The Hilbert Scheme of Space Curves: Ghost terms, linkage and obstructed curves

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Summary of talks. We consider space curves X with homogeneous ideal I and Rao module M which often satisfy ${}_0\mathrm{Ext}^2_R(M,M) = 0$. We find necessary and sufficient conditions for unobstructedness, and we compute the dimension of the Hilbert scheme, $\mathrm{H}(d,g)$, at (X) under the sufficient conditions. In the diameter one case ($M_v = 0$ for $v \neq$ some c), the necessary and sufficient conditions coincide, and the obstructedness turns out to be equivalent to the non-vanishing of certain graded Betti numbers, e.g. there are repeated direct free factors ("ghost-terms"), in the minimal free resolution of I. Moreover by taking suitable deformations we show how to kill some of the "ghost-terms" in the free resolution of a curve of arbitrary diameter. For Buchsbaum curves of diameter at most 2, we simplify in this way the minimal resolution further, allowing us to see when a *singular point* of $\mathrm{H}(d,g)$ sits in the intersection of several, or lies in a unique irreducible component of $\mathrm{H}(d,g)$. As a consequence we get that any irreducible component of $\mathrm{H}(d,g)$ is generically smooth in the diameter 1 case.

1 Introduction.

The main object of these notes is the Hilbert scheme $\mathrm{H}(d,g)$ of space curves. As a set the Hilbert scheme $\mathrm{H}(d,g)$ is

$$\mathrm{H}(d,g) = \{(X) \mid X \subseteq \mathbb{P}^3 \text{ a curve of degree } d \text{ and arithmetic genus } g\}$$

where we by a curve mean an equidimensional, locally Cohen-Macaulay (lCM) one-dimensional subscheme of the projective 3-space $\mathbb{P}^3$ defined over an algebraically closed field k. The Hilbert scheme $\mathrm{H}(d,g)$ is actually the representing object of a correspondingly defined functor of (flat) deformations. Its existence as a scheme was proved in the late fifties by Grothendieck [14].

In these notes we will focus on the structure of the Hilbert scheme $\mathrm{H}(d,g)$. Much is still unknown concerning questions related to irreducibility, number of components, dimension and smoothness of $\mathrm{H}(d,g)$. For particular classes of space curves, some results are known, e.g. that the open subset of $\mathrm{H}(d,g)$ of arithmetically Cohen-Macaulay curves is smooth of known dimension [8]. Further progress to the questions was made in [35], [19], [20], [2], [3], [36], [37], [25], [12], [26], [7], [10], [11], [31] and [22], to mention some paper of particular relevance for these notes. In the talks we will report on the work [22] with a special look to obstructedness of Buchsbaum curves, interpreted by their minimal resolutions, and the set of irreducible components of $\mathrm{H}(d,g)$ containing Buchsbaum and arithmetically CM curves.

In the following we will use these

Notations and terminologies

• $R = k[X_0, X_1, X_2, X_3]$ a polynomial ring over $k = \overline{k}$.

• $\mathfrak{m} = (X_0, .., X_3)$ is the irrelevant maximal ideal.

• $\mathcal{I}_X :=$ the ideal sheaf of X in $\mathbb{P} = \mathbb{P}^3$.

• $\mathcal{N}_X := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_X, \mathcal{O}_X)$ the normal sheaf of $X \subseteq \mathbb{P}^3$.

• $\mathrm{H}^i(\mathcal{F}) := \mathrm{H}^i(\mathbb{P}, \mathcal{F})$, where $\mathcal{F}$ is a coherent $\mathcal{O}_{\mathbb{P}}$-Module.

• $\mathrm{H}^i_*(\mathcal{F}) = \oplus_v \mathrm{H}^i(\mathcal{F}(v))$.

• $h^i(\mathcal{F}) := \dim \mathrm{H}^i(\mathcal{F})$, and $\chi(\mathcal{F}) = \Sigma(-1)^i h^i(\mathcal{F})$.

• $\Gamma_{\mathfrak{m}}(N) := \oplus_v \ker(N_v \to \Gamma(\mathbb{P}, \tilde{N}(v)))$, where N is a graded R module of finite type.

• $\mathrm{H}^i_{\mathfrak{m}}(N)$ is the right derived functor of $\Gamma_{\mathfrak{m}}(N)$.

• $I = I(X) := \mathrm{H}^0_*(\mathcal{I}_X)$ is the homogeneous ideal of $X \subseteq \mathbb{P}$.

• $M = M(X) := \mathrm{H}^1_*(\mathcal{I}_X)$ is the Hartshorne-Rao module of $X \subseteq \mathbb{P}$.

• $E = E(X) := \mathrm{H}^1_*(\mathcal{O}_X)$.

• The postulation of $X \subseteq \mathbb{P}$ is the function $\gamma(v) = h^0(\mathcal{I}_X(v))$ defined over the integers.

• $\rho(v) = \rho_X(v) := h^1(\mathcal{I}_X(v))$ is the deficiency function, and

• $\sigma(v) = \sigma_X(v) := h^1(\mathcal{O}_X(v))$ is the specialization function.

• ${}_v\mathrm{ext}(N_1, N_2) = \dim {}_v\mathrm{Ext}^i(N_1, N_2)$ etc., i.e. we use small letters for the k-dimension.

Attached to a curve $X \subseteq \mathbb{P}^3$ we define the following numbers;

• $s(X) := \min\{n \mid h^0(\mathcal{I}_X(n)) \neq 0\}$,

• $e(X) = \max\{n \mid h^1(\mathcal{O}_X(n)) \neq 0\}$,

• $c(X) = \max\{n \mid h^1(\mathcal{I}_X(n)) \neq 0\}$, provided $M \neq 0$.

• $b(X) = \min\{n \mid h^1(\mathcal{I}_X(n)) \neq 0\}$, provided $M \neq 0$.

• Then $\operatorname{diam} M = c(X) - b(X) + 1$ is the diameter of M, or of X (let $\operatorname{diam} M = 0$ if $M = 0$).

• If $M = 0$, we say X is arithmetically Cohen-Macaulay (ACM).

• If $M \neq 0$ and $c(X) < s(X)$, or if $M = 0$, we say X has maximal rank.

• If $M \neq 0$, $c(X) < s(X)$ and $e(X) < b(X)$ (or if $M = 0$, and $e(X) < s(X)$), we say X has seminatural cohomology.

• If $\mathfrak{m} \cdot M(X) = 0$, then X is a Buchsbaum curve.

For the Hilbert scheme and its strata we say that

• X is *unobstructed* if the Hilbert scheme, $\mathrm{H}(d, g)$, is smooth at the corresponding point (X), otherwise X is *obstructed*.

• $\mathrm{H}_{\gamma,\rho} \subseteq \mathrm{H}(d,g)$ is the stratum consisting of curves X with constant cohomology, i.e. γ_X and ρ_X do not vary with X [25].

• $\mathrm{H}_{\gamma} \subseteq \mathrm{H}(d,g)$ consists of curves with constant postulation γ.

• $\mathrm{H}(d,g)_S \subseteq \mathrm{H}(d,g)$ is the open subscheme consisting of smooth connected curves.

• The curve in a small open irreducible subset of $\mathrm{H}(d,g)$ (small enough to satisfy all the openness properties which we want it to have) is called a *generic* curve of $\mathrm{H}(d,g)$.

• If a generic curve has a certain property, then there is an non-empty open irreducible subset of $\mathrm{H}(d,g)$ of curves having this property.

• A *generization* X' of X in $\mathrm{H}(d,g)$ is a generic curve of some irreducible subset of $\mathrm{H}(d,g)$ containing (X). Then X is a *specialization* of X' in $\mathrm{H}(d,g)$.

2 Preliminaries.

In this section we shortly review some results which we will need later on and we include references for a more thorough reading.

2.1 Cohomology groups and deformations.

Let N, N_i be graded R-modules of finite type, and recall that the right derived functor ${}_v\mathrm{Ext}^i_{\mathfrak{m}}(N,-)$ of ${}_v\Gamma^0_{\mathfrak{m}}(\mathrm{Hom}_R(N,-))$ is equipped with a spectral sequence ([13], exp. VI)

$$E_2^{p,q} = {}_v\mathrm{Ext}^p_R(N_1, \mathrm{H}^q_{\mathfrak{m}}(N_2)) \quad \text{converging to} \quad {}_v\mathrm{Ext}^{p+q}_{\mathfrak{m}}(N_1,N_2) \tag{1}$$

and a duality isomorphism ([21], Thm. 1.1, [19] Thm. 2.1.4 and Rem. 2.1.5 for a full proof);

$${}_v\mathrm{Ext}^i_{\mathfrak{m}}(N_2,N_1) \cong {}_{-v-4}\mathrm{Ext}^{4-i}_R(N_1,N_2)^\vee \qquad \text{where } (-)^\vee = \mathrm{Hom}_k(-,k), \tag{2}$$

which generalizes the usual Gorenstein duality ${}_v\mathrm{H}^i_{\mathfrak{m}}(N) \simeq {}_{-v}\mathrm{Ext}^{4-i}_R(N,R(-4))^\vee$. These groups fit into a long exact sequence ([13], exp. VI)

$$\to {}_v\mathrm{Ext}^i_{\mathfrak{m}}(N_1,N_2) \to {}_v\mathrm{Ext}^i_R(N_1,N_2) \to \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}}}(\tilde{N}_1,\tilde{N}_2(v)) \to {}_v\mathrm{Ext}^{i+1}_{\mathfrak{m}}(N_1,N_2) \to \tag{3}$$

which in particular relates the deformation theory of the curve $X \subseteq \mathbb{P}^3$, described by

$$\mathrm{H}^{i-1}(\mathcal{N}_X) \cong \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_X,\mathcal{I}_X) \qquad \text{for } i=1,2 \tag{4}$$

(cf. [19], Rem. 2.2.6 for a proof of this isomorphism), to the deformation theory of the quotient $A := R/I(X)$, or equivalently of the homogeneous ideal $I = I(X)$, described by

$${}_0\mathrm{Ext}^i_R(I,I) \qquad \text{for } i=1,2$$

(cf. [19], Thm. 2.2.1 and [36], Thm. 2.3 where Walter manages to get rid of the "generically complete intersection" assumption of [19], § 2.2), in the following exact sequence

$$0 \to {}_v\mathrm{Ext}^1_R(I,I) \to \mathrm{H}^0(\mathcal{N}_X(v)) \to {}_v\mathrm{Hom}_R(I,M) \xrightarrow{\alpha} {}_v\mathrm{Ext}^2_R(I,I) \to \mathrm{H}^1(\mathcal{N}_X(v)) \to \ . \tag{5}$$

Note that $M = \mathrm{H}^2_{\mathfrak{m}}(I)$ and ${}_v\mathrm{Ext}^2_{\mathfrak{m}}(I,I) \cong {}_v\mathrm{Hom}_R(I,M)$. In this situation Walter also proved that the map $\alpha : {}_v\mathrm{Hom}_R(I,M) \to {}_v\mathrm{Ext}^2_R(I,I)$ of (5) factorizes via ${}_v\mathrm{Ext}^2_R(M,M)$ in a natural way ([36], Thm. 2.3), the factorization is in fact given by a certain edge homomorphism of the spectral sequence (1) with $N_1 = M$, $N_2 = I$ and $p+q=4$.

Let Def_A be the deformation functor of deforming $A = R/I$ as a graded quotient of R, defined on the category of local artinian k-algebras with residue field k. Let Hilb_X be the corresponding deformation functor of $X \subseteq \mathbb{P}^3$ (i.e the local Hilbert functor at X) defined on the same category. Note that (5) and the assumption ${}_0\mathrm{Hom}(I,M) = 0$ lead to

$$Def_A \cong \mathrm{Hilb}_X \tag{6}$$

because the assumption leads to an isomorphism of their tangent spaces and an injection of their obstruction spaces. This was first proved in [17] (Thm. 3.6 and Rem. 3.7) in this generality using a cohomological proof, but the case $M = 0$ was already proved in [8] with a direct proof.

