)$ on $\underline{l}_2$. Let Λ_2 be the universal lifting of Λ to S_{R_2}. If $\sigma_2 : T_2^2 \to k \to T_2^1$ is the composition,

then
$$R_2 = T_2^1 = T_2^1 \underset{T_2^2}{\otimes} k .$$

By induction we shall assume that

$$\sigma_i : T_i^2 \to T_i^1 \qquad ; 2 \leq i \leq n-1$$

are constructed such that

$$R_i = T_i^1 \underset{T_i^2}{\otimes} k$$

and such that Λ_2 is liftable to S_{R_i}.

Consider the following diagram

$$\begin{array}{ccccccc}
 & \nearrow & T_n^2 & \overset{\sigma_n}{- - - \to} & T_n^1 & & \\
H^{2*} & & \downarrow & & \downarrow \pi & & \\
 & \searrow & T_{n-1}^2 & \underset{\sigma_{n-1}}{\longrightarrow} & T_{n-1}^1 & \longrightarrow & R_{n-1} = T_{n-1}^1/\sigma_n
\end{array}$$

We shall try to construct $\sigma_n : T_n^2 \to T_n^1$ such that the diagram above commutes. In fact it is enough to define σ_n on H^{2*} as a k-linear map. Let

$$R_n' = T_n^1/\ \pi^{-1}(\alpha)$$

Then the diagram

$$\begin{array}{ccc} T_n^1 & \longrightarrow & R_n' \\ \pi \downarrow & & \downarrow \pi' \\ T_{n-1}^1 & \longrightarrow & R_{n-1} \end{array}$$

is commutative and $\ker \pi'$ is a k-module via $T_n^1 \longrightarrow k$.

Let A_{n-1} be any lifting of A_2 to $S_{R_{n-1}}$. The obstruction for lifting A_{n-1} to $S_{R_n'}$ is given by

$$\sigma(A_{n-1}) \in H^2(S_{R_{n-1}}, A_{n-1}, A_{n-1} \otimes \ker \pi') \simeq H^2(S,A,A) \underset{k}{\otimes} \ker \pi' = \mathrm{Hom}\,(H^{2*},\ \ker \pi')$$

Let σ_n be any k-linear map fitting into the commutative diagram

$$\begin{array}{ccccc} & & H^{2*} & & \\ & \swarrow^{\sigma_{n-1}} & \downarrow \sigma_n & \searrow^{\sigma(A_{n-1})} & \\ T_{n-1}^1 & \underset{\pi}{\longleftarrow} & T_n^1 & \longrightarrow\!\!\!\!\rightarrow & R_n' \supseteq \ker \pi' \end{array}$$

and put

$$R_n = R_n'/\mathrm{im}\,\sigma(A_{n-1})$$

Thus killing the obstruction of lifting, we conclude that A_{n-1} is liftable to S_{R_n}. Put

$$R(A) = \varprojlim R_n \quad \text{and} \quad \sigma = \varprojlim \sigma_n .$$

Laudal proves that this $R(A)$ is a hull for $\mathrm{Def}(A/S,-)$.
If $V \neq k$, just as in the general step, we let V_2 be the largest quotient of V/m_V2 to which $S \to A$ is liftable.
Any lifting $S_{V_2} \to A_2$ may serve as a zero point for the isomorphism

$$\mathrm{Def}(A/S,R) \simeq H^1(S,A,A) \underset{k}{\otimes} m_R$$

where $R \in \mathrm{obl}_2$ For the rest we may proceed as before.

<u>Corollary 2.2.a</u>

Let V be a regular local ring such that $S \to A$ is liftable to V/m_V2 . Then $R(A)$ is regular iff the composition

$$H^{2*} \longrightarrow T^2 \xrightarrow{\sigma} T^1$$

is zero.

<u>Proof.</u>

It follows from the fact that the image of the composition is in $m_T^2 1$. Q.E.D.

Similar results are true for $R^o(A)$. If

$$_oH^{i*} = \mathrm{Hom}_k^c({}_oH^i(S,A,A),k) \qquad i=1,2$$

and

$$^oT^i_A \text{ , or just } {}^oT^i \qquad \text{for } i=1,2$$

is the completion of $\mathrm{Sym}_V({}_oH^{i*})$ in the corresponding topology, then there is a morphism of complete local rings

$$\sigma_o = \sigma_o(A) : {}^oT^2 \longrightarrow {}^oT^1$$

such that

$$R^o(A) \simeq {}^oT^1 \underset{{}^oT^2}{\hat{\otimes}} \hat{V}$$

The canonical injections

$$ {}_oH^i = {}_oH^i(S,A,A) \longrightarrow H^i(S,A,A) = H^i $$

induces surjections

$$ T^i \longrightarrow {}^oT^i \qquad\qquad \text{for} \quad i = 1,2 $$

These surjections can be assumed to fit nicely into a commutative diagram

$$ \begin{array}{ccc} T^2 & \longrightarrow & {}^oT^2 \\ \sigma \downarrow & & \downarrow \sigma_o \\ T^1 & \longrightarrow & {}^oT^1 \end{array} $$

in such a way that the induced morphism

$$ R(A) \longrightarrow R^o(A) $$

makes the diagram

$$ \begin{array}{ccc} \mathrm{Def}^o(A/S,-) & \longrightarrow & \mathrm{Def}(A/S,-) \\ \uparrow & & \uparrow \\ \mathrm{Hom}(R^o(A),-) & \longrightarrow & \mathrm{Hom}(R(A),-) \end{array} $$

commutative.

We shall only sketch a proof of this commutativity since we will not use it much. We need an easy lemma, see (4.2.3) in [L1].

Lemma 2.3.

Consider the commutative diagram

$$ \begin{array}{ccc} R_1' & \longrightarrow & R_2' \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ R_1 & \longrightarrow & R_2 \\ & \searrow \; k \; \swarrow & \end{array} $$

whose objects and morphisms are in $\underline{l}$. Assume π_1 and π_2 surjective and $(\ker\pi_1)^2 = (\ker\pi_2)^2 = 0$. If A_1 is a lifting of A to S_{R_1}, and $A_2 = A_2 \underset{R_1}{\otimes} R_2$, then

$$H^2(S_{R_1}, A_1, A_1 \underset{R}{\otimes} \ker\pi_1) \longrightarrow H^2(S_{R_2}, A_2, A_2 \underset{R_2}{\otimes} \ker\pi_2)$$

maps the obstruction $\sigma(A_1)$ onto $\sigma(A_2)$.

Proof of the commulativity.

As in the "proof" of 2.2, let us assume $V = k$ and $H^1(S,A,A)$ finite as a k-vectorspace. We constructed R_n and σ_n in such a way that

$$R(A) = \varprojlim R_n \qquad \sigma = \varprojlim \sigma_n$$

In the graded case we shall use the notations

$$R^o(A) = \varprojlim {}^oR_n \qquad \sigma_o = \varprojlim (\sigma_o)_n$$

Now R_2 and ${}^oR_2$ represents this deformation functors on $\underline{l}_2$. If A_2 is the universal lifting of A to S_{R_2} , we easily see that $A_2 \underset{R_2}{\otimes} {}^oR_2$ is the graded universal lifting to So_{R_2} .

