THE HILBERT SCHEME OF SURFACES IN P^4 OF CONSTANT COHOMOLOGY.

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INTRODUCTION.

The study of particular surfaces in $\mathbf{P}^4$ has received some attention over the last years after Ellingsrud and Peskine managed to show that there are only finitely many surfaces of non-general type (i.e. finitely many such components of the Hilbert scheme), cf. [EP], [Ra], [DES] and [P]. If for instance the degree d > 66, there is no surface in $\mathbf{P}^4$ of non-general type at all [BC].

In the present paper we study the Hilbert scheme $H(d,p,\pi)$ of all surfaces of degree d and arithmetic (resp. sectional) genus p (resp. π) from a different angle. Recall that, for space curves $C \subseteq \mathbf{P}^3 = \mathrm{Proj}(R)$, Martin-Deschamps and Perrin have given a stratification $H(d,g)_{\gamma,\rho}$ of the Hilbert scheme $H(d,g)$ of space curves of degree d, genus g obtained by deforming curves with constant cohomology [MDP1]. They also proved the smoothness of the "morphism" $\varphi : H(d,g)_{\gamma,\rho} \to E_\rho$ = isomorphism classes of R-modules M of finite length, given by $(C \subseteq \mathbf{P}^3) \to M = \oplus H^1(I_C(v))$, they gave a scheme structure to $H(d,g)_{\gamma,M} = \varphi^{-1}(M)$ and computed its dimension. Earlier Rao proved that any R-module M of finite length determines the liaison class of a curve C, up to a shift in the grading. Note that Rao's result is related to the surjectivity of φ, while the smoothness implies infinitesimal surjectivity as well. For surfaces in $\mathbf{P}^4$ there is a recent result of Bolondi [B2], similar to that of Rao, telling that a triple $D = (M_1,M_2,b)$ of modules M_i of finite length and an extension $b \in {}_0Ext^2(M_2, M_1)$ determines the liason class of a surface X such that $M_i \cong \oplus H^i(I_X(v))$ modulo some shift in the grading. Therefore it is natural to consider the stratification $H_{\gamma,\rho} = H(d,p,\pi)_{\gamma,\rho}$ of $H(d,p,\pi)$, similar to the one in the curve case, and to ask if the corresponding $\varphi : H_{\gamma,\rho} \to V_\rho$ = isomorphism classes of R-modules M_1 and M_2 commuting with b, is smooth and irreducible. We prove in this paper that the answer is yes (theorem 1.1), thus extending Bolondi's result in this direction. It follows that the fiber $H(d,p,\pi)_{\gamma,D} := \varphi^{-1}(D)$ is smooth and irreducible and we compute its dimension (corollary 2.7). In section 3 we also determine the tangent space of $H_{\gamma,\rho}$ (resp. V_ρ) at $(X \subseteq \mathbf{P}^4)$ (resp. at D), from which we deduce a local isomorphism $H_{\gamma,\rho} \cong H(d,p,\pi)$ (theorem 3.7) under certain restrictive conditions (e.g. natural cohomology) and a smoothness criterion for V_ρ (proposition 3.4). The liaison result we prove in theorem 4.1 turns out to be helpful in determining the structure of $H_{\gamma,\rho}$ and its dimension. Note that the irreducibility of $H_{\gamma,D}$ follows from an earlier work of Bolondi [B1] while, in the special case of arithmetically Cohen- Macaulay surfaces (i.e. surfaces with $M_i = 0$ for i = 1,2), both the irreducibility and the smoothness of $H_{\gamma,D}$ follow from [E]. To see more generally how $H_{\gamma,\rho}$ determine $H(d,p,\pi)$, it is desireable to study the imbedding $H_{\gamma,\rho} \hookrightarrow H(d,p,\pi)$ in detail as we did in [K3] for the Hilbert scheme of curves. The corresponding problem for $H(d,p,\pi)$ will eventually be carried out in another paper. Indeed as we will see in what follows, the technical problems in describing the stratification of $H(d,p,\pi)$, the tangent spaces of $H_{\gamma,\rho}$ and V_ρ etc. are much more complicated than in the curve case, justifying this limitation.

We also limit the extent of this paper by omitting proving that $H(d,p,\pi)_{\gamma,D}$ is a scheme, although it seems quite natural to generalize the work of [MDP1] so far. Indeed the "morphism" $\varphi : H_{\gamma,\rho} \to V_\rho$ to the "scheme" (i.e. stack) V_ρ has a natural nice description in terms of the hulls of the local deformation functors at a given point ($X \subseteq \mathbf{P}^4$). In this local case $H(d,p,\pi)_{\gamma,D}$ corresponds to the hull of the local fiber functor. Even though we have not proved the existence of all this as schemes, we allow a thinking and a terminology as if they were schemes, knowing that the statements have a precise interpretation in terms of their corresponding "completed local rings", i.e. their hulls. Only the irreducibility is problematic from this local point of view, but this case is already taken care of in the literature by [B1] and [BM1]. To limit the size of this paper, we only sketch the proof of some other results (e.g the tangent spaces of $H_{\gamma,\rho}$ and V_ρ) as well.

Due to the importance of the works of Martin-Deschamps and Perrin and its consequences for the Hilbert scheme $H(d,g)$ of curves ([MDP1], [MDP2]), we hope the corresponding theory for the Hilbert scheme $H(d,p,\pi)$ of surfaces, of which we take a first main step, will turn fruitful. In our treatise we have frequently used a natural spectral sequence converging to the right derived functor $Ext_m{}^i(N, -)$ of $\Gamma_m(Hom_R(N, -))$ and the duality

$$ {}_vExt_m^i(N_2, N_1) \simeq {}_{-v-5}Ext_R^{5-i}(N_1, N_2)^v , $$

In [K3], section 1, we felt this allowed a simple treatment of $H(d,g)_{\gamma,\rho}$ because it avoided the extensive use of the various resolutions of [MPD1]. In this paper we have not been able to avoid such resolutions (e.g. (5) below) to prove our theorems for $H(d,p,\pi)_{\gamma,\rho}$, because we needed them to see the factorization of some main maps. We could probably present all the theory of this paper using different kinds of resolutions, as in [MDP1]. We are, however, not sure this would really simplify the treatise if one is familiar with spectral sequences, but it might be helpful in seeing the commutativity of some diagrams induced from the spectral sequence where we sometimes implicitely have assumed it by "naturality" of the maps.

The investigations of this paper started several years ago as a common project with prof. G. Bolondi at Sassari. As the reader will see, Bolondi's paper [B2] is a main source of idea for the work presented here. It was prof. G. Bolondi who introduced me to the idea of extending the results of [B2], as Martin-Deschamps and Perrin do for space curves, to get a stratified description of the Hilbert scheme $H(d,p,\pi)$, and who pointed out the interesting things to be proved. Parts of the paper are also a natural continuation of [BM1] and [BM2]. As time has pasted it is the author of this paper who has carried out the investigations, and as agreed upon by Bolondi, should be the sole author of this paper. I thank prof. G. Bolondi very much for many stimulating discussions while preparing this work and prof. E. Ballico and the University of Trento for their hospitality during my visits in June 1994 and May 1995. This paper was written in the context of EUROPROJ.

0. PRELIMINARIES AND TERMINOLOGY.

We need to recall and generalize some results of [K3] and [B2], but first we establish some terminology. A surface X is an *equidimensional, locally Cohen-Macaulay* subscheme of $\mathbf{P} = \mathbf{P}^4$ of dimension 2 with sheaf ideal I_X and normal sheaf $N_X = Hom_{O_\mathbf{P}}(I_X, O_X)$. If F is a coherent O_P -Module, we let $H^i(F) = H^i(\mathbf{P},F)$, $H_*^i(F) = \sum_v H^i(F(v))$, $h^i(F) = \dim H^i(F)$, and $\chi(F) = \Sigma\ (-1)^i\ h^i(F)$ is the Euler-Poincaré characteristic. The arithmetic genus p is defined by $p = \chi(O_X)-1$, while the sectional genus π is given by $\chi(O_X(1)) = d-\pi+1+\chi(O_X)$, leading to Riemann-Roch's theorem:

$$\chi(O_X(v)) = \tfrac{1}{2}dv^2 -(\pi-1-\tfrac{1}{2}d)v + \chi(O_X).$$

Moreover $M_i = M_i(X)$ is the "deficiency modules" $H_*^i(I_X)$ for i = 1,2 (playing the role as the Hartshorne-Rao module of a curve), E = E(X) is the module $H_*^2(O_X)$ and I = I(X) = $H_*^0(I_X)$ is the (saturated) homogeneous ideal. They are graded modules over the polynomial ring $R = k[X_0,X_1,..,X_4]$, where k is supposed to be an algebraically closed field of characteristic zero. The postulation γ (resp. deficiency $\rho = (\rho^1,\rho^2)$, resp. specialization σ) of X is the function defined over the integers Z by $\gamma(v) = \gamma_X(v) = h^0(I_X(v))$ (resp. $\rho(v) = \rho_X(v) = (\rho^1(v),\rho^2(v))$ where $\rho^i(v) = h^i(I_X(v))$ for i = 1,2, resp. $\sigma(v) = \sigma_X(v) = h^2(O_X(v))$). Put

$$s(X) = min\ \{ n \mid h^0(I_X(n)) \neq 0\},$$
$$e(X) = max\ \{ n \mid h^2(O_X(n)) \neq 0\},$$

A surface X is *unobstructed* if the Hilbert scheme $H(d,p,\pi)$ is smooth at the corresponding point ($X \subseteq \mathbf{P}$), otherwise X is obstructed. The open part of $H(d,p,\pi)$ of smooth surfaces is denoted by $H(d,p,\pi)_S$, while $H_{\gamma,\rho} = H(d,p,\pi)_{\gamma,\rho}$ (resp. H_γ, resp. $H_{\gamma,\rho 1}$, resp. $H_{\gamma,D}$ where D = (M_1,M_2,b)) denotes the subscheme of $H(d,p,\pi)$ of surfaces with constant cohomology given by γ and ρ, (resp. constant postulation γ, resp. constant γ and ρ_1, resp. constant postulation γ and deficiency modules isomorphic to M_i and commuting with $b \in {}_0Ext_R^2(M_2,M_1)$). Note that we can work with $H_{\gamma,D}$ as a locally closed subset of $H_{\gamma,\rho}$ (cf. the arguments of [BB], cor.2.2, and combine with (0.1) below), even though we have not proved the representability of the corresponding functor.