Lemma 1. *With $M = \mathrm{H}^1_*(\mathcal{I}_X)$ and $E = \mathrm{H}^1_*(\mathcal{O}_X)$ we have*

$${}_v\mathrm{Ext}^i_R(M,M) \cong {}_{-v-4}\,\mathrm{Ext}^{4-i}_R(M,M)^\vee \ , \quad i,v \in \mathbb{Z}.$$

Moreover if ${}_v\mathrm{Ext}^2_R(M,M) = 0$, then there is an exact sequence

$$0 \to {}_{-v-4}\mathrm{Ext}^1_R(M,M) \to {}_v\mathrm{Ext}^1_R(I,M)^\vee \to {}_{-v-4}\mathrm{Hom}_R(M,E) \to 0 \ .$$

Proof. The duality follows from (2) since M is artinian. By (2) we also have ${}_v\mathrm{Ext}^1_R(I,M)^\vee \cong {}_{-v-4}\mathrm{Ext}^3_{\mathfrak{m}}(M,I)$ and then we conclude by a standard exact sequence associated to (1). $\square$

2.2 Minimal resolutions, linkage and deformations.

Let X be a curve in $\mathbb{P}^3$. Related to a minimal resolution,

$$0 \to \oplus_i R(-i)^{\beta_{3,i}} \to \oplus_i R(-i)^{\beta_{2,i}} \to \oplus_i R(-i)^{\beta_{1,i}} \to I \to 0 \ , \tag{7}$$

of the homogeneous ideal I of X, we define the following invariant:

Definition 2.

$$\delta^j(v) = \sum_i \beta_{1,i} \cdot h^j(\mathcal{I}_X(i+v)) - \sum_i \beta_{2,i} \cdot h^j(\mathcal{I}_X(i+v)) + \sum_i \beta_{3,i} \cdot h^j(\mathcal{I}_X(i+v)) \ .$$

We also write $\beta_{j,i}(X) = \beta_{j,i}$. We get (see [22], Lem. 2.2, for a proof).

Lemma 3.

$${}_0\mathrm{ext}^1_R(I,I) - {}_0\mathrm{ext}^2_R(I,I) = 1 - \delta^0(0) = 4d + \delta^2(0) - \delta^1(0) = 1 + \delta^2(-4) - \delta^1(-4) \ .$$

Remark 4. *In [25] the numbers $1-\delta^0(0)$ and $\delta^1(-4)$ were called δ_γ and $\epsilon_{\gamma,\delta}$ respectively. By Lemma 3 it follows that the dimension of the tangent space to the Hilbert scheme $\mathrm{H}_{\gamma,\rho}$ at (X), which they show is $\delta_\gamma + \epsilon_{\gamma,\delta} - {}_0\mathrm{hom}(M,M) + {}_0\mathrm{ext}^1(M,M)$ (Thm. 4.2, page 173), is also equal to $1 + \delta^2(-4) - {}_0\mathrm{hom}(M,M) + {}_0\mathrm{ext}^1(M,M)$.*

In these notes to the talks we need the notion of linkage and the following proposition (see [20] , Prop. 3.12 and Prop. 3.8, for a proof). We consider $\mathcal{I}_{X/Y} := \mathcal{I}_X/\mathcal{I}_Y$ as the sheaf ideal of X in Y.

Definition 5. *Two curves X and X' are said to be (algebraically) CI-linked if there exists a complete intersection curve (a CI) Y such that*

$$\mathcal{I}_X/\mathcal{I}_Y \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{X'},\mathcal{O}_Y) \quad \text{and} \quad \mathcal{I}_{X'}/\mathcal{I}_Y \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X,\mathcal{O}_Y)\ .$$

Thus if Y is a CI of type (f,g), then we know that the dualizing sheaf satisfies $\omega_Y \cong \mathcal{O}_Y(f+g-4)$, and hence we get

$$\mathcal{I}_{X/Y} \cong \omega_{X'}(4-f-g) \qquad \text{and} \qquad \mathcal{I}_{X'/Y} \cong \omega_X(4-f-g)\ . \tag{8}$$

Proposition 6. *Let X and X' be curves in $\mathbb{P}^3$ which are linked (algebraically) by a complete intersection of two surfaces of degrees f and g. If*

$$\mathrm{H}^1(\mathcal{I}_X(v)) = 0 \quad \text{for } v = f, g, f-4 \text{ and } g-4,$$

then X is unobstructed (resp. generic) if and only if X' is unobstructed (resp. generic).

Now recall **Rao's theorem** concerning the form of a minimal resolution (7) of $I = I(X)$. Let

$$0 \to L_4 \xrightarrow{\sigma} L_3 \to L_2 \to L_1 \to L_0 \to M \to 0 \tag{9}$$

is the minimal resolution of $M = \mathrm{H}^1_*(\mathcal{I}_X)$. Then (7) and

$$0 \to L_4 \xrightarrow{\sigma\oplus 0} L_3 \oplus F_2 \to F_1 \to I \to 0 \tag{10}$$

are isomorphic. Here the composition of $L_4 \to L_3 \oplus F_2$ with the natural projection $L_3 \oplus F_2 \to F_2$ is zero ([34], Thm. 2.5).

It is well known that the Hartshorne-Rao module M is a biliaison invariant, up to twist ([34]). Moreover using (8) and the fact that $\omega_{X'} \cong \mathcal{E}xt^2(\mathcal{O}_{X'},\mathcal{O}_{\mathbb{P}}(-4))$ it is rather straightforward to find a resolution of $\mathcal{I}_{X'}$ in terms of the resolution (7) and some part of the resolution of the dual of M, by the mapping cone construction. Using the mapping cone construction twice, we get a nice relationship between the minimal resolution (10) of I and a free resolution of the homogeneous ideal of the bilinked curve Z. Indeed suppose we link X, first using a CI of type (f,g), then a CI of type (f',g'), and put $h = f'+g'-f-g$. Then

$$0 \to L_4(-h) \xrightarrow{\sigma(-h)\oplus 0} L_3(-h) \oplus F_{2Z} \to F_1(-h) \oplus R(-f') \oplus R(-g') \to I(Z) \to 0 \tag{11}$$

is a free resolution of $I(Z)$ where $F_{2Z} := F_2(-h) \oplus R(-f-h) \oplus R(-g-h)$ (see [27]). Note that this is not necessarily a minimal resolution. For instance if one of the hypersurfaces of the second linkage is the same as one of the hypersurfaces of the first linkage (e.g. $f = f'$), the direct factor $R(-g')$ is redundant.

Example 7. *(Sernesi [35], [6]) If X is "2 skew lines", and we link twice via CI's of type $(4,2)$ and $(4,6)$, using a common hypersurface of degree 4 in both linkages, then we get a curve Z of degree 18 and genus 39 and with minimal resolution*

$$0 \to R(-8) \to R(-8) \oplus R(-7)^4 \to R(-6)^4 \oplus R(-4) \to I(Z) \to 0\ ,$$

cf. (11). *If we compare it to the Rao form* (10)*, we see that $F_2 = R(-8)$ and that $0 \to L_4 = R(-8) \to L_3 = R(-7)^4$ is the leftmost part in the minimal resolution "the Koszul resolution" of $M(Z)$ where $M(Z) = R/\mathfrak{m}(-4)$.*

3 Sufficient conditions for unobstructedness.

In these notes we will focus on the structure of the Hilbert scheme $\mathrm{H}(d,g)$. As mentioned earlier, much is still unknown concerning questions related to irreducibility, number of components, dimension and smoothness of $\mathrm{H}(d,g)$. Some results are, however, known. In 1975 Ellingsrud [8] managed to prove that the open subset of $\mathrm{H}(d,g)$ of arithmetically Cohen-Macaulay curves (resp. with fixed postulation) is smooth (resp. irreducible), and he computed the dimension of the corresponding components. A generalization of this result in the direction of smoothness and dimension was given in [19] (see Theorem 8 (i) below) while the irreducibility was generalized by Bolondi [2]. Later, Martin-Deschamps and Perrin gave a stratification $\mathrm{H}_{\gamma,\rho}$ of $\mathrm{H}(d,g)$ obtained by deforming space curves with constant cohomology [25]. Their results lead rather immediately to (iii) in the result below. In [22] we made some further progress and proved among other things the following result.

Theorem 8. *Let X be a curve in $\mathbb{P}^3$ of degree d and arithmetic genus g, let $I = \mathrm{H}^0_*(\mathcal{I}_X)$, $M = \mathrm{H}^1_*(\mathcal{I}_X)$ and $E = \mathrm{H}^1_*(\mathcal{O}_X)$ and suppose at least one of the following conditions:*

(i) $\quad {}_v\mathrm{Hom}_R(I,M) = 0 \quad$ for $v = 0$ and $v = -4$,

(ii) $\quad {}_v\mathrm{Hom}_R(M,E) = 0 \quad$ for $v = 0$ and $v = -4$, or

(iii) $\quad {}_0\mathrm{Hom}_R(I,M) = 0$, ${}_0\mathrm{Hom}_R(M,E) = 0$ and ${}_0\mathrm{Ext}^2_R(M,M) = 0$.

Then $\mathrm{H}(d,g)$ is smooth at (X), i.e. X is unobstructed. Moreover if ${}_0\mathrm{Ext}^i_R(M,M) = 0$ for $i \geq 2$, then the dimension of the Hilbert scheme at (X) is

$$\dim_{(X)} \mathrm{H}(d,g) = 4d + \delta^2(0) + {}_{-4}\mathrm{hom}_R(I,M) + {}_{-4}\mathrm{hom}_R(M,E) \ .$$

The following proof is a simplification of the proof of Thm. 2.6 in [22].

Proof. Suppose (i). To see that X is unobstructed we combine (6) with

$${}_0\mathrm{Ext}^2_R(I,I) \cong {}_{-4}\mathrm{Ext}^2_{\mathfrak{m}}(I,I)^\vee \cong {}_{-4}\mathrm{Hom}(I,M)^\vee \ , \tag{12}$$

which we deduce from (1) and (2). It follows that Def_A is smooth by the assumption ${}_{-4}\mathrm{Hom}(I,M) = 0$, whence X is unobstructed. The proof is really nothing more than to interpret the exact sequence (5) in terms of deformation theory.

(iii) One may deduce the unobstructedness of X from results in [25] by combining Thm. 1.5, page 135 with their tangent space descriptions, pp. 155-166. We will, however, give a partially independent proof, using an exact sequence which we need later on. Indeed it suffices to prove, and interpret in terms of deformation theory, the exact sequence

$$0 \to T_{\gamma,\rho} \to {}_0\mathrm{Ext}^1_R(I,I) \to {}_0\mathrm{Hom}_R(M,E) \to {}_0\mathrm{Ext}^2_R(M,M) \to {}_0\mathrm{Ext}^2_R(I,I) \to \tag{13}$$

where $T_{\gamma,\rho}$ is the tangent space of the Hilbert scheme of constant cohomology $\mathrm{H}_{\gamma,\rho}$ at (X). To prove it we use the spectral sequence (1) and the duality (2) *twice* and Walter's remark on the factorization of α via ${}_0\mathrm{Ext}^2_R(M,M)$ in (5), see [22], Thm. 2.6 (iii) for details.

To see that X is unobstructed, we get by (13) and the vanishing of ${}_0\mathrm{Hom}_R(M,E)$ an isomorphism between the local Hilbert functor of constant cohomology at X and Def_A. The former functor is smooth because ${}_0\mathrm{Ext}^2(M,M)$ contains in a natural way the obstructions of deforming a curve in $\mathrm{H}_{\gamma,\rho}$ (cf. [25], Thm. 1.5, page 135). Hence we conclude by (6).

(ii) We claim that the unobstructedness of X follows from Proposition 6. Indeed if we take a complete intersection $Y \supseteq X$ of two surfaces of degrees f and g such that the *conditions* of Proposition 6 hold (such Y exists), then the corresponding linked curve X' satisfies

$$ {}_v\mathrm{Hom}_R(I(X'), M(X')) \cong {}_v\mathrm{Hom}_R(M(X), E(X)) \quad \text{for } v = 0 \text{ and } v = -4 \tag{14} $$

because $M(X')$ (resp. $I(X')/I(Y)$) is the dual of $M(X)(f+g-4)$ (resp. $E(X)(f+g-4)$) and ${}_v\mathrm{Hom}_R(I(Y), M(X')) = 0$ for $v = 0, -4$ by assumption. Hence we conclude by Proposition 6 and the proof of (i) above.