Let $n \geq 3$ and let A_{n-1} be a lifting of A_2 to $S_{R_{n-1}}$

By induction we may assume the commutativity of

$$\begin{array}{ccc} T^2_{n-1} & \longrightarrow & {}^oT^2_{n-1} \\ \Big\downarrow \sigma_{n-1} & & \Big\downarrow (\sigma_o)_{n-1} \\ T^1_{n-1} & \longrightarrow & {}^oT^1_{n-1} \end{array}$$

and that $A_{n-1} \underset{R_{n-1}}{\otimes} {}^oR_{n-1}$ is a graded lifting of $A_2 \underset{R_2}{\otimes} {}^oR_2$.

By 2.3 a commutative diagram

$$\begin{array}{ccc} H^{2*} & \longrightarrow & {}_{o}H^{2*} \\ \Big\downarrow \sigma_n & & \Big\downarrow (\sigma_o)_n \\ T^1{}_n & \longrightarrow & {}^{o}T^1{}_n \end{array}$$

is found, hence then is a commutative diagram

$$\begin{array}{ccc} R_n & \longrightarrow & {}^{o}R_n \\ \Big\downarrow \pi_n & & \Big\downarrow {}^{o}\pi_n \\ R_{n-1} & \longrightarrow & {}^{o}R_{n-1} \end{array}$$

Since

$$\ker \pi_n \longrightarrow \ker {}^{o}\pi_n$$

is a surjective map of k-vectorspaces, we deduce from the surjectivity of

$$H^1(S,A,A) \underset{k}{\otimes} \ker \pi_n \longrightarrow H^1(S,A,A) \underset{k}{\otimes} \ker {}^{o}\pi_n$$

(using 1.2 and 1.7) that there is a lifting A_n and A_{n-1} to S_{R_n} such that $A_n \underset{R_n}{\otimes} {}^{o}R_n$ is a graded lifting of $A_{n-1} \underset{R_{n-1}}{\otimes} {}^{o}R_{n-1}$.

The case $V \neq k$ makes no trouble.

Q.E.D.

From this we get

<u>Proposition 2.4</u>

Let V be a regular local ring such that $S \to A$ is liftable to $V/m_V 2$. Then

i) If $R(A)$ is a regular local ring, so is $R^o(A)$.

ii) If $R^o(A)$ is regular, then the morphism

$$R(A) \longrightarrow {}^oR(A)$$

splits

<u>Proof</u>

i) follows from the commutativity of the diagram

$$\begin{array}{ccc} H^{2*} & \longrightarrow & {}_oH^{2*} \\ \downarrow & & \downarrow \\ T^2 & \longrightarrow & {}_oT^2 \\ \sigma \downarrow & & \downarrow \sigma_o \\ T^1 & \longrightarrow & {}_oT^1 \end{array}$$

using 2.2.a

If $R^o(A)$ is regular, then ${}^oT^1 = R^o(A)$.

The obvious surjection

$$H^1 = H^1(S,A,A) = \amalg_\nu H^1(S,A,A) \longrightarrow {}_oH^1(S,A,A) = {}_oH^1$$

induces an injection

$${}^oT^1 \longrightarrow T^1$$

which defines a one-sided inverse of $R(A) \longrightarrow R^o(A)$.

Q.E.D.

The surjections

$$H^i \longrightarrow {}_oH^i \qquad \text{for } i = 1,2$$

induce morphisms

$$ {}^{o}T^{i} \longrightarrow T^{i} $$

If the corresponding diagram

$$\begin{array}{ccc} {}^{o}T^{2} & \longrightarrow & T^{2} \\ \sigma_{o}\downarrow & & \downarrow\sigma \\ {}^{o}T^{1} & \longrightarrow & T^{1} \end{array}$$

commutes , then $R(A)\longrightarrow R^{o}(A)$ splits. In general there seem to be no reasons for this diagram to commute. However imposing some rather natural conditions on the graded algebra A , the commutativity can be proved.

Definition 2.5

We say that $S\longrightarrow A$ has negative grading (resp. positive grading) if

$$ {}_{\nu}H^{1}(S,A,A) = 0 \qquad \text{for} \quad \nu > 0 $$

(resp. $$ {}_{\nu}H^{1}(S,A,A) = 0 \qquad \text{for} \quad \nu < 0 $$

If A has positive or negative grading, then the diagram above commutes, proving

Theorem 2.6

If $S \rightarrow A$ has negative or positive grading, then

$$ R(A) \longrightarrow R^{o}(A) $$

splits as a local V-homomorphism.

In the same direction We have the following more general result.

Theorem 2.7

Assume $S \rightarrow A$ has negative (resp. positive) grading and put $B = A[T]$ with $\deg T = 1$ (resp $\deg T = -1$). Then there is a V-isomorphism

$$R^o(B) \simeq R(A)$$

We shall need some preparations.

Let A and B be graded S-algebras and

$$\Psi : B \longrightarrow A$$

an S-algebra homomorphism, not necessarily graded. For every $i \geq 0$, Ψ induces maps

$$\Psi^i : H^i(S,A,A) \longrightarrow H^i(S,B,A)$$

$$\Psi i : H^i(S,B,B) \longrightarrow H^i(S,B,A)$$

Let $\Psi i/0$ be the composed map

$${}_oH^i(S,B,B) \longrightarrow H^i(S,B,B) \xrightarrow[\Psi i]{} H^i(S,B,A)$$

Lemma 2.8

If Ψ^i and $\Psi i/0$ are isomorphisms for $i = 1$ and injections for $i = 2$, then there is a local V-isomorphism

$$R(A) \simeq R^o(B)$$

Remark 2.8 a

Let $\pi : R \longrightarrow R'$ be a surjection in $\underline{1}$ such that $\ker \pi$ is a k-module via $R \longrightarrow k$. Look at

$$\begin{array}{ccccccc}
R & \longrightarrow & S_R & & & & \\
{\scriptstyle \pi}\downarrow & & \downarrow & & & & \\
R' & \longrightarrow & S_{R'} & \longrightarrow & B' & \xrightarrow{\Psi'} & A' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
k & \longrightarrow & S & \longrightarrow & B & \xrightarrow{\Psi} & A
\end{array}$$

where A' and Ψ' are liftings to $S_{R'}$ and B' is a graded lifting to $S_{R'}$. Consider the diagram

$$\begin{array}{ccccc}
 & & H^2(S,B,B) \underset{k}{\otimes} \ker\pi & \longleftarrow & {}_oH^2(S,B,B) \otimes \ker\pi \\
 & & \downarrow & \swarrow{\scriptstyle \Psi_2/0 \,\otimes\, \ker\pi} & \\
H^2(S,A,A) \underset{k}{\otimes} \ker\pi & \xrightarrow[\Psi^2 \otimes \ker\pi]{} & H^2(S,B,A) \underset{k}{\otimes} \ker\pi & &
\end{array}$$

By [L1], the obstructions for deforming A' and B' respectively map on the same element in $H^2(S,B,A) \otimes \ker\pi$.

