THE HILBERT SCHEME OF SPACE CURVES OF SMALL RAO MODULE WITH AN APPENDIX ON NON-REDUCED COMPONENTS.

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Introduction and main results.

The Hilbert scheme of space curves $H(d,g)$ has received much attention over the last years after Grothendieck showed its existence [G]. For so-called special curves it has turned out that the structure of $H(d,g)$ is difficult to describe in detail, and questions related to irreducibility and number of components, dimension and smoothness has been hard to solve.

For particular classes of space curves, some results are known. In 1975 Ellingsrud [El] managed to prove that the open subset of $H(d,g)$ of arithmetically Cohen Macaulay curves (with a fixed resolution of the sheaf ideal I_C) is smooth and irreducible, and he computed the dimension of the corresponding component. A generalization of this result in the direction of smoothness and dimension was already given in [K2] (see theorem 3i,i' of this paper) while the irreducibility was later nicely generalized by Bolondi [B]. More recently, Martin-Deschamps and Perrin have given a stratification $H_{\gamma,\rho}$ of $H(d,g)$ obtained by deforming space curves with constant cohomology [MDP1]. Their strata $H_{\gamma,\rho}$ is large in $H(d,g)$ provided the Hartshorne-Rao module $M = \oplus H^1(I_C(v))$ is small. Indeed if the graded module M is concentrated in at most two consecutive degrees (i.e. its diameter is two or less), then $H_{\gamma,\rho}$ is smooth and irreducible and its dimension is known. We get back Ellingsrud's result because $H_{\gamma,\rho} \cong H(d,g)$ at arithmetically Cohen Macaulay curves (i.e. at curves with $M = 0$). However, in the case the diameter is 1 or more, $H_{\gamma,\rho}$ can sit inside $H(d,g)$ in different ways, and we can not get a complete picture of $H(d,g)$ from $H_{\gamma,\rho}$ without studying the imbedding in detail.

This is what we do in this paper. We get complete results only in the diameter 1 case, thus generalizing Ellingsrud's results (and the study of the case $M \cong k$ of [MDP1]) to

Theorem *Let C be a curve in* $\mathbf{P}^3$, *let* $M = H_*^1(I_C)$ *and* $E = H_*^1(O_C)$ *and suppose M has diameter 1 or less. Then H(d,g) is smooth at C (i.e. C is unobstructed) if and only if (at least) one of the following conditions hold;*

i) ${}_vHom_R(I\,,\,M) = 0$ *for* $v = 0$ *and* $v = -4$
ii) ${}_vHom_R(M\,,\,E) = 0$ *for* $v = 0$ *and* $v = -4$, *or*
iii) ${}_0Hom_R(I\,,\,M) = 0$ *and* ${}_0Hom_R(M\,,\,E) = 0$.

Moreover if C is unobstructed, then the dimension of the Hilbert scheme at C is

$$dim_C H(d,g) = 4d + {}_0hom_R(I\,,\,E) + {}_{-4}hom_R(I\,,\,M) + {}_{-4}hom_R(M\,,\,E)$$

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(Once we know a minimal resolution of I_C, we easily express these *Hom* -groups as certain graded Betti numbers and we compute the dimension of $H(d,g)$ at C to be $4d + \delta^2(0) + r(a_1+b_2)$, cf. definition 1 and corollary 6).

More generally we show in section 2 that the impact of $H_{\gamma,\rho}$ in $H(d,g)$ is equipped with **three** Yoneda pairings (inducing corresponding cup-products, see propositions 7 and 8), some of which are essentially considered in [W1] and [F]. These are extremely easy to handle because they live on the Hom-level, and, as formulated here, they are given by just taking simple compositions. It turns out that the vanishing of *all three* pairings are necessary for unobstructedness, and quite close (resp. equivalent) to the sufficient conditions of theorem 3 in the diameter 2 case (resp. in the diameter 1 case). To appreciate such results, one should recall how hard it "classically" ([Mu] and [K2]) has been to prove obstructedness because one essentially had to compute a neighborhood of ($C \subseteq \mathbf{P}^3$) in $H(d,g)$ to conclude. The Yoneda pairings of proposition 7 and 8 are related to two other pairings, as indicated in (2.11). To get equivalent conditions when the diameter is 2 as well, we probably just need to include their Massey-products [L2] and the Massey-product which corresponds to the Yoneda pairing of proposition 7 and 8.

In theorem 3 we also compute $dim_C H(d,g)$. The technique used in section 1 to prove theorem 3, especially the following generalization of the local Gorenstein duality;

$$ {}_{v}Ext^{i}_{m}(N_2\ ,\ N_1) \simeq {}_{-v-4}Ext^{4-i}_{R}(N_1\ ,\ N_2)^{\vee}\ , $$

$(-)^{\vee} = Hom_k(-\ ,\ k)$, and the spectral (resp. exact) sequences involved (1.1), is important for the whole paper, and it can be used to give new proofs and additional informations of some main results in [MDP1] as well, cf. (1.6) and (1.21).

In section 3 we are concerned with curves which admit a generization or are generic in $H_{\gamma,\rho}$ (resp. in $H(d,g)$). Inspired by ideas of [MDP1] we prove a rather general theorem, telling that we can kill certain repetitions in a minimal resolution ("ghost-terms") of the homogeneous ideal I(C), under deformation. Hence curves with such simplified resolutions exist. One result of particular interest is

Theorem *If a curve C in $H_{\gamma,\rho}$ (or in $H(d,g)$) is general enough, then C admits a minimal free resolution of the form*

$$ 0 \to L_4 \xrightarrow{(\sigma,0)^t} L_3 \oplus F_1 \to F_0 \to I(C) \to 0\ , $$

where $\sigma\colon L_4 \to L_3$ is given by the minimal resolution of the Hartshorne-Rao module M, cf. (3.2) and where F_1 and F_0 are without repetitions (i.e. without common direct factors).

Restricting to general Buchsbaum curves, we prove that L_4 and F_1 (and L_4 and $F_0(-4)$ in some cases) have no common direct factor as well. Moreover we get a somewhat complete picture of the existing generizations of Buchsbaum curves, allowing us in many cases to decide whenever an obstructed curve is contained in a *unique* component of $H(d,g)$ or not. Moreover we show that any Buchsbaum curve whose Hartshorne-Rao module has diameter 2 or less, admits a generization in $H(d,g)$ to an unobstructed curve. It follows that any irreducible component of $H(d,g)$ is generically smooth in the diameter 1 case.

There are also other classes of curves where questions related to the structure of $H(d,g)$ is quite well understood. One such class consists of curves which sit on a smooth cubic surface. In the appendix we discuss the existence of non-reduced components of $H(d,g)$ in general, and we relate the discussion to a conjecture on non-reduced components for a maximal class of curves on a smooth cubic surface (cf. [K1] and [E]). We further prove the conjecture in most cases by generalizing the results of the mentioned papers to (see the appendix for notations):

Theorem *Let W be a maximal irreducible family of smooth connected space curves, whose general member sits on some smooth cubic surface and whose corresponding invertible sheaf is given by* $(\delta,m_1,..,m_6)$, $\delta \geq m_1 \geq .. \geq m_6$ *where* $\delta \geq m_1 + m_2 + m_3$. *Let* $\bar{W}$ *be the closure of W in* $H(d,g)$.

i) *If* $m_6 \geq 3$ *and* $(\delta,m_1,..,m_6) \neq (\lambda+9,\lambda+3,3,..3)$ *for any* $\lambda \geq 2$, *then* $\bar{W}$ *is a generically smooth, irreducible component of* $H(d,g)$.

ii) *If* $m_6 = 2$, $m_5 \geq 4$, $m_2 \geq 5$ *and* $d \geq 21$, *then* $\bar{W}$ *is a non-reduced irreducible component of* $H(d,g)$.

iii) *If* $m_6 = 1$, $m_5 \geq 6$, $m_2 \geq 7$ *and* $d \geq 35$, *then* $\bar{W}$ *is a non-reduced irreducible component of* $H(d,g)$.

Since the case $m_6 = 0$ is treated by Dolcetti and Pareshi in [DP], and the case $m_6 < 0$ cannot occur, we now have a pretty good picture of the structure of $H(d,g)$ for generic elements of this class of curves. However, as we may see from the theorem, for some values of $(\delta,m_1,..,m_6)$ (in fact infinitely many), the conjecture is still open.

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Notations and terminology. A space curve C is an *equidimensional, locally Cohen Macaulay* subscheme of $\mathbf{P} = \mathbf{P}^3$ of dimension 1 with sheaf ideal I_C and normal sheaf $N_C = Hom_{O_P}(I_C,O_C)$. If F is a coherent O_P-Module, we let $H^i(F) = H^i(\mathbf{P},F)$, $H_*^i(F) = \sum_v H^i(F(v))$, $h^i(F) = \dim H^i(F)$, and $\chi(F) = \Sigma\,(-1)^i\, h^i(F)$ is the Euler-Poincaré characteristic. Moreover $M = M(C)$ is the Hartshorne-Rao module $H_*^1(I_C)$ and $E = E(C)$ is the module $H_*^1(O_C)$. They are graded modules over the polynomial ring $R = k[X_0,X_1,X_2,X_3]$, where k is supposed to be an algebraically closed field of characteristic zero. The postulation γ (resp. deficiency ρ, resp. specialization σ) of C is the function defined over the integers Z by $\gamma(v) = \gamma_C(v) = h^0(I_C(v))$ (resp. $\rho(v) = \rho_C(v) = h^1(I_C(v))$, resp. $\sigma(v) = \sigma_C(v) = h^1(O_C(v))$). Put

$$s(C) = \min\{n \mid h^0(I_C(n)) \neq 0\},$$
$$c(C) = \max\{n \mid h^1(I_C(n)) \neq 0\},$$
$$e(C) = \max\{n \mid h^1(O_C(n)) \neq 0\},$$

where e(C) is the index of speciality (Put $c(C) = -\infty$ for arithmetically Cohen Macaulay curves). A curve C such that $mM(C) = 0$, $m = (X_0,..,X_3)$, is called a Buchsbaum curve. C is *unobstructed* if the Hilbert scheme of space curves of degree d and arithmetic genus g, $H(d,g)$, is smooth at the corresponding point ($C \subseteq \mathbf{P}$), otherwise C is obstructed. The open part of $H(d,g)$ of smooth connected space curves is denoted by $H(d,g)_S$, while $H_{\gamma,\rho} = H(d,g)_{\gamma,\rho}$ (resp. H_γ, resp. $H_{\gamma,M}$) denotes the subscheme of $H(d,g)$ of curves with constant postulation γ and deficiency ρ (resp. constant postulation γ, resp constant postulation γ and Rao module M), cf. [MDP1]. The curve in a small enough open irreducible subset of $H(d,g)$ is called a generic curve of $H(d,g)$, and accordingly, if we state that a generic curve has a certain property, then there is an open dense subset of $H(d,g)$ of curves having this property. A generization ($C' \subseteq \mathbf{P}$) of ($C \subseteq \mathbf{P}$) in the Hilbert scheme $H(d,g)$ may be thought of as the generic curve of some irreducible subset of $H(d,g)$ containing ($C \subseteq \mathbf{P}$).

For any graded R-module N, we have the right derived functors $H_m^i(N)$ and ${}_vExt_m^i(N\ ,\ -)$ of $\Gamma_m(N) = \sum_v ker(N_v \to \Gamma(P,\ N^{\sim}(v)))$ and $\Gamma_m(Hom_R(N\ ,\ -))_v$ respectively (cf. [SGA 2], exp. VI or [H2]) where $m = (X_0,..,X_3)$. We use small letters for the k-dimension and subscript v for the homogeneous part of degree v, i.e. ${}_vext_m^i(N_1,N_2) = dim\ {}_vExt_m^i(N_1,N_2)$.

1. Preliminaries. Sufficient conditions for unobstructedness.

In this section we recall the main technical tools of this paper, and we review some partially known results for unobstructedness of space curves. At most places we include proofs, also because we need parts of the arguments (e.g. the exact sequences which appear) later.

Let N_1 and N_2 be graded R-modules of finite type. Then there is a spectral sequence ([SGA 2], exp. VI)

(1.1) $$E_2^{p,q} = {}_vExt_R^p(N_1\ ,\ H_m^q(N_2)) \quad \text{converging to} \quad {}_vExt_m^{p+q}(N_1\ ,\ N_2),$$

and a duality isomorphism ([K2], th. 2.1.4)

(1.2) $${}_vExt_m^i(N_2\ ,\ N_1) \simeq {}_{-v-4}Ext_R^{4-i}(N_1\ ,\ N_2)^v\ ,$$

$(-)^v = Hom_k(-\ ,\ k)$, for any integer i and v. Moreover there is a long exact sequence ([SGA2], exp. VI)

(1.3) $$\to {}_vExt_m^i(N_1\ ,\ N_2) \to {}_vExt_R^i(N_1\ ,\ N_2) \to Ext_{O_P}^i(\tilde{N}_1\ ,\ \tilde{N}_2(v)) \to {}_vExt_m^{i+1}(N_1\ ,\ N_2) \to$$

which in particular relates the deformation theory of ($C \subseteq \mathbf{P}$), described by $H^{i-1}(N_C) \cong Ext_{O_P}^i(I^{\sim},\ I^{\sim})$ for i = 1,2, to the deformation theory of the homogeneous ideal I = I(C), described by ${}_0Ext_R^i(I,\ I)$, in the following exact sequence

(1.4) $$0 \to {}_vExt_R^1(I\ ,\ I) \to H^0(N_C(v)) \to {}_vExt_m^2(I\ ,\ I) \xrightarrow{\alpha} {}_vExt_R^2(I\ ,\ I) \to H^1(N_C(v)) \to {}_vExt_m^3(I\ ,\ I) \to 0$$

Observe that the map α: ${}_vExt_m^2(I,\ I) \cong {}_vHom(I,H_m^2(I)) \to {}_vExt_R^2(I,\ I)$ factorizes via ${}_vExt_R^2(M\ ,\ M)$ in a natural way ([W2], th. 2.3), the factorization is in fact given by certain edge homomorphisms of the spectral sequence (1.1) with $N_1 = H_m^2(I) = M$, $N_2 = I$ and $p+q = 4$, cf. (1.17) and (1.18) where this map of factorization occurs.

To compute the dimension of the components of $H(d,g)$, we have found it convenient to introduce the following invariant, defined in terms of the graded Betti numbers of a minimal resolution of the homogeneous ideal I of C:

(1.5) $$0 \longrightarrow \oplus_i R(-n_{3i}) \longrightarrow \oplus_i R(-n_{2i}) \longrightarrow \oplus_i R(-n_{1i}) \longrightarrow I \longrightarrow 0$$

Definition/lemma 1 *If C is any curve of degree d in $\mathbf{P}^3$, we let*

$$\delta^j(v) = \sum_i h^j(I_C(n_{1i}+v)) - \sum_i h^j(I_C(n_{2i}+v)) - \sum_i h^j(I_C(n_{3i}+v))$$

Then the following expressions are equal

$$ {}_0ext^1_R(I,\ I) - {}_0ext^2_R(I,\ I) = 1 - \delta^0(0) = 4d + \delta^2(0) - \delta^1(0) = 1 + \delta^2(-4) - \delta^1(-4)$$

(1.6) *Remark.* Those familiar with results and notations of [MDP1] will recognize $1-\delta^0(0)$ as δ_γ and $\delta^1(-4)$ as $\epsilon_{\gamma,\delta}$ in their terminology. By lemma 1 it follows that the dimension of the Hilbert scheme $H_{\gamma,M}$ of constant postulation and Rao module, which they show is $\delta_\gamma + \epsilon_{\gamma,\delta} - {}_0hom(M,M)$, is also equal to $1 + \delta^2(-4) - {}_0hom(M,M)$.