It remains to prove the dimension formula, i.e. to prove

$$ h^1(\mathcal{N}_X) = \delta^2(0) + {}_{-4}\mathrm{hom}_R(I, M) + {}_{-4}\mathrm{hom}(M, E) \ . $$

Since this follows from the next lemma, we are done. □

Lemma 9. *Let X be any curve in $\mathbb{P}^3$ such that ${}_0\mathrm{Ext}^i_R(M, M) = 0$ for $i \geq 2$. Then*

$$ h^1(\mathcal{N}_X) = \delta^2(0) + {}_{-4}\mathrm{hom}_R(I, M) + {}_{-4}\mathrm{hom}(M, E) \ . \tag{15} $$

Proof. The main observation for proving the lemma is the fact that ${}_0\mathrm{Ext}^2_R(M, M) = 0$ implies $\alpha = 0$ in the sequence (5) for $v = 0$. Since the cokernel of the rightmost map in (5) is ${}_0\mathrm{Ext}^3_{\mathfrak{m}}(I, I)$, it follows that $h^1(\mathcal{N}_X) = {}_0\mathrm{ext}^2_R(I, I) + {}_0\mathrm{ext}^3_{\mathfrak{m}}(I, I)$. Moreover using the spectral sequence (1) which converging to ${}_0\mathrm{Ext}^3_{\mathfrak{m}}(I, I)$, we get ${}_0\mathrm{ext}^3_{\mathfrak{m}}(I, I) = {}_0\mathrm{hom}_R(I, E) + {}_0\mathrm{ext}^1_R(I, M)$ because

$$ {}_0\mathrm{Ext}^2_R(I, M) \cong {}_{-4}\mathrm{Ext}^2_{\mathfrak{m}}(M, I)^\vee \cong {}_{-4}\mathrm{Hom}(M, M)^\vee \cong {}_0\mathrm{Ext}^4_R(M, M) = 0. $$

Finally since ${}_0\mathrm{ext}^1_R(I, M) = {}_{-4}\mathrm{hom}(M, E)$ by Lemma 1 and $\delta^2(0) = {}_0\mathrm{hom}_R(I, E)$ by Remark 7 of [20], we conclude by (12). □

Corollary 10. *Let X be any curve in $\mathbb{P}^3$, let $\operatorname{diam} M \leq 2$ and suppose $e(X) < s(X)$. If $\operatorname{diam} M \neq 0$, suppose also $e(X) \leq b(X)$ and $c(X) \leq s(X)$. Then*

$$ \mathrm{H}^1(\mathcal{N}_X) = 0 \ . $$

Proof. This follows from (15). We leave the details to the reader (or see [22], Cor. 2.8). □

4 Comparison of deformation functors

In this section we will use cohomological methods to investigate when the immersions

$$ \mathrm{H}_{\gamma,\rho} \overset{(1)}{\hookrightarrow} \mathrm{H}_\gamma \overset{(2)}{\hookrightarrow} \mathrm{H}(d, g) $$

are isomorphisms around (X). Indeed by the semicontinuity of $h^i(\mathcal{I}_X(v))$ it is rather obvious that (2) (resp. the composition of (1) and (2)) is an isomorphism at (X) provided X has maximal rank (resp. seminatural cohomology). We shall, however, show the isomorphisms under weaker assumptions. As a consequence we get some interesting information on the semicontinuity of the graded Betti number for curves X in $\mathbb{P}^3$.

Proposition 11. *Let* $M = \mathrm{H}^1_*(\mathcal{I}_X)$ *and* $E = \mathrm{H}^1_*(\mathcal{O}_X)$. *Then*

(a) $\quad {}_0\mathrm{Hom}_R(I, M) = 0 \quad \Rightarrow \quad \mathrm{H}_\gamma \cong \mathrm{H}(d, g)$ are isomorphic as schemes at (X).

(b) $\quad {}_0\mathrm{Hom}_R(M, E) = 0 \quad \Rightarrow \quad \mathrm{H}_{\gamma,\rho} \cong \mathrm{H}_\gamma$ are isomorphic as schemes at (X).

Proof. (a) By interpreting the exact sequence

$$0 \to {}_0\mathrm{Ext}^1_R(I, I) \to \mathrm{H}^0(\mathcal{N}_X) \to {}_0\mathrm{Hom}_R(I, M) \xrightarrow{\alpha} {}_0\mathrm{Ext}^2_R(I, I) \to \mathrm{H}^1(\mathcal{N}_X) \to \tag{16}$$

in terms of deformation theories, as done in (6), we get the conclusion.

(b) was established in the proof of Theorem 8 (iii) by interpreting the exact sequence

$$0 \to T_{\gamma,\rho} \to {}_0\mathrm{Ext}^1_R(I, I) \to {}_0\mathrm{Hom}_R(M, E) \to {}_0\mathrm{Ext}^2_R(M, M) \to {}_0\mathrm{Ext}^2_R(I, I) \to \tag{17}$$

in terms of deformation theories. $\square$

To see that the assumption above is by much weaker than claiming X to have seminatural cohomology, we include a result for *generic* curves:

Proposition 12. *Let* X *be a curve in* $\mathbb{P}^3$, *and suppose* X *is generic in the Hilbert scheme* $\mathrm{H}(d, g)$ *and satisfies* ${}_0\mathrm{Ext}^2_R(M, M) = 0$. *Then* X *is unobstructed if and only if*

$${}_0\mathrm{Hom}_R(I, M) = 0 \quad \text{and} \quad {}_0\mathrm{Hom}_R(M, E) = 0 \ . \tag{18}$$

Proof. One way is clear from Theorem 8. Now suppose X is unobstructed and generic with postulation γ and deficiency ρ. By generic flatness we see that $\mathrm{H}_{\gamma,\rho} \cong \mathrm{H}_\gamma \cong \mathrm{H}(d, g)$ near (X) from which we deduce an isomorphism of tangent spaces $\mathrm{T}_{\gamma,\rho} \cong {}_0\mathrm{Ext}^1_R(I, I) \cong \mathrm{H}^0(\mathcal{N}_X)$. We therefore conclude by the exact sequences (17) and (16), recalling that the map α of (16) factorizes via ${}_0\mathrm{Ext}^2_R(M, M)$, whence $\alpha = 0$. $\square$

As a surprising consequence of Proposition 11, we get some "new" semicontinuity results which we heavily use in section 6.

Corollary 13. *Inside* H_γ *and hence inside* $\mathrm{H}_{\gamma,\rho}$ *the graded Betti numbers obey semicontinuity, i.e. if* X' *is a generization of* X *in* H_γ, *then*

$$\beta_{i,j}(X') \le \beta_{i,j}(X) \quad \text{for any } i, j \ .$$

In particular if ${}_0\mathrm{Hom}_R(I(X), M(X)) = 0$, *then* $\beta_{i,j}(X') \le \beta_{i,j}(X)$ *for any i,j and every generization* X' *of* X *in* $\mathrm{H}(d, g)$.

Proof. Apply Nakayama's lemma to the syzygy modules of (7) (see Rem. 7 of [23] and [33]). Then combine with Proposition 11 (a). $\square$

Example 14. *It is known that the curve* Z *of Example 7 sits in the intersection of two irreducible components of* $\mathrm{H}(18, 39)_S$ *and moreover that the generic curve* $\tilde{Z}$ *of one of the components is ACM with minimal resolution*

$$0 \to R(-8) \oplus R(-6)^2 \to R(-5)^4 \to I(\tilde{Z}) \to 0 \ ,$$

see [35], [6]. Looking to the minimal resolution of $I(Z)$ *in Example 7, we get* $\beta_{1,5}(Z) = 0$ *while* $\beta_{1,5}(\tilde{Z}) = 4$, *i.e. we don't have semicontinuity for* $\beta_{1,5}$. *In this example Corollary 13 does not apply to generizations of* Z *in* $\mathrm{H}(d, g)$ *because* ${}_0\mathrm{Hom}_R(I(Z), M(Z)) \neq 0$*!*

5 Necessary conditions for unobstructedness.

First note that we can easily reformulate a part of Theorem 8 as

Proposition 15. *Let X be a space curve with Rao module M satisfying ${}_0\mathrm{Ext}^2_R(M,M)=0$. If X is obstructed, then either*

(a) ${}_0\mathrm{Hom}_R(I,M)\neq 0$ and ${}_0\mathrm{Hom}_R(M,E)\neq 0$, **or**

(b) ${}_{-4}\mathrm{Hom}_R(I,M)\neq 0$ and ${}_0\mathrm{Hom}_R(M,E)\neq 0$, **or**

(c) ${}_0\mathrm{Hom}_R(I,M)\neq 0$ and ${}_{-4}\mathrm{Hom}_R(M,E)\neq 0$.

There is a partial converse to this result, at least for Buchsbaum curves X, or more generally for curves admitting "a Buchsbaum factor" which we denote as $M_{[t]}$.

Definition 16. *$M=M(X)$ admits "a Buchsbaum factor" $M_{[t]}$ in degree t if*

$$M\simeq M'\oplus M_{[t]}$$

as R-modules where $M_{[t]}$ supported in degree t and has diameter 1.

If $M\simeq M'\oplus M_{[t]}$, then M has a minimal resolution of the form

$$0\to L_4'\oplus R(-t-4)^r\xrightarrow{\sigma}L_3'\oplus R(-t-3)^{4r}\to L_2'\oplus R(-t-2)^{6r}...\to L_0'\oplus R(-t)^r\to M\to 0$$

where

$$0\to L_4'\to L_3'\to L_2'\to L_1'\to L_0'\to M'\to 0$$

is the minimal resolution of M' and

$$0\to R(-t-4)^r\xrightarrow{\sigma_{[t]}}R(-t-3)^{4r}\to R(-t-2)^{6r}\to R(-t-1)^{4r}\to R(-t)^r\to M_{[t]}\to 0 \qquad (19)$$

is "r times" the Koszul resolution of the R-module $k\cong R/(X_0,X_1,X_2,X_3)$.

Remark 17. *Suppose $M=M(X)$ admits "a Buchsbaum factor", $M\simeq M'\oplus M_{[t]}$.*
(a) *If $M'=0$ then X is Buchsbaum curve of diameter one.*
(b) *Every Buchsbaum curve (not ACM) has a Buchsbaum factor. Every curve obtained from Liaison addition where one of curves is Buchsbaum, admits a Buchsbaum factor ([27]).*

Combining with Rao's theorem (10), we get a minimal resolution

$$0\to L_4\xrightarrow{\sigma\oplus 0}L_3\oplus F_2\to F_1\to I\to 0\ ,$$

of $I=I(X)$ where

$$L_4\cong L_4'\oplus R(-t-4)^r\ ,\qquad L_3\cong L_3'\oplus R(-t-3)^{4r}\ .$$

Moreover to define what we will call the fundamental 5-tuple, we write F_i as

$$F_2\cong P_2\oplus R(-t-4)^{b_1}\oplus R(-t)^{b_2}\ ,\qquad F_1\cong P_1\oplus R(-t-4)^{a_1}\oplus R(-t)^{a_2} \qquad (20)$$

where P_i, for $i=1,2$ are supposed to contain no direct factor of degree t and $t+4$.

Definition 18. *The fundamental 5-tuple (with respect to obstructedness) associated to a curve X with Buchsbaum factor $M_{[t]}$ in degree t is*

$$(a_1, a_2, b_1, b_2, r)$$

Note that $(a_1, a_2) = (\beta_{1,t+4}, \beta_{1,t})$ are 1^{st} graded Betti numbers of $I = I(X)$ while

$$(\beta'_{2,t+4}, \beta'_{2,t}, \beta'_{3,t+4}) := (b_1, b_2, r)$$

are graded "Betti number of I off L'_3 and L'_4".