Let X be a surface in $\mathbf{P}^4$ and let

(1)
$$0 \to P_5 \xrightarrow{\sigma5} P_4 \xrightarrow{\sigma4} P_3 \xrightarrow{\sigma3} .. \to P_0 \xrightarrow{\sigma0} M_1 \to 0\ ,$$
$$0 \to Q_5 \xrightarrow{\tau5} Q_4 \xrightarrow{\tau4} Q_3 \xrightarrow{\tau3} .. \to Q_0 \xrightarrow{\tau0} M_2 \to 0$$

(for short $\sigma_\bullet : P_\bullet \to M_1 \to 0$ and $\tau_\bullet : Q_\bullet \to M_2$) be minimal free resolutions over R. Let $K_\bullet$ and $L_\bullet$ be the *i*th syzygies of M_1 and M_2 respectively, i.e. $K_i = \ker \sigma_i$ and $L_i = \ker \tau_i$. Recall that syzygies has nice cohomological properties, for instance

(2)
$$M_1 = H_*^1(\tilde{K}_1) \quad and \quad H_*^2(\tilde{K}_1) = H_*^3(\tilde{K}_1) = 0\ ,$$
$$M_2 = H_*^3(\tilde{L}_3) \quad and \quad H_*^1(\tilde{L}_3) = H_*^2(\tilde{L}_3) = 0$$

These resolutions have some strong connections to the minimal resolutions of I = I(X);

(3)
$$0 \to \oplus_i R(-n_{4i}) \to \oplus_i R(-n_{3i}) \to \oplus_i R(-n_{2i}) \to \oplus_i R(-n_{1i}) \to I \to 0$$

and the following minimal resolutions of $A = H_*^0(O_X)$;

(4)
$$0 \to P_3' \xrightarrow{\sigma3'} P_2' \xrightarrow{\sigma2'} P_1' \xrightarrow{\sigma1'} P_0 \oplus R \to A \to 0$$

where the morphism $P_0 \oplus R \to A$ of (4) is naturally deduced from $P_0 \to M_1$, recalling the exact sequence $R \to A \to M_1 \to 0$. The connections we have in mind can be formulated and proved for a family of surfaces with constant cohomology (at least locally which is the case we frequently need later), e.g. we can replace the field k by a local k-algebra S. Now, in [B2], Bolondi uses some ideas of Horrocks [Ho] to define the element $b \in {}_0Ext_R^2(M_2, M_1)$, and conversely, given $D = (M_1,M_2,b)$ where M_i are R-modules of finite length, he constructs a surface X by defining some shift I(h), $h \in Z$, of $I = I(X)$ in terms of an exact sequence $0 \to L_3' \to K_1' \to I(h) \to 0$ where L_3'(resp. K_1') is isomorphic to the syzygy L_3 (resp. K_1) up to some R-free module F_L (resp. F_K). Up to liaison this construction is the inverse to the first approach which defines (M_1,M_2,b) from a given X. To prove the main smoothness theorem of the next section in an easy way, we need to adapt the treatise above slightly by determining F_L and F_K more explicitly. Using ideas of Rao's paper [R], we can prove

Proposition 0.1 *Let X be a surface (i.e. locally Cohen-Macaulay and equidimensional) in P_S^4, flat over a local noetherian k-algebra S, and let $M_1 = M_1(X)$, $M_2 = M_2(X)$ and $I(X)$ be flat S-modules. Then there exist minimal R-free resolutions of M_i, $I(X)$ and $A = H_*^0(O_X)$ (with $R = S[X_0,X_1,..,X_4]$), as in* (1), (3) *and* (4). *Moreover let $L_3' = ker\ \sigma_1'$ and let K_1' be the kernel of the composition of σ_1' and the natural projection $P_0 \oplus R \to P_0$, cf.* (4). *Then there is an exact sequence*

$$(5) \qquad 0 \to L_3' \xrightarrow{b'} K_1' \longrightarrow I(X) \to 0$$

of flat graded S-modules and a surjective morphism $d : {}_0Hom_R(L_3', K_1') \to {}_0Ext_R^2(M_2, M_1)$, defining a triple (M_1,M_2,b) where $b = d(b')$ (coinciding with the uniquely defined "Horrocks' triple" of [Ho] or [B2]). Moreover L_3' (resp. K_1') is the direct sum of the 3. *syzygy of M_2 (resp.* 1. *syzygy of M_1) up to a direct free factor, i.e. there exist R-free modules F_L and F_K such that the horizontal exact sequences in*

$$\begin{array}{ccccccc} 0 \to & K_1' & \longrightarrow & P_1' & \longrightarrow & P_0 \\ & \wr\| & \circ & \wr\| & \circ & \| \\ 0 \to & K_1 \oplus F_K & \longrightarrow & P_1 \oplus F_K & \xrightarrow{\sigma_1 \oplus 0} & P_0 \end{array}$$

are isomorphic. Similarly, the exact sequences $0 \to Q_5 \xrightarrow{(\tau_5,0)} Q_4 \oplus F_L \to L_3 \oplus F_L \to 0$ and $0 \to P_3' \to P_2' \to L_3' \to 0$ are isomorphic as well.

Remark 0.2 The proposition above, defining the "Horrocks' triple" (M_1,M_2,b) from a given X, can be regarded as our definition of the "morphism" $\varphi : H_{\gamma,\rho} \to V_\rho$ = isomorphism classes of R-modules M_1 and M_2 commuting with b.

Proof We obviously have minimal resolutions of $M_i \otimes_S k$, $I(X) \otimes_S k$ and $A \otimes_S k$ as described above with $R = k[X_0,X_1,..,X_4]$, cf. (1), (3) and (4), and these resolutions can easily be lifted to the minimal resolution of the proposition by cutting into short exact sequences and using the flatness of the modules involved.

By the definition of K_1' there is a commutative diagram

$$\begin{array}{ccccccccc} 0 \to P_3' & \longrightarrow & P_2' & \longrightarrow & P_1' & \longrightarrow & P_0 \oplus R & \to & A \to 0 \\ & & \downarrow & \circ & \| & \circ & \downarrow & \circ & \downarrow \\ & 0 \longrightarrow & K_1' & \longrightarrow & P_1' & \longrightarrow & P_0 & \longrightarrow & M_1 \to 0 \end{array}$$

and we get easily the exact sequence (5) by the snake lemma. Comparing the lower exact sequence in the diagram above with the following part of the *minimal* resolution of M_1; $\to P_1 \to P_0 \to M_1 \to 0$, we get the commutative diagram of the proposition because K_1 is the 1. syzygy of M_1.

To prove the corresponding commutative diagram for L_3' and L_3, we sheafify (5), and we get $M_2 \cong H_*^3(\mathcal{L}_3')$. Recalling the definition of L_3', we have the exact sequence

$$H_*^4(\mathcal{P}_2')^\vee \to H_*^4(\mathcal{P}_3')^\vee \to M_2^\vee \cong Ext_R^5(M_2, R(-5)) \to 0$$

which we compare to the *minimal* resolution

$$Q_4^\vee \longrightarrow Q_5^\vee \longrightarrow Ext_R^5(M_2, R) \longrightarrow 0$$

obtained by applying $Hom_R(-, R)$ to the resolution $Q_\bullet \to M_2$. Recalling $H_*^4(\mathcal{P}_i')^\vee(5) \cong P_i'^\vee$, we easily get the conclusion, as in the proof of th. 2.5 of [R].

Finally to define the morphism d and to see that the defined triple (M_1,M_2,b) is the one given by Horrocks' construction (seen to be unique by [Ho]), one may consult [B2] (for the case S = k which, however, easily generalize to a local ring S). The important part is as follows. The definition of K_1' and K_0 imply immediately $Ext^2(M_2,M_1) \cong Ext^3(M_2, K_0) \cong Ext^4(M_2,K_1')$. Next, by Gorenstein duality, we know $Ext_R^i(M_2,R) = 0$ for $i \neq 5$. Hence the definition of the syzygies L_i lead easily to $Ext^4(M_2, K_1') \cong Ext^3(L_0, K_1') \cong Ext^1(L_2, K_1')$ and to a diagram

$$\begin{array}{ccccccc} {}_0Hom_R(Q_3, K_1') & \to & {}_0Hom(L_3, K_1') & \to & {}_0Ext^1(L_2, K_1') & \to & 0 \\ & & \downarrow & & \wr\Vert & & \\ & & {}_0Hom(L_3', K_1') & & {}_0Ext_R^2(M_2,M_1) & & \end{array} \tag{6}$$

where the horizontal sequence is exact and the first vertical map is injective and split. We let d be the natural composition, and we get the conclusions of the proposition.