Proof of lemma 1. To see the equality to the left, we apply ${}_vHom_R(-\ ,\ I)$ to the resolution (1.5). Since $Hom_R(I\ ,\ I) \cong R$ and since the alternating sum of the dimension of the terms in a complex equals the alternating sum of the dimension of its homology groups, we get

(1.7) $$dim\ R_v - {}_vext_R{}^1(I\ ,\ I) + {}_vext_R{}^2(I\ ,\ I) = \delta^0(v)\ ,\ v \in Z$$

If v = 0 we get the equality of lemma 1 to the left. The equality in the middle follows from [K2], lemma 2.2.11. We will, however, indicate how we can prove this and the last equality from (1.2) and (1.3). Indeed by (1.2), ${}_vext_m{}^{4-i}(I\ ,\ I) = {}_{-v-4}ext_R{}^i(I\ ,\ I)$, hence

(1.8) $${}_vext_m{}^2(I\ ,\ I) - {}_vext_m{}^3(I\ ,\ I) + dim\ R_{-v-4} = \delta^0(-v-4)\ ,\ v \in Z$$

by (1.7). Combining (1.7) and (1.8) with the exact sequence (1.4), we get

(1.9) $$\binom{v+3}{3} - \chi(N_C(v)) = \delta^0(v) - \delta^0(-v-4) \quad , \quad v \in Z$$

because $dim\ R_v - dim\ R_{-v-4} = \binom{v+3}{3}$. Therefore it suffices to prove

(1.10) $$\delta^0(-v-4) = \delta^1(v) - \delta^2(v) \quad , \quad v \geq -4$$

Indeed using (1.9) and (1.10) for v = 0 we get the equality of lemma 1 in the middle because $\chi(N_C) = 4d$ holds for *any* curve (cf. rem. 1.13) while (1.10) for v = -4 takes care of the last equality appearing in lemma 1.

To prove (1.10) we use the spectral sequence (1.1) together with (1.8). Recalling $M = H_m{}^2(I)$ and $E = H_m{}^3(I)$ we get ${}_vExt_m{}^2(I\ ,\ I) \cong {}_vHom(I\ ,\ M)$ and ${}_vExt^2(I\ ,\ E) \cong {}_vExt_m{}^5(I\ ,\ I) = 0$ and an exact sequence

(1.11) $$0 \to {}_vExt^1_R(I,M) \to {}_vExt^3_m(I,I) \to {}_vHom(I,E) \to {}_vExt^2_R(I,M) \to {}_vExt^4_m(I,I) \to {}_vExt^1(I,E) \to 0$$

where we have used that v ≥ -4 implies ${}_vHom(I,\ H_m{}^4(I)) = 0$. As argued for (1.7), applying ${}_vHom(-,\ M)$ (resp. ${}_vHom(-,\ E)$) to the resolution (1.5), we get

$$(1.12)\quad \delta^1(v) = \sum_{i=0}^{2}(-1)^i \cdot {}_v ext^i(I\ ,\ M)\quad ,\quad (\ resp.\ \ \delta^2(v) = \sum_{i=0}^{2}(-1)^i \cdot {}_v ext^i(I\ ,\ E)\ )$$

So $\delta^1(v)-\delta^2(v)$ equals $\sum_{i=2}^{4}(-1)^i \cdot {}_v ext_m^i(I,I)$ by (1.11), and since ${}_vExt_m^{\ 4}(I\ ,\ I) \cong {}_{-v-4}Hom(I\ ,\ I)^v$ $\cong R_{-v-4}{}^v$ we get (1.10) from (1.8), and the proof of lemma 1 is complete.

(1.13) *Remark.* In [K2], lemma 2.2.11 we proved $\chi(N_C(v)) = 2dv + 4d$ for *any* curve and any integer v by computing $\delta^0(v)$ for v >> 0. Indeed using the definition of $\delta^0(v)$, the sequence $0 \to I_C \to O_P \to O_C \to 0$ and $\sum_j \sum_i (-1)^j n_{ji} = 0$ and applying Riemann-Roch to $\chi(O_C(n_{ji}+v))$ we get easily

$$(1.14)\qquad \delta^0(v) = \sum_j \sum_i (-1)^j \chi(O_P(n_{ji} + v)) - (dv + 1 - g)\ ,\ \ v >> 0$$

while duality on **P** and (1.5) show that the double sum of (1.14) equals $-\chi(I_C(-v-4))$. Hence

$$\delta^0(v) = \binom{v+3}{3} + \chi(O_C(-v-4)) - (dv + 1 - g)\ ,\ \ v >> 0$$

and now $\chi(N_C(v)) = 2dv + 4d$ follows by combining with (1.9).

Proposition 2 *Let C and C' be curves in P^3 which are linked (algebraically) by a complete intersection of two surfaces of degree f and g. If*

$$H^1(I_C(v)) = 0 \ \ for\ v = f,\ g\ ,\ f - 4 \ \ and\ \ g - 4,$$

then C is unobstructed if and only if C' is unobstructed.

One may find a proof in [K3], prop. 3.2. The proposition allows us to complete the proof of the following result:

Theorem 3 *If C is any curve in P^3 of degree d and arithmetic genus g, satisfying (at least) one of the following conditions:*

i) ${}_vHom_R(I\ ,\ M) = 0$ *for* $v = 0$ *and* $v = -4$
ii) ${}_vHom_R(M\ ,\ E) = 0$ *for* $v = 0$ *and* $v = -4$, *or*
iii) ${}_0Hom_R(I\ ,\ M) = 0\ ,\ {}_0Hom_R(M\ ,\ E) = 0$ *and* ${}_0Ext_R{}^2(M\ ,\ M) = 0$,

then C is unobstructed. Moreover, in each case, the dimension of the Hilbert scheme H(d,g) at (C ⊆ P) is given by

i') $\dim_C H(d,g) = 4d + \delta^2(0) - \delta^1(0)$, *provided i) holds*
ii') $\dim_C H(d,g) = 4d + \delta^2(0) - \delta^1(0) + {}_{-4}hom_R(I,M) + {}_0hom_R(I,M) - {}_0ext_R{}^2(M,M)$, *provided ii) holds*
iii') $\dim_C H(d,g) = 4d + \delta^2(0) - \delta^1(0) + {}_{-4}hom_R(I,M)$, *provided iii) holds.*

Proof. i) To see that C is unobstructed we just need, thanks to the duality (1.2), to interpret the exact sequence (1.4) in terms of deformation theory. Indeed by (1.1) and (1.2);

(1.15) $_0Ext_m^{\,2}(I\,,\,I) \cong {}_0Hom(I\,,\,M)$, and $_0Ext_R^{\,2}(I\,,\,I) \cong {}_{-4}Ext_m^{\,2}(I\,,\,I)^v \cong {}_{-4}Hom(I\,,\,M)^v$

Now (1.4) and the vanishing of the first group of (1.15) imply that the local Hilbert functor at C, $Hilb_C$, and the deformation functor of deforming the homogeneous ideal I as a graded R-module, Def_I, are isomorphic. The functor, Def_I, (which one may see is isomorphic to the local Hilbert functor of constant postulation at C introduced in [MDP1]) is smooth because $_0Ext_R^{\,2}(I\,,\,I)$ vanishes, cf. [K2], prop. 2.2.12 or [W1] for details. This proves i), and then i') follows at once from lemma 1.

iii) One may deduce the unobstructedness of C from results in [MDP1]. However, since we need the basic exact sequences below later (for which we have no reference), we give a proof. Indeed for any curve we claim there is an exact sequence:

$$(1.16)\quad 0 \to T_{\gamma,\rho} \to {}_0Ext_R^1(I\,,\,I) \xrightarrow{\beta} {}_0Hom_R(M\,,\,E) \to {}_0Ext_R^2(M\,,\,M) \to {}_0Ext_R^2(I\,,\,I)$$

where $T_{\gamma,\rho}$ is the tangent space of the Hilbert scheme of constant cohomology $H_{\gamma,\rho}$ at C. To prove it we use the spectral sequence (1.1) and the duality (1.2) *twice*, i.e. we get an isomorphism, resp. a surjection

$$(1.17)\quad \begin{aligned} &{}_0Ext_R^{\,2}(I\,,\,I) \cong {}_{-4}Ext_m^{\,2}(I\,,\,I)^v \cong {}_{-4}Hom(I\,,\,M)^v \cong {}_0Ext_m^{\,4}(M\,,\,I) \\ &\beta_1 : {}_0Ext_R^{\,1}(I\,,\,I) \cong {}_{-4}Ext_m^{\,3}(I\,,\,I)^v \twoheadrightarrow {}_{-4}Ext_R^{\,1}(I\,,\,M)^v \cong {}_0Ext_m^{\,3}(M\,,\,I) \end{aligned}$$

Now replacing I by M as the first variable in (1.11) or using (1.1) directly, we get

$$(1.18)\quad 0 \to {}_0Ext_R^1(M\,,\,M) \to {}_0Ext_m^3(M\,,\,I) \xrightarrow{\beta_2} {}_0Hom(M\,,\,E) \to {}_0Ext_R^2(M\,,\,M) \to {}_0Ext_m^4(M\,,\,I)$$

which combined with (1.17) yields (1.16) because the composition β of β_1 (arising from duality used twice) and β_2 must be the natural one, i.e. the one which sends an extension of $_0Ext^1(I\,,\,I)$, (i.e. a short exact sequence) onto the corresponding connecting homomorphism $M = H_m^{\,2}(I)) \to E = H_m^{\,3}(I)$. And we get the claim by [MDP1], prop.2.1, page 157, which tells $ker\ \beta = T_{\gamma,\rho}$.

To see that C is unobstructed, we get by (1.16) and the vanishing of $_0Hom_R(M\,,\,E)$ an isomorphism between the local Hilbert functor of constant cohomology at C and Def_I. The latter functor Def_I is isomorphic to $Hilb_C$ because $_0Hom_R(I\,,\,M) = 0$ (cf. the proof of i)), while the former functor is smooth because $_0Ext^2(M\,,\,M)$ contains in a natural way the obstructions of deforming a curve in $H_{\gamma,\rho}$ (cf. [MDP1], th. 1.5, page 135). This leads easily to the conclusion of iii). Moreover note that we now get iii') from lemma 1 because $h^0(N_C) = {}_0ext^1{}_R(I,\,I)$ and $_0ext^2{}_R(I,\,I) = {}_{-4}hom_R(I\,,\,M)$.

ii) The unobstructedness of C follows from proposition 2. Indeed if we take a complete intersection Y $\supseteq$ C of two surfaces of degree f and g such that the *conditions* of proposition 2 hold (such Y exists), then the corresponding linked curve C' satisfies $_vHom_R(I(C')\,,\,M(C')) \cong {}_vHom_R(M(C)\,,\,E(C))$ for v = 0 and v = -4 and we conclude by proposition 2 and theorem 3i).

It remains to prove the dimension formula in ii'). For this we claim that the image of the map $\alpha : {}_0Ext_m^2(I , I) \cong {}_0Hom_R(I , M) \rightarrow {}_0Ext_R^2(I , I)$ which appears in (1.4) for v = 0, is isomorphic to ${}_0Ext_R^2(M , M)$. Indeed α factorizes via ${}_0Ext_R^2(M , M)$ in a natural way, and the factorization is given by a certain map of (1.16). Now ${}_vHom_R(M , E) = 0$ for v = 0 and -4 implies that the maps ${}_vExt_R^2(M , M) \rightarrow {}_vExt_R^2(I , I)$ of (1.16) are injective for v = 0 and v = -4. Dualizing one of them (the map for v = -4) we get a surjective composition;

$${}_0Hom_R(I , M) \cong {}_{-4}Ext_R^2(I , I)^{\vee} \rightarrow {}_{-4}Ext_R^2(M , M)^{\vee} \cong {}_0Ext_R^2(M , M)$$

which composed with the other injective map above is precisely α. This proves the claim. Now by (1.4) and the proven claim;

$$h^0(N_C) = {}_0ext_R^1(I , I) + dim\ ker\ \alpha = {}_0ext_R^1(I , I) + {}_0hom_R(I , M) - {}_0ext_R^2(M , M)$$

and we get the dimension formula by lemma 1 and we are done.

(1.19) *Remark.* a) (1.15), (1.16), (1.17) and (1.18) are valid for *any* curve in $\mathbf{P}^3$. Moreover if $M_{-4} = 0$, we get ${}_0Hom(M , H_m^4(I))$ and one may see that the spectral sequence which converges to ${}_0Ext_m^4(M , I)$ (cf. (1.17) and (1.18)) consists of at most two non-vanishing terms. Hence we can continue the exact sequences (1.18) and (1.16) to the right with

$${}_0Ext_m^4(M , I) \cong {}_0Ext^2(I , I) \rightarrow {}_0Ext_R^1(M , E) \rightarrow {}_0Ext_R^3(M , M).$$

b) The proof of theorem 3 implies also the following result, valid for any curve C. With notations as in the proof, we have:

i) ${}_0Hom_R(I(C) , M(C)) = 0$ implies $Def_{I(C)} \cong Hilb_C$, i.e. $H_\gamma \cong H(d,g)$ at C
ii) ${}_0Hom_R(M(C) , E(C)) = 0$ implies $H_{\gamma,\rho} \cong H_\gamma$ (i.e. as schemes) at C

One objective of this paper to prove that the conditions i), ii), iii) of theorem 3 are necessary for unobstructedness provided M has diameter 1. Moreover, we shall in section 3 see what happens to the unobstructedness of C when we impose on C different conditions of being "general enough". One result is already now clear, and it points out that the condition iii) of theorem 3 is the most important one for *generic* curves:

Proposition 4 *Let C be a curve in $\mathbf{P}^3$, and suppose C is generic in the Hilbert scheme H(d,g) and satisfies ${}_0Ext_R^2(M , M) = 0$. Then C is unobstructed if and only if*

$${}_0Hom_R(I , M) = 0 \text{ and } {}_0Hom_R(M , E) = 0$$

Proof One way is clear from theorem 3. Now suppose C is unobstructed and generic with postulation γ and deficiency ρ. By generic flatness we see that $H_{\gamma,\rho} \cong H_\gamma \cong H(d,g)$ near C from which we deduce an isomorphism of tangent spaces $T_{\gamma,\rho} \cong {}_0Ext_R^1(I , I) \cong H^0(N_C)$. We therefore conclude by the exact sequences (1.16) and (1.4), recalling that $\alpha : {}_0Ext_m^2(I , I) \rightarrow {}_0Ext_R^2(I , I)$, which appears in (1.4) for v = 0 factorizes via ${}_0Ext_R^2(M , M)$, i.e. $\alpha = 0$.

(1.20) *Remark.* In section 3 we will encounter the open subscheme $H_2(d,g)$ of $H(d,g)$ of curves C satisfying diam M(C) $\leq$ 2, and the subset

$$U = \{ C \in H_2(d,g) \mid {}_0Hom_R(I , M) = 0 \text{ and } {}_0Hom_R(M , E) = 0 \}$$

Since the isomorphisms of (1.19b) extend to isomorphisms in some neighborhood, hence to isomorphisms of their tangent spaces which in turn lead to the vanishing of the *Hom*-groups appearing in U (cf. the proof of proposition 4), we see that U is an *open* subset of $H_2(d,g)$. By theorem 3, $H(d,g)$ is smooth along U and, by proposition 4, U has a non-empty intersection with every reduced component of $H_2(d,g)$.