Remark 19.
(a) *If $M' = 0$, then $\beta'_{i,j} = \beta_{i,j}$ are the usual Betti numbers of $I = I(X)$.*
(b) *Note that the 5-tuple (a_1, a_2, b_1, b_2, r) was written as (r, a_1, a_2, b_1, b_2) in [22].*

Theorem 20. *Let $X \subseteq \mathbb{P}^3$ be a curve, and suppose ${}_0\mathrm{Ext}^2_R(M, M) = 0$ and $M \cong M' \oplus M_{[t]}$ as R-modules. Then X is obstructed if either*

(a) ${}_0\mathrm{Hom}_R(I, M_{[t]}) \neq 0$ and ${}_0\mathrm{Hom}_R(M_{[t]}, E) \neq 0$, **or**

(b) ${}_{-4}\mathrm{Hom}_R(I, M_{[t]}) \neq 0$ and ${}_0\mathrm{Hom}_R(M_{[t]}, E) \neq 0$, **or**

(c) ${}_0\mathrm{Hom}_R(I, M_{[t]}) \neq 0$ and ${}_{-4}\mathrm{Hom}_R(M_{[t]}, E) \neq 0$.

The proof of (a) relies on the study of a morphism appearing in the Proposition below. It is straightforward to show Proposition 21 by using [12], Prop. 2.13. We will, however, give the proof of Prop. 3.6 of [22] which relies only on (16) and (17). For the proof recall that $\mathrm{H}^0(\mathcal{N}_X) \cong \mathrm{Ext}^1_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_X, \mathcal{I}_X)$ by (4).

Proposition 21. *Let $X \subseteq \mathbb{P}^3$ be a curve, and suppose ${}_0\mathrm{Ext}^2_R(M, M) = 0$. If the obvious morphism*

$${}_0\mathrm{Hom}_R(I, M) \times {}_0\mathrm{Hom}_R(M, E) \longrightarrow {}_0\mathrm{Hom}_R(I, E)$$

(given by the composition) is non-zero, then X is obstructed.

Proof. It is well known (cf. [24]) that if the Yoneda pairing (inducing the cup product)

$$< -, - > \ : \quad \mathrm{Ext}^1_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_X, \mathcal{I}_X) \times \mathrm{Ext}^1_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_X, \mathcal{I}_X) \to \mathrm{Ext}^2_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_X, \mathcal{I}_X), \tag{21}$$

given by composition of resolving complexes, satisfies $< \lambda, \lambda > \neq 0$ for some λ, then X is obstructed. If we let $p_1 : \mathrm{Ext}^1_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_X, \mathcal{I}_X) \to {}_0\mathrm{Hom}_R(I, M)$ and $p_2 : \mathrm{Ext}^1_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_X, \mathcal{I}_X) \to {}_0\mathrm{Hom}_R(M, E)$ be the maps induced by sending an extension onto the corresponding connecting homomorphisms (cf. the maps appearing in (16) and (17)), then <-,-> fits into a commutative diagram of natural maps

$$\begin{array}{ccccc} \mathrm{Ext}^1_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_X, \mathcal{I}_X) & \times & \mathrm{Ext}^1_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_X, \mathcal{I}_X) & \longrightarrow & \mathrm{Ext}^2_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_X, \mathcal{I}_X) \\ \downarrow p_1 & & \downarrow p_2 & & \downarrow \\ {}_0\mathrm{Hom}_R(I, M) & \times & {}_0\mathrm{Hom}_R(M, E) & \longrightarrow & {}_0\mathrm{Hom}_R(I, E) \end{array} \tag{22}$$

By (16), ${}_0\mathrm{Ext}^1_R(I,I) = \ker p_1$, and p_1 is surjective because $\alpha = 0$. Moreover since the composition ${}_0\mathrm{Ext}^1_R(I,I) \hookrightarrow \mathrm{Ext}^1(I_X, I_X) \to {}_0\mathrm{Hom}_R(M,E)$ is surjective by (17), there exists $(\lambda_1,\lambda_2) \in \mathrm{Ext}^1(\mathcal{I}_X,\mathcal{I}_X) \times {}_0\mathrm{Ext}^1_R(I,I)$ such that the composed map $p_2(\lambda_2)p_1(\lambda_1)$ is non-zero by assumption. Using $\lambda_2 \in {}_0\mathrm{Ext}^1_R(I,I) = \ker p_1$, we get

$$p_2(\lambda_1+\lambda_2)p_1(\lambda_1+\lambda_2) = p_2(\lambda_1)p_1(\lambda_1) + p_2(\lambda_2)p_1(\lambda_1)$$

i.e. either $< \lambda_1+\lambda_2\ ,\ \lambda_1+\lambda_2 >$ or $< \lambda_1\ ,\ \lambda_1 >$ are non-zero, whence X is obstructed. □

Remark 22. *Assume* ${}_0\mathrm{Ext}^2_R(M,M) = 0$. *From* (16) *and* (17), *we see that* ${}_0\mathrm{Hom}_R(I,M) \neq 0$ *and* ${}_0\mathrm{Hom}_R(M,E) \neq 0$ *if and only if we have the following strict inclusions of tangent spaces*

$$T_{\gamma,\rho} \subsetneq {}_0\mathrm{Ext}^1_R(I,I) \subsetneq \mathrm{H}^0(\mathcal{N}_X) \tag{23}$$

By Proposition 21, X is obstructed if (23) *holds, cf. [25], ch. X, Prop. 5.9. for the case $M \cong k$.*

Before proving Theorem 20, we remark that

Lemma 23. *Let $X \subseteq \mathbb{P}^3$ be a curve, and suppose $M \cong M' \oplus M_{[t]}$ as R-modules. If ${}_v\mathrm{Ext}^2_R(M_{[t]},M) = 0$ for $v = 0$ and -4, then*

(a) ${}_0\mathrm{hom}_R(I, M_{[t]}) = ra_2$ and ${}_{-4}\mathrm{hom}_R(I, M_{[t]}) = ra_1$.

(b) ${}_0\mathrm{hom}_R(M_{[t]}, E) = rb_1$ and ${}_{-4}\mathrm{hom}_R(M_{[t]}, E) = rb_2$.

Proof. (a) If we apply ${}_v\mathrm{Hom}(-, M_{[t]})$ to (10), we get ${}_v\mathrm{Hom}(I, M_{[t]}) \simeq {}_v\mathrm{Hom}(F_1, M_{[t]})$ because $\mathfrak{m} \cdot M_{[t]} = 0$ and we conclude by (20).

(b) is the dual result. We *claim* that ${}_{-v-4}\mathrm{Hom}_R(F_2, M_{[t]})^\vee \cong {}_v\mathrm{Hom}_R(M_{[t]}, E)$ for $v = 0$ and -4. Indeed, exactly as in Lemma 1, we get by duality (2) and the spectral sequence (1) (which converges to ${}_{-v-4}\mathrm{Ext}^3_{\mathfrak{m}}(M_{[t]}, I)$) an exact sequence

$$0 \to {}_{-v-4}\mathrm{Ext}^1_R(M_{[t]}, M) \to {}_v\mathrm{Ext}^1_R(I, M_{[t]})^\vee \to {}_{-v-4}\mathrm{Hom}_R(M_{[t]}, E) \to 0$$

for $v = 0$ and -4. We get ${}_{-v-4}\mathrm{Ext}^1_R(M_{[t]}, M)^\vee \cong {}_v\mathrm{Ext}^3_R(M, M_{[t]})$ by (2) and (1) and ${}_v\mathrm{Ext}^3_R(M, M_{[t]}) \cong {}_v\mathrm{Hom}_R(L_3, M_{[t]})$ by (9). Interpreting ${}_v\mathrm{Ext}^1_R(I, M_{[t]})$ similarly via the minimal resolution (10) of I, we get ${}_v\mathrm{Ext}^1_R(I, M_{[t]}) \cong {}_v\mathrm{Hom}_R(L_3 \oplus F_2, M_{[t]})$. Hence we get the claim and we conclude by (20). □

Proof (of Theorem 20). (a) Since ${}_0\mathrm{Hom}_R(I, M_{[t]}) \cong {}_0\mathrm{Hom}_R(R(-t)^{a_2}, M_{[t]})$ by (10) and (20), we get that the composition

$${}_0\mathrm{Hom}_R(I, M_{[t]}) \times {}_0\mathrm{Hom}_R(M_{[t]}, E) \longrightarrow {}_0\mathrm{Hom}_R(I, E)$$

is non-zero. Since ${}_0\mathrm{Hom}(I, M) \to {}_0\mathrm{Hom}(I, M_{[t]})$ is surjective by the existence of the R-split morphism $M \to M_{[t]}$, the corresponding composition of Proposition 21 is also non-zero, and (a) is proved.

(b) In [22], there is a result (Prop. 3.8) with a quite complicated proof which implies (b) and (c) of Theorem 20. Here we just remark that we immediately get (b) from (a) and a result of Walter ([37]) in the case $M' = 0$. Indeed using [37], Thm. 0.5 we get the obstructedness of X provided ${}_0\mathrm{Hom}_R(I, M) = 0$ and the assumptions of (b) holds. Since (a) takes care of the case ${}_0\mathrm{Hom}_R(I, M) \neq 0$ under the assumptions of (b), we get (a)

(c) The conclusion follows from Proposition 6 and (b) exactly as in the proof of Theorem 8 (ii). $\square$

Combining Lemma 23 with Theorem 20 and using Proposition 15 for the final conclusion, we get

Corollary 24. *Suppose* ${}_0\mathrm{Ext}^2_R(M, M) = 0$ *and* $M \cong M' \oplus M_{[t]}$ *as R-modules. Then X is obstructed if*

$$a_2 \cdot b_1 \neq 0 \quad \text{or} \quad a_1 \cdot b_1 \neq 0 \quad \text{or} \quad a_2 \cdot b_2 \neq 0$$

where $(a_1, a_2, b_1, b_2) = (\beta_{1,t+4}, \beta_{1,t}, \beta'_{2,t+4}, \beta'_{2,t})$.

Moreover if $M' = 0$, *i.e.* $M \cong M_{[c]}$, *then X is obstructed if and only if*

$$\beta_{1,c} \cdot \beta_{2,c+4} \neq 0 \quad \text{or} \quad \beta_{1,c+4} \cdot \beta_{2,c+4} \neq 0 \quad \text{or} \quad \beta_{1,c} \cdot \beta_{2,c} \neq 0 \ .$$

Remark 25. *Let* $r \neq 0$. *For the fundamental 5-tuple* (a_1, a_2, b_1, b_2, r), *we have that*

$$a_2 \cdot b_1 = 0 \text{ and } a_1 \cdot b_1 = 0 \text{ and } a_2 \cdot b_2 = 0$$

is equivalent to requiring that the 5-tuple is of the form

$$(0, 0, b_1, b_2, r) \ , \quad (a_1, 0, 0, b_2, r) \quad \text{or} \quad (a_1, a_2, 0, 0, r) \ .$$

Hence if X is unobstructed, then there are "two consecutive 0's in the 1^{st} *four coordinates of the 5-tuple". This is equivalent to unobstructedness if* $\operatorname{diam} M = 1$.

Example 26. (a) *(Sernesi, [6]). Take the curve bilinked to 2 skew lines, as in Example 7. The minimal resolution is*

$$0 \to R(-8) \to R(-8) \oplus R(-7)^4 \to R(-6)^4 \oplus R(-4) \to I \to 0 \ ,$$

whence $c = 4$. *The corresponding 5-tuple,* $(\beta_{1,c+4}, \beta_{1,c}, \beta_{2,c+4}, \beta_{2,c}, \beta_{3,c+4})$, *is*

$$(0, 1, 1, 0, 1) \ .$$

It follows that the curve X of $\mathrm{H}(18, 39)_S$ *is obstructed by Corollary 24 or Remark 25.*

(b) *Start with the generic curve of* $\mathrm{H}(8, 5)_S$. *It has 2-dimensional Rao module M and* $\operatorname{diam} M = 1$ *([15]). Link with* $(4, 6)$, *then with* $(6, 8)$ *using a common hypersurface of degree 6 in both linkages. The minimal resolution is*

$$0 \to R(-10)^2 \to R(-10) \oplus R(-9)^8 \to R(-8)^7 \oplus R(-6) \to I \to 0 \ ,$$

whence $c = 6$. *The corresponding 5-tuple is* $(\beta_{1,c+4}, \beta_{1,c}, \beta_{2,c+4}, \beta_{2,c}, \beta_{3,c+4}) = (0, 1, 1, 0, 2)$, *i.e. the curve X of* $\mathrm{H}(32, 109)_S$ *is obstructed by Remark 25.*

(c) *([3] and [37]). There is a curve in* $\mathrm{H}(33, 117)_S$ *of maximal rank and* $\operatorname{diam} M = 1$ *with minimal resolution*

$$0 \to R(-9) \to R(-10)^2 \oplus R(-9) \oplus R(-8)^4 \to R(-9) \oplus R(-8) \oplus R(-7)^5 \to I(X) \to 0 \ .$$

The corresponding 5-tuple is $(1, 0, 1, 0, 1)$, *i.e. X is obstructed by Remark 25.*

In the next section, we shall see that the curve of Example 26 (a) sits in the intersection of two irreducible components, while the curve of Example 26 (b) belongs to a unique irreducible component, by studying the possible generizations of the curves.