For any graded R-module N, we have the right derived functors $H_m^i(N)$ and ${}_vExt_m^i(N, -)$ of $\Gamma_m(N) = \bigoplus_v \ker(N_v \to \Gamma(\mathbf{P}, \tilde{N}(v)))$ and $\Gamma_m(Hom_R(N, -)))_v$ respectively (cf. [SGA 2], exp. VI or [H]) where $m = (X_0,..,X_4)$. We use small letters for the k-dimension and subscript v for the homogeneous part of degree v, e.g. ${}_vext_m^i(N_1,N_2) = dim\ {}_vExt_m^i(N_1,N_2)$.

Let N_1 and N_2 be graded R-modules of finite type. As in [K3] (cf. [W2] or [F] for a related treatise), we frequently need the spectral sequence ([SGA 2], exp. VI)

$$E_2^{p,q} = {}_vExt_R^p(N_1, H_m^q(N_2)) \Rightarrow {}_vExt_m^{p+q}(N_1, N_2) \tag{7}$$

(⇒ means "converging to") and the duality isomorphism ([K2], th. 2.1.4)

$${}_vExt_m^i(N_2, N_1) \simeq {}_{-v-5}Ext_R^{5-i}(N_1, N_2)^\vee , \tag{8}$$

valid for any integer i and v. Moreover there is a long exact sequence ([SGA2], exp. VI)

$$\to {}_vExt_m^i(N_1, N_2) \to {}_vExt_R^i(N_1, N_2) \to Ext_{O_P}^i(\tilde{N}_1, \tilde{N}_2(v)) \to {}_vExt_m^{i+1}(N_1, N_2) \to \tag{9}$$

which in particular relates the deformation theory of $(X \subseteq \mathbf{P})$, described by $H^{i-1}(N_X) \cong$

$Ext_{O_P}{}^i(\mathcal{I}, \mathcal{I})$ for i = 1,2, to the deformation theory of the homogeneous ideal I = I(X), described by ${}_0Ext_R{}^i(I, I)$, in an exact sequence

(10) $$0 \to {}_vExt^1_R(I, I) \to H^0(N_X(v)) \to {}_vExt^2_m(I, I) \to {}_vExt^2_R(I, I) \to H^1(N_X(v)) \to {}_vExt^3_m(I, I) \to$$

To compute the dimension of the components of $H(d,p,\pi)$, we introduce the following invariant, defined in terms of the graded Betti numbers of a minimal resolution (3) of I(X)

Definition/proposition 0.3 *If X is any surface in P^4 of degree d and sectional genus π, we let*

$$\delta^j(v) = \sum_i h^j(\mathcal{I}_X(n_{1i}+v)) - \sum_i h^j(\mathcal{I}_X(n_{2i}+v)) - \sum_i h^j(\mathcal{I}_X(n_{3i}+v)) + \sum_i h^j(\mathcal{I}_X(n_{4i}+v))$$

Then the following expressions are equal

$${}_0ext_R{}^1(I, I) - {}_0ext_R{}^2(I, I) + {}_0ext_R{}^3(I, I) = 1 - \delta^0(0) =$$

$$\chi(N_X) - \delta^3(0) + \delta^2(0) - \delta^1(0) = 1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5)$$

Moreover

$$\chi(N_X(v)) = dv^2 + 5dv + 5(2d + \pi - 1) - d^2 + 2\chi(O_X)$$

Indeed, the first upper equality follows easily by applying ${}_vHom_R(- , I)$ to the resolution (3) because $Hom_R(I, I) \cong R$ and because the alternating sum of the dimension of the terms in a complex equals the alternating sum of the dimension of its homology groups. The other equalities involving $\delta^j(v)$ follow from (7), (8) and (9) as outlined in [K3], lemma 1 in the curve case (the surface case is technically more complicated because the spectral sequence of the proof; $E_2^{p,q} = {}_vExt_R{}^p(I, H_m{}^q(I))$, contains one more non-vanishing term. The principal parts of the proof are, however, the same). Similarly the arguments of [K3], remark 1.13, lead to the formula

$$\chi(N_X(v)) = \chi(O_X(v)) + \chi(O_X(-v-5)) - d^2$$

for *any* surface X (i.e. locally Cohen-Macaulay and equidimensional), from which the final formula of proposition 0.3 follows easily. We omit proving the final formula because our main application is smooth surfaces where the known formula

$$\chi(N_X(v)) = dv^2 + 5dv + 5(d - \pi + 1) - 2K^2 + 14\chi(O_X)$$

and the double point formula $d^2 - 10d - 5H.K - 2K^2 + 12\chi(O_X) = 0$ imply the result of the proposition.

Finally we will use the spectral sequence (7) and the duality (8) to give an interpretation of $\delta^1(-5)$ and $\delta^2(-5)$ provided X satisfies some natural variant of having "natural cohomology". More precisely we have

Proposition 0.4 *Let X be any surface in P^4, and suppose its modules M_2, E and $H_m{}^5(R)$ are* *supported to the left of* M_1 *(i.e if* $(M_1)_v \neq 0$ *for some v, then* $(M_2)_\mu = 0$ *for* $\mu \geq v$, $E_v = 0$ *and* $v > -5$*). Then we have*

$$ {}_{-5}Ext_R^i(I, M_1)^\vee \cong {}_0Ext_R^{3-i}(M_1, M_1)\ , \quad \textit{for any } i $$

In particular ${}_0Ext_R^i(M_1, M_1) = 0$ *for* $i \geq 4$ *and*

$$ \delta^1(-5) = \sum_{i=0}^{3} (-1)^{i+1} \cdot {}_0ext_R^i(M_1, M_1) $$

Proof We have ${}_{-5}Ext_R^i(I, M_1)^\vee \cong {}_0Ext_m^{5-i}(M_1, I) \Leftarrow {}_0Ext_R^p(M_1, H_m^q((I)))$ for p+q = 5-i. Since $H_m^3(I) \cong M_2$, $H_m^4(I) \cong E$ and $H_m^5(I) \cong H_m^5(R)$, and ${}_0Ext_R^p(M_1, F) = 0$ for $p \geq 0$ and F = M_2, E and $H_m^5(R)$ by assumption, the spectral sequence above degenerates and we get

$$ {}_{-5}Ext_R^i(I, M_1)^\vee \cong {}_0Ext_R^{3-i}(M_1, H_m^2(I)) $$

Applying ${}_vHom_R(- , M_1)$ to the resolution (3), we get $\delta^1(-5) = \Sigma_i (-1)^i\ {}_{-5}ext_R^i(I, M_1)$, and we conclude easily.

Proposition 0.5 *Let X be any surface in* $\mathbf{P}^4$*, and suppose its modules E and* $H_m^5(R)$ *are supported to the left of* M_2 *(i.e* $(M_2)_v \neq 0$ *implies* $E_v = 0$ *and* $v > -5$*). Then we have* ${}_{-5}Ext_R^3(I, M_2)^\vee \cong {}_0Hom_R(M_2, M_1)$ *and there is an exact sequence*

$$ 0 \to {}_0Ext^1(M_2, M_1) \to {}_{-5}Ext^2(I, M_2)^\vee \to {}_0Hom(M_2, M_2) \xrightarrow{d_2} {}_0Ext^2(M_2, M_1) \to {}_{-5}Ext^1(I, M_2)^\vee $$

$$ \to {}_0Ext^1(M_2, M_2) \xrightarrow{e_2} {}_0Ext^3(M_2, M_1) \to {}_{-5}Hom(I, M_2)^\vee \to {}_0Ext^2(M_2, M_2) \to {}_0Ext^4(M_2, M_1) \to 0 $$

Moreover ${}_0Ext_R^3(M_2, M_2) \cong {}_0Ext_R^5(M_2, M_1)$ *and* ${}_0Ext_R^i(M_2, M_2) = 0$ *for* $i \geq 4$*. In particular*

$$ \delta^2(-5) = \sum_{i=0}^{2} (-1)^i \cdot {}_0ext_R^i(M_2, M_2) - \sum_{i=0}^{4} (-1)^i \cdot {}_0ext_R^i(M_2, M_1) $$

Proof If we replace M_1 by M_2 in the spectral sequence in the proof of proposition 0.4, we get a spectral sequence with two non-vanishing terms from which we get the long exact sequence of the proposition and the other statements as well. We conclude by combining with $\delta^2(-5) = \Sigma_i(-1)^i\ {}_{-5}ext^i(I, M_2)$.

Remark 0.6 The isomorphism ${}_{-5}Ext^3(I, M_2)^\vee \cong {}_0Hom(M_2, M_1)$ and the exactness of the first five non-vanishing terms of the long exact sequence of proposition 0.5 are valid for any surface X. In particular we have in general that

$$\sum_{i=2}^{3} (-1)^{i+1} \cdot {}_{-5}ext^i(I, M_2) = \sum_{i=0}^{2} (-1)^i \cdot {}_0ext^i(M_2, M_1) - {}_0hom(M_2, M_2) - dim\, coker\, d_2$$

Proposition 0.7 *Let X be any surface in* P^4. *Then we have* ${}_{-5}Ext_R^i(I, E) = 0$ *for* $i = 2,3$ *and*

$$\delta^3(-5) = {}_{-5}hom_R(I, E) - {}_{-5}ext_R^1(I, E)$$

Moreover if $M_1 \neq 0$ *or* $M_2 \neq 0$, *then* ${}_{-5}Ext_R^1(I, E) \cong {}_{-5}Ext_R^3(I, M_2)$.