Proof of 2.8

We shall use the notation

$$T^i_\Psi \qquad\qquad i = 1,2$$

for the completion of

$$\mathrm{Sym}_V\ (H^i(S,B,A)^*)$$

The morphisms

$$\Psi^i\ :\ H^i(S,A,A) \longrightarrow H^i(S,B,A)$$

$$\Psi i/o\ ;\ {}_oH^i(S,B,B) \longrightarrow H^i(S,B,A)$$

induce morphisms

$$T^i_A \longleftarrow T^i_\Psi$$

$${}^oT^i_B \longleftarrow T^i_\Psi.$$

which by the "proof" of 2.2 and by 2.8.a fit into a commutative diagram

$$\begin{array}{ccccc}
{}^oT^2_B & \twoheadleftarrow & T^2_\Psi & \twoheadrightarrow & T^2_A \\
\sigma_o \downarrow & & \downarrow & & \downarrow \sigma \\
{}^oT^1_B & \xleftarrow{\sim} & T^1_\Psi & \xrightarrow{\sim} & T^1_A
\end{array}$$

The horizontal maps are surjections and isomorphisms by the assumptions of 2.8. Q.E.D

Remark 2.8 b

If Ψ^i is an isomorphism for $i = 1$ and an injection for $i = 2$ the morphism

$$(\Psi^1)^{-1}\Psi_1 : H^1(S,B,B) \longrightarrow H^1(S,A,A)$$

induces a morphism

$$R(A) \longrightarrow R(B)$$

Now we turn to the proofs of 2.6 and 2.7

Proof of 2.7

Let

$$\Psi : B \longrightarrow A$$

be the composition

$$B = A[T] \longrightarrow A[T]/(T-1) \simeq A$$

and let j be the canonical injection

$$j : A \longrightarrow B$$

Let M be any B-module.

j induces maps

$$j_M^i : H^i(S,B,M) \longrightarrow H^i(S,A,M)$$

Using the exact sequence

$$\longrightarrow H^i(A,B,M) \longrightarrow H^i(S,B,M) \xrightarrow{j_M^i} H^i(S,A,M) \longrightarrow H^{i+1}(A,B,M) \longrightarrow$$

and the fact that

$$H^i(A,B,M) = 0 \qquad \text{for } i \geq 1$$

we deduce that j^i_M are isomorphisms for $i \geq 1$. However Ψ^i are the inverse maps of j^i_A for $i = 1,2$.
Hence by 2.8 it is enough to prove that

$$\Psi i/o \, : \, {}_oH^i(S,B,B) \longrightarrow H^i(S,B,A)$$

is an isomorphism for $i = 1$ and an injection for $i = 2$.
Look at the diagram

$$\begin{array}{ccccc}
{}_oH^i(S,B,B) & \longrightarrow & H^i(S,B,B) & \longrightarrow & H^i(S,B,A) \\
\downarrow j^i_B & & \downarrow j^i_B & & \downarrow j^i_A \\
{}_oH^i(S,A,B) & \longrightarrow & H^i(S,A,B) & \longrightarrow & H^i(S,A,A) \\
 & & \| & & \| \\
 & & H^i(S,A,A) \underset{k}{\otimes} k[T] & \longrightarrow & H^i(S,A,A)
\end{array}$$

where the lower horizontal map is induced by sending T to 1. If $\deg T = 1$ and if $i \geq 1$, $(\Psi^i)^{-1}\Psi i/o$ is given by the composition

$${}_oH^i(S,B,B) \simeq \coprod_{\nu=-\infty}^{o} {}_\nu H^i(S,A,A)T^{-\nu} \simeq \coprod_{\nu=-\infty}^{o} {}_\nu H^i(S,A,A) \longrightarrow H^i(S,A,A)$$

which is an injection for all $i \geq 1$. If A has negative grading, then by definition Ψ_1/o is an isomorphism. The case $\deg T = -1$ is similar.

Q.E.D.

<u>Proof of 2.6</u>

Let

$$\varphi \, : \, B \longrightarrow B/(T) = A$$

be the canonical surjection. Then

$$\varphi^i \, : \, {}_oH^i(S,A,A) \longrightarrow {}_oH^i(S,B,A)$$

are isomorphisms for $i \geq 1$. By 2.8 b there is a morphism

$$R^o(A) \longrightarrow R^o(B)$$

deduced form the commutative diagram

$$\begin{array}{ccc} {}^oT^2_A & \longrightarrow & {}^oT^2_B \\ \sigma_o(A)\downarrow & & \downarrow\sigma_o(B) \\ {}^oT^1_A & \longrightarrow & {}^oT^1_B \end{array}$$

The horizontal maps are induced by $(\varphi^i)^{-1} \circ \varphi i$. Moreover by 2.7 and its proof, the isomorphism

$$R^o(B) \simeq R(A)$$

is deduced from the commutative diagram

$$\begin{array}{ccc} {}^oT^2_B & \twoheadleftarrow & T^2_A \\ \sigma_o(B)\downarrow & & \downarrow\sigma(A) \\ {}^oT^1_B & \xleftarrow{\simeq} & T^1_A \end{array}$$

The horisontal maps are induced by

$$(\Psi^i)^{-1}o(\Psi i/o) \; : \; {}_oH^i(S,B,B) \longrightarrow H^i(S,A,A)$$

However if $\deg T = 1$, this morphism is given by

$${}_oH^i(S,B,B) \simeq \prod_{\nu=-\infty}^{o} {}_\nu H^i(S,A,A) \longrightarrow H^i(S,A,A)$$

which splits. Using a one-sided inverse, i.e. a projection for $i = 2$, a commutative diagram

$$\begin{array}{ccc} {}^oT^2_B & \longrightarrow & T^2_A \\ \sigma_o(B)\downarrow & & \downarrow\sigma(A) \\ {}^oT^1_B & \xrightarrow{\simeq} & T^1_A \end{array}$$

is found, inducing the isomorphism $R^o(B) \simeq R(A)$ We claim that the composed map

$$R^o(A) \longrightarrow R^o(B) \simeq R(A)$$

is a one-sided inverse of $R(A) \longrightarrow R^o(A)$. This is trivial if we look at the diagram

$$\begin{array}{ccccc} {}^oT_A^2 & \longrightarrow & {}^oT_B^2 & \longrightarrow & T_A^2 \\ \sigma_o(A) \Big\downarrow & o & \Big\downarrow & o & \Big\downarrow \sigma(A) \\ {}^oT_A^1 & \longrightarrow & {}^oT_B^1 & \xrightarrow{\sim} & T_A^1 \end{array}$$

The composition of the horizontal maps are induced by the obvious projections

$${}_oH^i(S,A,A) \longleftarrow H^i(S,A,A)$$

since $(\varphi^i)^{-1} \circ \varphi i$ are given by

$${}_oH^i(S,B,B) \simeq \prod_{\nu=-\infty}^{o} H^i(S,A,A)T^{-\nu} \longrightarrow {}_oH^i(S,A,A)$$

sending T to 0 . The case $\deg T = -1$ is similarily treated.

Q.E.D.

Theorem 2.7 can be generalized in the following way. Let

$$C = A[T]_T = B_T$$

be the localization of B in the multiplicative system $\{1,T,T^2,\ldots\}$ and put $\deg T = 1$. Then for any finitely generated S-algebra A , then is an isomorphism

$$R^o(C) \simeq R(A)$$

We omit details of a proof.