(1.21) *Remark.* Combining (1.16) and (1.18) we get a *surjective* map $T_{\gamma,\rho} \to {}_0Ext_R{}^1(M , M)$. Moreover dualizing the exact sequence of (1.11) (with v = -4), the surjective map above fits into the exact sequence

$$k \to {}_0Hom_R(M , M) \to {}_{-4}Hom_R(I , E)^{\vee} \to T_{\gamma,\rho} \to {}_0Ext^1_R(M , M) \to 0$$

and $k \to {}_0Hom_R(M , M)$ is injective provided M $\neq$ 0. We can use this surjectivity (and some considerations on the obstructions involved) to give a new proof of the smoothness of the morphism from $H_{\gamma,\rho}$ to the "scheme" of Rao modules ([MDP1], th. 1.5, page 135). Since ${}_{-4}hom(I , E) = \delta^2(-4)$, cf. (1.12), the exact sequence above also leads to the dimension formula of $H_{\gamma,M}$ we pointed out (1.6), cf. [BK] for the generalization of this argument to the Hilbert scheme of surfaces in $\mathbf{P}^4$.

2. Necessary conditions for unobstructedness.

In this section we will prove that the conditions i), ii), iii) of theorem 3 are both necessary and sufficient for unobstructedness provided M has diameter 1. In terms of obstructed curves, we can state the result as

Theorem 5 *Let C be a curve in $\mathbf{P}^3$, let $M = H_*^1(I_C)$ and $E = H_*^1(O_C)$ and suppose M has diameter 1 (or less). Then C is obstructed if and only if (at least) one of the following conditions hold*

a) ${}_0Hom_R(I , M) \neq 0$ and ${}_0Hom_R(M , E) \neq 0$, or
b) ${}_0Hom_R(M , E) \neq 0$ and ${}_{-4}Hom_R(I , M) \neq 0$, or
c) ${}_0Hom_R(I , M) \neq 0$ and ${}_{-4}Hom_R(M , E) \neq 0$.

Moreover if C is ***unobstructed****, then the dimension of the Hilbert scheme at C is*

$$\dim_C H(d,g) = 4d + \delta^2(0) + {}_{-4}hom_R(I , M) + {}_{-4}hom_R(M , E)$$

Even though we prove theorem 5 only when the Rao module has diameter 1, the most important contributions for proving the "converse" of theorem 3 are given by proposition 7 and 8 which are valid in the diameter 2 case as well. In that case, however, we have not been able to establish coincidentally necessary and sufficient conditions.

To state theorem 5 in a simple way, we rephrase the result in terms of some invariants of a minimal resolution of the homogeneous ideal I = I(C). To do this, observe that the assumption on the diameter is equivalent to requiring $M = H^1(\mathcal{I}_C(c))$ for some integer c. So C is a Buchsbaum curve and the minimal resolution of M is well known ([MDP1], page 41). Moreover using Rao's result ([R], theorem 2.5), the minimal free (graded) resolution of I has following form

$$(2.1)\quad 0 \to R(-c-4)^{\oplus r} \to R(-c-4)^{\oplus b_1} \oplus R(-c)^{\oplus b_2} \oplus P_1 \to R(-c-4)^{\oplus a_1} \oplus R(-c)^{\oplus a_2} \oplus P_0 \to I \to 0$$

where P_i, for i = 0,1, is supposed to contain no direct factor of degree c and c+4.

Corollary 6 *Let C be a curve in P^3 whose Rao module $M \neq 0$ is concentrated in degree c, and let a_1 and a_2 (resp. b_1 and b_2) be the number of minimal generators (resp. relations) of degree c+4 and c respectively, cf. (2.1). Then C is obstructed if and only if*

$$a_2 b_1 \neq 0 \ \text{ or } a_1 b_1 \neq 0 \ \text{ or } a_2 b_2 \neq 0$$

Moreover if C is ***unobstructed****, then the dimension of the Hilbert scheme at C is*

$$\dim_C H(d,g) = 4d + \delta^2(0) + r(a_1 + b_2)$$

Proof. Applying ${}_v Hom_R(- , M)$ to the minimal resolution (2.1) we get at once

$$(2.2)\qquad {}_0 hom_R(I , M) = ra_2 \ \text{ and } \ {}_{-4} hom_R(I , M) = ra_1$$

while the duality (1.2) and the spectral sequence (1.1) (which degenerates for $v \neq -1$ and -2) lead to

$$(2.3)\qquad {}_{-v-4}Ext_R^{\,1}(I , M) \cong {}_v Ext_m^{\,3}(M , I)^{\vee} \cong {}_v Hom_R(M , E)^{\vee} \quad , \quad v \neq -1,-2$$

Interpreting ${}_{-v-4}Ext_R^{\,1}(I , M)$ via the minimal resolution of I as in (2.2), we get

$$(2.4)\qquad {}_0 hom_R(M , E) = rb_1 \ \text{ and } \ {}_{-4} hom_R(M , E) = rb_2$$

and we conclude easily.

Proposition 7 *Let C be a curve in P^3, let $M = H_*^1(I_C)$ and $E = H_*^1(O_C)$ and suppose ${}_0 Ext_R^{\,2}(M , M) = 0$. If the obvious morphism*

$${}_0 Hom_R(I , M) \times {}_0 Hom_R(M , E) \dashrightarrow {}_0 Hom_R(I , E)$$

(given by the composition) is non-zero, then C is obstructed. In particular if the diameter of M is 1, then C is obstructed provided

$${}_0 Hom_R(I , M) \neq 0 \ \text{ and } \ {}_0 Hom_R(M , E) \neq 0$$

Proof. It is well known (cf. [L1]) that if the Yoneda pairing (inducing the cup-product)

$$<-,-> \; : \; Ext_{OP}^{1}(I_C\,,\, I_C) \times Ext_{OP}^{1}(I_C\,,\, I_C) \rightarrow Ext_{OP}^{2}(I_C\,,\, I_C),$$

given by composition of resolving complexes, satisfies $<\lambda,\lambda> \neq 0$ for some λ, then C is obstructed. If we let p_1: $Ext_{OP}^{1}(I_C\,,\, I_C) \rightarrow {}_0Hom_R(I,\, M)$ and p_2: $Ext_{OP}^{1}(I_C\,,\, I_C) \rightarrow {}_0Hom_R(M,\, E)$ be the maps induced by sending an extension onto the corresponding connecting homomorphisms, then $<-,->$ fits into a commutative diagram

$$(2.5) \qquad \begin{array}{ccccc} Ext^1(I_C\,,\, I_C) & \times & Ext^1(I_C\,,\, I_C) & \rightarrow & Ext^2(I_C\,,\, I_C) \\ \downarrow p_1 & & \downarrow p_2 & \circ & \downarrow \\ {}_0Hom_R(I\,,\, M) & \times & {}_0Hom_R(M\,,\, E) & \rightarrow & {}_0Hom_R(I\,,\, E) \end{array}$$

where the lower horizontal map is given as in proposition 7 (cf. [F]). By (1.4), ${}_0Ext^1{}_R(I\,,\, I) = ker\; p_1$, and p_1 is surjective because $\alpha = 0$ for $v = 0$. Moreover since the composition ${}_0Ext_R^{1}(I\,,\, I) \hookrightarrow Ext^1(I_C\,,\, I_C) \rightarrow {}_0Hom_R(M\,,\, E)$ is surjective by the important sequence (1.16), there exists $(\lambda_1\,,\lambda_2) \in Ext^1(I_C\,,\, I_C) \times {}_0Ext_R^{1}(I\,,\, I)$ such that the composed map $p_2(\lambda_2)p_1(\lambda_1)$ is non-zero by assumption. Using $\lambda_2 \in {}_0Ext_R^{1}(I\,,\, I) = ker\; p_1$, we get

$$p_2(\lambda_1 + \lambda_2)p_1(\lambda_1 + \lambda_2) = p_2(\lambda_1)p_1(\lambda_1) + p_2(\lambda_2)p_1(\lambda_1)$$

i.e. either $<\lambda_1 + \lambda_2, \lambda_1 + \lambda_2>$ or $<\lambda_1\,,\lambda_1>$ are non-zero, and C is obstructed.

Finally if $M = M_c$ has diameter 1, the minimal resolution (2.1) of I leads to ${}_0Hom_R(I\,,\, M) \cong M_c^{\oplus a2}$. Therefore $a_2 \neq 0$ and there exists a non-trivial map $\psi \in {}_0Hom_R(M\,,\, E)$ by assumption. It follows that $\psi(m) \neq 0$ for some $m \in M_c$, i.e. the element

$$((m,0,..,0),\psi) \in {}_0Hom_R(I\,,\, M) \times {}_0Hom_R(M\,,\, E)$$

maps to a non-trivial element of ${}_0Hom_R(I\,,\, E)$, and we conclude by the first part of the proof.

(2.6) *Remark.* Let C be a curve in $\mathbf{P}^3$ whose Rao module has *diameter* 1. From (1.4) and (1.16), cf. the proof above, we see at once that ${}_0Hom_R(I\,,\, M) \neq 0$ and ${}_0Hom_R(M\,,\, E) \neq 0$ if and only if we have the following strict inclusions of tangent spaces

$$(*) \qquad T_{\gamma,\rho} \subsetneq {}_0Ext_R^{1}(I\,,\, I) \subsetneq H^0(N_C)$$

where ${}_0Ext_R^{1}(I\,,\, I)$ is the tangent space of the functor Def_I, i.e. of the Hilbert scheme of constant postulation H_γ at C. By proposition 7, *C is obstructed if (*) holds*, cf. [MDP1], page 193 for the case $M \cong k$.

Along the same lines we are able to generalize a result of Walter ([W1], theorem 0.5) to curves whose Rao module has diameter 2 or less. Indeed if the diameter of M is 1, Walter proves proposition 8a) below and he computes the completion of $O_{H(d,g),C}$ in detail under the extra assumption ${}_0Hom_R(I\,,\, M) = 0$.

Proposition 8 *Let C be a curve in* $\mathbf{P}^3$, *let* $M = H_*^1(I_C)$ *and* $E = H_*^1(O_C)$ *and suppose* ${}_0Ext_R^2(M\ ,\ M) = 0$.

a) If the obvious morphism

$$ {}_{-4}Hom_R(I\ ,\ M) \times {}_0Hom_R(M\ ,\ E) \dashrightarrow {}_{-4}Hom_R(I\ ,\ E) $$

(given by the composition) is non-zero, then C is obstructed. In particular if the diameter of M is 1, then C is obstructed provided

$$ {}_{-4}Hom_R(I\ ,\ M) \neq 0 \ \text{ and } \ {}_0Hom_R(M\ ,\ E) \neq 0 $$

b) If the morphism

$$ {}_0Hom_R(I\ ,\ M) \times {}_{-4}Hom_R(M\ ,\ E) \dashrightarrow {}_{-4}Hom_R(I\ ,\ E) $$

(given by the composition) is non-zero, then C is obstructed. In particular if the diameter of M is 1, then C is obstructed provided

$$ {}_0Hom_R(I\ ,\ M) \neq 0 \ \text{ and } \ {}_{-4}Hom_R(M\ ,\ E) \neq 0 $$

Proof. Step 1. We start by proving a) under the extra temporary assumption $M_{-4} = 0$. Denote by p_2' the restriction of p_2 (see (2.5)) to ${}_0Ext_R^1(I\ ,\ I)$ via the natural inclusion ${}_0Ext_R^1(I\ ,\ I) \hookrightarrow Ext^1(I_C\ ,\ I_C)$ and consider the commutative diagram

$$ (2.7) \qquad \begin{array}{ccccc} <-,->_0 : {}_0Ext_R^1(I\ ,\ I) & \times & {}_0Ext_R^1(I\ ,\ I) & \to & {}_0Ext_R^2(I\ ,\ I) \\ \uparrow & & \downarrow p_2' & \circ & \downarrow i \\ T_{\gamma,\rho} & \times & {}_0Hom_R(M\ ,\ E) & \to & {}_0Ext_R^1(M\ ,\ E) \end{array} $$

where $<-,->_0$ is the Yoneda pairing. Indeed the restriction of ${}_0Ext_R^1(I\ ,\ I)$ to $T_{\gamma,\rho}$ in (2.7) makes the lower horizontal arrow well-defined in the commutative diagram above because of the natural map $T_{\gamma,\rho} \to {}_0Ext_R^1(M\ ,\ M)$ of (1.21). Due to the exact sequence (1.16), continued as in (1.19a), the map p_2' is surjective and i is injective by the assumption ${}_0Ext_R^2(M,\ M) = 0$. Hence the pairing $<-,->_0$ of factorizes via

$$ (2.8) \qquad \varphi' : T_{\gamma,\rho} \times {}_0Hom_R(M\ ,\ E) \to {}_0Ext_R^2(I\ ,\ I) $$

and vanishes if we restrict φ' to ${}_{-4}Hom_R(I\ ,\ E)^{\vee} \times {}_0Hom_R(M\ ,\ E)$ via the map of remark 1.21 (using the identity on ${}_0Hom_R(M\ ,\ E)$), because ${}_{-4}Hom_R(I\ ,\ E)^{\vee}$ maps to zero in ${}_0Ext_R^1(M\ ,\ M)$.

To prove a) it suffices to prove $<\lambda,\lambda>_0 \neq 0$ for some λ. We do this, we *claim that* there is another pairing $\varphi \neq 0$, commuting with $<-,->_0$, which essentially corresponds to φ' above except for the exchange of variables, i.e.

$$ (2.9) \qquad \varphi : {}_0Hom_R(M\ ,\ E) \times T_{\gamma,M} \to {}_0Ext_R^2(I\ ,\ I) $$

where $T_{\gamma,M} = {}_{-4}Hom_R(I\ ,\ E)^\vee / {}_0Hom_R(M\ ,\ M)$, cf. (1.21). Indeed as in (2.5) there is a commutative diagram

$$\begin{array}{ccccc}
{}_{-4}Ext_m^{\,2}(I\ ,\ I) & \times & {}_0Ext_R^{\,1}(I\ ,\ I) & \to & {}_{-4}Ext_m^{\,3}(I\ ,\ I) \\
\wr\big\downarrow & & \big\downarrow p_2' & \circ & \big\downarrow \\
{}_{-4}Hom_R(I\ ,\ M) & \times & {}_0Hom_R(M\ ,\ E) & \to & {}_{-4}Hom_R(I\ ,\ E) \\
 & & & & \big\downarrow d_{2,-1} \\
 & & & & {}_{-4}Ext_R^{\,2}(I\ ,\ M)
\end{array}$$

where three of the vertical arrows are given by the spectral sequence (1.1) (cf. (1.11)) and where the lower pairing is the *non-vanishing* map of proposition 8. Dualizing, we get the commutative diagram

$$\begin{array}{ccccc}
{}_0Ext_R^{\,1}(I\ ,\ I) & \times & {}_{-4}Ext_m^{\,3}(I\ ,\ I)^\vee & \to & {}_{-4}Ext_m^{\,2}(I\ ,\ I)^\vee \\
\big\downarrow p_2' & & \big\uparrow & \circ & \wr\big\uparrow \\
{}_0Hom_R(M\ ,\ E) & \times & (\ker d_{2,-1})^\vee & \to & {}_{-4}Hom_R(I\ ,\ M)^\vee
\end{array}$$

where the *non-vanishing* lower arrow can be identified with the map φ of (2.9) because ${}_{-4}Ext_R^{\,2}(I\ ,\ M)^\vee \cong {}_0Ext_m^{\,2}(M\ ,\ I) \cong {}_0Hom_R(M\ ,\ M)$. Using the duality (1.2), we see φ commutes with the Yoneda pairing $<-,->_0$, and the claim follows easily.