6 On the minimal resolution of a generic curve.

In this section we consider generizations of a space curve, i.e. deformations to a more general curve, by simplifying their minimal resolutions ("killing ghost terms"). We shall prove a general result (Theorem 27) valid for any space curve and a more special result (Proposition 31) which holds for curves with a Buchsbaum factor under some conditions. These results are inspired by the corresponding results in [25] which treat the case $M \cong k$.

Theorem 27. *Let X be any curve, let*

$$0 \to L_4 \xrightarrow{\sigma} L_3 \to L_2 \to L_1 \to L_0 \to M(X) \to 0$$

be a minimal free resolution of $M = M(X)$ and look to the minimal free resolution

$$0 \to L_4 \xrightarrow{\sigma \oplus 0} L_3 \oplus F_2 \to F_1 \to I(X) \to 0 \tag{24}$$

given by Rao's theorem. If F_1 and F_2 has a common free factor;

$$F_2 = F_2' \oplus R(-i) \ , \quad F_1 = F_1' \oplus R(-i),$$

then there is a generization X' of X in $\mathrm{H}(d,g)$ with constant postulation and Rao module and with minimal resolution

$$0 \to L_4 \xrightarrow{\sigma \oplus 0} L_3 \oplus F_2' \to F_1' \to I(X') \to 0 \ .$$

The proof is rather straightforward once we have proven a key lemma. We delay the proof of the lemma and Theorem 27 until the end of this section.

Suppose $M = M(X)$ admits an *R-module* decomposition $M = M' \oplus M_{[t]}$ where the diameter of $M_{[t]}$ is 1 (e.g. X is Buchsbaum). Using Theorem 27 we get immediately

Corollary 28. *Let X be a curve and suppose $M(X) \cong M' \oplus M_{[t]}$ as R-modules. Let*

$$(\beta_{1,t+4}, \beta_{1,t}, \beta'_{2,t+4}, \beta'_{2,t}, \beta'_{3,t+4}) = (a_1, a_2, b_1, b_2, r)$$

be the corresponding 5-tuple of X.
(a) *If $a_1 \cdot b_1 \neq 0$, then there is a generization X' of X in $\mathrm{H}(d,g)$ with constant postulation and Rao module whose 5-tuple is*

(C1) $\quad (a_1 - 1, a_2, b_1 - 1, b_2, r)$.

(b) *If $a_2 \cdot b_2 \neq 0$, then there is a generization X' of X in $\mathrm{H}(d,g)$ with constant postulation and Rao module whose 5-tuple is*

(C2) $\quad (a_1, a_2 - 1, b_1, b_2 - 1, r)$.

Remark 29. *Repeated use of Corollary 28 results in a curve (a generization of X) with Buchsbaum factor $M_{[t]}$ whose corresponding 5-tuple satisfies*

$$a_1 \cdot b_1 = 0 \ \text{ and } \ a_2 \cdot b_2 = 0 \ .$$

Remark 30. *In the following we need a slight extension of the notion of a 5-tuple to a special case where $r = 0$, i.e. we need to consider generizations X' of a curve X with a Buchsbaum factor in degree t (i.e. $M(X) \cong M' \oplus M_{[t]}$) for which $M(X') \cong M'$. For such X' we still have a number t and a minimal resolution of the form* (10). *Then we use the free summands of* (20) *appearing in the minimal resolution of $I(X')$ to define (a_1, a_2, b_1, b_2, r) with $r = 0$.*

Proposition 31. *Let $M(X) \cong M' \oplus M_{[t]}$ as R-modules and suppose L'_2 has no factor in degree t and $t + 4$. Let (a_1, a_2, b_1, b_2, r) be the corresponding 5-tuple of X.*

(a) *If $r \cdot b_1 \neq 0$, then there is a generization X' of X in $\mathrm{H}(d, g)$ with constant postulation and constant M' whose 5-tuple is*

(P1) $$(a_1, a_2, b_1 - 1, b_2, r - 1)\ .$$

(b) *If $r \cdot a_2 \neq 0$, then there is a generization X' of X in $\mathrm{H}(d, g)$ with constant specialization (function) and constant M' whose 5-tuple is*

(P2) $$(a_1, a_2 - 1, b_1, b_2, r - 1)\ .$$

Remark 32. *Repeated use of Proposition 31 results in a generization X' with 5-tuple satisfying* $r \cdot a_2 = 0$ and $r \cdot b_1 = 0$. *Using also Theorem 27, we get a curve X' whose corresponding 5-tuple is of the form*

$$(a_1, 0, 0, b_2, r) \ \text{ with } \ r \neq 0, \qquad \textbf{or} \qquad (a_1, a_2, b_1, b_2, 0) \ \text{ with } a_1 \cdot b_1 = a_2 \cdot b_2 = 0. \tag{25}$$

Note that if $M' = 0$ then the curve X' is unobstructed by Corollary 24 (and Theorem 8).

Remark 33. *We give details of the proof of Proposition 31 later. For a complete proof, we refer to the proof of Prop. 4.2 of [22]. There is an inaccuracy in the statement of Prop. 4.2 (a) of [22] which states that the resolution for the generization in which r and b_1 are replaced by $r - 1$ and $b_1 - 1$, is minimal. Looking to the proof of [22] the resolution may be non-minimal at one and only one spot. Indeed if F_1 contains a direct factor $R(-t - 3)$, this factor may become redundant (we try in the proof of this paper to clarify. I intend to put a revised version, correcting Prop. 4.2, for the corresponding paper on the arXiv). In conclusion the graded Betti numbers of a generization as in (a) do not change except for the change which corresponds to the replacement of r and b_1 by $r - 1$ and $b_1 - 1$ respectively and for a possible change in which $\beta_{2,t+3}$ and $\beta_{1,t+3}$ decrease by the same number i, $i \leq 4$. Note that this remark does not affect the conclusion on the 5-tuple in Proposition 31 (a).*

Remark 34. *Suppose* $\operatorname{diam} M = 1$, *i.e.* $M(X) \cong M_{[t]}$ *and* $t = c$.

(a) *By Remark 33 there is a generization as in Proposition 31 (a) whose graded Betti numbers do not change except for $\beta_{3,c+4}$ and $\beta_{2,c+4}$, which both decrease by 1, and for $\beta_{1,c+3}$ and $\beta_{2,c+3}$ which may decrease by at most 4, keeping, however, $\beta_{1,c+3} - \beta_{2,c+3}$ unchanged. Moreover if we combine with Theorem 27, we can suppose $\beta_{2,c+3}$ decreases by exactly* $\min\{\beta_{1,c+3}, 4\}$ *after a further generization.*

(b) *Extending some ideas of the proof of Proposition 31 (b) (which we hope to do in a forthcoming paper), we can describe the possible changes of the graded Betti numbers under the generization (P2) in detail. In particular we get that the graded Betti numbers of a generization as in Proposition 31 (b) do not change except $\beta_{3,c+4}$ and $\beta_{1,c}$, which both decrease by 1, and $\beta_{1,v}$ and $\beta_{2,v}$ for $v \in \{c + 3, c + 2, c + 1\}$ for which $\beta_{1,c+1} - \beta_{2,c+1}$ increases by 4, $\beta_{2,c+2} - \beta_{1,c+2}$ increases by 6 and $\beta_{1,c+3} - \beta_{2,c+3}$ increases by 4. Moreover combining with Theorem 27, we can take $\beta_{1,c+1} \cdot \beta_{2,c+1} = 0$ and $\beta_{1,c+2} \cdot \beta_{2,c+2} = 0$ after a further generization.*

Example 35. (a) *([35]). The 5-tuple of X is*

$$(0, 1, 1, 0, 1)\ .$$

Using (P1) and (P2) the curve X admits two generizations to two curves with 5-tuples

$$(0, 1, 0, 0, 0) \ \text{ and } \ (0, 0, 1, 0, 0)\ .$$

(b) *The curve X with 5-tuple* $(0, 1, 1, 0, 2)$ *admits two generizations to two curves with 5-tuples*

$$(0, 1, 0, 0, 1) \ \text{ and } \ (0, 0, 1, 0, 1).$$

Both these curves, however, admit generizations to a curve with 5-tuple $(0, 0, 0, 0, 0)$*, indicating that the curve X, which is obstructed by Corollary 24, may not sit in an intersection of two irreducible components of* $\mathrm{H}(d, g)$*!!*

(c) *([3] and [37]). The 5-tuple of X is* $(1, 0, 1, 0, 1)$*, i.e. the curve X admits two generizations to two curves with 5-tuples:*

$$(1, 0, 0, 0, 0) \ \text{ and } \ (0, 0, 0, 0, 1)$$

To see that the 2 directions of generizations in Example 35 (a) and (c) correspond to two different components, while those in Example 35 (b) lead to a unique component, we prove Proposition 37 below which relies on the semicontinuity of the graded Betti numbers (cf. Corollary 13). E.g. we will in Example 35 (c) see that Proposition 37 "separates irreducible components" where the usual semicontinuity of $h^i(\mathcal{I}_X(v))$ fails! Indeed since both generizations in Example 35 (c) have the same postulation as X, the semicontinuity of $h^i(\mathcal{I}_X(v))$ can not be used to prove that the two generizations belong to two different irreducible components!

For the rest of this section we suppose $M' = 0$, i.e. we consider curves of diameter at most 1 for which we attach a 5-tuple (cf. Remark 30) consisting of the usual graded Betti numbers.

Definition 36. (a) *A minimal 5-tuple is a 5-tuple which doesn't allow further reductions by using either (C1), (C2), (P1) or (P2), i.e. the 5-tuple is of the form* (25) *or as in the proposition below.*

(b) *The set of* all *graded Betti numbers is called minimal provided it doesn't allow further reductions by using either (P1), (P2) or reductions coming from cancellations of a common direct free factor as in Theorem 27 for some i (in particular $\beta_{1,i} \cdot \beta_{2,i} = 0$ for every $i \neq t+3$ if $M = M_{[t]}$). We denote the minimal set of all graded Betti numbers by $\beta_{tot}(X)$, or just β_{tot}.*

Proposition 37. *Let t be an integer such that $M(X)_v = 0$ for $v \neq t$, and let $\underline{\beta} = \underline{\beta}(X)$ be the minimal 5-tuple of X, i.e.*

$$\underline{\beta} = (\beta_{1,t+4}, 0, 0, \beta_{2,t}, \beta_{3,t+4}) \text{ with } \beta_{3,t+4} \neq 0 \text{ and } t = c,$$

or

$$\underline{\beta} = (\beta_{1,t+4}, \beta_{1,t}, \beta_{2,t+4}, \beta_{2,t}, 0) \text{ with } \beta_{1,i} \cdot \beta_{2,i} = 0 \text{ for } i = t, t+4.$$

Then X is unobstructed. Let $V(\underline{\beta})$ be the unique irreducible component of $\mathrm{H}(d,g)$ *containing X and let X_i be curves as X above with minimal 5-tuples $\underline{\beta}_i$, $i=1,2$ and with the same t. Then*

(i) *The generic curve of $V(\underline{\beta})$ has 5-tuple $=\underline{\beta}$*

(ii) *If $\underline{\beta}_1 \neq \underline{\beta}_2$, then $V(\underline{\beta}_1) \neq V(\underline{\beta}_2)$.*

(iii) *$V(\underline{\beta})$ depends only on β_{tot} or, equivalently, only on the postulation γ_X and $\beta_{3,t+4}$, not on X.*

Proof. (i) X is unobstructed by Corollary 24 (and Theorem 8). Let X' be the generic curve of $V(\underline{\beta})$. We have ${}_0\mathrm{Hom}_R(I(X), M(X)) = 0$, cf. Lemma 23, and we get

$$\mathrm{H}_\gamma \cong \mathrm{H}(d,g) \quad \text{at} \quad (X)$$

by Proposition 11. It follows that X' has the same postulation as X, whence

$$\beta_{1,v}(X') - \beta_{2,v}(X') + \beta_{3,v}(X') = \beta_{1,v}(X) - \beta_{2,v}(X) + \beta_{3,v}(X) \tag{26}$$

for every v, cf. [9], [32]. Moreover $\underline{\beta}(X') \leq \underline{\beta}(X)$ by Corollary 13. Since $\underline{\beta}(X)$ is minimal, see the form of $\underline{\beta}$ of this proposition, we get $\underline{\beta}(X') = \underline{\beta}(X)$, and (i) is proved.