The conditions of negative and positive grading on $S \to A$ are only reasonable if the graded ring S sits in degree zero However, if $S \to A$ is any graded morphism and S is S_o smooth, then

$$R(A) \longrightarrow R^o(A)$$

splits if $S_o \to A$ has negative or positive grading. In beeing more precise we shall assume that the "choice" of the liftings of S_o and S are compatible, i.e. for any $R \in \mathrm{ob}\underline{l}$, there is a morphism $(S_o)_R \to S_R$ such that if $R \to R'$ is in $\underline{l}$, then there is a commutative diagram

$$\begin{array}{ccc} (S_o)_R & \longrightarrow & S_R \\ \downarrow & & \downarrow \\ (S_o)_{R'} & \longrightarrow & S_{R'} \end{array}$$

Then the maps

$$\mathrm{Def}^o(A/S,-) \longrightarrow \mathrm{Def}^o(A/S_o,-)$$

$$\mathrm{Def}\ (A/S,-) \longrightarrow \mathrm{Def}\ (A/S_o,-)$$

are well defined and they are easily seen to be smooth. Therefore the morphisms

$$R^o(A/S) \longleftarrow R^o(A/S_o)$$

$$R\ (A/S) \longleftarrow R\ (A/S_o)$$

are still smooth. These maps fit into a commutative diagram

$$\begin{array}{ccc} R^o(A/S) & \longleftarrow & R^o(A/S_o) \\ \uparrow & & \uparrow \\ R\ (A/S) & \longleftarrow & R\ (A/S_o) \end{array}$$

The right hand vertical morphism splits because $S_o \longrightarrow A$ has negative or positive grading. By definition of smoothness the left hand vertical morphism also splits.

CHAPTER 3

Relations to projective geometry.

As we know the graded theory of algebras are closely related to projective geometry. In what follows we shall compare the groups ${}_{\nu}H^i(S,A,M)$ with $A^i(S,X,\tilde{M}(\nu))$ when $X = \mathrm{Proj}(A)$. Moreover if

$$\varphi : B \longrightarrow A$$

is a surjective graded morphism and

$$f : \mathrm{Proj}(A) \longrightarrow \mathrm{Proj}(B)$$

is the induced embedding, we shall relate the groups ${}_{\nu}H^i(B,A,M)$ to $A^i(S,f,\tilde{M}(\nu))$.

Let X be any S-scheme, $\underline{M}$ any quasicoherent O_X-Module and let $f : X \to Y$ be a morphism of S-schemes. Then there are groups

$$A^i(S,X,\underline{M}) \quad \text{and} \quad A^i(S,f,\underline{M})$$

for every $i \geq 0$. Using [L1] we shall summarize some properties needed in the sequel.

i) (3.1.12) in [L1]states that $A^i(S,X,\underline{M})$ is the abutment of a spectral sequence given by the term

$$E_2^{p,q} = H^p(X,\underline{A}^q(S,\underline{M}))$$

If $U = \mathrm{Spec}(A)$ is an open affine subscheme of X, the O_X-Module $\underline{A}^q(S,\underline{M})$, or just $\underline{A}^q(\underline{M})$, is given by

$$\underline{A}^q(\underline{M})(U) = H^q(S,A,\underline{M}(U))$$

If X is affine, say $X = \mathrm{Spec}(A)$, and $\underline{M} = \widetilde{M}$ for some A-module M, we deduce

$$A^i(S,X,\underline{M}) = H^i(S,A,M)$$

If X is S-smooth, we find

$$A^i(S,X,\underline{M}) = H^i(X,\theta_X \underset{S}{\otimes} \underline{M})$$

where $\theta_X = \underline{A}^o(O_X)$ is the sheaf of S-derivations.

ii) By (3.1.14) in [L1] $A^i(S,f,\underline{M})$ is the abutment of the spectral sequence given by

$$E_2^{p,q} = H^p(Y,\underline{A}^q(f,\underline{M}))$$

If $V = \mathrm{Spec}(B)$ is any open affine subscheme of Y, then by definition

$$A^q(f,\underline{M})(V) = A^q(B,f^{-1}(V),\underline{M})$$

Therefore if f is affine, say $f^{-1}(V) = \mathrm{Spec}(A)$,

$$A^q(B,f^{-1}(V),\underline{M}) = H^q(B,A,H^o(f^{-1}(V),\underline{M}))$$

iii) Let $Z \subseteq X$ be locally closed. By (3.1.16) there is an exact sequence

$$\longrightarrow A_Z^n(S,X,\underline{M}) \longrightarrow A^n(S,X,\underline{M}) \longrightarrow A^n(S,X-Z,\underline{M}) \longrightarrow A_Z^{n+1}(S,X,\underline{M}) \longrightarrow$$

where the groups $A_Z^n(S,X,\underline{M})$ is the abutment of a spectral sequence given by the term

$$E_2^{p,q} = A^p(S,X,\underline{H}_Z^q(\underline{M}))$$

If $X = \mathrm{Spec}(A)$ and $Z = V(I)$ for a suitable ideal $I \subseteq A$ we write

$$H_I^n(S,A,M) = A_Z^n(S,X,\widetilde{M})$$

iv) Let $f : X \to Y$ be an affine morphism of S-schemes. By (3.2.3) there is a long exact sequence

$$\longrightarrow A^n(S,f,\underline{M}) \longrightarrow A^n(S,X,\underline{M}) \longrightarrow A^n(S,Y,f_*\underline{M}) \longrightarrow A^n(S,f,\underline{M}) \longrightarrow$$

Let S be noetherian and let A and B be finitely generated, positively graded S-algebras generated by its elements of degree 1. Assume $A_o = B_o = S$. Let

$$\varphi : B \longrightarrow A$$

be a surjective graded S-algebra morphism and let

$$f : X = \mathrm{Proj}(A) \longrightarrow \mathrm{Proj}(B) = Y$$

be the corresponding embedding. Put

$$m = \coprod_{\nu=1}^{\infty} A_\nu \qquad \text{and} \qquad X' = \mathrm{Spec}(A) - V(m)$$

Let

$$\pi : X' \longrightarrow X$$

be the obvious morphism. π is an affine smooth surjection.

If M is a graded A-module, we shall denote by M_a the localization of M in $\{1,a,a^2,\dots\}$. Let $M_{(a)}$ be the homogeneous piece of M_a of degree zero.

Let $b \in B$ such that $a = \varphi(b)$. Since

$$B_{(b)} \longrightarrow B_b$$

is flat, a theorem from [A] gives the isomorphism

$$H^q(B_{(b)},A_{(a)},M_a) \simeq H^q(B_b,A_{(a)} \underset{B_{(b)}}{\otimes} B_b,M_a)$$

However

$$A_{(a)} \otimes_{B_{(b)}} B_b \simeq A_a$$

Therefore

$$H^q(B_{(b)},A_{(a)},M_a) \simeq H^q(B_b,A_a,M_a) \simeq H^q(B,A,M)_a$$

Hence

$$H^q(B_{(b)},A_{(a)},M_{(a)}) \simeq H^q(B,A,M)_{(a)}$$

Put

$$D_+(b) = \mathrm{Spec}(B_{(b)}) \subseteq Y$$

Then by (ii)

$$\underline{A}^q(f,\widetilde{M}(\nu))(D_+(b)) = H^q(B_{(b)},A_{(a)},M(\nu)_{(a)})$$

proving

$$A^q(F,\widetilde{M}(\nu)) \simeq \widetilde{H^q(B,A,M)}(\nu)$$

Using (i) we find

$$\underline{A}^q(B,\widetilde{M})(D(a)) = H^q(B,A_a,M_a) = H^q(B,A,M)_a$$

Therefore

$$\underline{A}^q(B,\widetilde{M}) \simeq \widetilde{H^q(B,A,M)}$$

This proves

$$\pi_*(\underline{A}^q(B,\widetilde{M})) \simeq \coprod_\nu A^q(f,\widetilde{M}(\nu))$$

<u>Lemma 3.1</u>

With notations as above there is an isomorphism

$$A^i(S,f,\widetilde{M}(\nu)) \simeq {}_\nu A^i(B,X',\widetilde{M})$$

where ${}_\nu A^i(B,X',\widetilde{M})$ is the homogeneous piece of $A^i(B,X',\widetilde{M})$ of degree ν.