Proof The arguments of this proof require a more sophisticated use of the spectral sequence (7) that earlier. Indeed one knows that $E_2^{p,q} = {}_{-5}Ext_R^p(I, H_m^q(I)) \Rightarrow {}_{-5}Ext_m^{p+q}(I, I)$ converges to zero if $p+q \geq 6$ by the duality (8). Recalling $E = H_m^4(I)$ and pd I ≤ 3, we get two surjective connecting homomorphisms

$$E_2^{1,5} = {}_{-5}Ext_R^1(I, H_m^5(I)) \longrightarrow E_2^{3,4} = {}_{-5}Ext_R^3(I, E)\ , \quad E_2^{0,5} \longrightarrow E_2^{2,4}$$

which leads to ${}_{-5}Ext^i(I,E) = 0$ for i = 2,3 because $H_m^5(I) \cong H_m^5(R)$ vanish in degree v > -5. Moreover combining with $\delta^3(-5) = \Sigma_i(-1)^i\ {}_{-5}ext^i(I,E)$, we get the expression of $\delta^3(-5)$.

Finally we consider the connecting homomorphism

$$E_2^{1,4} = {}_{-5}Ext_R^1(I, E) \longrightarrow E_2^{3,3} = {}_{-5}Ext_R^3(I, M_2).$$

This map is surjective (i.e. its cokernel $E_3^{3,3} = 0$) because $E_4^{3,3} = 0$ by the duality (8) and we know $E_4^{3,3}$ is the cokernel of $E_3^{0,5} \longrightarrow E_3^{3,3}$ where $E_3^{0,5} = 0$. To see the injectivity (i.e. that $E_3^{1,4} \cong E_\infty^{1,4} = 0$), we consider the spectral sequence $E_2^{p,q} \Rightarrow {}_{-5}Ext_m^{p+q}(I,I)$ above for p+q = 5 and the duality (8) which tells

$$E_\infty^{3,2} \oplus E_\infty^{2,3} \oplus E_\infty^{1,4} \cong {}_{-5}Ext_m^5(I, I) \cong {}_0Hom(I, I)^\vee \cong k$$

If $M_1 \neq 0$, one checks that $E_\infty^{3,2} \cong E_4^{3,2} \cong k$, while the case $M_1 = 0$, $M_2 \neq 0$ leads to $E_\infty^{2,3} \cong E_3^{2,3} \cong k$ (these isomorphisms will become quite clear in section 2, cf. (2.6)), and we are done.

1. THE SMOOTHNESS OF THE "MORPHISM" $\varphi : H_{\gamma,\rho} \to V_\rho$.

In this section we prove the smoothness of φ (locally). We shall see that the preparations we have made in the preceding section (e.g. proposition 0.1) allow a rather easy proof of

Theorem 1.1 *The "morphism"* $\varphi : H_{\gamma,\rho} \to V_\rho$ *= isomorphism classes of R-modules* M_1 *and* M_2 *commuting with b, is smooth (i.e. for any surface X in* P_k^4, *the corresponding local deformation functor of* φ *is smooth at* ($X \subseteq P^4$)).

Proof Let T → S → k be surjections of local Artin k-algebras with residue fields k such that ker(T→S) is a k-module via T → k. Let $X_S \subseteq \mathbf{P}_S^4$ (defining the "Horrocks' triple" (M_{1S},M_{2S},b_S) as in proposition 0.1), resp. (M_{1T},M_{2T},b_T), be given deformations of $X \subseteq \mathbf{P}^4$ to S, resp. of (M_{1S},M_{2S},b_S) to T. To prove the smoothness at $(X \subseteq \mathbf{P}^4)$, we must show the existence of a deformation $X_T \subseteq \mathbf{P}_T^4$ of $X_S \subseteq \mathbf{P}_S^4$, whose corresponding "Horrocks' triple" is precisely (M_{1T},M_{2T},b_T).

Since $X_S \subseteq \mathbf{P}_S^4$ is flat over S, we have by proposition 0.1 minimal resolutions of M_{iS}, $I(X_S)$ and A_S over $R_S = S[X_0,X_1,..,X_4]$ as in (1)-(4), flat S-modules L_{iS}, K_{iS}, L_{3S}', K_{1S}' fitting into the exact sequence (5) and a surjection d defined as the composition (cf. (6))

$$(12)\qquad \begin{array}{ccccccc} {}_0Hom_R(L_{3S}',K_{1S}') & \longrightarrow & {}_0Hom_R(L_{3S}, K_{1S}') & \longrightarrow & {}_0Ext_R^1(L_{2S}, K_{1S}') & \cong & {}_0Ext_R^2(M_{2S},M_{1S}) \\ \ni & & \ni & & \ni & & \ni \\ b_S' & & \beta_S & & b_S & & b_S \end{array}$$

"on the S-level" (β_S is simply the image of b_S' via the map of (12)) which lifts the corresponding resolutions/modules/sequences on the "k-level". Since M_{iT} are *given* deformations of M_{iS}, we can lift the minimal resolutions $\sigma_{\bullet S} : P_{\bullet S} \to M_{1S}$ and $\tau_{\bullet S} : Q_{\bullet S} \to M_{2S}$ further to T, thus proving the existence of deformations L_{iT}, K_{iT}, L_{3T}', K_{1T}' of L_{iS}, K_{iS}, L_{3S}', K_{1S}' resp. (the free submodules F_{LS} and F_{KS} of L_{3S}' and K_{1S}' are lifted trivially). So we have a diagram (6) and hence a sequence (12) "on the T-level" where the elements b_T' and β_T are not yet defined. The element $b_T \in {}_0Ext^1(L_{2T}, K_{1T}') \cong {}_0Ext_R^2(M_{2T},M_{1T})$ is, however, given and if we consider the diagram (cf. (6))

$$\begin{array}{ccccccc} {}_0Hom_R(Q_{3T},K_{1T}') & \to & {}_0Hom_R(L_{3T}, K_{1T}') & \to & {}_0Ext_R^1(L_{2T}, K_{1T}') & \to & 0 \\ \downarrow & \circ & \downarrow\alpha & \circ & \downarrow & & \\ {}_0Hom_R(Q_{3S},K_{1S}') & \to & {}_0Hom_R(L_{3S}, K_{1S}') & \to & {}_0Ext_R^1(L_{2S}, K_{1S}') & \to & 0 \end{array}$$

of exact horizontal sequences and surjective vertical maps, we easily get a morphism $\beta_T \in {}_0Hom(L_{3T}, K_{1T}')$ such that $\alpha(\beta_T) = \beta_S$. Since $L_{3S}' \cong L_{3S} \oplus F_{LS}$ we can decompose the map b_S' as $(\beta_S,\gamma_S) \in {}_0Hom(L_{3S}', K_{1S}')$, and taking any lifting $\gamma_T : F_{LT} \to K_{1T}'$ of γ_S, we get a map $b_T' = (\beta_T,\gamma_T) \in {}_0Hom(L_{3T}', K_{1T}')$ fitting into a commutative diagram.

$$\begin{array}{ccccc} L_{3T} \oplus F_{LT} \cong & L_{3T}' & \xrightarrow{bT'} & K_{1T}' \\ & \downarrow & \circ & \downarrow \\ L_{3S} \oplus F_{LS} \cong & L_{3S}' & \xrightarrow{bS'} & K_{1S}' \end{array}$$

Once having proved the existence of such a commutative diagram, we can define a surface X_T of $\mathbf{P}^4$ with the desired properties, thus proving the claimed smoothness. Indeed it is straightforward to see that coker b_T' is a (flat) deformation of coker $b_S' = I(X_S)$ to T. However, in codimension 2 one knows that an $R_T = T[X_0,X_1,..,X_4]$-module coker b_T' which lifts a graded *ideal* $I(X_S)$ is again a graded *ideal* I_T (we can deduce this information by interpreting the isomorphisms $H^{i-1}(N_X) \cong Ext_O^i(\tilde{I}\ ,\ \tilde{I}\ )$ for i = 1,2 in terms of their deformation theory from which we see that coker $\tilde{b}_T'$ is a sheaf ideal, and we conclude by taking global sections, cf. [K3] of [W1] for further details). Hence we have proved the existence of a surface $X_T = Proj(R_T/I_T)$, flat over T which via T → S reduces to X_S. By the construction above the corresponding "Horrocks' triple" is precisely the given triple (M_{1T},M_{2T},b_T), and we are done.

Remark 1.2 Theorem 1.1 implies the smoothness of the fiber $H_{\gamma,D} = \varphi^{-1}((M_1,M_2,b))$, D = (M_1,M_2,b) while [BM1] implies its irreducibility as well. Indeed [BM1], cor. 3.2 tells that the family of surfaces in $\mathbf{P}^4$ belonging to the same shift of the same liaison class, with fixed postulation, form an irreducible family, from which we see that $H_{\gamma,D}$ is irreducible.