(ii) Suppose $V(\underline{\beta}_1) = V(\underline{\beta}_2)$, i.e. we can suppose their generic curves X_1' and X_2' coincide. Then (i) leads to a contradiction because

$$\underline{\beta_1} = \underline{\beta}(X_1') = \underline{\beta}(X_2') = \underline{\beta_2}\ .$$

(iii) By a result of Bolondi (cf. [2]) the subset of $\mathrm{H}(d,g)$ of curves of constant postulation and Rao module structure is irreducible. For curves of diameter at most one there is only one structure on the module $M(X)$, namely the trivial structure, in which case "constant Rao module structure" may be replaced by requiring $\dim M(X)$ or $\beta_{3,t+4}$ to be constant. It follows that if the curves X_1 and X_2 of the proposition, in addition to having the same $\underline{\beta}$, also satisfy $\gamma_{X_1} = \gamma_{X_2}$, then they define the same irreducible component of $\mathrm{H}(d,g)$ and we get the equivalent statement by (26). □

Once more we revisit Example 35.

Example 38. *In* (a) *the 5-tuples of the two generizations are minimal, both with $t=4$ and both ACM, and they correspond to two* different *irreducible components of* $\mathrm{H}(18,39)_S$ *containing* (X) *by Proposition 37. Note that we for this example may separate the two components by the usual semicontinuity because the triple* $(h^0(\mathcal{I}_Z(4)), h^1(\mathcal{I}_Z(4)), h^1(\mathcal{O}_Z(4)))$ *equals* $(1,1,1)$ *for $Z=X$, while it is* $(1,0,0)$ *and* $(0,0,1)$ *for the two generizations.*

(c) *Again the 5-tuples of the two generizations are minimal, both with $t=5$, and they correspond to two different irreducible components of* $\mathrm{H}(33,117)_S$ *containing* (X) *by Proposition 37. In this case one of the generizations is ACM, the other is Buchsbaum of diameter 1. In this case we can not separate the two components by the usual semicontinuity of $h^i(\mathcal{I}_Z(v))$.*

(b) *To see that the curve X with 5-tuple* $(0,1,1,0,2)$, *which we know is obstructed, belongs to a unique component, we use Proposition 37 (iii). To conclude we need to verify that the postulation of X_1 and X_2 coincides where X_1 and X_2 are two curves with a common 5-tuple* $(0,0,0,0,0)$ *and having X as a common specialization. Since X is Buchsbaum of* *diameter 1 and $t = c = 6$, it follows that $\gamma_{X_1}(v) = \gamma_{X_2}(v)$ for $v \neq 6$ by the semicontinuity of $h^i(\mathcal{I}_X(v))$ since $h^0(\mathcal{I}_X(v)) - h^1(\mathcal{I}_X(v)) + h^1(\mathcal{O}_X(v)) = \chi(\mathcal{I}_X(v))$ is a constant under generization. Moreover we easily see that $\gamma_{X_1}(6) = \gamma_{X_2}(6) = 0$, and we are done.*

Question on possible generizations: If we use (P1), (P2), (C1), (C2) and more generally the removal of "ghost terms" allowed by Theorem 27 (shortly called (Ci) in the following), we find generizations of a diameter one curve X which "allow simplifications", i.e. except for (P2) which remove "ghost terms", in the minimal resolution of $I(X)$. The question is: have we found "all" generizations of X in $\mathrm{H}(d,g)$, at least from the point of view of its minimal resolution ("all" should be taken modulo those (Ci) which do not change the 5-tuple, cf. Remark 34)? In particular can we find the minimal free resolution of a generic curve of **every** irreducible component of $\mathrm{H}(d,g)$ containing (X) as a combination of (P1),(P2) and (Ci)?

Case (1). To answer, we first suppose the 5-tuple (a_1, a_2, b_1, b_2, r) of X satisfies $r \cdot a_2 = 0$. As in the proof of Proposition 37 we have ${}_0\mathrm{Hom}_R(I(X), M(X)) = 0$ and hence

$$\mathrm{H}_\gamma \cong \mathrm{H}(d,g) \ \text{ at } \ (X).$$

It follows that every generization of X has postulation γ, i.e. that only consecutive cancellations in the minimal resolution are allowed ([32]). So the answer is **YES** in this case. Indeed (P1) and (Ci) suffices! Note that, compared with [32], we have by Theorem 27 and Proposition 31 also proved the existence of the generizations!

Case (2). Also in the case $r \cdot b_1 = 0$, the answer is **YES**. Indeed this case follows from the case $r \cdot a_2 = 0$ by linkage. Note that if we link X to X_l as in Proposition 6, we get, by combining (14) and Lemma 23, that the 5-tuple (a_1, a_2, b_1, b_2, r) of X is equal to $(b_2(X_l), b_1(X_l), a_2(X_l), a_1(X_l), r(X_l))$ where $(a_1(X_l), a_2(X_l), b_1(X_l), b_2(X_l), r(X_l))$ is the 5-tuple of X_l, see the last part of the proof of Proposition 31 for details. In particular we get $a_2(X_l) = 0$ in the 5-tuple of X_l and since we know "all" generizations of a curve X_l with $a_2(X_l) = 0$ by the case (1), one may check this gives "all" expected generizations of X. Note that this also nicely explains the somewhat strange generizations of X given by (P2). Indeed a generization given by (P2) corresponds to a generization (P1) for the linked curve, i.e. to the removal of a "ghost term" in the minimal resolution of the linked curve!

Case (3). Finally for the remaining unsolved case $a_2 \cdot b_1 \neq 0$ the answer seems to be YES, which we hope to treat in a forthcoming paper. Here we only partially give the answer. To simplify, we restrict to curves which satisfy $a_1 \cdot b_1 = 0$ and $a_2 \cdot b_2 = 0$ (letting $a_1 = \beta_{1,c+4}$, $a_2 = \beta_{1,c}$, $b_1 = \beta_{2,c+4}$ and $b_2 = \beta_{2,c}$). Let us a little more generally consider the case

$$\underline{\beta}(X) := (0, a_2, b_1, 0, r) \ \text{ and } \ (a_2 \neq 0 \text{ or } b_1 \neq 0) \tag{27}$$

where proper generizations as in Proposition 31 occur, to give a picture of the existing generizations in $\mathrm{H}(d,g)$. Thanks to [2] we remark that any curve D satisfying $\underline{\beta}(D) = \underline{\beta}(X)$ and $\gamma_D(v) = \gamma_X(v)$ for $v \neq c$, belongs to the same *irreducible* family as $\overline{(X)}$, i.e. further generizations of X and D with constant postulation and Rao module structure lead to the "same" generic curve. By Proposition 31, we have

$$\begin{gathered}\text{For any pair } (i,j) \text{ of non-negative integers such that } r - i - j \geq 0, \\ a_2 - i \geq 0 \text{ and } b_1 - j \geq 0, \text{ there exists a generization } X_{ij} \text{ of } X \\ \text{in } \mathrm{H}(d,g) \text{ such that } \underline{\beta}(X_{ij}) = (0, a_2 - i, b_1 - j, 0, r - i - j).\end{gathered} \tag{28}$$

Example 39. *As an example, let* $\underline{\beta}(X) = (0, 3, 2, 0, 4)$ *(such curves exist by Example 41). By* (28) *we have* 10 *different generizations* X_{ij} *among which two curves correspond to the 5-tuples* $\underline{\beta}(X_{22}) = (0, 1, 0, 0, 0)$ *and* $\underline{\beta}(X_{31}) = (0, 0, 1, 0, 0)$*, i.e. they correspond to two unobstructed ACM curves with different postulation. Hence they belong to two different irreducible components of* $\mathrm{H}(d, g)$ *having (X) in their intersection.*

Pushing the arguments of Example 38 (c) and Example 39 a little further, we get at least

Proposition 40. *Suppose the Rao module* $M(X) \neq 0$ *is* r*-dimensional and concentrated in degree* c*, let* $a_1 = \beta_{1,c+4}$ *and* $a_2 = \beta_{1,c}$ *(resp.* $b_1 = \beta_{2,c+4}$ *and* $b_2 = \beta_{2,c}$*) be the number of minimal generators (resp. minimal relations) of degree* $c + 4$ *and* c *respectively, and suppose*

$$a_1 = 0,\ b_2 = 0 \quad \text{and} \quad a_2 \cdot b_1 \neq 0\ .$$

(a) *If* $r < a_2 + b_1$*, then the generic curve of every irreducible component containing* (X) *is ACM. Moreover the number,* $n(comp, X)$*, of irreducible components containing* (X) *satisfies*

$$n(comp, X) \geq \min\{a_2, r\} + \min\{b_1, r\} - r + 1 \geq 2\ ,$$

and in the case $s(X) = e(X) = c$*, we have equality to the left.*

(b) *If* $r \geq a_2 + b_1$ *and* $s(X) = e(X) = c$*, then* X *is an obstructed curve which belongs to a unique irreducible component of* $\mathrm{H}(d, g)$*.*

Proof. For the details of a proof, see [22], Prop. 4.6. □

It is not difficult to find examples of smooth connected curves satisfying even the restrictive condition $s(X) = e(X) = c(X)$ of the proposition (see Example 3.12 of [22] for examples in which $a_2 > 1$). See also [1].

Example 41. ($char\, k = 0$)*. We claim that for any pair of positive integers* (r, b) *there exist* smooth connected *curves* X *of* $\operatorname{diam} M(X) = 1$ *with minimal resolution as in* (10) *and* (20) *such that*

$$s(X) = e(X) = c\ , \quad \text{and}$$

$h^0(\mathcal{I}_X(c)) = 1$*,* $h^1(\mathcal{I}_X(c)) = r$*,* $h^1(\mathcal{O}_X(c)) = b$ *and* $c = 1 + b + 2r$ *and such that the corresponding 5-tuple is* $(0, 1, b, 0, r)$*. Note that such curves are obstructed by Corollary 24. To get the examples, we use Chang's results ([5] or [37], Thm. 4.1) to show the existence of smooth connected curves with* Ω*-resolution*

$$0 \to \mathcal{O}_{\mathbb{P}}(-2)^{3r-1} \oplus \mathcal{O}_{\mathbb{P}}(-4)^b \to \mathcal{O}_{\mathbb{P}} \oplus \Omega^r \oplus \mathcal{O}_{\mathbb{P}}(-3)^{b-1} \to \mathcal{I}_X(c) \to 0.$$

Here Ω *is by definition given by the exact sequences*

$$0 \to \Omega \to \mathcal{O}_{\mathbb{P}}(-1)^4 \to \mathcal{O}_{\mathbb{P}} \to 0 \quad \text{and} \quad 0 \to \mathcal{O}_{\mathbb{P}}(-4) \to \mathcal{O}_{\mathbb{P}}(-3)^4 \to \mathcal{O}_{\mathbb{P}}(-2)^6 \to \Omega \to 0 \quad (29)$$

which we deduce from the Koszul resolution of the regular sequence $\{X_0, X_1, X_2, X_3\}$*. Using the mapping cone construction we find the minimal resolution of* $I = I(X)$ *to be*

$$0 \to R(-4)^r \to R(-4)^b \oplus R(-3)^{4r} \to R(-2)^{3r+1} \oplus R \oplus R(-3)^{b-1} \to I(c) \to 0\ .$$

Moreover the degree of X is $d = \binom{c+4}{2} - 3r - 7$ and the genus is $g = (c+1)d - \binom{c+4}{3} + 5$.