Now since $\varphi \neq 0$ and p_2' is surjective, there exists $(\lambda_2, \lambda_1) \in {}_0Hom_R(M\ ,\ E) \times T_{\gamma,M}$ and $\lambda_2' \in {}_0Ext_R^{\,1}(I\ ,\ I)$ such that $p_2'(\lambda_2') = \lambda_2$ and such that $<\lambda_2'\ ,\lambda_1>_0 = \varphi(\lambda_2\ ,\lambda_1) \neq 0$. Note that $<\lambda_1\ ,\lambda>_0 = 0$ for any $\lambda \in {}_0Ext_R^{\,1}(I\ ,\ I)$ because $<\lambda_1\ ,\lambda>_0 = \varphi'(\lambda_1\ ,p_2'(\lambda)) = 0$ by (2.8). It follows that

$$<\lambda_1 + \lambda_2',\lambda_1 + \lambda_2'>_0 = \ <\lambda_2',\lambda_1>_0 + <\lambda_2',\lambda_2'>_0$$

i.e. either $<\lambda_1 + \lambda_2',\lambda_1 + \lambda_2'>_0$ or $<\lambda_2',\lambda_2'>_0$ are non-zero. Finally since the map α of (1.4) factors via ${}_0Ext^2_R(M\ ,\ M)$ for $v = 0$, it follows that the map ${}_0Ext_R^{\,2}(I\ ,\ I) \to Ext^2(I_C\ ,\ I_C)$ is injective and maps obstructions to obstructions, i.e. the Yoneda pairing $<-,->_0$ and the corresponding pairing $<-,->$ of (2.5) commute and vanish simultaneously. C is therefore obstructed.

b) To prove b) we use step 1 and proposition 2. Indeed let C be a curve as in b) and let $Y \supseteq C$ be a complete intersection of two surfaces of degree f and g such that the *conditions* of proposition 2 hold and such that $H^1(\mathcal{I}_C(f+g)) = 0$, $H^1(\mathcal{O}_C(f-4)) = 0$ and $H^1(\mathcal{O}_C(g-4)) = 0$ (such Y exists). Then we claim that the corresponding linked curve C' satisfies the conditions given in step 1. Indeed using

$$\begin{array}{ll}
 & {}_0Hom_R(I(C)\ ,\ M(C)) \cong {}_0Hom_R(M(C')\ ,\ E(C')) \\
(2.10) & {}_{-4}Hom_R(M(C)\ ,\ E(C)) \cong {}_{-4}Hom_R(I(C')\ ,\ M(C')) \\
 & {}_{-4}Hom_R(I(C)\ ,\ E(C)) \cong {}_{-4}Hom_R(I(C)/I(Y)\ ,\ E(C)) \cong {}_{-4}Hom_R(I(C')/I(Y)\ ,\ E(C'))
\end{array}$$

we get the claim because ${}_{-4}Hom_R(I(C')/I(Y)\,,\,E(C')) \to {}_{-4}Hom_R(I(C')\,,\,E(C'))$ is injective and $H^1(\mathcal{I}_C(f+g)) \cong H^1(\mathcal{I}_{C'}(-4))$. It follows that C' is obstructed by step 1, and so is C by proposition 2. Moreover if the diameter of M is 1, we conclude easily by arguing as in the very end of the proof of proposition 7.

a) Finally using the same idea as in b), we prove that b) and proposition 2 imply a). Indeed by proposition 2 we can see that a) and b) are equivalent by making a suitable linkage, and the proof is complete.

(2.11) *Remark.* i) Since it is well known that the tangent space, resp. the obstruction space of the functor Def_I of deforming the homogeneous ideal I, is ${}_0Ext_R^1(I\,,\,I)$, resp. sits in ${}_0Ext_R^2(I\,,\,I)$, we have by step 1 of the proof above that Def_I is smooth (i.e. the Hilbert scheme of constant postulation, H_γ, is smooth at C) provided $M_{-4} = 0$ and the conditions of proposition 8a) hold.

ii) Note that the pairing of proposition 8a) *and* the Yoneda pairing $<-,->_0$ of (2.7) vanish provided ${}_{-4}Hom_R(I\,,\,M) = 0$. Hence ${}_{-4}Hom_R(I\,,\,M) = 0$ leads to more than the vanishing of the pairing of proposition 8a) do. Indeed since

$$T_{\gamma,\rho} \oplus {}_0Hom_R(M\,,\,E) \cong {}_0Ext_R^1(I\,,\,I) \quad \text{as k-vectorspaces,}$$

the vanishing of $<-,->_0$ implies also that, properly interpreted, the cup-product of two elements from ${}_0Hom_R(M\,,\,E)$ is zero. By linkage, we prove a corresponding "dual" result, replacing ${}_vHom_R(I\,,\,M)$ *by* ${}_vHom_R(M\,,\,E)$ and visa versa everywhere in the argument.

Proof of theorem 5. If a), b) or c) are satisfied, then C is obstructed by proposition 7 and 8. Conversely if C is obstructed, then we conclude easily by theorem 3.

It remains to find $dim_C\,H(d,g)$. By the first part of the proof, C is unobstructed if and only if i), ii) or iii) of theorem 3 hold. So it suffices to compute $dim_C\,H(d,g)$ in each of these 3 cases, for which we use the last part of theorem 3 and (1.12). We get

$$\delta^1(0) = {}_0hom(I\,,\,M) - {}_0ext_R^1(I\,,\,M) + {}_0ext_R^2(I\,,\,M)$$

Computing ${}_0Ext_R^2(I\,,\,M)$ via the minimal resolution (2.1) of I, we see that the group is vanishes, while (2.3) implies ${}_0ext_R^1(I\,,\,M) = {}_{-4}hom(M\,,\,E)$. So in the diameter 1 case we have

$$\delta^1(0) = {}_0hom(I\,,\,M) - {}_{-4}hom(M\,,\,E)\,, \tag{2.12}$$

and now the dimension formulas of i'), ii') and iii') of theorem 3 lead all to the same dimension formula, i.e. the one given in theorem 5. This completes the proof.

(2.13) *Remark.* We have by proposition 7 and 8 the following three Yoneda pairings

$${}_0Hom_R(I\,,\,M) \times {}_0Hom_R(M\,,\,E) \dashrightarrow {}_0Hom_R(I\,,\,E)$$
$${}_0Hom_R(I\,,\,M) \times {}_{-4}Hom_R(I\,,\,E)^\vee \to {}_{-4}Hom_R(M\,,\,E)^\vee$$
$${}_0Hom_R(M\,,\,E) \times {}_{-4}Hom_R(I\,,\,E)^\vee \to {}_{-4}Hom_R(I\,,\,M)^\vee$$

To see how the right hand sides contribute to $H^1(\mathcal{N}_C)$, we suppose ${}_0Ext_R^i(M\,,\,M)$ for $i \geq 2$ to simplify, and we recall

$$ {}_0Ext_R^{\,2}(I,M) \cong {}_{-4}Ext_m^{\,2}(M,I)^{\vee} \cong {}_{-4}Hom(M,M)^{\vee} \cong {}_0Ext_R^{\,4}(M,M) = 0 $$

and ${}_0Ext_R^{\,1}(I,M) \cong {}_{-4}Ext_m^{\,3}(M,I)^{\vee} \cong {}_{-4}Hom(M,E)^{\vee}$ because ${}_{-4}Ext_R^{\,i}(M,M) \cong {}_0Ext_R^{\,4-i}(M,M) = 0$ for i = 1,2. Now (1.4), resp (1.11), leads to the exactness of the horizontal, resp. vertical, sequence in the diagram

$$
\begin{array}{ccccccc}
 & & & & {}_0Ext_R^{\,1}(I\,,\,M) \cong {}_{-4}Hom(M\,,\,E)^{\vee} & & \\
 & & & & \downarrow & & \\
0 \to & {}_0Ext_R^{\,2}(I\,,\,I) \to & H^1(N_C) \to & & {}_0Ext_m^{\,3}(I\,,\,I) & \to & 0 \\
 & \| & & & \downarrow & & \\
 & {}_{-4}Hom(I\,,\,M)^{\vee} & & & {}_0Hom(I\,,\,E) & &
\end{array}
$$

In the proof of theorem 5 we could have used this diagram to find $dim_C H(d,g)$ once we have proved that C is unobstructed.

We will end this section by showing that there exists smooth connected space curves in any of the three cases a), b) and c) of theorem 5. The case b) is treated in [W1], where Walter manage to find obstructed curves of maximal rank (see also [BKM]). By linkage we can transfer the treatise in [W1] to the case c) and we get the existence of obstructed curves of maximal corank, whose local ring $O_{H,C}$ can be described exactly as in [W1] . However, since we in the next section will see that a sufficiently general curve of $H_{\gamma,\rho}$ does not verify neither b) nor c), the case a) deserves special attention. We shall now see that there exist many smooth connected curves satisfying the conditions a);

(2.14) *Example.* We claim that for any triple (r,a_2,b_1) of positive integers there exists a smooth connected curve C with minimal resolution of the form (2.1) and diam M(C) = 1, such that s(C) = e(C) = c , $h^0(I_C(c)) = a_2$, $h^1(I_C(c))$ = r , $h^1(O_C(c)) = b_1$ and $a_1 = 0$, $b_2 = 0$. Hence

$$ {}_0hom_R(I\,,\,M) = ra_2 \neq 0 \text{ and } {}_0hom_R(M\,,\,E) = rb_1 \neq 0 $$

(cf. (2.2) and (2.4)). By corollary 6 these curves are obstructed. To see the existence, put a = a_2 and b = b_1. If a = 1, we consider curves with Ω-resolution

(2.15) $$ 0 \to O(-2)^{\oplus 3r-1} \oplus O(-4)^{\oplus b} \to O \oplus \Omega^{\oplus r} \oplus O(-3)^{\oplus b-1} \to I_C(c) \to 0 $$

By Chang's results ([C] or [W1], th. 4.1) there exists smooth connected curves having Ω-resolution as above. Moreover $c = 1 + b + 2r$, the degree $d = \binom{c+4}{2} - 3r - 7$ and the genus $g = (c+1)d - \binom{c+4}{3} + 5$. If a > 1, curves with Ω-resolution

(2.16) $$ 0 \to O(-1)^{\oplus a-2} \oplus O(-2)^{\oplus 3r} \oplus O(-4)^{\oplus b} \to O^{\oplus a} \oplus \Omega^{\oplus r} \oplus O(-3)^{\oplus b-1} \to I_C(c) \to 0 $$

exist, they are smooth and connected ([C] or [W1], th. 4.1), $c = a + b + 2r + 1$, $d = \binom{c+4}{2} - 3a - 3r - 6$ and the genus $g = (c+1)d - \binom{c+4}{3} + 3a + 3$. We leave the verification of details to the reader, recalling only the exact sequences we frequently used in the verification;

(2.17) $0 \to \Omega \to O(-1)^{\oplus 4} \to O \to 0$, and $0 \to O(-4) \to O(-3)^{\oplus 4} \to O(-2)^{\oplus 6} \to \Omega \to 0$

(Putting the two sequences together, we get the Koszul resolution of the regular sequence $\{X_0,X_1,X_2,X_3\}$).

We will analyze these curves a little further, using Laudal's description of the completion of $O_{H(d,g),C}$ [L1]. This completion is $k[[H^0(N_C)^\vee]]/o(H^1(N_C)^\vee)$, where o is a certain obstruction morphism (giving essentially the cup- and Massey-products). Now, consulting for instance the proof of proposition 8, we see that the dual spaces of ${}_0Hom_R(I\ ,\ M)^\vee$ and ${}_0Hom_R(M\ ,\ E)^\vee$ inject into $H^0(N_C)^\vee$ and their intersection is empty. This implies

$$H^0(N_C)^\vee \cong T_{\gamma,\rho}{}^\vee \oplus {}_0Hom_R(I\ ,\ M)^\vee \oplus {}_0Hom_R(M\ ,\ E)^\vee \quad \text{as k-vectorspaces,}$$

and we can represent $k[[H^0(N_C)^\vee]]$ as $k[[Y_1,..Y_m,Z_{11},..,Z_{ar},W_{11},..,W_{rb}]]$, letting $\{Y_1,..Y_m\}$, resp. $\{Z_{11},..,Z_{ar}\}$, resp. $\{W_{11},..,W_{rb}\}$ correspond to a basis of $T_{\gamma,\rho}{}^\vee$, resp. ${}_0Hom_R(I\ ,\ M)^\vee$, resp. ${}_0Hom_R(M\ ,\ E)^\vee$. Since $a_1 = 0$, $b_2 = 0$, we get

$${}_{-4}Hom_R(I\ ,\ M) = 0 \quad \text{and} \quad {}_{-4}Hom_R(M\ ,\ E) = 0$$

(cf. (2.2) and (2.4)). By (2.13) $h^1(N_C) = {}_0hom_R(I\ ,\ E) = a_2b_1$, and we can use proposition 8 and its proof, see (2.11), to conclude that, modulo m_O^3 (m_O the maximal ideal of the completion of $O_{H(d,g),C}$), we have

(2.18) $$O_{H(d,g),C}/m_O^3 = k[[Y_1,..Y_l,Z_{11},..,Z_{ar},W_{11},..,W_{rb}]]/\mathcal{L}$$

where the ideal $\mathcal{L}$ is generated by the components of matrix given by the product

(2.19) $$\begin{bmatrix} Z_{11},\dots,Z_{1r} \\ Z_{21},\dots,Z_{2r} \\ \cdot \qquad \cdot \\ Z_{a1},\dots,Z_{ar} \end{bmatrix} \begin{bmatrix} W_{11},\dots,W_{1b} \\ W_{21},\dots,W_{2b} \\ \cdot \qquad \cdot \\ W_{r1},\dots,W_{rb} \end{bmatrix}$$

Note that (2.19) corresponds precisely to the composition given by the pairing of proposition 7. (We believe that the Massey products corresponding to (2.19) vanish, i.e. the right-hand side of (2.18) is exactly the completion of $O_{H(d,g),C}$).

The simplest case is $(r,a_2,b_1) = (1,1,1)$, which yields curves C with $s(C) = 4$, $d = 18$ and $g = 39$ (Sernesi's example [Se] or [EF]), while the case $(r,a_2,b_1) = (2,1,1)$ yields curves C with $s(C) = 6$, $d = 32$ and $g = 109$. More generally, the curves of the case $(r,1,1)$ satisfy $h^1(N_C) = a_2b_1 = 1$, i.e. the ideal $\mathcal{L}$ of (2.18) is generated by the single element

(2.20) $$\sum_{i=1}^{r} Z_{1i}W_{i1}$$

For Sernesi's example ($r = 1$), we recognize the known fact that this curve sit in the intersection of two irreducible components of $H(d,g)$, while for $r > 1$, the irreducibility of (2.20) can be used to show that C belongs to a unique irreducible component of $H(d,g)$. In the next section we prove the irreducibility/reducibility by studying in detail the possible generizations of a Buchsbaum curve.