2. THE FIBER OF THE "MORPHISM" $\varphi : H_{\gamma,\rho} \rightarrow V_\rho$.

In this section we describe the fibers $H_{\gamma,D} = H(d,p,\pi)_{\gamma,D} = \varphi^{-1}(D)$, $D = (M_1,M_2,b)$, of φ locally at $(X \subseteq \mathbf{P}^4)$ and we compute its dimension. As in the preceding section the exact sequence (5) and proposition 0.1 play an important role. Indeed, let $X_S \subseteq \mathbf{P}_S^4$ be a deformation of $X \subseteq \mathbf{P}^4$ to the dual numbers $S = k[\epsilon]$. Then we have the fundamental exact sequence (5);

$$0 \rightarrow L_{3S}' \xrightarrow{bS'} K_{1S}' \longrightarrow I(X) \rightarrow 0$$

To describe the fiber of φ, we suppose $M_{iS} \cong M_i \otimes_k S$ and $b_S = b \otimes_k id_S$ are the trivial deformations of M_i, i = 1,2, and b respectively. Since L_{3S}' and K_{1S}' are syzygies of M_{2S} and M_{1S} resp. up to some free factor, we can suppose L_{3S}' and K_{1S}' are trivial deformations of L_3' and K_1' resp. as well. Hence we expect that the fiber of φ is essentially given by ${}_0Hom_R(L_{3S}', K_{1S}')$ modulo isomorphisms and that its tangent space $T_{\gamma,D} = T(d,p,\pi)_{\gamma,D}$ at $(X \subseteq \mathbf{P}^4)$ is correspondingly described by ${}_0Hom_R(L_3', K_1')$ modulo automorphisms. More precisely we have

Proposition 2.1 *Let X be a surface in* $\mathbf{P}^4$, *and let* L_3', K_1', $b': L_3' \longrightarrow K_1'$ *and the morphism* $d: {}_0Hom_R(L_3', K_1') \rightarrow {}_0Ext_R^2(M_2, M_1)$ *be as in proposition* 0.1 (*with* $S = k$). *Then the tangent space* $T_{\gamma,D}$ *of the fiber of* φ *at* $(X \subseteq \mathbf{P}^4)$ *is given by*

$$T_{\gamma,D} \cong ker\ d \,/\, ({}_0Hom_R(L_3', L_3')^\circ + {}_0Hom_R(K_1', K_1')^\circ) \bigcap ker\ d$$

where we have denoted the natural image of a set H in ${}_0Hom_R(L_3', K_1')$ *by* H°.

Remark 2.2 If M_ρ "parametrizes" pairs (M_1, M_2) modulo isomorphisms, then it is rather clear from the proof of proposition 2.1 that

$${}_0Hom_R(L_3', K_1') \,/\, ({}_0Hom_R(L_3', L_3')^\circ + {}_0Hom_R(K_1', K_1')^\circ)$$

is the tangent space of the fiber of the composition $H_{\gamma,\rho} \xrightarrow{\varphi} V_\rho \rightarrow M_\rho$ where the natural $V_\rho \rightarrow M_\rho$ is essentially given by $(M_1,M_2,b) \rightarrow (M_1,M_2)$.

Proof Let α be any element of ${}_0Hom_R(L_3', K_1')$ which maps to zero in ${}_0Ext_R^2(M_2, M_1)$ via d, and consider the morphism $b'+\epsilon\alpha : L_3' \otimes_k S \rightarrow K_1' \otimes_k S$ where $S = k[\epsilon] \cong k \oplus k\epsilon$. By for instance the very last part of the proof of theorem 1.1 it is clear that $coker(b'+\epsilon\alpha)$ is the graded ideal $I(X_S)$ of a deformation $X_S \subseteq \mathbf{P}_S^4$ of $X \subseteq \mathbf{P}^4$ satisfying $M_{1S} \cong H_m^2(I(X_S)) \cong H_m^2(K_1' \otimes_k S) \cong M_1 \otimes_k S$ and $M_{2S} \cong M_2 \otimes_k S$. Hence there is a well-defined map $\Psi: ker\ d \rightarrow T_{\gamma,D}$. If $W = {}_0Hom_R(L_3',L_3')^\circ + {}_0Hom_R(K_1',K_1')^\circ$, it is rather straightforward to see that $\Psi(W \bigcap ker\ d) = 0$ (cf. the cohomological argument below) and that Ψ is surjective. It remains to prove that the induced map

$$\Psi' : ker\ d/(W \cap ker\ d) \longrightarrow T_{\gamma,D}$$

is injective for which we use a cohomological argument. Indeed it suffices to prove that the composition of Ψ' with $T_{\gamma,D} \hookrightarrow {}_0Ext_R^1(I(X), I(X))$ is injective (We will see later that the tangent space T_γ of H_γ at X is precisely ${}_0Ext_R^1(I(X), I(X))$, an observation which we do not really need in this proof). Hence it is sufficient to prove that the map

$${}_0Hom_R(L_3', K_1') / ({}_0Hom_R(L_3', L_3')^* + {}_0Hom_R(K_1', K_1')^*) \longrightarrow {}_0Ext_R^1(I(X), I(X))$$

(which is well-defined by (13) below) is injective. Now the exact sequence (5) "on the k-level" leads to a commutative diagram

$$\begin{array}{lccccccc} & & \to & {}_0Hom_R(K_1', K_1') & \to & {}_0Hom_R(K_1', I(X)) & \to & {}_0Ext_R^1(K_1', L_3') \\ & & & \downarrow & & \downarrow & & \downarrow \\ (13) & 0 \to {}_0Hom_R(L_3', L_3') & \to & {}_0Hom_R(L_3', K_1') & \to & {}_0Hom_R(L_3', I(X)) & \to & {}_0Ext_R^1(L_3', L_3') \\ & & & \downarrow & & \downarrow & & \\ & & & {}_0Ext_R^1(I(X), K_1') & \to & {}_0Ext_R^1(I(X), I(X)) & & \end{array}$$

of exact horizontal and vertical sequences. It suffices therefore to prove that ${}_0Ext_R^1(K_1', L_3') = 0$. By (7) and (8), however, we have $E_2^{p,q} = {}_{-5}Ext_R^p(L_3', H_m^q(K_1')) \Rightarrow {}_{-5}Ext_m^4(L_3', K_1') \cong {}_0Ext_R^1(K_1', L_3')^v$ where $p+q = 4$. Since pd $L_3' \leq 1$ we need only check the vanishing of the terms $E_2^{p,4-p}$ for $p \leq 1$, for which we recall that K_1 is a first syzygy of M_1, leading to $H_*^3(K_1) = H_*^4(K_1) = 0$ by (2) and hence to $H_m^3(K_1') = H_m^4(K_1') = 0$ and we are done.

Remark 2.3 We can suppose the morphism d: ${}_0Hom_R(L_3', K_1') \to {}_0Ext_R^2(M_2, M_1)$ of proposition 2.1 coincides with a certain connecting homomorphism of the spectral sequence $E_2^{p,q} = {}_{-5}Ext_R^p(K_1', H_m^q(L_3'))$, cf. (7) and the proof above. Indeed since $H_m^q(L_3') = 0$ for $q \leq 3$ and $H_m^4(L_3') = M_2$ by (2), the spectral sequence $E_2^{p,5-p} \Rightarrow {}_{-5}Ext_m^5(K_1', L_3') \cong {}_0Hom_R(L_3', K_1')^v$ has two non-vanishing terms, leading to the exact sequence

$$0 \to {}_{-5}Ext_R^1(K_1', M_2) \to {}_0Hom_R(L_3', K_1')^v \to {}_{-5}Hom_R(K_1', H_m^5(L_3')) \to {}_{-5}Ext_R^2(K_1', M_2) \to 0$$

(the surjection to the right follows from (8) which implies ${}_{-5}Ext_m^6(K_1', L_3') = 0$). Dualizing and using ${}_{-5}Ext_R^1(K_1', M_2)^v \cong {}_0Ext_m^4(M_2, K_1') \cong {}_0Ext_R^2(M_2, H_m^2(K_1'))$ and $M_1 \cong H_m^2(K_1')$, cf. (2), and similarly ${}_{-5}Ext_R^2(K_1', M_2)^v \cong {}_0Ext_R^1(M_2, M_1)$, we get an exact sequence

$$0 \to {}_0Ext_R^1(M_2, M_1) \to {}_{-5}Hom_R(K_1', H_m^5(L_3'))^v \to {}_0Hom_R(L_3', K_1') \xrightarrow{d} {}_0Ext_R^2(M_2, M_1) \to 0$$

where we have identified the morphism to the right with d.

Applying the spectral sequence argument of remark 2.3 also to ${}_0Hom_R(L_3', L_3')$, ${}_0Hom_R(L_3', I)$ and ${}_0Hom_R(L_3', K_1')$, we can express $T_{\gamma,D}$ in terms of the local cohomology groups of $I = I(X)$ only. Indeed we have

Theorem 2.4 *Let X be a surface in* $\mathbf{P}^4$, *and let* L_3', K_1', $b': L_3' \longrightarrow K_1'$ *and the surjection*

d: ${}_0Hom_R(L_3', K_1') \to {}_0Ext_R^2(M_2, M_1)$ be as in proposition 0.1 *(with $S = k$). Then there exists natural surjective maps*

$$\lambda_1 : {}_0Hom_R(K_1', K_1') \longrightarrow {}_0Hom_R(M_1, M_1)$$
$$\lambda_2 : {}_0Hom_R(L_3', L_3') \longrightarrow {}_0Hom_R(M_2, M_2)$$

and two homomorphisms

$$d_2 : {}_0Hom_R(M_2, M_2) \longrightarrow {}_0Ext_R^2(M_2, M_1)$$
$$d_1 : {}_0Hom_R(M_1, M_1) \longrightarrow {}_0Ext_R^2(M_2, M_1)/im\ d_2$$

such that $d_i \bullet \lambda_i = d$ composed with some obvious maps. Moreover if $M_1 \neq 0$, then the tangent space $T_{\gamma,D}$ of the fiber of φ at ($X \subseteq \mathbf{P}^4$) fits into the following two exact sequences

$$0 \to T_{\gamma,D} \to {}_0Hom_R(L_3', K_1')/({}_0Hom_R(L_3', L_3')^\circ + {}_0Hom_R(K_1', K_1')^\circ) \to coker\ d_1 \to 0$$

$$0 \to k \to ker\ d_1 \to {}_{-5}Hom_R(I, E)^\vee / {}_{-5}Ext_R^2(I, M_2)^\vee \to T_{\gamma,D} \to 0$$

where we have denoted the natural image of a set H in ${}_0Hom_R(L_3', K_1')$ by H°.