The simplest case is $(r,b) = (1,1)$, which yields curves X with $s(X) = 4, d = 18$ and $g = 39$ (Sernesi's example), while $(r,b) = (2,1)$ yields curves X with $s(X) = 6, d = 32$ and $g = 109$ (Example 35 (b)). More generally, all curves X above satisfy $h^1(\mathcal{N}_X) = \delta^2(0) = b$ by Lemma 9, and if $r > b$, then X belongs to a unique *irreducible component of* $\mathrm{H}(d,g)$ *by Proposition 40. Other examples of singularities of* $\mathrm{H}(d,g)$ *which belong to a unique irreducible component are known ([18], Rem. 3b) and [11], Thm. 3.10).*

To prove Theorem 27 and Proposition 31 we need the following lemma for deforming a module N, which basically tells that if we can lift a (three term) resolution with augmentation N to a *complex*, then the complex defines a flat deformation of N which in the case $N = I(X)$ has to be a deformation of $I(X)$ as an ideal! Below we mainly follow the arguments of [22], Lem. 4.8.

Lemma 42. *Let X be a curve in $\mathbb{P}^3$ whose homogeneous ideal $I(X)$ has a minimal resolution of the following form*

$$(L^\bullet) \qquad 0 \to \bigoplus_i R(-i)^{\beta_{3,i}} \xrightarrow{\varphi} \bigoplus_i R(-i)^{\beta_{2,i}} \xrightarrow{\psi} \bigoplus_i R(-i)^{\beta_{1,i}} \to I(X) \to 0\ .$$

Let A be a finitely generated k-algebra, B the localization of A in a maximal ideal $\wp$, and suppose there exists a complex

$$(L_B^\bullet) \qquad \bigoplus_i R_B(-i)^{\beta_{3,i}} \xrightarrow{\varphi_B} \bigoplus_i R_B(-i)^{\beta_{2,i}} \xrightarrow{\psi_B} \bigoplus_i R_B(-i)^{\beta_{1,i}} \quad , \quad R_B = R \otimes_k B\ ,$$

such that $L_B^\bullet \otimes_B (B/\wp) \cong L^\bullet$. Then $(L_B^\bullet)$ is acyclic, φ_B is injective and the cokernel of ψ_B is a flat deformation of $I(X)$ as an ideal (so $\operatorname{coker}(\psi_B) \subseteq R_B$ *defines a flat deformation of $X \subseteq \mathbb{P}^3$ with constant postulation). Moreover for some $a \in A - \wp$, we can extend this conclusion to A_a via* $\mathrm{Spec}(B) \hookrightarrow$ *Spec*(A_a)*, i.e. there exists a flat family of curves $X_{\mathrm{Spec}(A_a)} \subseteq \mathbb{P}^3 \times \mathrm{Spec}(A_a)$ whose homogeneous ideal $I(X_{A_a})$ has a resolution (not necessarily minimal) of the form*

$$(L_{A_a}^\bullet) \qquad 0 \to \bigoplus R_{A_a}(-i)^{\beta_{3,i}} \to \bigoplus R_{A_a}(-i)^{\beta_{2,i}} \to \bigoplus R_{A_a}(-i)^{\beta_{1,i}} \to I(X_{A_a}) \to 0\ .$$

Proof (sketch). If $E = \operatorname{coker}\varphi$ and $E_B = \operatorname{coker}\varphi_B$, then one proves that $E_B \otimes_B (B/\wp) = E$, $\mathrm{Tor}_1(E_B, B/\wp) = 0$ and that φ_B is injective. By the local criterion of flatness, E_B is a flat deformation of E. Letting $Q_B = \operatorname{coker}(E_B \to \oplus_i R_B(-i)^{\beta_{1,i}})$, we can argue as we did for E_B to see that Q_B is a flat deformation of $I(X)$ and that $L_B^\bullet$ augmented by Q_B is exact.

To prove that Q_B is an ideal in R_B, we can use (4) to see that a deformation of the $\mathcal{O}_{\mathbb{P}}$-Module $\mathcal{I}_X$ (such as $\tilde{Q_B}$) corresponds to a deformation of the curve X in the usual way, i.e. via the cokernel of $\tilde{i}$: $\tilde{Q_B} \to \tilde{R_B}$. We get in particular a morphism $\mathrm{H}^0_*(\tilde{i})$: $Q_B \to R_B$ which proves what we want (or use [37]). One may give a direct proof using Hilbert-Burch theorem (cf. [25], page 37-38).

Finally we easily extend the morphism i and any morphism of the resolution $L_B^\bullet$ to be defined over $A_{a'}$, for some $a' \in A - \wp$ (such that $L_{A_{a'}}^\bullet$ is a complex). By shrinking $\operatorname{Spec} A_{a'}$ to $\operatorname{Spec} A_a$, $a \in A - \wp$, we get the exactness of the complex and the flatness of $I(X_{A_a})$ because these properties are open. $\square$

Proof (of Theorem 27). In the resolution

$$0 \to L_4 \xrightarrow{\sigma\oplus 0\oplus 0} L_3 \oplus F_2' \oplus\ R(-i) \xrightarrow{\psi} F_1' \oplus\ R(-i) \to I(X) \to 0$$

the matrix of ψ has a zero in the entry which corresponds to $R(-i) \to R(-i)$. As in the "Lemma de générisation simplifiantes" ([25], page 189), we can replace the mentioned zero by some indeterminate λ of degree zero. Keeping $\sigma \oplus 0 \oplus 0$ unchanged, we still have a complex which by Lemma 42 implies the existence a flat family of curves over $\operatorname{Spec}(A_a)$, $A = k[\lambda]$, for some $a \in A-(\lambda)$. Since any curve X' of the family given by $\operatorname{Spec}(A_{\lambda a})$ has a resolution where $R(-i)$ is redundant ($R(-i)$, and only $R(-i)$, is missing in its *minimal* resolution), and since we may still interpret the Rao module $M(X')$ as $\ker \mathrm{H}^3_*(\tilde{\sigma} \oplus 0 \oplus 0)$ with $\sigma \oplus 0 \oplus 0$ as above (so the whole family given by $\operatorname{Spec}(A_a)$ has constant Rao modules), we get the theorem. $\square$

Proof (of Proposition 31). (a) By the assumption $M(X) \cong M' \oplus M_{[t]}$, the minimal resolution (9) of M is given as the direct sum of the resolution of M' and the one of $M_{[t]}$. Let $\underline{X} = (X_0, X_1, X_2, X_3)^T$ and recall (10) and (20). Let η: $R(-t-4) \to L_4' \oplus R(-t-4)^r$ be the map into, say, the last direct factor of $R(-t-4)^r$, and let

$$\pi : L_3' \oplus R(-t-3)^{4r} \oplus P_2' \oplus R(-t-4)^{b_1} \to R(-t-4)$$

be the projection onto the last factor of $R(-t-4)^{b_1}$. As observed by Martin Deschamps and Perrin in the case $M \cong k$ ([25], page 189) we can change the 0 component in the matrix of $\sigma \oplus 0$ which corresponds to $\pi(\sigma \oplus 0)\eta$: $R(-t-4) \to R(-t-4)$, to some indeterminate of degree zero. To use Lemma 42 we must change four columns of the matrix A associated to $L_3 \oplus F_2 \to F_1$, to get a complex. Indeed let $r_1 := \operatorname{rank} F_1$ and look to the last column (a_k), $1 \le k \le r_1$, of A. Put $a_k = \sum_{n=0}^{3} \gamma_{k,n}^i X_n$ for every $1 \le k \le r_1$. Since the resolution is minimal, such $\gamma_{k,n}^i$ exist. Since the last column of the matrix of $\sigma \oplus 0$ consists of only 0's and one $\underline{X}$ there are precisely four columns $[H_{k,0}^j, H_{k,1}^j, H_{k,2}^j, H_{k,3}^j]$, $1 \le k \le r_1$, of A satisfying $\sum_{n=0}^{3} H_{k,n}^j X_n = 0$ for every k, which may contribute when we take the product of A with the last column of $\sigma \oplus 0$. Now if we change the trivial map $\pi(\sigma \oplus 0)\eta$ to the multiplication by an indeterminate λ and simultaneously change the four columns $[H_{k,0}^1, H_{k,1}^1, H_{k,2}^1, H_{k,3}^1]$ of A to $[H_{k,0}^1 - \gamma_{k,0}^1\lambda, H_{k,1}^1 - \gamma_{k,1}^1\lambda, H_{k,2}^1 - \gamma_{k,2}^1\lambda, H_{k,3}^1 - \gamma_{k,3}^1\lambda]$, leaving the rest of A unchanged, we get the desired complex. By Lemma 42 we get a flat irreducible family of curves having the same (not necessarily minimal) resolution of the homogeneous ideal, hence the same postulation, as X. Since we can remove redundant factors of the resolution in an open set, we have a generization X' with properties as claimed in Proposition 31. For more details, we refer to the proof of Prop. 4.2 of [22].

Note that in the above proof the Rao module $M(X')$ will satisfy $M(X') \cong M' \oplus M_{[t-1]}$. In particular the module F_2 (resp. L_3) in the minimal resolution (10) of $I(X')$ increases (resp. decreases) by $R(-t-3)^4$. If F_1 contains a direct factor $R(-t-3)^i$, then some of its summands may become redundant, because when we concretely make the row and column equivalent operations to get the isomorphic resolution in which $R(-t-4)$ directly may be deleted, the submatrix of A which corresponds to the map $R(-t-3)^4 \to R(-t-3)^i$, may have positive rank, say equal to p. Then we remove these p redundant factors as in the proof of Theorem 27. This takes care of the correction in Remark 33.

(b) We will prove (b) by linking X to an X_l via a CI of type (f,g) satisfying $\mathrm{H}^1(\mathcal{I}_X(v))=0$ for $v=f,g,f-4$ and $g-4$, and then apply (a) to X_l. To see that X_l satisfies the assumptions of (a), let $M'(X_l):=\mathrm{Ext}^4_R(M',R)(-f-g)$. Then $M(X_l)$ admits a decomposition $M(X_l)\cong M'(X_l)\oplus M_{[f+g-4-t]}$ as R-modules because of the duality

$$M(X_l)\cong \mathrm{Ext}^4_R(M,R)(-f-g)\cong \mathrm{Hom}_k(M,k)(-f-g+4)\ , \tag{30}$$

and the self-duality of $M_{[t]}(t)$. Moreover with $L^*:=\mathrm{Hom}_R(L,R)$ we have an exact sequence $\to (L_2')^*\to (L_3')^*\to (L_4')^*\to \mathrm{Ext}^4_R(M',R)\cong M'(X_l)(f+g)\to 0$. Since L_2' has no direct free factor of degree t and $t+4$, it follows that $(L_2')^*(-f-g)$ has no direct free factor of degree $f+g-t$ and $f+g-t-4$, and visa versa, see the proof of Prop. 4.2 of [22] for details.

Now since X_l satisfies the assumptions of (a), we need to see that the direct free part F_1 generated in degree t in the resolution of $I(X)$, is equal (at least dimensionally) to the corresponding part of $F_2(X_l)(4)$ in the minimal resolution of $I(X_l)$ of the linked curve X_l. Indeed since the isomorphism of (14) is given by the duality used in (30), it must commute with their decomposition as R-modules, i.e. we have

$${}_0\mathrm{Hom}_R(I(X),M_{[t]})\cong {}_0\mathrm{Hom}_R(M_{[f+g-4-t]},E(X_l))\ . \tag{31}$$

Then we conclude by Lemma 23 because

$${}_v\mathrm{Ext}^2_R(M_{[t]},M)^\vee\cong {}_{-v-4}\,\mathrm{Ext}^2_R(M,M_{[t]})\cong {}_{-v-4}\mathrm{Hom}_R(L_2',M_{[t]})=0$$

for $v=0$ and 4 by the assumption on L_2'. Note that ${}_v\mathrm{Ext}^2_R(M_{[f+g-4-t]},M(X_l))=0$ for $v=0$ and 4 by the same reason, i.e. because ${}_{-v-4}\mathrm{Hom}_R((L_2')^*(-f-g),M_{[f+g-4-t]})=0$.