3. The minimal resolution of a general curve.

In this section we study generizations of space curves C and how we can simplify the minimal resolution (1.5) of I(C). The general philosophy should be that a sufficiently general curve of any irreducible component of $H(d,g)$ has as few repeated direct factors "as possible" in consecutive terms of the minimal resolution. We prove below a general result in this direction (theorem 9), cancelling direct free factors in the middle part of (1.5). In particular, if the Rao module $M = H^1(I_C(c))$ for some integer c, i.e. if the minimal resolution of I = I(C) has following form, cf. (2.1);

$$(3.1)\quad 0 \to R(-c-4)^{\oplus r} \to R(-c-4)^{\oplus b_1} \oplus R(-c)^{\oplus b_2} \oplus P_1 \to R(-c-4)^{\oplus a_1} \oplus R(-c)^{\oplus a_2} \oplus P_0 \to I \to 0$$

(the P_i's contain no factor of degree c and c+4), we get $a_1b_1 = 0$ and $a_2b_2 = 0$ for a general curve of $H_{\gamma,\rho}$. Moreover, for *Buchsbaum* curves, we prove another result (proposition 13) cancelling direct free factors to the left in the resolution (1.5). In particular, a general curve of $H(d,g)$ with minimal resolution (3.1) must satisfy $rb_1 = 0$ and the "dual" $ra_2 = 0$ (theorem 15), and this observation implies that any irreducible component of $H(d,g)$ is generically smooth in the diameter 1 case. Finally we remark that the proofs of theorem 9 and proposition 13 are quite close to the proofs of "les lemmas de générisation simplifiantes" appearing in [MDP1], page 189, although they treat the very special case $M \cong k$.

To give theorem 9 an appropriate interpretation, recall that once we have a minimal free graded resolution of the Rao module M = M(C);

$$(3.2)\qquad 0 \to L_4 \xrightarrow{\sigma^t} L_3 \to L_2 \to L_1 \to L_0 \to M \to 0\ ,$$

there exists a minimal resolution of the homogeneous ideal I(C) of free graded R-modules of the following form

$$(3.3)\qquad 0 \to L_4 \xrightarrow{(\sigma,0)^t} L_3 \oplus F_1 \to F_0 \to I(C) \to 0\ ,$$

i.e. where the composition of $L_4 \to L_3 \oplus F_1$ with the natural projection $L_3 \oplus F_1 \to F_1$ onto the second factor is zero. cf. [R], theorem 2.5. Note that since minimal resolutions are isomorphic, any minimal resolution of I(C) of the form (3.3) have well-defined modules F_1 and F_0. Our result tells that, after a generization in the Hilbert scheme $H_{\gamma,M}$ of *constant postulation and Rao-module*, F_1 and F_0 have no common direct factor (in which case γ and M *determines* F_1 *and* F_0, e.g. the graded Betti numbers of the whole minimal resolution).

Theorem 9 *Let U be the set of curves C of the Hilbert scheme $H_{\gamma,M}$ whose modules F_1 and F_0 of the minimal resolution (3.3) of I(C) are without common direct free factors. Then U is an open dense irreducible subset of $H_{\gamma,M}$.*

Corollary 10 *Let C be a curve in P^3 whose Rao module $M \neq 0$ is concentrated in one degree only, and let a_i and b_i be the numbers given by (3.1). If $C \subseteq P^3$ is generic in $H_{\gamma,\rho}$ (or in H_γ or in $H(d,g)$), then*

$$a_1b_1 = 0 \qquad \text{and} \qquad a_2b_2 = 0$$

In particular, C is obstructed if and only if $a_2b_1 \neq 0$.

We get the corollary as a simple consequence of the theorem and corollary 6. The corollary generalizes [BKM], prop. 1.1 which tells that a curve of maximal rank (or maximal corank) of diam M(C) = 1, which is generic in $H_{\gamma,\rho}$, is unobstructed.

To prove the theorem and a later proposition, we need a lemma for deforming a module N, which basically is known. Loosely speaking it tells that if we can lift a three term resolution with augmentation N to a *complex*, then the complex defines a flat deformation of N. In the case N = I(C) where C has *codimension* 2 in $\mathbf{P}^3$, we also know that a deformation of an ideal I(C) is again an ideal, i.e.

Lemma 11 *Let C be a curve in* $P^3 = P_k^3$ *whose homogeneous ideal I(C) has a minimal resolution of the following form*

$$(L^\circ) \qquad 0 \to \oplus R(-n_{3i}) \xrightarrow{\varphi} \oplus R(-n_{2i}) \xrightarrow{\psi} \oplus R(-n_{1i}) \to I(C) \to 0$$

Let A be a finitely generated k-algebra, B the localization of A in a prime ideal $\wp$, $B/\wp B \cong$ *k the residue field, and suppose there exists a complex*

$$(L_B^\circ) \quad \oplus R_B(-n_{3i}) \xrightarrow{\varphi_B} \oplus R_B(-n_{2i}) \xrightarrow{\psi_B} \oplus R_B(-n_{1i}) \quad , \quad R_B = R\otimes_k B$$

such that $L_B^\circ\otimes_B(B/\wp B) \cong L^\circ$. *Then* (L_B°) *is exact,* φ_B *is injective and the cokernel of* ψ_B *is a flat deformation of I(C) (as an ideal, so* $coker(\psi_B) \subseteq R_B$ *defines a flat deformation of C* $\subseteq P^3$*). Moreover for some* $a \in A$-$\wp$, *we can extend this conclusion to* A_a *via* $Spec(B) \hookrightarrow Spec(A_a)$, *i.e. there exists a flat family of curves* $C_{Spec(Aa)} \subseteq P^3 \times Spec(A_a)$ *whose homogeneous ideal* $I(C_{Aa})$ *has a resolution (not necessarily minimal) of the form*

$$(L_{A_a}^\circ) \qquad 0 \to \oplus R_{A_a}(-n_{3i}) \to \oplus R_{A_a}(-n_{2i}) \to \oplus R_{A_a}(-n_{1i}) \to I(C_{A_a}) \to 0$$

Proof (sketch). If $E = coker\ \varphi$ and $E_B = coker\ \varphi_B$, then one proves easily that $E_B\otimes_B(B/\wp B) = E$, $Tor_1(E_B,B/\wp B) = 0$ and that φ_B is injective. By the local criterion of flatness, E_B is a flat deformation of E. Letting $Q_B = coker\ (E_B \to R_B(-n_{1i}))$, we can argue as we did for E_B to see that Q_B is a flat deformation of I(C) and that L_B° is exact.

To prove that Q_B is an ideal, we can use the isomorphisms $H^{i-1}(N_C) \cong Ext_{O_P}^i(I^\sim, I^\sim)$ for i = 1,2 (cf. [K2], 2.2.6), interpreted via deformation theory, to see that we essentially get the existence of the desired morphisms $\tilde{\imath}$: $Q^\sim_B \to R^\sim_B$ and $i = H_*^0(\tilde{\imath})$. For this important observation, we remark that one may give a direct proof using Hilbert-Burch theorem (cf. [MDP1], page 37). Indeed if $F_B = \oplus^{r+1}R_B(-n_{1i})$ and $n = -\sum n_{1i}$, then one may deduce the existence of $\tilde{\imath}$ from the complex

$$\tilde{E}_B \to \tilde{F}_B \simeq (\wedge^r\tilde{F}_B)^\vee(n) \to (\wedge^r\tilde{E}_B)^\vee(n) \simeq O_{\tilde{P}_B} \simeq \tilde{R}_B$$

because one knows that the corresponding complex over R (i.e. with B = k) and the sequence $0 \to E^\sim \to F^\sim \to I(C)^\sim \hookrightarrow R^\sim$ ($F = \oplus^{r+1}R(-n_{1i})$) essentially coincide (Hilbert-Burch).

Finally we easily extend the morphism i and any morphism of the resolution L_B° to be defined over $A_{a'}$, for some a' $\in$ A-$\wp$ (such that $L_{Aa'}^{\bullet}$ is a complex). By shrinking $Spec(A_{a'})$ to $Spec(A_a)$, a $\in$ A-$\wp$, we get the exactness of the complex and the flatness of $I(C_{A_a})$ because these properties are open.

(3.4) *Remark.* We can simplify some later proofs by observing that the composition of two generizations (the first one starting with $C \subseteq \mathbf{P}^3$) is again a generization of $C \subseteq \mathbf{P}^3$ in the following sense. Let $C_{Spec(T)} \subseteq \mathbf{P}_T^3$ ($Spec(T)$ integral and K = Q(T) the quotient field) be a flat family of curves over k containing the member ($C \subseteq \mathbf{P}^3$) and let $C_K \subseteq \mathbf{P}_K^3$ be the pullback of $C_{Spec(T)} \subseteq \mathbf{P}_T^3$ via $Spec(K)$ -> $Spec(T)$ (i.e. essentially the first generization). The second generization which is a deformation of ($C_K \subseteq \mathbf{P}_K^3$), corresponds to a flat family of curves $C_{Spec(A)} \subseteq \mathbf{P}_A^3$ ($Spec(A)$ an integral K-scheme of finite type; A = $K[\lambda_1,..,\lambda_l]$) over K which we suppose is obtained as in lemma 11 by deforming the minimal resolution of $I(C_K)$ to $I(C_A)$ flatly. As in the very last part of the proof of lemma 11 one extends morphisms, the exactness of the complex and the flatness of $I(C_A)$ to the integral domain $S := T_t[\lambda_1,..,\lambda_l]_a$ for some t $\in$ T and a $\in$ $T[\lambda_1,..,\lambda_l]$, i.e. we have an irreducible flat family of curves over $Spec(S)$ and ($C \subseteq \mathbf{P}^3$) belongs to the closure of the image of $Spec(S)$ in $H(d,g)$.

Proof of theorem 9. If $H_{\gamma,M}$ is non-empty, we first *claim* that U is non-empty, i.e. that there exists a curve C' whose F_0 and F_1 of any minimal resolution of I(C') of the form (3.3) are without common direct factors. To prove the claim, we can use the ideas of the proof of the "lemma de générisation simplifiantes" appearing in [MDP1], page 189. Indeed *if U is empty* and if C is a curve of $H_{\gamma,M}$ with as few common direct factors among F_0 and F_1 of rank one as possible, we consider

$$0 \to L_4 \xrightarrow{(\sigma,0)^t} L_3 \oplus F_1' \oplus R(-m) \xrightarrow{\psi} F_0' \oplus R(-m) \to I(C) \to 0$$

Then we can change the number 0 of the component of the matrix of ψ which corresponds to R(-m) -> R(-m) to some λ (λ an indeterminate of degree zero). Keeping $(\sigma,0)^t$ unchanged, we still have a complex which by lemma 11 implies the existence a flat family of curves over $Spec(A_a)$, A = $k[\lambda]$, for some a $\in$ A-(λ). Since any curve C' of the family given by $Spec(A_{\lambda a})$ has a minimal resolution with fewer common direct factors among F_0 and F_1 that C had, and since we may interpret the Rao module M(C') as $ker\ H_*^3((\sigma^\sim,0)^t)$ (e.g. the whole family given by $Spec(A_a)$ has isomorphic Rao modules), we get a contradiction, and the claim is proved.

Since $H_{\gamma,M}$ is irreducible ([B] or [MDP1], page 134), it suffices to prove that U is open in $H_{\gamma,M}$. Take an arbitrary curve C of U. Let $Spec(D)$ be a neighborhood of ($C \subseteq \mathbf{P}^3$) in $H_{\gamma,M} \hookrightarrow H(d,g)$, and let C_D be the restriction of the universal curve of $H(d,g)$ to $Spec(D)$. Using that $I(C_D)$ is D-flat, we can deform (lift) the minimal resolution (3.3) of I(C) to a resolution of $I(C_{D_d})$ for some d $\in$ D ([MDP1], page 140). Hence for any C' of $Spec(D_d)$ we get a resolution (not yet known to be minimal);

(3.5) $$0 \to L_4 \xrightarrow{\varphi_4} L_3 \oplus F_1 \to F_0 \to I(C') \to 0$$

Now, since any curve C' of $Spec(D_d)$ has the same Rao module M and since we by construction know that F_0 and F_1 are without common free factors, we *claim* that the resolution (3.5) is minimal for any C' of $Spec(D_d)$ (and that $\varphi_4 = (\sigma,0)^t$ up to an isomorphism of $L_3 \oplus F_1$). Indeed if it is not minimal, we can make it minimal by "removing" redundant factors. But we can not remove a free factor from L_4 because by Rao's theorem, cf. (3.3), I(C') admits a minimal resolution having L_4 to the very left. Similarly if the modules L_3 and F_0 of (3.5) have a common redundant (maximal) free factor $F \neq \emptyset$, we deduce a minimal resolution of the form

$$0 \to L_4 \xrightarrow{\varphi} (L_3/F) \oplus F_1 \to F_0/F \to I(C') \to 0$$

Arguing as in the proof of Rao's theorem, i.e. observing that

$$0 \to L_0^{\vee} \to L_1^{\vee} \to L_2^{\vee} \to L_3^{\vee} \to L_4^{\vee} \to {}_0Ext_R^4(M\ ,\ R) \to 0$$

is a minimal resolution of ${}_0Ext_R^4(M\ ,\ R)$ and that $(L_3/F)^{\vee} \oplus F_1^{\vee} \xrightarrow{\varphi^{\vee}} (L_4)^{\vee} \to {}_0Ext_R^4(M\ ,\ R) \to 0$ is exact (the entries of φ are in (X_0,X_1,X_2,X_3)), we get

$$(L_3/F) \oplus F_1 \ \cong L_3 \oplus F', \qquad \text{for some free module } F',$$

(and that $\varphi \cong (\sigma,0)^t$) which leads to a contradiction because F_1 does not contain F while L_3 does. This proves that $F = \emptyset$ (and that $\varphi \cong (\sigma,0)^t$), i.e. the resolution of (3.5) is minimal, as claimed. It follows by the definition of U that $Spec(D_d) \subseteq U$, i.e. that U is open in $H_{\gamma,M}$, and we are done.

(3.6) *Remark.* We can easily extend the arguments of the first part of the proof above (replacing λ by a diagonal matrix of indeterminates or just using remark 3.4) to get:

Let C be a curve in P^3 with postulation γ and Rao module M and suppose the homogeneous ideal I(C) has a minimal free resolution;

$$0 \to P_2 \xrightarrow{(\sigma',0)^t} P_1 \oplus F_1 \to F_0 \to I(C) \to 0$$

If there exists a direct free factor F satisfying $F_1 \cong F_1' \oplus F$ and $F_0 \cong F_0' \oplus F$, then there is a generization $C' \subseteq P^3$ of $C \subseteq P^3$ in the Hilbert scheme $H_{\gamma,M}$ whose homogeneous ideal I(C') has a minimal free resolution of the following form

$$0 \to P_2 \xrightarrow{(\sigma',0)^t} P_1 \oplus F_1' \to F_0' \to I(C') \to 0$$

Now we restrict to Buchsbaum curves, in which case we will be able to, taking suitable generizations in $H(d,g)$, simplify the minimal resolution of I(C) further. To see that this simplification is related to the vanishing of some important groups studied in the preceding sections, we start with a lemma. We recall first the notation

$$H_i(d,g) = \{\ C \in H(d,g) \mid diam\ M(C) \leq i\ \} \quad \text{and} \quad H_i(d,g)_B = H_i(d,g) \cap H(d,g)_B$$

where $H(d,g)_B$ (resp. $H_{\gamma,B}$) is the closed subset of $H(d,g)$ (resp. H_γ) consisting of Buchsbaum curves. Note that $H_i(d,g)$ is considered an *open subscheme* of $H(d,g)$. Moreover recall also that, in the Buchsbaum case, the minimal resolution (3.2) of M is just given as the direct sum of the Koszul resolution associated with the regular sequence $\{X_0, X_1, X_2, X_3\}$. The matrix associated to σ will have the form

$$(3.7) \qquad \sigma = \begin{bmatrix} \underline{X},0,\ldots,0 \\ 0,\underline{X},\ldots,0 \\ \cdot\ \cdot \qquad \cdot \\ 0,0,\ldots,\underline{X} \end{bmatrix}$$

where $\underline{X}$ is $(X_0,X_1,X_2,X_3)^t$ and each "row" is a $4\times r$ matrix, $r = \Sigma r_i$ and $r_i = \dim M(C)_i$.