<u>Proof</u>

Going back to the definitions of $A^i(S,f,\widetilde{M}(\nu))$ and

$A^i(B,X',\widetilde{M})$ in [L1] we deduce a morphism

$$A^i(S,f,\widetilde{M}(\nu)) \longrightarrow {}_\nu A^i(B,X',\widetilde{M})$$

The corresponding morphism of spectral sequences

$$H^p(X,A^q(f,\widetilde{M}(\nu))) \longrightarrow {}_\nu H^p(X',\underline{A}^q(B,\widetilde{M})) \simeq {}_\nu H^p(X,\pi_*\underline{A}^q(B,\widetilde{M}))$$

is an isomorphism for every p and q

Q.E.D.

Theorem 3.2

If $\varphi : B \to A$ is surjective and if

$$\text{depth}_m M \geq n \qquad n \text{ an integer}$$

then the morphisms

$${}_\nu H^i(B,A,M) \longrightarrow A^i(S,f,\widetilde{M}(\nu))$$

are isomorphisms for $i < n$ and injections for $i = n$

Proof

By iii) there is a long exact sequence

$$\longrightarrow H^i_m(B,A,M) \longrightarrow H^i(B,A,M) \longrightarrow A^i(B,X',\widetilde{M}) \longrightarrow H^{i+1}_m(B,A,M) \longrightarrow$$

Since $\text{depth} M \geq n$, we conclude that

$$H^q_m(M) = 0 \qquad \text{for} \quad q \leq n-1$$

Moreover $H^0(B,A,-) = 0$ since φ is surjective. By the spectral sequence of iii) we deduce

$$H^i_m(B,A,M) = 0 \qquad \text{for } i \leq n$$

Q.E.D.

Corollary 3.3

If $\text{depth}_m A \geq 2$

$${}_\nu H^1(B,A,A) \simeq A^1(S,f,O_X(\nu))$$

$$_{\nu}H^2(B,A,A) \hookrightarrow A^2(S,f,O_X(\nu))$$

are isomorphisms and injections respectively.

Let us apply this result to the case $S = k$, k a field. We denote by

$$\mathrm{Hilb}_f(-)$$

the local Hilbert functor relative to Y at f, defined on the category $\underset{\sim}{1}$. (See the beginning of chapter 2 and use $V = k$). Let

$$\mathrm{Def}^o(\varphi,-)$$

by the functor $\mathrm{Def}^o(A/B,-)$ defined in chapter 2 using trivial liftings of B.

Corollary 3.4

If $\mathrm{depth}_{\underset{\sim}{m}} A \geq 2$, then there is an isomorphism of functor

$$\mathrm{Def}^o(\varphi,-) \simeq \mathrm{Hilb}_f(-)$$

on $\underset{\sim}{1}$.

Proof

Both functors are prorepresentable. By (2.2) $\mathrm{Def}^o(\varphi,-)$ is prorepresented by

$$\mathrm{Sym}({}_oH^1(B,A,A)^*)^\wedge \ \hat{\underset{\mathrm{Sym}({}_oH^2(B,A,A)^*)^\wedge}{\otimes}} k$$

Using (5.1.1) in [L1], $\mathrm{Hilb}_f(-)$ is prorepresented by the object

$$\mathrm{Sym}(A^1(f,O_X)^*)^\wedge \ \hat{\underset{\mathrm{Sym}(A^2(f,O_X)^*)^\wedge}{\otimes}} k$$

The natural morphism of functors

$$\mathrm{Def}^o(\varphi,-) \longrightarrow \mathrm{Hilb}_f(-)$$

corresponds to a morphism between their prorepresenting objects. This is nothing but the morphism induced by the natural maps in (3.2)

Q.E.D.

Assume B to be k-free and

$$f : X \subseteq \mathbb{P}^n_k$$

to be the induced embedding. In this case $\mathrm{Hilb}_f(-)$ is also denoted by $\mathrm{Hilb}_X(-)$. Both $\mathrm{Hilb}_X(-)$ and $\mathrm{Def}^o(\varphi,-)$ are easily defined on $\underline{l}$ for V arbritary, and by the same arguments as before there is an isomorphism of functors

$$\mathrm{Def}^o(\varphi,-) \xrightarrow{\simeq} \mathrm{Hilb}_X(-)$$

on $\underline{l}$ whenever depth $A \geq 2$. Even if depth $A \geq 1$ we deduce this isomorphism in some cases. In fact, the sequence

$$0 \longrightarrow {}_oH^1(B,A,A) \longrightarrow A^1(k,f,O_X) \longrightarrow {}_oH^1(B,A,H^1_m(A)) \longrightarrow {}_oH^2(B,A,A) \longrightarrow A^2(k,f,O_X)$$

is exact. The isomorphism therefore follows from

$${}_oH^1(B,A,H^1_m(A)) = 0$$

Recall that if $I = \ker \varphi \subseteq B$

$${}_oH^1(B,A,H^1_m(A)) = {}_o\mathrm{Hom}_A(I/I^2,H^1_m(A))$$

Furthermore $X \subseteq \mathbb{P}^n_k = P$ and if $n \geq 2$

$$H^1_m(A) \simeq \coprod H^1(P,\tilde{I}(\nu))$$

If we define c by

$$c = \max\{\nu \mid H^1(P,\tilde{I}(\nu)) \neq 0\}$$

and

$$s = \min \{\deg f_i \mid \{f_i, -, f_r\} \text{ is a minimal set of generators}$$

of I}

then

$${}_{o}H^{i}(B,A,H^{1}_{m}(A)) = 0 \qquad \text{for } c < s$$

In [E] we find more or less a direct proof of (3.4).

So far we have concentrated on deformations of embeddings. One may ask for the relationship between the groups

$${}_{\nu}H^{i}(S,A,M)$$

and

$$A^{i}(S,X,\widetilde{M}(\nu))$$

This is given by our next theorem

Theorem 3.5

There are canonical morphisms

$${}_{\nu}H^{i}(S,A,M) \longrightarrow A^{i}(S,X,\widetilde{M}(\nu))$$

for any $i \geq o$ and any ν. If $n \geq 1$ and if $\operatorname{depth}_m M \geq n+2$, then the morphisms above are bijective for $1 \leq i < n$ and injective for $i = n$.