Remark 2.5. The lower fundamental exact sequence of theorem 2.4 has an interesting analogue in the Hilbert scheme $H_{\gamma,\rho} = H(d,g)_{\gamma,\rho}$ of space curves of constant cohomology which we can relate to the "morphism" $\varphi : H_{\gamma,\rho} \to E_\rho$ = isomorphism classes of R-modules M, given by (C $\subseteq \mathbf{P}^3$) $\to$ M = $H_*^1(I_C)$. Indeed if E = $H_*^1(O_C)$ and M $\neq$ 0, then the tangent space $T_{\gamma,\rho}$ of $H_{\gamma,\rho}$ at (C $\subseteq \mathbf{P}^3$) is determined by an exact sequence (cf. [K3], rem.1.22 for details)

$$0 \to k \to {}_0Hom_R(M, M) \to {}_{-4}Hom_R(I, E)^\vee \to T_{\gamma,\rho} \to {}_0Ext_R^1(M, M) \to 0$$

and the tangent space $T_{\gamma,M}$ of the fiber of φ at (C $\subseteq \mathbf{P}^3$), is just ker($T_{\gamma,\rho} \to {}_0Ext_R^1(M, M)$).

Proof We will first describe ${}_0Hom_R(L_3', K_1')/{}_0Hom_R(L_3', L_3')^\circ$ (as imφ_2 in (16) below). Indeed we observe that ${}_0Hom_R(L_3', N) \cong {}_{-5}Ext_m^5(N, L_3')^\vee$ for any R-module N of finite type. As in remark 2.3 we use the spectral sequence (7) to get an exact sequence

$$0 \to {}_{-5}Ext_R^2(N, M_2)^\vee \to {}_{-5}Hom_R(N, H_m^5(L_3'))^\vee \to {}_0Hom_R(L_3', N) \to {}_{-5}Ext_R^1(N, M_2)^\vee \to 0$$

and we have ${}_0Ext_R^i(N, M_2)^\vee \cong {}_{-5}Ext_m^{5-i}(M_2, N)$ for i = 1,2. Letting N successively be L_3', K_1 and I = I(X) in the exact sequence $0 \to L_3' \to K_1' \to I \to 0$, we get a big commutative diagram (where we have given names to some morphisms)

$$\begin{array}{ccccccccc}
0 \to & {}_0Ext_m^3(M_2, L_3') & \to & {}_{-5}Hom_R(L_3', H_m^5(L_3'))^\vee & \to & {}_0Hom_R(L_3', L_3') & \to & {}_0Ext_m^4(M_2, L_3') & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
(14)\ 0 \to & {}_0Ext_m^3(M_2, K_1') & \to & {}_{-5}Hom_R(K_1', H_m^5(L_3'))^\vee & \to & {}_0Hom_R(L_3', K_1') & \to & {}_0Ext_m^4(M_2, K_1') & \to 0 \\
& \downarrow & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow & \\
0 \to & {}_{-5}Ext_R^2(I, M_2)^\vee & \to & {}_{-5}Hom_R(I, H_m^5(L_3'))^\vee & \to & {}_0Hom_R(L_3', I) & \to & {}_0Ext_m^4(M_2, I) & \to 0
\end{array}$$

Note that ${}_0Ext_m^4(M_2, L_3') \cong {}_0Hom_R(M_2, H_m^4(L_3')) \cong {}_0Hom_R(M_2, M_2)$ and ${}_0Ext_m^4(M_2, K_1') \cong {}_0Ext_R^2(M_2, H_m^2(K_1')) \cong {}_0Ext_R^2(M_2, M_1)$ because $H_m^i(L_3')$ for $i \leq 3$ and $H_m^j(K_1')$ for $3 \leq j \leq 4$ vanish. The right upper corner of the big diagram (14) can therefore be identified as

$$\begin{array}{ccc} {}_0Hom_R(L_3', L_3') & \xrightarrow{\lambda_2} & {}_0Hom(M_2, M_2) \\ \downarrow & \circ & \downarrow d_2 \\ {}_0Hom_R(L_3', K_1') & \xrightarrow{d} & {}_0Ext_R^2(M_2, M_1) \end{array} \tag{15}$$

with d as in proposition 2.1. Denoting by λ_2 and d_2 two of the morphisms of (15), we remark that the composition $d_2 \bullet \lambda_2$ is as claimed in the theorem. Now observing that the map φ_1 of (14) is surjective, we easily deduce that $\mathrm{im}\varphi_2 \cong {}_0Hom_R(L_3', K_1')/{}_0Hom_R(L_3', L_3')^\circ$ fit into an exact sequence

$$0 \to {}_{-5}Ext_R^2(I, M_2)^\vee \to {}_{-5}Hom_R(I, H_m^5(L_3'))^\vee \to im\ \varphi_2 \to coker\ d_2 \to 0 \tag{16}$$

Next we will show that $T := {}_0Hom_R(L_3', K_1')/({}_0Hom_R(L_3', L_3')^\circ + {}_0Hom_R(K_1', K_1')^\circ)$ fit into an exact sequence (cf. (19) below) which leads to the exact sequences of the theorem. To do this, we consider the diagram (13) of the proof of (2.1), recalling that ${}_0Hom_R(K_1', K_1') \to {}_0Hom_R(K_1', I)$ is surjective. We get

$$T \cong im\varphi_2 \,/\, {}_0Hom_R(K_1', K_1')im\varphi_2 \cong im\varphi_2 \,/\, {}_0Hom_R(K_1', I)im\varphi_2$$

and that ${}_0Hom(K_1', I) \to {}_0Hom(L_3', I)$ (cf.(13)) factors via $\mathrm{im}\varphi_2 \hookrightarrow {}_0Hom(L_3', I)$. Note that ${}_0Hom(K_1', I) \cong {}_{-5}Ext_m^5(I, K_1')^\vee$ and using the spectral sequence (7) converging to ${}_{-5}Ext_m^5(I, K_1')$, we get the upper horizontal exact sequence in the following diagram

$$\begin{array}{ccccccccc} & & & & {}_0Hom_R(I, I) & & & & \\ & & & & \downarrow & & & & \\ & 0 & \longrightarrow & {}_{-5}Hom_R(I, H_m^5(K_1'))^\vee & \to {}_0Hom_R(K_1', I) & \to & {}_{-5}Ext_R^3(I, M_1)^\vee & \to & 0 \\ & & & \downarrow \quad \circ & \downarrow & \circ & \vdots\downarrow & & \\ 0 \to {}_{-5}Ext_R^2(I, M_2)^\vee & \to & & {}_{-5}Hom_R(I, H_m^5(L_3'))^\vee & \longrightarrow im\ \varphi_2 & \longrightarrow & coker\ d_2 & \to & 0 \\ & & & \downarrow & & & & & \\ & & & {}_{-5}Hom_R(I, H_m^4(I))^\vee & & & & & \end{array} \tag{17}$$

The lower horizontal sequence is found earlier and so the dotted arrow in the diagram (17) exists. Moreover the exactness of the vertical sequence to the left follows from $0 \to L_3' \to K_1' \to I \to 0$ which induces an exact sequence

$$0 \longrightarrow H_m^4(I) \longrightarrow H_m^5(L_3') \longrightarrow H_m^5(K_1') \tag{18}$$

We observe that ${}_{-5}Ext_R^3(I, M_1)^\vee \cong {}_0Ext_m^2(M_1, I) \cong {}_0Hom_R(M_1, M_1)$ and that the right side of the commutative diagram (17) can be identified with

$$\begin{array}{ccc} {}_0Hom_R(K_1', I) & \xrightarrow{\lambda} & {}_0Hom(M_1, M_1) \\ \downarrow & \circ & \downarrow d_1 \\ im\ \varphi_2 & \longrightarrow & coker\ d_2 \cong {}_0Ext_R^2(M_2, M_1)/d_2({}_0Hom(M_2, M_2)) \end{array}$$

where d_1 corresponds to the dotted arrow and where λ maps $\gamma : K_1' \to I$ onto $H_m^2(\lambda) : M_1 \cong H_m^2(K_1') \to H_m^2(I) \cong M_1$. Since ${}_0Hom_R(I, I) \cong k$, it follows that the composed map

$${}_0Hom_R(I, I) \longrightarrow {}_0Hom_R(K_1', I) \xrightarrow{\lambda} {}_0Hom(M_1, M_1)$$

is injective if and only if $M_1 \neq 0$,in which case it follows that the composition ${}_{-5}Ext_R^2(I, M_2)^\vee \to {}_{-5}Hom_R(I, H_m^5(L_3'))^\vee \to {}_{-5}Hom_R(I, H_m^4(I))^\vee$ of maps of (17) is injective. Mainly by the snake lemma, we get the exact sequence (provided $M_1 \neq 0$)

(19) $\quad 0 \to {}_0Hom_R(I, I) \to ker\ d_1 \to {}_{-5}Hom_R(I, E)^\vee / {}_{-5}Ext_R^2(I, M_2)^\vee \to T \xrightarrow{d'} coker\ d_1 \to 0$

where $E = H_m^4(I)$ and d' is induced by d.