Lemma 12 *Let C be a curve of $H_2(d,g)_B$ whose homogeneous ideal $I = I(C)$ has a minimal free resolution of the form (3.3);*

$$0 \to L_4 \xrightarrow{(\sigma,0)^t} L_3 \oplus F_1 \to F_0 \to I \to 0 ,$$

with σ as in (3.7). Then L_4 and $F_0(-4)$ (resp. L_4 and F_1) are without common direct free factors if and only if ${}_0Hom_R(I , M) = 0$ (resp. ${}_0Hom_R(M , E) = 0$).

Proof. Indeed since C is Buchsbaum, we get easily from the resolution (3.3) that

$${}_0Hom_R(I , M) \cong {}_0Hom_R(F_0 , M).$$

The latter group vanishes if and only if L_4 and $F_0(-4)$ are without common direct free factors because $M \cong ker\ H_*^3((\sigma^\sim,O))$ with σ as in (3.7). Moreover we have by the duality (1.2); ${}_0Ext_R^1(M , M)^v \cong {}_{-4}Ext_R^3(M , M)$. Using (3.2), (3.3) and the trivial module structure of M, we get

$${}_{-4}Ext_R^3(M , M) \cong {}_{-4}Hom_R(L_3 , M) \text{ , and } {}_{-4}Ext_R^1(I , M) \cong {}_{-4}Hom_R(L_3 \oplus F_1 , M)$$

Combining with the exact sequence (1.18), cf. (1.17), we get an isomorphism

$${}_{-4}Hom_R(F_1 , M)^v \cong {}_0Hom_R(M , E),$$

i.e. also ${}_0Hom_R(M , E)$ vanishes as claimed in the lemma.

Proposition 13 *Let C be a Buchsbaum curve in H(d,g) with postulation γ and suppose the homogeneous ideal I(C) has a minimal free resolution of the form (3.3);*

$$0 \to L_4 \xrightarrow{(\sigma,0)^t} L_3 \oplus F_1 \to F_0 \to I(C) \to 0 ,$$

where σ is given as in (3.7). If L_4 and F_1 have a common free direct factor F; $L_4 \cong L_4' \oplus F$ and $F_1 \cong F_1' \oplus F$ and F_1' and F_0 have a common direct factor G, then there is a generization $C' \subseteq P^3$ of $C \subseteq P^3$ in $H_{\gamma,B}$ such that I(C') has a minimal free resolution of the following form;

$$0 \to L_4' \to L_3 \oplus (F_1'/G) \to (F_0/G) \to I(C') \to 0$$

Moreover, if U is the subset of $H_{\gamma,B}$ consisting of curves C' whose module F_1 of the minimal resolution (3.3) of I(C') has no direct factor in common with L_4 and F_0 (i.e. with $L_4 \oplus F_0$), then U is an open dense subset $H_{\gamma,B}$, and for any curve C' of U we have an exact sequence

$$0 \to {}_0Hom_R(M(C') , E(C')) \to {}_0Ext_R^2(M(C') , M(C')) \to {}_0Ext_R^2(I(C') , I(C'))$$

Proof Thanks to remark 3.4 and remark 3.6, it suffices to prove the existence of a generization C' as in the proposition where F = R(-m) and G = $\emptyset$. As observed by Martin Deschamps and Perrin ([MDP1], page 189) we can easily change the 0 component in the matrix of $(\sigma,0)^t$ which corresponds to R(-m) → R(-m), to some indeterminate λ, changing four columns of the matrix A associated to $L_3 \oplus F_1 \to F_0$ (by adding $\lambda\gamma$ where $-\gamma\underline{X}$ is the column of A which corresponds to the image of R(-m) in F_0) as well to get a complex (cf. [MDP1], page 189, for details). By lemma 11 we get a flat irreducible family of curves C' over some $Spec(k[\lambda]_{\lambda a})$ having the same graded Betti numbers, hence the same postulation, as C. Moreover $M(C) \cong ker\, H_*^3((\sigma^{\sim},0))$ and using the corresponding expression for M(C'), we see that the module structure of M(C') is trivial. Since λ is invertible in $Spec(k[\lambda]_{\lambda a})$, we can remove a redundant factor of the resolution of I(C'), i.e. we have a generization C' with properties as claimed in proposition 13.

To see that U is an *open dense* subset of the closed subscheme $H_{\gamma,B}$ (which we here give the reduced scheme structure) of H_γ, we observe that the first part of the proof implies, for each irreducible component V_i of $H_{\gamma,B}$, the existence of a curve C_i of V_i (which does not belong to any other component) such that $C_i \in U$. Hence it suffices to prove that U is an *open* subset of $H_{\gamma,B}$. If we restrict to $H_2(d,g)_B$, the openness follows easily from lemma 12 and remark 1.20. To prove this part more generally, we take, as in the proof of theorem 9, an arbitrary curve C of U and we deform (lift) the minimal resolution of I(C) to some open non-empty set U' of $H_{\gamma,B}$ such that any C' of U' is a Buchsbaum curve with a resolution (not yet known to be minimal) of the form

$$0 \to L_4 \xrightarrow{\varphi_4} L_3 \oplus F_1 \to F_0 \to I(C') \to 0 \qquad (3.8)$$

It suffices to prove that the resolution (3.8) is minimal. To prove the minimality, we use that C' is Buchsbaum and that there exists a minimal resolution of the form; $0 \to L_4(C') \xrightarrow{(\sigma,0)^t} L_3(C') \oplus F_1(C') \to F_0(C') \to I(C') \to 0$, with σ as in (3.7). We get

$$L_3(C')(-1) \cong (L_4(C'))^{\oplus 4} \qquad (3.9)$$

By construction we know that (3.8) have no redundant factors cancelling factors of L_4 against F_1, or F_1 against F_0. From (3.9) we then get that L_3 and L_4, and L_3 and F_0, have no redundant factors, because if they had, we could remove them to get a minimal resolution. But the cancellation a factor R(-m) of L_3, m as large as possible, against the same factor of either L_4 or F_0 will imply the existence of a direct factor R(-m-1) of $L_4(C')$ which contradicts (3.9) because F_1 does not contain R(-m). The resolution (3.8) is therefore minimal.

Finally the exact sequence follows from (1.18) by the arguments of the proof of lemma 12 because F_1 and L_4 have no direct factor in common, i.e. ${}_4Hom_R(F_1\, , M(C')) = 0$, and we are done.

Corollary 14 *Let C be a curve in $\mathbf{P}^3$ whose Rao module $M \neq 0$ is concentrated in one degree only, and let a_i , b_i and r be the numbers given by (3.1). If $C \subseteq \mathbf{P}^3$ is generic in H_γ, then*

$$rb_1 = 0\,, \quad a_1b_1 = 0 \quad \text{and} \quad a_2b_2 = 0$$

In particular, the schemes H_γ and H(d,g) are smooth at C. Moreover if C is generic in H(d,g), then we also have $ra_2 = 0$.

Proof. If C is generic in H_γ, then C belongs to the set U of proposition 13. Hence we immediately have $a_1b_1 = 0$, $a_2b_2 = 0$ and $rb_1 = 0$. Moreover, by corollary 10 we see that $H(d,g)$ is smooth at C. By lemma 12 and remark 1.19b we get an isomorphism between $H_{\gamma,\rho}$ and H_γ at C. The former scheme is smooth because ${}_0Ext_R^2(M\ ,\ M) = 0$. Hence we get the corollary because $ra_2 = 0$ follows from (2.2) and remark (3.10) below.

(3.10) *Remark.* By [K3], cor. 3.6, and proposition 2 of this paper we get that the linked curve C_1' of a generization C' of C is a generization of the linked curve C_1 of C, provided we link the whole family of curves by complete intersections of type (f,g), with f and g as in proposition 2. Using this, we get:

Let C be a Buchsbaum curve with diam $M(C) \leq 2$ and deficiency ρ. Keep the notations and assumptions of proposition 13 and suppose furthermore that L_4' and $(F_0/G)(-4)$ have a common direct factor G'. If

$$F \cong R(-c-4)^{\oplus\beta_1}\oplus R(-c-3)^{\oplus\beta_2} \quad , \quad G' \cong R(-c)^{\oplus\alpha_1}\oplus R(-c+1)^{\oplus\alpha_2}$$

then there exists a generization C'' in $H(d,g)_B$ whose postulation $\gamma_{C''}$ (resp. deficiency $\rho_{C''}$) is given by $\gamma_{C''}(c-i) = \gamma_C(c-i) - \alpha_{i+1}$ (resp. $\rho_{C''}(c-i) = \rho_C(c-i) - \alpha_{i+1} - \beta_{i+1}$)) for $i = 0$ and 1. In particular there exists a generization $C'' \subseteq P^3$ of $C \subseteq P^3$ satisfying

$${}_0Hom_R(I(C'')\ ,\ M(C'')) = 0 \quad \text{and} \quad {}_0Hom_R(M(C'')\ ,\ E(C'')) = 0$$

Indeed if we combine the first formula of (2.10) with two formulas of the proof of lemma 12 where the groups of (2.10) appear, we see that the free direct part of F_0/G with generators in degree c and c-1 in the resolution of I(C'), is equal to the corresponding part of $F_1(C_1')(4)$ in the minimal resolution of $I(C_1')$ of the linked curve C_1'. Applying proposition 13 to the linked curve C_1', we get a generization of C_1' (hence a generization of C_1 by (3.4)) with constant postulation where G' is "removed" in its minimal resolution. A further linkage, using a complete intersection of the same type as in the linkage above (such a complete intersection exists by [K3], cor. 3.7), gives the desired curve C'', leaving some easy details on the verification of $\gamma_{C''}$ and $\rho_{C''}$ to the reader. We remark that the second formula of (2.10) also give some information, briefly mentioned in the arguing of (3.15).

Even though we can extend the next theorem to Buchsbaum curves satisfying ${}_0Ext_R^2(M\ ,\ M) = 0$, we have chosen to formulate it for the somewhat more natural set $H_2(d,g)_B$ of Buchsbaum curves C with diam $M(C) \leq 2$. Since $H_1(d,g) = H_1(d,g)_B \subseteq H_2(d,g)_B$, the theorem below remains true if we replace $H_2(d,g)_B$ by $H_1(d,g)$, and it is probably this *restricted version to $H_1(d,g)$* which is the most striking.

Theorem 15 *Let U be the subset of $H_2(d,g)_B$ consisting of curves whose minimal free resolution (3.3) are such that the modules of all three sets*

$$\{F_1, F_0\} \quad , \quad \{L_4, F_1\} \quad \textit{and} \quad \{L_4, F_0(-4)\}$$

are without common direct factors. Moreover let $U_{\gamma,\rho}$ be the subset of U of curves with postulation γ and deficiency ρ. Then U (resp. $U_{\gamma,\rho}$) is an open dense (resp. open irreducible) subset of $H_2(d,g)_B$ and $H(d,g)$ is smooth along U.

Proof. With the preparations we now have done, the proof is a straightforward application of remark 1.20, lemma 12, proposition 13 and remark 3.10. Indeed since the set U of theorem 15 is the intersection of the corresponding set of (1.20) with $H(d,g)_B$ by lemma 12, it follows that $H(d,g)$ is smooth along U and that U is open in $H_2(d,g)_B$ by (1.20). Moreover U is dense in $H_2(d,g)_B$ by (3.10).

To see that $U_{\gamma,\rho}$ is open as well, we observe that we can define $U_{\gamma,\rho}$ as a subset of the set U of proposition 13, in which case we see that $U_{\gamma,\rho}$ is open in $H_{\gamma,B}$ by its proof (any curve of the set U' of the proof of proposition 13 has (3.8) as its minimal resolution, hence the same deficiency ρ, and we get the openness from $U' \subseteq U_{\gamma,\rho}$). It follows that the set $U_{\gamma,\rho}$ of theorem 15 is open in $H_2(d,g)_B$ because by (1.19b) and lemma 12, H_γ and $H(d,g)$, hence $H_{\gamma,B}$ and $H(d,g)_B$, are isomorphic for any curve of $U_{\gamma,\rho}$. Since the irreducibility of $U_{\gamma,\rho}$ follows from the irreducibility of $H_{\gamma,M}$ where M is Buchsbaum of deficiency ρ, we get the theorem.

(3.11) *Example.* In [BKM] we proved the existence of an obstructed curve of $H(33,117)_S$ of maximal rank with 1-dimensional Rao module. Since the degrees of the minimal generators of I(C) are given in [BKM] and M = $H^1(I_C(5))$, we easily find the minimal resolution to be

$$0 \to R(-9) \to R(-10)^{\oplus 2} \oplus R(-9) \oplus R(-8)^{\oplus 4} \to R(-9) \oplus R(-8) \oplus R(-7)^{\oplus 5} \to I(C) \to 0$$

By proposition 13 there exists two different generizations C_1 (resp. C_2) of C, obtained by removing the direct factor R(-9) from L_4 and F_1 (resp. from F_1 and F_0). The curves C_i belong to the open set U of theorem 15 and to separate $U_{\gamma,\rho}$'s. Taking the closure of $U_{\gamma,\rho}$, we easily find two different components of $H(33,117)_S$ whose generic curves have the "same minimal resolution" as C_1 and C_2 (any curve of $U_{\gamma,\rho}$ has the same graded Betti numbers) and whose intersection contain C, i.e. we get the main example of [BKM] from theorem 15.

We should have liked to generalize theorem 15 to the arbitrary case of diameter 2 by dropping the Buchsbaum assumption (e.g. to prove that the set U of remark 1.20 is *dense* in $H_2(d,g)$). In particular if we could prove a result analogous to proposition 13 for curves whose Rao module M is the generic module of diameter 2 (cf. [MDP2] for existence and minimal resolution), we would be able to answer affirmatively the

(3.12) *Question.* Is any irreducible component of $H(d,g)$ whose Rao module of its generic curve is concentrated in at most two consecutive degrees, generically smooth?