Proof

Consider the following two exact sequences

$$\longrightarrow H^{i}_{m}(S,A,M) \longrightarrow H^{i}(S,A,M) \longrightarrow A^{i}(S,X',\widetilde{M}) \longrightarrow H^{i+1}_{m}(S,A,M) \longrightarrow$$

$$\longrightarrow A^{i}(S,\pi,\widetilde{M}) \longrightarrow A^{i}(S,X',\widetilde{M}) \longrightarrow A^{i}(S,X,\pi_{*}\widetilde{M}) \longrightarrow A^{i+1}(S,\pi,\widetilde{M})$$

with

$$\pi : X' = \operatorname{Spec}(A) - V(m) \longrightarrow X = \operatorname{Proj}(A)$$

as before. The spectral sequence given by

$$E_2^{p,q} = H^p(X, \underline{A}^q(\pi, \widetilde{M}))$$

converges to

$$A^{p+q}(S,\pi,M)$$

$\underline{A}^q(\pi,\widetilde{M})$ is defined by

$$\underline{A}^q(\pi,\widetilde{M})(D_+(a)) = A^q(A_{(a)},A_a,M_a)$$

and it is easy to see that

$$\underline{A}^q(\pi,\widetilde{M}) = \begin{cases} 0 & q \neq 0 \\ \pi_*\widetilde{M} & q = 0 \end{cases}$$

Since depth $M \geq n+2$ then

$$A^i(S,\pi,M) = H^i(X,\pi_*\widetilde{M}) = \amalg H^i(X,M(\nu)) = 0 \qquad \text{for } 1 \leq i \leq n$$

Furthermore

$$H^i_m(M) = 0 \qquad \text{for } i \leq n+1$$

implying that

$$H^i_m(S,A,M) = 0 \qquad \text{for } i \leq n+1$$

The theorem now follows from the two exact sequences stated at the beginning of this proof.

Q.E.D.

Corollary 3.6

If $\text{depth}_m A \geq 3$ and

$$A^1(S,X,O_X(\nu)) = 0$$

for every ν , then

$$H^1(S,A,A) = 0$$

In the smooth case

$$A^1(S,X,O_X(\nu)) \simeq H^1(X,\theta_X(\nu))$$

and (3.6) reduces to a rigidity theorem of Schlessinger; see (2.2.6) in [K,L] . See also [Sv].

Corollary 3.7

If $\mathrm{depth}_m A \geq 4$ and

$$A^2(S,X,O_X(\nu)) = 0$$

for every ν, then

$$H^2(S,A,A) = 0$$

If X has only a finite number of nonsmooth points, then

$$H^1(X,A^1(O_X(\nu)) = 0$$

Moreover if the non-smooth points are complete intersections

$$H^0(X,\underline{A}^2(O_X(\nu)) = 0$$

In this case we conclude

$$A^2(S,X,O_X(\nu)) \simeq H^2(X,\theta_X(\nu))$$

We will end this chapter by proving a geometic variant of (2.7) due to Pinkham [P]. We also need (3.4).

Let R be k-free and $\varphi : R \longrightarrow A$ be surjective, corresponding to $X = \mathrm{Proj}(A) \subseteq \mathbb{P}^n_k$. Look at the diagram

$$\begin{array}{ccc} R[T] & \longrightarrow & R \\ \Big\downarrow \bar{\varphi} & & \Big\downarrow \varphi \\ B = A[T] & \longrightarrow & A \end{array}$$

where $\bar{\varphi} = \varphi \underset{k}{\otimes} \mathrm{id}_{k[T]}$ and where the horizontal maps are induced by sending T to 1. Put $\deg T = 1$.

Clearly

$$\mathrm{Def}^{o}(\bar{\varphi},-) \longrightarrow \mathrm{Def}^{o}(B/k,-)$$

is smooth. Hence

$$R^{o}(B) \longrightarrow R^{o}(\bar{\varphi})$$

is smooth. If A has negative grading

$$R(A) \simeq R^{o}(B)$$

The composition

$$R(A) \simeq R^{o}(B) \longrightarrow R^{o}(\bar{\varphi})$$

is therefore smooth.

Moreover if depth $A \geq 1$ then depth $B \geq 2$. Using (3.4) we find

$$\mathrm{Def}^{o}(\bar{\varphi},-) \simeq \mathrm{Hilb}_{\bar{X}}(-)$$

whenever $\bar{X} = \mathrm{Proj}(B)$. This proves

<u>Theorem 3.8</u>

Let X be a closed subscheme of $\mathbb{P}^n_k$ and let A be its minimal cone. If

$$\bar{X} = \mathrm{Proj}(A[T])$$

is its projective cone in $\mathbb{P}^{n+1}_k$ and if A has negative grading, then there is a smooth morphism of functors

$$\mathrm{Hilb}_{\bar{X}}(-) \longrightarrow \mathrm{Def}(A/k,-)$$

on $\underline{1}$ (V arbritary)

CHAPTER 4

Positive and negative grading

In this paragraph we shall see that if

$$X \subseteq \mathbb{P}_S^N$$

is closed and satisfies some weak conditions, then after a suitable twisting the minimal cone of the corresponding embedding will have positive of negative grading.

Suppose S noetherian and let

$$A = \coprod_{\nu=0}^{\infty} A_\nu$$

be a graded $A_o = S$ algebra of finite type, generated by A_1. Denote by $\mathfrak{m}$ the angmentation ideal of A; i.e.

$$\mathfrak{m} = \coprod_{\nu=1}^{\infty} A_\nu$$

Assume moreover

$$\operatorname{depth}_{\mathfrak{m}} A \geq 1$$

Let M be any graded A-module and put

$$M_{(d)} = \coprod_{\nu=-\infty}^{\infty} M_{d\nu}$$

In what follows we shall relate the groups

$$H^i(S,A,M)$$

to the groups

$$H^i(S,A_{(d)},M_{(d)})$$

Lemma 4.1

If $X' = \operatorname{Spec}(A) - V(m)$ and $X'_{(d)} = \operatorname{Spec}(A_{(d)}) - V(m_{(d)})$ then the groups

$$A^i(S,X',\widetilde{M})_{(d)} \simeq A^i(S,X'_{(d)},\widetilde{M}_{(d)})$$

are isomorphic for every i

Proof

The canonical morphism $A_{(d)} \hookrightarrow A$ induces a morphism of schemes

$$X' \longrightarrow X'_{(d)}$$

thus a homomorphism

$$A^i(S,X',\widetilde{M}) \longrightarrow A^i(S,X'_{(d)},\widetilde{M})$$

It suffices to prove that the corresponding morphism of spectral sequences

$$H^p(X',\underline{A}^q(\widetilde{M}))_{(d)} \longrightarrow H^p(X'_{(d)},\underline{A}^q(\widetilde{M}_{(d)}))$$

is an isomorphism for every p and q.

Consider the commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X'_{(d)} \\ \downarrow \pi & & \downarrow \pi \\ \mathrm{Proj}(A) = X & \xrightarrow{\sim} & X_{(d)} = \mathrm{Proj}(A_{(d)}) \end{array}$$

Then

$$H^p(X',\underline{A}^q(\widetilde{M}))_{(d)} \simeq H^p(X,\pi_*\underline{A}^q(\widetilde{M}))_{(d)} \simeq \coprod_\nu H^p(X,A^q(\widetilde{M})(d\nu)) \simeq$$

$$\coprod_\nu H^p(X_{(d)},A^q(\widetilde{M}_{(d)})(\nu)) \simeq H^p(X_{(d)},\pi_*\underline{A}^q(\widetilde{M}_{(d)})) \simeq H^p(X'_{(d)},\underline{A}^q(\widetilde{M}_{(d)}))$$

This will prove

Theorem 4.2

Let n be an integer and assume $\mathrm{depth}_{\mathfrak{m}} M \geq n+2$

Then

$$H^i(S,A,M)_{(d)} \simeq H^i(S,A_{(d)},M_{(d)})$$

are isomorphic for $i \leq n$ and for every $d \geq 1$.