Now we get the two exact sequences of the theorem if we can show that $T_{\gamma,D}$=ker d'. Letting λ_1 be the composition of the surjective ${}_0Hom(K_1',K_1') \to {}_0Hom(K_1',I)$ with λ, we see that λ_1 and λ_2 are both surjective. The reader may now easily prove that $T_{\gamma,D}$ is as claimed and that $d_1 \bullet \lambda_1$ "commutes with d", and we are done.

Remark 2.6 i) If $M_1 = 0$, we get by the proof above an exact sequence

$${}_0Hom_R(I, I) \to {}_{-5}Ext_R^2(I, M_2)^\vee \to {}_{-5}Hom_R(I, E)^\vee \to T_{\gamma,D} \to 0$$

where ${}_0Hom_R(I,I) \to {}_{-5}Ext_R^2(I,M_2)^\vee$ is isomorphic to $k \to {}_0Hom(M_2,M_2)$, a map which is injective if $M_2 \neq 0$. If, however, $M_2 = 0$ as well, then $T_{\gamma,D} \cong {}_{-5}Hom_R(I, E)^\vee$.

ii) Consider the spectral sequence

(20) $$E_2^{p,q} = {}_{-5}Ext_R^p(I, H_m^q(I)) \Rightarrow {}_{-5}Ext_m^{p+q}(I, I)$$

which we frequently will use in the next section. Looking to i) we recognize the morphism in "the middle" as the dual of a certain connecting homomorphism of $E_2^{p,q}$, i.e. the exact sequence of i) is simply the dual of $0 \to E_3^{0,4} \to E_2^{0,4} \to E_2^{2,3} \to E_3^{2,3} \to 0$ (provided $M_1 = 0$ and $M_2 \neq 0$). If $M_1 \neq 0$, the middle term of (19) is still $E_3^{0,4\vee}$. We *claim* that ker $d_1 = (E_3^{3,2})^\vee$. Indeed by (14) and (15), coker $d_2 \subseteq {}_0Ext_m^4(M_2, I) \cong {}_{-5}Ext^1(I, M_2)^\vee$. Then we recognize the dual of the connecting morphism $d_{2,-1} : E_2^{1,3} \to E_2^{3,2}$ as the composition ${}_{-5}Ext_R^3(I, M_1)^\vee \cong {}_0Hom(M_1, M_1) \xrightarrow{d_1}$ coker $d_2 \subseteq {}_{-5}Ext^1(I, M_2)^\vee$, whence the claim. Pushing these arguments just a little further, we see that the final exact sequence of theorem 2.4 is just $0 \to E_4^{3,2\vee} \to E_3^{3,2\vee} \to E_3^{0,4\vee} \to E_4^{0,4\vee} \to 0$ (provided $M_1 \neq 0$). In any case the discussion above shows

$$T_{\gamma,D} \cong (E_4^{0,4})^\vee$$

Corollary 2.7. *Let X be a surface in* $\mathbf{P}^4$, *and let*

$$d_1 : {}_{-5}Ext_R^3(I, M_1)^\vee \cong {}_0Hom_R(M_1, M_1) \longrightarrow {}_0Ext_R^2(M_2, M_1)/d_2({}_0Hom(M_2, M_2))$$

be the map of theorem 2.4. *Then*

$$dim\ T_{\gamma,D} = 1 + \delta^3(-5) + {}_{-5}ext_R^3(I, M_2) - {}_{-5}ext_R^2(I, M_2) - dim\ ker\ d_1 =$$

$$1 + \delta^3(-5) + \sum_{i=0}^{2} (-1)^i \cdot {}_0ext^i_R(M_2, M_1) - \sum_{j=1}^{2} {}_0hom_R(M_j, M_j) - dim\, coker\, d_1$$

Proof. If $M_1 \neq 0$ or $M_2 \neq 0$, we have by theorem 2.4 and remark 2.6i);

$$dim\, T_{\gamma,D} = {}_{-5}hom_R(I, E) - {}_{-5}ext^2_R(I, M_2) - dim\, ker\, d_1 + 1$$

Then first formula follows therefore at once from proposition 0.7 while the second follows from remark 0.6. If $M_1 = 0$ and $M_2 = 0$, we conclude by remark 2.6i) and proposition 0.7 because, by (7) and (8), ${}_{-5}ext_R{}^1(I, E) = {}_{-5}ext_m{}^5(I, I) = 1$ in this case.

Now we consider an example of a surface X of $\mathbf{P}^4$ where V_ρ is trivial at the corresponding (M_1,M_2,b). Since X also has natural cohomology as in proposition 0.4 and 0.5, it follows from results of the next section that $H_{\gamma,D}$ is actually the whole Hilbert scheme $H(d,p,\pi)$, i.e. $H_{\gamma,D} \cong H_{\gamma,\rho} \cong H(d,p,\pi)$ at $(X \subseteq \mathbf{P}^4)$.

Example 2.8 Let X be the smooth rational surface with invariants $d = 7$, $\pi = 4$ and $K^2 = -2$, cf. [DES] for existence. In this case the graded module M_1 is 1-dimensional and supported in degree 2, $M_2 = 0$, and $I = I(X)$ admits a minimal resolution

$$0 \to R(-7) \to R(-6)^{\oplus 5} \to R(-5)^{\oplus 10} \to R(-4)^{\oplus 6} \oplus R(-3) \to I \to 0$$

We easily get ${}_0hom(M_1, M_1) = 1$ and coker $d_1 = 0$ because ${}_0Ext^2(M_2, M_1) = 0$. Recalling $h^2(O_X(-1)) = \pi = 4$ and $h^2(O_X(-2)) = d + 2\pi - 1 = 14$, we get by corollary 2.7 and the definition 0.3 of $\delta^3(-5)$;

$$dim\, T_{\gamma,D} = 1 + \delta^3(-5) - 1 = h^2(O_X(-2)) + 6h^2(O_X(-1)) - 10h^2(O_X) = 38$$

Lazardsfield and Rao [LR] have shown that space curves C satisfying $e(C) < s(C)-4$ are minimal and unique in their biliaison class, a result which rather easily generalizes to surfaces (cf. [PRa], lemma 4.1). For curves with $e(C) < s(C) - 4$ we see that the group ${}_{-4}Hom_R(I, E)$ of remark 2.5 is zero. By the same remark we get at once $T_{\gamma,M} = 0$ (and ${}_0Hom_R(M, M) \cong k$ provided $M = H_*^1(I_C) \neq 0$) which is the infinitesimal variant of the uniqueness of Lazardsfield and Rao. Theorem 2.4 shows that the same infinitesimal variant is true for surfaces in $\mathbf{P}^4$. Including also the corresponding information on the deficiency modules M_i, we get

Corollary 2.9. *Let X be a surface in $\mathbf{P}^4$, let $D = (M_1,M_2,b)$ be its "Horroks' triple" and suppose $e(X) < s(X) - 5$. Then ${}_0Ext_R{}^i(M_2, M_1) = 0$ for $i = 0,1$ and*

$$T_{\gamma,D} = 0.$$

Moreover if $M_1 \neq 0$, then d_2 is injective and $ker\, d_1 \cong k$. If $M_1 = 0$ and $M_2 \neq 0$, then $ker\, d_2 = {}_0Hom(M_2, M_2) \cong k$.

Proof. Indeed e < s-5 leads to ${}_{-5}Ext_R^i(I, E) = 0$ for i ≥ 0 by using the minimal resolution (3). The last exact sequence of theorem 2.4 implies therefore $T_{\gamma,D} = 0$. Moreover if $M_1 \neq 0$, we have ${}_{-5}Ext_R^2(I, M_2) = 0$ and ker $d_1 \cong k$ by theorem 2.4 and ${}_{-5}Ext_R^3(I, M_2) = 0$ by proposition 0.7. We conclude easily by remark 0.6. If $M_1 = 0$ and $M_2 \neq 0$, we combine remark 0.6 and remark 2.6i) to conclude, while the case $M_1 = 0$ and $M_2 = 0$ is trivial.

3. THE TANGENT AND OBSTRUCTION SPACES OF $H_{\gamma,\rho}$ AND V_ρ .

In this section we determine the tangent spaces of $H_{\gamma,\rho}$ and V_ρ at ($X \subseteq \mathbf{P}^4$) and (M_1,M_2,b) respectively and we compute its dimensions. We will give a criterion for V_ρ to be smooth at (M_1,M_2,b). Since $\varphi : H_{\gamma,\rho} \to V_\rho$ is smooth by theorem 1.1, this leads to a criterion for $H_{\gamma,\rho}$ to be smooth at ($X \subseteq \mathbf{P}^4$). At some places we only sketch the proofs. Indeed the proofs require varied, at most places standard but technical, use of the spectral sequence (7), as in the preceding section. In general the spectral sequence does not degenerate, the part of the sequence we consider can consist of *three* or even more non-vanishing terms (which is a good reason for skipping bothering technical details), but still it formalizes the information we need to prove our theorems. Some consequences (related to $H(d,p,\pi)$) of our results may also be deduced from [K5], section 1. We end this section by considering an example.