We have tried, using general deformation theory, to get a generalization of theorem 15, and we briefly mention the following result, the details of which we leave to the reader;

(3.13) *Remark.* If we define the size of obstructedness, $l_\gamma(C)$, of H_γ at C to be $l_\gamma(C) = {}_0ext^1(I, I) - \dim_C H_\gamma$, we have by deformation theory;

i) $0 \le l_\gamma(C) \le {}_{-4}hom_R(I, M)$, and
ii) H_γ *is smooth at C if and only if* $l_\gamma(C) = 0$.

because ${}_0Ext_R^2(I, I) \cong {}_{-4}Hom_R(I, M)$. Suppose ${}_0Ext_R^2(M, M) = 0$. Using (1.16), we get

$$\dim_C H_{\gamma,\rho} = \dim_C H_\gamma \quad \text{if and only if} \quad {}_0hom_R(M, E) \le l_\gamma(C)$$

By linkage we get a corresponding "dual" result. In particular we get back proposition 4, and moreover that a generic curve of $H(d,g)$ which satisfies ${}_0Ext_R{}^2(M\ ,\ M) = 0$ must also satisfy

$${}_0hom_R(M\ ,\ E) \leq {}_{-4}hom_R(I\ ,\ M) \quad \text{and} \quad {}_0hom_R(I\ ,\ M) \leq {}_{-4}hom_R(M\ ,\ E)$$

As we see from theorem 15 and lemma 12, the two smallest groups of these inequalities vanish, provided the curve is *Buchsbaum* (and general enough) which certainly is a better result under this extra assumption.

For the rest of this section we restrict to curve whose Rao module has diameter one. Since the module L_4 of (3.2) is equal to $R(-c-4)^{\oplus r}$, we have by proposition 13 a good grasp on the existing generizations of C in $H(d,g)$. We can for instance use the preceding results to find many existing generizations of a non-generic curve of $H_{\gamma,M}$ (cf. example 3.11), indicating that our results so far can be used to analyze the case of a non-generic curve of $H_{\gamma,M}$ far beyond the treatise of [W1] and [BKM]. As an illustration of the main results of this section, we will, however, restrict to curves which are generic in $H_{\gamma,M}$, or more generally to curves which satisfy $a_1b_1 = 0$ and $a_2b_2 = 0$, e.g. to the case

(3.14) $\qquad a_1 = 0\ ,\ b_2 = 0 \quad \text{and} \quad (a_2 \neq 0\ \ or\ \ b_1 \neq 0)$

where proper generizations as in remark 3.10 occur, to give a rather complete picture of the existing generizations in $H(d,g)$ (caused by simplifications of the minimal resolution). Let $n(C) = (r,a_1,a_2,b_1,b_2)$ be an associated 5-tuple. Only for curves satisfying $a_1 = 0$ and $b_2 = 0$ we allow the writing $n(C) = (r,a_2,b_1)$ as a triple. We remark that any curve D satisfying $n(D) = n(C)$ and $\gamma_D(v) = \gamma_C(v)$ for $v \neq c$, belongs to the same irreducible family $H_{\gamma,M}$ as C, i.e. a further generization of C and D in $H_{\gamma,M}$ lead to the "same" generic curve. Now given a curve C with $n(C) = (r,a_2,b_1)$, we have by (3.10):

(3.15) *For any pair (i,j) of non-negative integers such that $r-i-j \geq 0$, $a_2-i \geq 0$ and $b_1-j \geq 0$, there exists a generization C_{ij} of C in $H(d,g)$ such that $n(C_{ij}) = (r-i-j,a_2-i,b_1-j)$.*

Note that if we link C to C_l as in proposition 2, we get, by combining (2.10), (2.2) and (2.4) that the 5-tuple $n(C_l) = (r(C_l),a_1(C_l),a_2(C_l),b_1(C_l),b_2(C_l))$ is equal to (r,b_2,b_1,a_2,a_1) where $n(C) = (r,a_1,a_2,b_1,b_2)$. In particular if C satisfies (3.14), then the linked curve C_l does (the equalities among some of the other integers which we get, are already proved in (3.10)).

As an example, let $n(C) = (4,3,2)$ (such curves exist by (2.14)). By (3.15) we have 10 different generizations C_{ij} among which two curves correspond to the triples $n(C_{22}) = (0,1,0)$ and $n(C_{31}) = (0,0,1)$, i.e. they correspond to two *generic* curves by theorem 15. Pushing this argument further, we get

Proposition 16 *Let C be a curve in $\mathbf{P}^3$ whose Rao module $M \neq 0$ is concentrated in degree c, let a_1 and a_2 (resp. b_1 and b_2) be the number of minimal generators (resp. relations) of degree $c+4$ and c respectively, cf. (3.1), and suppose*

$$a_1 = 0\ ,\ b_2 = 0 \ \text{ and } \ a_2b_1 \neq 0$$

a) *If $r < a_2 + b_1$, then C sits in the intersection of at least two irreducible components of H(d,g). Moreover, the generic curve of any component containing C is arithmetically Cohen Macaulay, and the number n(comp,C) of irreducible components containing C satisfies*

$$min\{a_2,r\} + min\{b_1,r\} - r + 1 \leq n(comp,C) \leq r + 1$$

In the case $s(C) = e(C) = c$, we have equality to the left.
b) *If $r \geq a_2 + b_1$ and $s(C) = e(C) = c$, then C is an obstructed curve which belongs to a unique irreducible component of H(d,g).*

Proof. We firstly prove b). Since s(C) = c and since the number s(C) increases under generization by the semicontinuity of $h^0(I_C(v))$, we easily get $s(C') \geq c$. Hence $h^0(I_{C'}(c)) = a_2'$ and $b_2' = 0$ by using (3.1) for any generization C' of C in $H(d,g)$ where $n(C') = (r',a_1',a_2',b_1',b_2')$ and where we allow $r' = 0$ to correspond to the arithmetically Cohen Macaulay case of C'. Similarly e(C) = c implies $h^1(O_{C'}(c)) = b_1'$ and $a_1' = 0$. Applying this considerations to C' = C, we get $\chi(I_C(c)) \leq 0$ by the assumption $r \geq a_2 + b_1$.

Now let C' be the generic curve of an irreducible component containing C. By corollary 14 we get $r'a_2' = 0$ and $r'b_1' = 0$ which combined with $\chi(I_{C'}(c)) \leq 0$ yields $a_2' = 0$ and $b_1' = 0$. Hence $n(C') = (r-a_2-b_1,0,0,0,0)$ for any generic curve of $H(d,g)$. Since $\gamma_{C'}(v) = \gamma_C(v)$ for $v \neq c$ by semicontinuity and the vanishing of $H^1(I_C(v))$, any such C' belongs to the same irreducible component of $H(d,g)$ by the irreducibility of $H_{\gamma',M(C')}$. Moreover C is obstructed by corollary 6, and b) is proved.

a) Suppose $r < a_2 + b_1$. To get the lower bound of n(comp,C) (which in fact is ≥ 2), we use (3.15) to produce several generic curves of $H(d,g)$ which are generizations of C. Indeed let $m(a) = min\{a_2,r\}$ and $m(b) = min\{b_1,r\}$. By (3.15) there exist generizations $C_0, C_1,..,C_{m(a)+m(b)-r}$ such that $n(C_0) = (0,a_2-m(a),b_1+m(a)-r)$, $n(C_1) = (0,a_2-m(a)+1,b_1+m(a)-r-1)$,.., $n(C_{m(a)+m(b)-r}) = (0,a_2+m(b)-r,b_1-m(b))$. Since the curves C_i are arithmetically Cohen Macaulay and have different postulations, they belong to $m(a) + m(b) - r + 1$ different components, and we get the minimum number of irreducible components as stated in the proposition.

To see that the generic curve C' of *any* component containing C is arithmetically Cohen Macaulay, we recall that $r'a_2' = 0$ and $r'b_1' = 0$ by corollary 14 as in the first part of the proof. Suppose $r' \neq 0$. Then $a_2' = 0$ and $b_1' = 0$. To get a contradiction, we remark that the terms of the minimal resolution of I(C) and I(C') which determines $\gamma_{C'}(v) = \gamma_C(v)$ for $v < c$ are equal, from which we get $h^0(I_{C'}(c)) + b_2' = h^0(I_C(c)) - a_2$. Hence $h^0(I_{C'}(c)) \leq h^0(I_C(c)) - a_2$ and similarly we have the "dual" result $h^1(O_{C'}(c)) \leq h^1(O_C(c)) - b_1$. Adding the inequalities, we get

$$\chi(I_{C'}(c)) + h^1(I_{C'}(c)) \leq \chi(I_C(c)) + h^1(I_C(c)) - a_2 - b_1 < \chi(I_C(c)),$$

i.e. a contradiction because $\chi(I_{C'}(c)) = \chi(I_C(c))$.

Now using the fact that the generic curve C' of any irreducible component containing C is arithmetically Cohen Macaulay and that $H_{\gamma',M(C')}$ is irreducible, we prove easily that $n(comp,C) \leq r + 1$ because there are at most r + 1 different postulations $\gamma_{C'}$. Indeed since $M(C') = 0$, $\gamma_{C'}(v) = \gamma_C(v)$ for $v \neq c$ and

$$\gamma_{C'}(c) + \sigma_{C'}(c) = \chi(I_{C'}(c)) = \chi(I_C(c)) = \gamma_C(c) + \sigma_C(c) - r$$

where $\sigma_C(v) = h^1(\mathcal{O}_C(v))$, we see that the different choices of $\gamma_{C'}$ are realized in degree v = c only, and that they are given by $\gamma_{C'}(c) = \gamma_C(c)$-i where i is chosen among {0,1,2,..,r}.

Suppose s(C) = e(C) = c. Since in this case $\gamma_C(c) = a_2$ and $\sigma_C(c) = b_1$ by arguments as in the first part of the proof, we can easily limit the (at most) r+1 different choices of the postulation $\gamma_{C'}(c) = \gamma_C(c)$-i above by choosing

$$m(a) \leq i \leq r - m(b)$$

i.e. n(comp,C) equals precisely m(a) + m(b) - r + 1, and we are done.

(3.16) *Example.* Now we reconsider some particular cases of example 2.14, even though proposition 16 is well adapted to treat the whole example in detail. Recall that for any triple (r,a_2,b_1) of natural numbers, there exists a curve C with $n(C) = (r,a_2,b_1)$ and s(C) = e(C) = c(C) by (2.14). In particular

a) For every integer r > 0 there exists a smooth connected curve C, with triple n(C) = (r,r,r), of degree d and genus g as in (2.14), which is contained in r + 1 irreducible components of $H(d,g)_S$. Moreover the generic curves of all the components containing C are arithmetically Cohen Macaulay.

b) For every r > 0 there exists an obstructed, smooth connected curve with triple (r,a_2,b_1) = (2t,t,t) or (2t+1,t,t), of degree d and genus g as given by (2.14), which belongs to a unique irreducible component of $H(d,g)_S$ by proposition 16. In particular the obstructed curve C with (r,a_2,b_1) = (2,1,1) belongs to a *unique* irreducible component of $H(32,109)_S$, confirming what we saw in (2.14).

Appendix: Non-reduced components of $H(d,g)_S$

So far we have studied curves C with strong restrictions of the Rao module M(C). We have proved (theorem 15) that if the diameter of M(C) of a generic curve C of *H(d,g)* is ≤ 1 (and some weaker statement in the diameter 2 case), then the corresponding irreducible component is reduced (i.e. generically smooth). So there are no non-reduced components under these assumptions, and *maybe no non-reduced component in the diameter ≤ 2 case at all* (*cf. Question* (3.12))?

In the following we shall see that the diameter ≤ 2 case is special with regard to the existence of such components because once the diameter is greater, there are lots of non-reduced components ([Mu], [K2], [K1], [GP2], [E] and [F]). Indeed the Rao module in the well-known example of Mumford has diameter 3, and any diameter ≥ 3 occurs for this phenomena (combine (4.5) below with theorem 17 to conclude). The mentioned papers deal mostly with curves on a cubic surface, but one may, as pointed out in [K2], use linkage (proposition 2 of this paper) to find non-reduced components whose generic curve C sits on a smooth surface of any degree s(C) ≥ 6. In the paper of Fløystad [F],there is a nice treatise of the case where M is a complete intersection from the point of view of cup-products.

In this appendix we will, however, extend the results of [K1] and [E] considerably by making some computations. Indeed these papers prove the following conjecture under some assumptions;

(4.1) **Conjecture**. Let W be a maximal irreducible family of smooth connected, *linearly* *normal* space curves of degree d and genus g, whose general member C is contained in a smooth cubic surface, and let $\overline{W}$ be the closure in $H(d,g)$. *Then* $\overline{W}$ *is a non-reduced irreducible component of* $H(d,g)$ *if and only if*

$$d \geq 14, \quad 3d - 18 \leq g \leq (d^2-4)/8 \quad \text{and} \quad H^1(\mathcal{I}_C(3)) \neq 0$$

This conjecture, originating in [K2], is here presented by modifications proposed by Ellia [E] and Dolcetti, Pareshi [DP], because they found countrexamples which heavily depended on the fact the generic curves were *not* linearly normal (i.e. the curves satisfied $H^1(\mathcal{I}_C(1)) \neq 0$). Note that in (4.1) the maximality of W is taken with respect to the degree of the cubic surface, i.e. the general curve C' of any proper closed subset of $H(d,g)$ containing $\overline{W}$ satisfies s(C') > 3.

Now recall that a smooth cubic surface S is obtained by blowing up $\mathbf{P}^2$ in six general points [H1]. Taking the linear equivalence classes of the inverse image of a line in $\mathbf{P}^2$ and $-E_i$ (minus the exceptional divisors), i = 1,..,6, as a basis for Pic(S), we can associate a curve C on S and its corresponding invertible sheaf $O_S(C)$ with a 7-tuple of non-negative integers $(\delta, m_1, .., m_6)$ satisfying

(4.2) $$\delta \geq m_1 \geq .. \geq m_6 \quad \text{and} \quad \delta \geq m_1 + m_2 + m_3$$

The degree and the (arithmetic) genus of the curve are given by

$$d = 3\delta - \sum_{i=1}^{6} m_i \ , \quad g = \binom{\delta-1}{2} - \sum_{i=1}^{6} \binom{m_i}{2}$$

The explicit size of the interval where M(C) is non-vanishing is known in terms of $(\delta, m_1, .., m_6)$ (cf. [K4], page 314 or [GM], rem. 2.7). Using this, one may verify the following facts for a curve C whose corresponding 7-tuple $(\delta, m_1, .., m_6)$ *satisfies* (4.2);

(4.3) If $m_6 \geq 3$ and $(\delta, m_1, .., m_6) \neq (\lambda+9, \lambda+3, 3, ..3)$ for any $\lambda \geq 2$, then $H^1(\mathcal{I}_C(3)) = 0$. In particular if a curve (effective divisor) on a smooth cubic satisfies $g > (d^2-4)/8$, then

$$H^1(\mathcal{I}_C(3)) = 0$$

([K1], lemma 16 and corollary 17).