Proof

Consider the exact sequences

$$\longrightarrow H^i_m(S,A,M)_{(d)} \longrightarrow H^i(S,A,M)_{(d)} \longrightarrow A^i(S,X',\widetilde{M})_{(d)} \longrightarrow H^{i+1}_m(S,A,M)_{(d)} \longrightarrow$$

$$\longrightarrow H^i_{m_{(d)}}(S,A_{(d)},M_{(d)}) \longrightarrow H^i(S,A_{(d)},M_{(d)}) \longrightarrow A^i(S,X'_{(d)},\widetilde{M}_{(d)}) \longrightarrow H^{i+1}_{m_{(d)}}(S,A_{(d)},$$

$$M_{(d)}) \longrightarrow$$

Since $\operatorname{depth} M \geq n+2$ is equivalent to the conditions

$$M \simeq \coprod_{\nu} H^0(X,\widetilde{M}(\nu))$$

$$\coprod_{\nu} H^i(X,\widetilde{M}(\nu)) = 0 \qquad \text{for } 1 \leq i \leq n$$

we easily deduce

$$\operatorname{depth} M_{(d)} \geq n+2$$

Hence

$$H^i_m(S,A,M) = H^i_{m_{(d)}}(S,A_{(d)},M_{(d)}) = 0 \qquad \text{for } i \leq n+1$$

Q.E.D.

We are specially interested in (4.2) for the case $n = 1$ and $M = A$. Let R be a graded S-free algebra, generated by R_1, such that

$$A = R/I$$

Put
$$P = \mathbb{P}^N_S = \operatorname{Proj}(R)$$

If $N \geq 2$

$$H^1_m(A) = \coprod_t H^1(P,\tilde{I}(t)) = \coprod_t H^1(\tilde{I}(t))$$

Futhermore by assumption depth $A \geq 1$, thus

$$H^o_m(A) = 0$$

Recall also

$$H^{i+1}_m(A) = \coprod_t H^i(X,O_X(t)) = \coprod_t H^i(O_X(t)) \qquad \text{for } i \geq 1$$

<u>Proposition 4.3</u>

Let $d \geq 1$ and assume

$$H^1(X,O_X(t+1)) = 0 \qquad H^1(P,\tilde{I}(t+1)) = 0$$

for all $t \geq d$. Then there is a natural isomorphism

$${}_{d\nu}H^1(S,A,A) \simeq {}_{\nu}H^1(S,A_{(d)},A_{(d)})$$

for $\nu \geq 1$

<u>Proof</u>

Consider the long exact sequences

$$\longrightarrow H^1_m(S,A,A)_{(d)} \longrightarrow H^1(S,A,A)_{(d)} \longrightarrow A^1(S,X',O_{X'})_{(d)} \longrightarrow H^2_m(S,A,A)_{(d)} \longrightarrow$$

$$\longrightarrow H^1_{m_{(d)}}(S,A_{(d)},A_{(d)}) \longrightarrow H^1(S,A_{(d)},A_{(d)}) \longrightarrow A^1(S,X'_{(d)},O_{X'_{(d)}}) \longrightarrow$$

$$H^2_{m_{(d)}}(S,A_{(d)},A_{(d)}) \longrightarrow$$

By assumption we have

$${}_{d\nu}H^1_m(S,A,A) = {}_{d\nu}H^o(S,A,H^1_m(A)) = {}_{d\nu}\mathrm{Der}_S(A,\coprod_t H^1(\tilde{I}(t))) = 0$$

$${}_{\nu}H^1_{m_{(d)}}(S,A_{(d)},A_{(d)}) = {}_{\nu}\mathrm{Der}_S(A_{(d)},\coprod_t H^1(\tilde{I}(dt))) = 0$$

since $\nu \geq 1$.

Futhermore

$$ {}_{d\nu}H^{o}(S,A,H^{2}_{m}(A)) = {}_{d\nu}\mathrm{Der}_{S}(A,\coprod_{t} H^{1}(O_{X}(t))) = 0 $$

$$ {}_{\nu}H^{o}(S,A_{(d)},H^{2}_{m_{(d)}}(A)) = {}_{\nu}\mathrm{Der}_{S}(A_{(d)},\coprod_{t} H^{1}(O_{X}(dt))) = 0 $$

Since

$$ H^{1}(R,A,H^{1}_{m}(A)) \longrightarrow H^{1}(S,A,H^{1}_{m}(A)) $$

is surjective and since

$$ {}_{d\nu}H^{1}(R,A,H^{1}_{m}(A)) = {}_{d\nu}\mathrm{Hom}_{A}(I/I^{2},\coprod_{t} H^{1}(\tilde{I}(t))) = 0 $$

we find

$$ {}_{d\nu}H^{1}(S,A,H^{1}_{m}(A)) = 0 \qquad \text{for } \nu \geq 1 $$

Similarely we prove that

$$ {}_{\nu}H^{1}(S,A_{(d)},H^{1}_{m_{(d)}}(A_{(d)})) = 0 \qquad \text{for } \nu \geq 1. $$

Hence

$$ {}_{d\nu}H^{2}_{m}(S,A,A) = 0 \qquad \text{for } \nu \geq 1 $$

$$ {}_{\nu}H^{2}_{m_{(d)}}(S,A_{(d)},A_{(d)}) = 0 \qquad \text{for } \nu \geq 1 $$

The exact sequences above together with (4.1) prove the proposition

Q.E.D.

Corollary 4.4

If $\mathrm{depth}_{m}A \geq 2$ and if ν is an integer such that

$$ H^{1}(X,O_{X}(d\nu+1)) = 0 $$

then

$$ {}_{d\nu}H^{1}(S,A,A) = 0 \quad \text{implies} \quad {}_{\nu}H^{1}(S,A_{(d)},A_{(d)}) = 0 $$

Proof

By assumption

$$H^1_m(A) = H^1_{m(d)}(A_{(d)})$$

Moreover

$$_{(d\nu+1)}H^2_m(A) = H^1(X,O_X(d\nu+1)) = 0$$

Thus

$$_{d\nu}H^2_m(S,A,A) = {}_{d\nu}\mathrm{Der}_S(A,H^2_m(A)) = 0$$

Using the long exact sequences of the proof of (4.3) we find a diagram

$$\begin{array}{ccc} 0 \longrightarrow {}_{d\nu}H^1(S,A,A) & \xrightarrow{\sim} & {}_{d\nu}A^1(S,X',O_{X'}) \longrightarrow 0 \\ & & \wr\wr \\ 0 \longrightarrow {}_{\nu}H^1(S,A_{(d)},A_{(d)}) & \longrightarrow & {}_{\nu}A^1(S,X'_{(d)},O_{X'_{(d)}}) \end{array}$$

which proves 4.4

Q.E.D.