Now applying (a) to the linked curve X_l, we get a generization of X_l' with constant postulation where $R(-f-g+4)$ is "removed" in its minimal resolution. A further linkage, using a complete intersection of the same type as in the linkage above (such a complete intersection exists by [20], Cor. 3.7) and formula (31) combined with Lemma 23 (replacing X and X_l by X' and X_l' in (31), in the case $r-1=0$ one explicitly constructs the resolution of $I(X')$ by the mapping cone construction), gives the desired generization X', and we are done. $\square$

Remark 43. *Note that the proof of Proposition 31 (b) simplifies in an obvious way in the case of most interest to us, i.e. where the diameter is 1* ($M\cong M_{[c]}$).

7 Consequences, remarks and questions.

An interesting consequence of Theorem 27 and Proposition 31 is the following

Corollary 44. *Let X be Buchsbaum (or ACM) of* $\operatorname{diam} M(X)\le 2$. *Then there exists a generization X' of X in* $\mathrm{H}(d,g)$ *such that X' is Buchsbaum (or ACM) and such that the modules of the three sets*

$$\{F_2,F_1\}\ ,\ \{L_4,F_2\}\ \ \textit{and}\ \ \{L_4,F_1(-4)\}$$

in the minimal resolution (10) *of* $I(X')$ *are without common direct free factors. In particular*

$${}_0\mathrm{Hom}_R(I(X'),M(X'))={}_0\mathrm{Hom}_R(M(X'),E(X'))=0$$

and $\mathrm{H}(d,g)$ *is smooth at* (X').

Proof. Since the module structure of $M = M(X)$ is trivial, we get

$$_0\mathrm{Hom}_R(I, M) \cong {}_0\mathrm{Hom}_R(F_1, M).$$

Since $M \cong M_{[c]} \oplus M_{[c-1]}$ as R-modules and $M_{[t]}$ is given by (19), it follows that the latter group vanishes if and only if L_4 and $F_1(-4)$ are without common free factors. Moreover, as in the proof of Lemma 23, we have

$$_{-4}\mathrm{Hom}_R(F_2, M)^\vee \cong {}_0\mathrm{Hom}_R(M, E)$$

where the former group vanishes if and only if L_4 and F_2 are without common free factors. Now, using $M \cong M_{[c]} \oplus M_{[c-1]}$, we can successively apply Proposition 31 to $M_{[c]}$ and $M_{[c-1]}$ to see that $\{L_4, F_2\}$ and $\{L_4, F_1(-4)\}$ are without common direct free factors. Finally since $\{F_2, F_1\}$ are without common direct free factors after performing a suitable generization (Theorem 27), we are done. □

All general curves of $\operatorname{diam} M(X) = 1$ are Buchsbaum. Hence

Corollary 45. *Every irreducible component of* $\mathrm{H}(d, g)$ *whose generic curve* X *satisfies* $\operatorname{diam} M = 1$ *is generically smooth.*

Question. Is any irreducible component of $\mathrm{H}(d, g)$ whose Rao module of its generic curve is concentrated in at most two consecutive degrees, generically smooth?

We believe the answer is YES. Indeed if we as in Corollary 44 can show the existence of a generization X' of X such that

$$_0\mathrm{Hom}_R(I(X'), M(X')) = {}_0\mathrm{Hom}_R(M(X'), E(X')) = 0$$

for *every* curve X of $\operatorname{diam} M(X) = 2$, then we can use Theorem 8 to see that $\mathrm{H}(d, g)$ is smooth at (X'), whence we get an affirmative answer to the question!

Remark 46. *If we simply call an irreducible component* V *of* $\mathrm{H}(d, g)$ *reduced if* $\mathrm{H}(d, g)$ *is generically smooth along* V*, then we know that* V *is reduced if the Rao module* M *of its generic curve satisfies* $\operatorname{diam} M \leq 1$*. Moreover recall that in Mumford's well known example ([30]) the diameter of* M *of the obstructed generic curve is* 3*. Using [18] we find a large number of non-reduced components among which there are examples of any diameter* $\operatorname{diam} M \geq 3$*. Indeed, as is well known, a smooth cubic surface* $X \subset \mathbb{P}^3$ *satisfies* $\mathrm{Pic}(X) \simeq \mathbb{Z}^{\oplus 7}$*. It follows from the main theorem of [18] (or directly from [31]) that the general curve which comes from the linear system* $(3\alpha, \alpha^5, 2) \in \mathbb{Z}^{\oplus 7}$ *is the generic curve of a non-reduced component of* $\mathrm{H}(d, g)$ *for every* $\alpha \geq 4$ *(and* $\alpha = 4$ *is Mumford's example). Its diameter is* $2\alpha - 5$*. In the same way the general curve which corresponds to* $(3\alpha+1, \alpha^5, 2) \in \mathbb{Z}^{\oplus 7}$ *is the generic curve of a non-reduced component of* $\mathrm{H}(d, g)$ *of* $\operatorname{diam} M = 2\alpha - 4$ *for every* $\alpha \geq 4$*. So the question above represents the only open case concerning reducedness of irreducible components of* $\mathrm{H}(d, g)$ *with respect to the diameter of the Rao module of its generic curve.*

References

[1] M. Amasaki. Examples of Nonsingular Irreducible Curves Which Give Reducible Singular Points of red($\mathrm{H}_{d,g}$). *Publ. RIMS, Kyoto Univ.* 21 (1985), 761-786.

[2] G. Bolondi. Irreducible families of curves with fixed cohomology *Arch. der Math.*, 53 (1989), 300-305

[3] G. Bolondi, J.O. Kleppe and R.M. Miro-Roig. Maximal rank curves and singular points of the Hilbert scheme *Compositio Math.*, 77 (1991), 269-291.

[4] G. Bolondi, J. Migliore. On curves with natural cohomology and their deficiency modules *Annales de l'institut Fourier* 43 (2) (1993), 325-357.

[5] M-C. Chang. A Filtered Bertini-type Theorem. *J. reine angew. Math.* 397, (1989), 214-219.

[6] Ph. Ellia, M. Fiorentini. Défaut de postulation et singularités du Schéma de Hilbert. *Annali Univ. di Ferrara* 30 (1984), 185-198.

[7] Ph. Ellia, A. Hirschowitz, E. Mezzetti On the number of irreducible components of the Hilbert scheme of smooth space curves *International J. of Math.* Vol.3, No. 6,(1992) 799-807.

[8] G. Ellingsrud. Sur le schéma de Hilbert des variétés de codimension 2 dans $\mathbb{P}^e$ a cône de Cohen-Macaulay *Ann. Scient. Éc. Norm. Sup.* 8 (1975), 423-432.

[9] D. Eisenbud. Commutative algebra. With a view toward algebraic geometry. *Graduate Texts in Math.*, volume 150, Springer–Verlag, New York, 1995.

[10] S. Ginouillac. Sur le nombre de composant du schéma de Hilbert des courbes ACM de $\mathbb{P}^3_k$. *C. R. Acad. Sci. Paris Sér. I, Math.* 329 (1999), 857-862.

[11] S. Guffroy. Sur l'incomplétude de la série linéaire caractéristique d'une famille de courbes planes a nœuds et a cusps. *Nagoya Math. J.* 171 (2003), 51-83.

[12] G. Fløystad. Determining obstructions for space curves, with application to non-reduced components of the Hilbert scheme. *J. reine angew. Math.* 439 (1993), 11-44.

[13] A. Grothendieck. Cohomologie Locale des Faisceaux Cohérents et Théorèmes de Lefschetz Locaux et Globaux. Augmenté d'un exposé par M. Raynaud. (*SGA* 2). *Advanced Studies in Pure Mathematics* Vol. 2. North-Holland, Amsterdam (1968).

[14] A. Grothendieck. Les schémas de Hilbert. *Séminaire Bourbaki*, exp. 221 (1960).

[15] L. Gruson, Chr. Peskine. Genre des courbes de l'espace projectif, in: *Proc. Tromsø 1977, Lectures Notes in Math.* Vol. 687, Springer–Verlag, New York, (1978).

[16] R. Hartshorne. Algebraic Geometry. *Graduate Texts in Math.* Vol. 52 Springer–Verlag, New York, 1983.

[17] J. O. Kleppe. Deformations of graded algebras. *Math. Scand.* 45 (1979), 205–231.

[18] J. O. Kleppe. Non-reduced components of the Hilbert scheme of smooth space curves, in *Proc. Rocca di Papa 1985 Lectures Notes in Math.* Vol. 1266 Springer–Verlag, New York, (1987).

[19] J. O. Kleppe. The Hilbert-flag scheme, its properties and its connection with the Hilbert scheme. Applications to curves in 3-space. Preprint (part of thesis), March 1981, Univ. of Oslo. http://www.iu.hio.no/~jank/papers.htm.

[20] J. O. Kleppe. Liaison of families of subschemes in $\mathbb{P}^n$, in *"Algebraic Curves and Projective Geometry, Proceedings (Trento, 1988)," Lectures Notes in Math.* Vol. 1389 Springer–Verlag, New York, (1989).

[21] J. O. Kleppe. Concerning the existence of nice components in the Hilbert scheme of curves in $\mathbb{P}^n$ for $n = 4$ and 5, *J. reine angew. Math.* 475 (1996), 77-102.

[22] J.O. Kleppe. The Hilbert Scheme of Space Curves of small diameter. *Annales de l'institut Fourier* 56 no. 5 (2006), p. 1297-1335.

[23] J.O. Kleppe. Families of Artinian and one-dimensional algebras. *J. Algebra* 311 (2007), 665-701.

[24] A. Laudal. Formal Moduli of Algebraic Structures. *Lectures Notes in Math.* Vol. 754, Springer–

Verlag, New York, (1979).

[25] M. Martin-Deschamps, D. Perrin. Sur la classification des courbes gauches, *Asterisque*, 184-185 (1990).

[26] M. Martin-Deschamps, D. Perrin Courbes Gauches et Modules de Rao, *J. reine angew. Math.* 439 (1993), 103-145.

[27] J. Migliore. Introduction to liaison theory and deficiency modules. *Progress in Math.*, Vol. 165, Birkhäuser Boston, Inc., Boston, MA, 1998.

[28] J. Migliore, U. Nagel. Tetrahedral Curves. *Int. Math. Res. Not.* (2005), no. 15, p. 899-939.

[29] R. M. Miró-Roig. Unobstructed Arithmetically Buchsbaum Curves. in *"Algebraic Curves and Projective Geometry, Proceedings (Trento, 1988)," Lectures Notes in Math.* Vol. 1389 Springer–Verlag, New York, (1989).

[30] D. Mumford. Further pathologies in algebraic geometry, *Amer. J. Math.*, 84, 1962, 642-648.

[31] H. Nasu. Obstructions to deforming space curves and non-reduced components of the Hilbert scheme. *Publ. RIMS, Kyoto Univ. (2006), no 42, p. 117-141.*

[32] I. Peeva. Consecutive cancellations in Betti Numbers. *Proc. Amer. Math. Soc.* 132 no. 12 (2004), 3503-3507.

[33] Ragusa and Zappala. On the reducibility of the postulation Hilbert scheme. *Rend. Circ. Mat. Palermo*, Serie II, no LIII (2004), 401-406.

[34] A. P. Rao. Liaison Among Curves. *Invent. Math.* 50 (1979), 205-217.

[35] E. Sernesi. Un esempio di curva ostruita in $\mathbb{P}^3$. *Sem. di variabili Complesse, Bologna* (1981), 223-231.

[36] C. Walter. Horrocks theory and algebraic space curves. Preprint (1990).

[37] C. Walter. Some examples of obstructed curves in $\mathbb{P}^3$ *In: Complex Projective Geometry. London Math. Soc. Lecture Note Ser.* 179 (1992).

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