(4.4) If $m_6 \geq 1$ and $(\delta, m_1, .., m_6) \neq (\lambda+3, \lambda+1, 1, ..1)$ for any $\lambda \geq 2$, then $H^1(\mathcal{I}_C(1)) = 0$. Moreover, in the range $d \geq 14$ and $g \geq 3d-18$, we have

$$H^1(\mathcal{I}_C(3)) \neq 0 \text{ and } H^1(\mathcal{I}_C(1)) = 0 \quad \text{if and only if} \quad 1 \leq m_6 \leq 2.$$

(4.5) If $(\delta, m_1, .., m_6) \neq (\lambda+3t, \lambda+t, t, ..t)$ and $(3t, t, t, .., t, t-\lambda)$ for any $\lambda \geq 2$ and any $t \geq 0$, then the diameter of M(C) is

$$\mathrm{diam}\, M(C) = 2\delta - m_2 - m_3 - m_4 - m_5 - 2m_6 - 2 \ (+\ 1 \text{ in the exceptional cases})$$

Using (4.3), the fact that $H^1(\mathcal{I}_C(3)) = 0$ implies unobstructedness ([K1], theorem 1) and dim W = d+g+18 (for d > 9), one may easily see that the conditions of (4.1) are necessary for $\overline{W}$ to be a non-reduced component. The conjecture therefore really deals with the converse, and by (4.4), we may as well suppose $m_6 = 1$ or 2. For both values the main theorem of this section tells that the conjecture is true under weak assumptions, thus generalizing the results of the papers [E] and [K1] to:

Theorem 17 *Let W be a maximal irreducible family of smooth connected space curves, whose general member sits on a smooth cubic surface S and corresponds to the 7-tuple* $(\delta,m_1,..,m_6)$, $\delta \geq m_1 \geq .. \geq m_6$ *and* $\delta \geq m_1 + m_2 + m_3$, *of Pic(S). Let* $\overline{W}$ *be the closure of W in H(d,g). Then*

i) $\overline{W}$ *is a generically smooth, irreducible component of H(d,g) provided*

$$m_6 \geq 3 \quad \text{and} \quad (\delta,m_1,..,m_6) \neq (\lambda+9,\lambda+3,3,..3) \quad \text{for any } \lambda \geq 2.$$

ii) $\overline{W}$ *is a non-reduced irreducible component of H(d,g) provided;*

a) $m_6 = 2$, $m_5 \geq 4$, $m_2 \geq 5$ *and* $d \geq 21$, *or*
b) $m_6 = 1$, $m_5 \geq 6$, $m_2 \geq 7$ *and* $d \geq 35$, *or*
c) $m_6 = 1$, $m_5 = 5$, $m_4 \geq 7$, $m_2 \geq 8$ *and* $d \geq 36$.

In the exceptional case $(\lambda+9,\lambda+3,3,..3)$ of i) and in some other cases where $m_6 = 1$ or 2 and ii) is elsewhere not satisfied, we have $H^1(\mathcal{O}_C(3)) = 0$, in which case $\overline{W}$ is contained a unique irreducible component of $H(d,g)$ and the codimension of $\overline{W}$ in $H(d,g)$ is $h^1(\mathcal{I}_C(3))$, cf. [K1], th. 1. For the case $m_6 = 0$, see [E], rem. VI.6 and [DP].

To prove theorem 17, we will need the following two results;

Proposition 18 (Ellia) *Let d and g be integers such that* $d \geq 21$ *and* $g \geq 3d - 18$, *let W be as in theorem 17 and suppose the general curve C of W satisfies* $H^1(I_C(1)) = 0$. *If C' is a generization of C in H(d,g) satisfying* $H^0(I_{C'}(3)) = 0$, *then* $H^0(I_{C'}(4)) = 0$.

Proof. See [E], prop. VI.2.

We remark that Ellia uses the important proposition 18 to prove the conjecture provided $d \geq 21$ and $g > G(d,5)$ where G(d,s) is the maximum genus of smooth connected curves of degree d not contained in a surface of degree s-1. His result is in most cases better that the one in [K1] which requires $g > 7 + (d-2)^2/8$, $d \geq 18$, because $G(d,5) = d^2/10 + d/2 + \epsilon$, ϵ a correction term, cf. [GP1]. There is, however, quite a lot of cases where theorem 17 imply the conjecture while this result of Ellia does not.

Lemma 19 *Let C be a curve sitting on a smooth cubic surface S, whose corresponding invertible sheaf is given by* $(\delta,m_1,..,m_6)$, $\delta \geq m_1 \geq .. \geq m_6$ *and* $\delta \geq m_1 + m_2 + m_3$. *If v is a non-negative integer such that* $v < m_2$ *and* $v \leq m_3$, *then*

$$h^0(I_C(v)) - h^1(I_C(v)) = \binom{v}{3} - \sum_{r+1}^{6} \binom{m_i - v}{2}$$

where r is the unique integer such that $v > m_{r+1}$, $v \leq m_r$ *(So* $3 \leq r \leq 6$ *and* $r = 6$ *means* $v \leq m_i$ *for any i).*

Proof. It is not difficult to prove the lemma using the proof of [K1], lemma 18. Indeed one shows $h^1(O_C(v)) = h^0(O_S(C)(-v-1)) = h^0(L)$ where L corresponds to $(\delta-3v-3, m_1-v-1, .., m_r-v-1, -1, .., -1)$. Since the number of sections of L is easily found, and $\chi(L)$ is known, we get the lemma, cf. [K1] for details. One may continue the proof and see that $h^0(I_C(v)) = \binom{v}{3}$. One may also get the lemma from [Gi], rem. 2.7.

Proof of theorem 17 i) is a special case of [K1], theorem 1.

ii) By [K1], (2.7) and lemma 13, one may see that

$$\dim W + h^1(I_C(3)) = h^0(N_C). \tag{4.6}$$

Since $h^1(I_C(3)) \neq 0$, it suffices to prove that $\overline{W}$ is an irreducible *component* of $H(d,g)$ because if it is, then $\dim \overline{W} < h^0(N_C)$ implies that the general curve C of $\overline{W}$ is obstructed, i.e. $\overline{W}$ is non-reduced.

a) To get a contradiction, suppose $\overline{W}$ is *not* a component. Since W is a maximal family of curves on a cubic surface, there exists a generization C' of C satisfying $h^0(I_{C'}(3)) = 0$. By semicontinuity, $h^1(O_{C'}(4)) \leq h^1(O_C(4))$. Combining with $\chi(I_{C'}(4)) = \chi(I_C(4))$, it follows that $h^0(I_{C'}(4)) - h^1(I_{C'}(4)) \geq h^0(I_C(4)) - h^1(I_C(4))$. However, by lemma 19, we have $h^0(I_C(4)) - h^1(I_C(4)) = 1$, hence $h^0(I_{C'}(4)) \geq 1$. Since the curve is linearly normal by (4.4), this inequality contradicts the conclusion of proposition 18.

b) Again it suffices to prove that $\overline{W}$ is an irreducible *component* of $H(d,g)$. To get a contradiction we suppose there is a generization C' of C satisfying $h^0(I_{C'}(3)) = 0$. By semicontinuity of $h^1(O_C(v))$ and lemma 19, we get

$$h^0(I_{C'}(v)) - h^1(I_{C'}(v)) \geq h^0(I_C(v)) - h^1(I_C(v)) = \tbinom{v}{3} - \tbinom{v}{2} \text{ for } 1 \leq v \leq 6,$$

Hence $h^0(I_{C'}(6)) - h^1(I_{C'}(6)) \geq 5$. Since $s(C') \geq 5$ by proposition 18 and (4.4), C' is contained in a complete intersection of bidegree (5,6) or (6,6). Hence $d \leq 36$ and we have a contradiction except when $d = 35$ or 36. In the case $d = 36$, C' is a complete intersection satisfying $h^0(I_{C'}(6)) \geq 5$, and if $d = 35$, we can link C' to a line C'' satisfying $h^1(O_{C''}(2)) \neq 0$, i.e. we get a contradiction in both cases, and we are done.

c) The proof is similar to b), remarking only that in this case $h^0(I_{C'}(6)) \geq 4$ and $h^0(I_{C'}(7)) \geq 11$ by lemma 19, i.e. C' is contained in a complete intersection of bidegree (5,7) or (6,6), and we conclude as in b).

(4.7) *Remark.* We see from the proof that once we have proved that $\overline{W}$ is a component, it is automatically non-reduced by the assumption $H^1(I_C(3)) \neq 0$ (e.g. it is non-reduced if and only if $H^1(I_C(3)) \neq 0$).

Also the proof of [K1] deserves renewed attention because it admits some interesting extensions. Indeed, in [K1], we showed that the set $\bar{W}$ of theorem 17 was a component by proving that $34-g+d^2/4$ was an upper bound for the dimension of any component of $H(d,g)_S$ whose general curve is contained in a *quartic integral* surface F (cf. [K1], prop. 20 where unfortunately the weak assumption on the singular locus "Sing(F)$\cap$C is finite" is missing). We deduced there that $\bar{W}$ was a component in the case $d+g+18 > 33-g+d^2/4$. An interesting observation concerning this proof is, however, that it is rather straightforward to generalize it to get;

Proposition 20 *Let V be an irreducible component of $H(d,g)_S$ containing a curve C which sits on some integral surface F of degree $s \geq 4$. If Sing(F)$\cap$C is a finite set and if $g \leq d/2(d/s+s-4)$, then*

$$dim\,V \leq \max\{\frac{d^2}{s} - g + \binom{s+3}{3} - 1\ ,\ \frac{d^2}{2s} + \binom{s+3}{3} - 1\}$$

We get a somewhat better bound for dim V if we include the Clifford index of the normal bundle of C in S, S a desingularization of F. Now combining this result with proposition 18 we can prove that the set $\bar{W}$ of theorem 17 is a component in the case

$$g \geq \max\{\frac{d^2}{10} - \frac{d}{2} + 18\ ,\ G(d,6)\}\ ,\quad d \geq 26$$

i.e. the conjecture 4.1 holds in this range by remark 4.7 (and we can weaken the part $g > G(d,6)$ of the assumption above by using proposition 20 for $s \geq 6$). More importantly, we can by exactly the same proof treat some other maximal irreducible families than those of curves on a smooth cubic surface. For instance, for families of curves on a *smooth quartic* surface, one knows that (4.6) holds (replacing 3 by 4, cf. [K1], lemma 13). Therefore we can use proposition 20 for $s = 5$ (the case $s \geq 6$ can be excluded because the genus turns out to satisfy $g > G(d,6)$) to prove that the corresponding $\bar{W}$ is a component in the case given by;

Proposition 21 *Let W be a maximal irreducible family of smooth connected space curves, whose general member C sits on a smooth quartic surface, and let $\bar{W}$ be the closure of W in H(d,g). If*

$$d \geq 31 \quad and \quad g > 21 + d^2/10,$$

then $\bar{W}$ is an irreducible component of H(d,g). Moreover $\bar{W}$ is non-reduced if and only if $H^1(I_C(4)) \neq 0$.

References:

[B] *Bolondi G.*: Irreducible families of curves with fixed cohomology, Arch. der Math., 53 (1989), 300-305.

[BK] *Bolondi G., Kleppe J.O.* To appear.

[BKM] *Bolondi G., Kleppe J.O. and Miro-Roig R.M.*: Maximal rank curves and singular points of the Hilbert scheme, Compositio Math., 77 (1991), 269-291.

[BM] *Bolondi G., Migliore J. C.*: Classification of maximal rank curves in the liaison class $\mathbf{L}_n$, Math. Ann. 277 (1987), 585-603.

[C] *Chang, M-C.*: A Filtered Bertini-type Theorem, J. reine angew. Math. 397, (1989), 214-219.

[DP] *Dolcetti A., Pareschi G.*: On Linearly Normal Space Curves, Math. Z. 198, (1988), 73-82.

[E] *Ellia Ph.*: D'autres composantes non réduites de Hilb $\mathbf{P}^3$, Math. Ann. 277, (1987) 433-446.

[EF] *Ellia Ph., Fiorentini, M.*: Défaut de postulation et singularités du Schema de Hilbert. Annali Univ. di Ferrara 30 (1984), 185-198.

[El] *Ellingsrud G.*: Sur le schéma de Hilbert des variétés de codimension 2 dans $\mathbf{P}^e$ a cône de Cohen-Macaulay. Ann. Scient. Éc. Norm. Sup. 8 (1975), 423-432.

[F] *Fløystad G.*: Determining obstructions for space curves, with application to non-reduced components of the Hilbert scheme. J. reine angew. Math. 439 (1993), 11-44.

[G] *Grothendieck A.*: Technique de descente et theoreme d'existence en geometrie algebrique, in Sem. Bourbaki, Vol 1960/1961, exp. 221.

[Gi] *Giuffrida S.*: Graded Betti numbers and Rao modules of curves lying on a smooth cubic surface in **P3**, Queen's Papers in Pure and Applied Math. 88 (1991) A1-A61.

[GM] *Giuffrida S., Maggioni R.*: On the Rao module of a curve lying on a smooth cubic surface in **P3**, Comm. Algebra 18 (1990), 2039-2061.

[GP1] *Gruson L., Peskine Chr.*: Genre des courbes de l'espace projectif, In: Proc. Tromsø 1977, Lecture Notes in Math. 687 (1978), Springer.

[GP2] *Gruson L., Peskine Chr.*: Genre des courbes de l'espace projectif (II), Ann. Sci. Éc. Norm. Sup. (4), 15 (1982), 401-418.

[H1] *Hartshorne R.*: Algebraic geometry. Springer Verlag (1977).

[H2] *Hartshorne R* (notes).: Local Cohomology. Lecture Notes in Math. 41 (1967), Springer.

[K1] *Kleppe J. O.*: Non-reduced components of the Hilbert scheme of smooth space curves. In: Proc. Rocca di Papa 1985. Lecture Notes in Math. 1266 (1987), Springer.

[K2] *Kleppe J. O.*: The Hilbert-flag scheme, its properties and its connection with the Hilbert scheme. Applications to curves in 3-space. Preprint (part of thesis), March 1981, Univ. of Oslo.

[K3] *Kleppe J. O.*: Liaison of families of subschemes in $\mathbf{P}^n$. In: Proc. Trento 1988. Lecture Notes in Math. 1389 (1989), Springer.

[K4] *Kleppe J. O.*: On the existence of Nice Components in the Hilbert Scheme *H(d,g)* of Smooth Connected Space Curves. Boll. U.M.I.(7) 8-B (1994), 305-326.

[K5] *Kleppe J. O.*: Concerning the existence of nice components in the Hilbert scheme curves in $\mathbf{P}^n$ for n = 4 and 5. To appear in J. reine angew. Math.

[L1] *Laudal O. A.*: Formal Moduli of Algebraic Structures. Lecture Notes in Math. 754 (1979), Springer.

[L2] *Laudal O. A.*: Matrix Massey Products and Formal Moduli I, In: Algebra, Algebraic Topology and Their Interactions Structures. Lecture Notes in Math. 1183, 218-240 (1986), Springer.

[Mu] *Mumford D.*: Further pathologies in algebraic geometry, Amer. J. Math., 84, 1962, 642-648.

[MDP1] *Martin-Deschamps M., Perrin D.*: Sur la classification des courbes gauches, Asterisque, 184-185 (1990).

[MDP2] *Martin-Deschamps M., Perrin D.*: Courbes Gauches et Modules de Rao, Preprint LMENS-91-5 (1991).

[R] *Rao A. P.*: Liaison Among Curves. Invent. Math. 50 (1979), 205-217.

[Se] *Sernesi E.*: Un esempio di curva ostruita in $\mathbf{P}^3$ Sem. di variabili Complesse, Bologna (1981), 223-231.

[SGA2] *Grothendieck A.*: Cohomologie locale des faisceaux cohérents et Théorèmes de Lefschetz locaux et globaux, North Holland, Amsterdam, 1968.

[W1] *Walter C.*: Some examples of obstructed curves in $\mathbf{P}^3$. In: Complex Projective Geometry. London Math. Soc. Lecture Note Ser. 179 (1992).

[W2] *Walter C.*: Horrocks theory and algebraic space curves. Preprint (1990).

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