Corollary 4.5 (Negative grading of $A_{(d)}$)

Assume $\mathrm{depth}_m A \geq 1$ and suppose there is a $d \geq 1$ such that

$$H^1(X,O_X(t+1)) = 0 \qquad H^1(P,\tilde{I}(t+1)) = 0$$

$$_tH^1(S,A,A) = 0 \qquad \text{for } t \geq d$$

Then

$$_{\nu}H^1(S,A_{(d)},A_{(d)}) = 0 \qquad \text{for } \nu \geq 1$$

Proof

Use 4.3 for $\nu = 1,2,\ldots.$

Q.E.D.

<u>Corollary 4.6</u> (Positive grading of $A_{(d)}$)

Assume depth $A \geq 2$. Suppose there is a $d \geq 1$ such that

$$_{(-t)}H^1(S,A,A) = 0 \qquad H^1(X,O_X(-t+1)) = 0 \qquad \text{for } t \geq d$$

then

$$_{\nu}H^1(S,A_{(d)},A_{(d)}) = 0 \qquad \text{for } \nu < 0$$

<u>Proof</u>

Use 4.4 for $\nu = -1,-2,\ldots$

Q.E.D.

Let us put 4.5 and 4.6 together in the following theorem

<u>Theorem 4.7</u>

Let $X = \mathrm{Proj}(A)$

a) If X is S-smooth, then there is a graded S-algebra B having negative grading such that

$$X \simeq \mathrm{Proj}(B)$$

b) If depth $A \geq 2$ and if there is an integer n such that

$$H^1(X,O_X(t)) = 0 \qquad \text{for } t \leq n$$

then there is an S-algebra B having positive grading such that

$$X \cong \mathrm{Proj}(B)$$

c) If X satisfies the conditions of a) and b) then

$$X \cong \mathrm{Proj}(B)$$

for an S-algebra B which has both positive and negative grading

<u>Proof</u>

If X is S-smooth, then

$${}_{\nu}H^1(S,A,A) = 0$$

for large ν. In fact the sequence

$$\longrightarrow H^o(S,A,H^1{}_m(A)) \longrightarrow H^1(S,A,A) \longrightarrow A'(S,X',O_{X'}) \longrightarrow$$

is exact and

$${}_{\nu}H^o(S,A,H^1{}_m(A)) = {}_{\nu}\mathrm{Der}_S(A, {}_tH^1(\tilde{I}(t))) = 0$$

$${}_{\nu}A^1(S,X',O_{X'}) = {}_{\nu}H^1(X',\theta_{X'}) = {}_{\nu}H^1(X,\pi_*\theta_{X'}) = 0$$

for large ν. Thus (4.5) proves a). (4.6) proves b) since

$${}_{\nu}H^1(S,A,A) = 0$$

for small ν. This follows from the surjection

$$H^1(R,A,A) \longrightarrow H^1(S,A,A)$$

and from the fact that

$${}_{\nu}H^1(R,A,A) = {}_{\nu}\mathrm{Hom}_A(I/I^2,A) = 0$$

for small ν.

Q.E.D.

For similar results, see [S3] and [M].

CHAPTER 5

The existence of a k-algebra which is unliftable to characteristic zero.

In [Se] Serre gives an example of a k-smooth projective variety X in characteristic p which cannot be lifted to characteristic zero. This means that for any complete local ring Λ of characteristic zero such that $\Lambda/m_{\Lambda}=k$, it is impossible to lift X to Λ. His varitey is of the form

$$X = Y/G$$

when Y is a complete intersection of dimention 3 and G is a finite group operating on Y without fixpoints. Futhermore the order of G divides p.

By (4.7 a) there exists a graded k-algebra B with negative grading such that

$$X = \mathrm{Proj}(B)$$

Hence (2.6) proves that B cannot be lifted to any (noetherian) complete local ring Λ of characteristic zero. In fact the example of Serre satisties even (4.7 c), thus proving the existence of a graded k-algebra C satisfying ${}_{\nu}H^1(k,C,C) = 0$ for $\nu \neq 0$, such that $X = \mathrm{Proj}(A)$. (2.6) reduces to the almost trivial result

$$R^0(C) \simeq R(C)$$

Clearly C is unliftable to any complete local ring Λ of characteristic zero.

The reason why Serre's example works is obviously that p, the characteristic of k, divides the order of G. To see this, let us prove

Theorem 5.1

Let $B \longrightarrow A$ be an S-algebrahomomorphism having a B-linear retraction. Let $I \subseteq A$ be an ideal such that the comosed morphism

$$U = \mathrm{Spec}(A) - V(I) \hookrightarrow \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(B)$$

is étale. If $\mathrm{depth}_I A \geq n+2$, then there is an injection

$$H^i(S,B,B) \hookrightarrow H^i(S,A,A)$$

for $i \leq n$.

Proof

By étaleness $A^i(B, U, O_U) = 0$ for all i, and the depth condition implies

$$H^i_I(B,A,A) = 0 \qquad \text{for } i \leq n+1$$

Using the exact sequence

$$\longrightarrow H^i_I(B,A,A) \longrightarrow H^i(B,A,A) \longrightarrow A^i(B,U,O_U) \longrightarrow$$

we conclude

$$H^i(B,A,A) = 0 \qquad \text{for } i \leq n+1$$

However, there is an exact sequence

$$\longrightarrow H^i(B,A,A) \longrightarrow H^i(S,A,A) \longrightarrow H^i(S,B,A) \longrightarrow H^{i+1}(B,A,A) \longrightarrow$$

Hence

$$H^i(S,A,A) \xrightarrow{\simeq} H^i(S,B,A) \qquad i \leq n$$

Since the injection $B \longrightarrow A$ has a B-linear retraction

$$H^i(S,B,B) \longrightarrow H^i(S,B,A)$$

is injective for any i

Q.E.D.

Apply (5.1) to the following situation. Let

$$Y = \mathrm{Proj}(A)$$

be a projective k-scheme, and let G be a finite group acting on A such that the graded injection

$$A^G \longrightarrow A$$

induces $Y \longrightarrow Y/G = X$. Assume $Y \longrightarrow X$ étale and suppose that the order of G does not divide the characteristic of the field k.

Corollary 5.2

a) If $\mathrm{depth}_m A \geq 3$ then

$$H^1(k,A,A) = 0 \quad \text{implies} \quad H^1(k, A^G, A^G) = 0$$

b) If $\mathrm{depth}_m A \geq 4$ then

$$H^2(k,A,A) = 0 \quad \text{implies} \quad H^2(k, A^G, A^G) = 0$$

Proof

Clearly $A^G \longrightarrow A$ has a retraction by the assumption on ord(G). Moreover the morphism

$$\mathrm{Spec}(A) - V(m) \longrightarrow \mathrm{Spec}(A^G)$$

is étale. We use (5.1)

Q.E.D.

Assume Y to be a complete intersection

$$Y = \mathrm{Proj}(A)$$

with $\mathrm{depth}_m A \geq 4$. Under the same conditions as in (5.2) we deduce

$$H^2(k, A^G, A^G) = 0$$

Clearly $X = \mathrm{Proj}(A^G)$ behaves as the example of Serre except for the condition on ord(G).

Remark

Clearly (5.2) is true not only for graded k-algebras A. In this case we suppose the condition on ord(G) and that the morphism

$$\mathrm{Spec}(A^G) - V(m) \longrightarrow \mathrm{Spec}(A)$$

is étale. For (5.2 a) , see [S2].

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