Let X be a surface in $\mathbf{P}^4$ with graded ideal I = I(X) and let D = (M_1,M_2,b), $M_i = H_*^i(\tilde{I})$, be its "Horroks' triple". Recall ([K2], section 2.2) that ${}_0Ext_R^1(I,I)$ is the tangent space of H_γ at ($X \subseteq \mathbf{P}^4$) because a deformation in H_γ keeps the postulation constant, i.e. it corresponds precisely to a graded deformation of I. Moreover there exists maps

$$\varphi_i : {}_0Ext_R^1(I, I) \longrightarrow {}_0Hom_R(H_*^i(\tilde{I}), H_*^{i+1}(\tilde{I}))$$

taking an extension 0 → I → E → I → 0 of ${}_0Ext_R^1(I, I)$ onto the connecting homomorphism $\delta = \delta^i$ in the exact sequence

$$H_*^i(\tilde{E}) \to H_*^i(\tilde{I}) \xrightarrow{\delta} H_*^{i+1}(\tilde{I}) \to H_*^{i+1}(\tilde{E})$$

cf. [MDP1]. For graded homogeneous ideals $I = H_*^0(\tilde{I})$ we see that the composition E → $H_*^0(\tilde{E}) \to H_*^0(\tilde{I})$ is surjective, i.e. we get $\varphi_0 = 0$. Moreover note that if δ^{i-1} and δ^i are both zero for some i, then the exact sequence 0 → I → E → I → 0 above defines an extension

$$0 \longrightarrow H_*^i(\tilde{I}) \longrightarrow H_*^i(\tilde{E}) \longrightarrow H_*^i(\tilde{I}) \longrightarrow 0$$

Hence there is a well-defined morphism Ψ_i :

$$\ker\left({}_0Ext_R^1(I,I) \xrightarrow{(\varphi_{i-1},\varphi_i)} {}_0Hom(H_*^{i-1}(\tilde{I}),H_*^i(\tilde{I}))\times{}_0Hom(H_*^i(\tilde{I}),H_*^{i+1}(\tilde{I}))\right) \to {}_0Ext_R^1(H_*^i(\tilde{I}),H_*^i(\tilde{I}))$$

Definition/proposition 3.1 *The tangent space of* H_γ *(resp.* $H_{\gamma,\rho 1}$*, resp.* $H_{\gamma,\rho}$*) at* ($X \subseteq P^4$), *resp. of* V_ρ *at* $D = (M_2,M_1,b)$, *is*

$${}_0Ext_R^1(I, I), \quad (\ resp.\ {}_0Ext_R^1(I, I)_{\rho 1} := ker\ \varphi_1,\quad resp.\ {}_0Ext_R^1(I, I)_\rho := ker\ (\varphi_1,\varphi_2)\ ),$$

resp. $T_{V_\rho,D} = coker(T_{\gamma,D} \to {}_0Ext_R^1(I, I)_\rho)$.

where $\rho = (\rho_1,\rho_2)$. *Hence there exists natural maps*

$$\Psi_1 : {}_0Ext_R^1(I, I)_{\rho 1} \to {}_0Ext_R^1(M_1, M_1)$$
$$\Psi_2 : {}_0Ext_R^1(I, I)_\rho \to {}_0Ext_R^1(M_2, M_2) \ ,$$

as defined above. Abusing the language, we let Ψ_1 *(or* Ψ_{1_ρ}*) denote the restriction of* Ψ_1 *to* ${}_0Ext_R^1(I, I)_\rho$ *as well.*

Proof. The first three tangent spaces are proved by the base change theorem as in [MDP1] while the last is due to theorem 1.1 of this paper (cf. theorem 2.4) because a smooth morphism is surjective on its tangent spaces. We shall make $T_{V_\rho,D}$ more explicit later.

Remark 3.2 In the preceding section we frequently used the spectral sequence ${}_vE_2^{p,q}(M) = {}_vExt_R^p(M,H_m^q(I))$ of (7) and the duality in (8), especially

$$E_2^{p,q} = {}_{-5}E_2^{p,q}(I) = {}_{-5}Ext_R^p(I, H_m^q(I)) \Rightarrow {}_{-5}Ext_m^{p+q}(I, I) \cong {}_0Ext_R^{5-p-q}(I, I)^v$$

If $H_m^q(I)$ is of finite type (i.e. $q \le 3$), we can use (8) *once more* to get ${}_{-5}Ext_R^p(I, H_m^q(I))^v \cong {}_0Ext_m^{5-p}(H_m^q(I), I) \Leftarrow {}_0Ext_R^i(H_m^q(I), H_m^{5-p-i}(I))$. This implies the set-up

$$\begin{array}{ccc}
{}_0Ext_R^1(I, I) & \overset{a}{\cdots\cdots\to} & {}_0Ext_R^p(H_m^q(I), H_m^{q+1-p}(I)) \\
\| & & \Downarrow \\
{}_{-5}Ext_m^4(I, I)^v & & \\
\Downarrow & & \\
(E_2^{4-q,q})^v = {}_{-5}Ext_R^{4-q}(I, H_m^q(I))^v & \cong & {}_0Ext_m^{q+1}(H_m^q(I), I)
\end{array}$$

As we partially have seen by considering resolutions in the preceding section, the dotted arrow $a = a^{p,q}$ is well-defined on appropriate quotients of subspaces of ${}_0Ext_R^1(I, I)$ given by its spectral sequence, in which case the restriction/factorization of $a^{p,q}$ is nothing but the obvious map. In particular we have

$$a^{0,2} = \varphi_1 \qquad a^{0,3} = \varphi_2 \quad \text{(i.e. commutes with } \varphi_2)$$
(21) $$a^{1,2} \text{ (restricted to } {}_0Ext_R^1(I,I)_{\rho 1}) = \Psi_1 \quad \text{(cf. proof of 3.3) etc.}$$

In the preceding section we studied the fiber of φ: $H_{\gamma,\rho} \to V_\rho$ at ($X \subseteq \mathbf{P}^4$) and in (2.6) we characterized it as $(E_2^{0,4})^v$, with $E_2^{p,q}$ as in (3.2). We could, however, proved this characterization and theorem 2.4 using the spectral sequence and the local duality twice (properly interpreted, cf. (3.2)). The direct approach we chose to prove theorem 2.4 (via lemma 2.1 and the exact sequence $0 \to L_3' \to K_1' \to I \to 0$) led, however, to some extra information, namely to the factorization of ${}_{-5}Ext_R^3(I, M_1)^v \to {}_{-5}Ext_R^1(I, M_2)^v$ via d_1. Similarly, interpreting (3.2) in terms of deformation theory, we can get a quicker proof of the main parts of theorem 3.5 below, but the approach via $0 \to L_3' \to K_1' \to I \to 0$ still give some additional information, related to the factorization of some main maps. We will need

Proposition 3.3 *Let X be a surface in* $\mathbf{P}^4$.

i) Then there exists morphisms e_i *(induced from the spectral sequence* (3.2)) *fitting into a commutative diagram*

$$\begin{array}{ccc} {}_0Ext_R^1(I, I)_\rho & \xrightarrow{\Psi_2} & {}_0Ext_R^1(M_2, M_2) \\ \downarrow \Psi_1 & \circ & \downarrow e_2 \\ {}_0Ext_R^1(M_1, M_1) & \xrightarrow{e_1} & {}_0Ext_R^3(M_2, M_1) \end{array}$$

such that the induced map ${}_0Ext_R^1(I, I)_\rho \longrightarrow {}_0Ext_R^1(M_1, M_1) \times_{Ex} {}_0Ext_R^1(M_2, M_2)$ *where* $Ex = {}_0Ext_R^3(M_2, M_1)$, *is surjective.*

ii) The map e_1 *fit into a commutative diagram*

$$\begin{array}{ccc} {}_0Ext_R^1(M_1, M_1) & \xrightarrow{e_1} & {}_0Ext_R^3(M_2, M_1) \\ \downarrow & \circ & \downarrow \\ {}_{-5}Ext_R^2(I, M_1)^\vee & \longrightarrow & {}_{-5}Hom_R(I, M_2)^\vee \end{array}$$

of natural maps of the spectral sequence of (3.2) *where the left (resp. right) vertical map is an edge homomorphism of the spectral sequence* ${}_0E_2^{p,2-p}(M_1)$ *(resp.* ${}_0E_2^{p,5-p}(M_2)$ *to which also* e_2 *belongs) while the lower horizontal map is the dual of a connecting homomorphism of* ${}_{-5}E_2^{p,q}(I)$. *In particular if* $\bar{e}_1$ *(resp.* e_1'*) is the composition of* e_1 *with the natural* ${}_0Ext_R^3(M_2, M_1) \to$ *coker* e_2 *(resp.* ${}_0Ext_R^3(M_2, M_1) \to {}_{-5}Hom_R(I, M_2)^\vee$*), then*

$$\Psi_1({}_0Ext_R^1(I, I)_{\rho 1}) = \ker e_1' \quad \text{and} \quad \Psi_1({}_0Ext_R^1(I, I)_\rho) = \ker \bar{e}_1$$

Moreover if the natural map ${}_0Hom_R(M_2, E) \to {}_0Ext_R^2(M_2, M_2)$ of ${}_0E_2^{p,q}(M_2)$ is injective, then

$$\text{coker } e_2 \subseteq {}_{-5}Hom_R(I, M_2)^\vee$$

iii) *Finally let* $T \to S \to k$ *be surjections of local Artin k-algebras with residue fields k such that* $ker(T \to S) \cong k$ *(via* $T \to k$*), let* (M_{1S}, M_{2S}, b_S) *be a deformation of* (M_1, M_2, b) *to S and suppose we can deform* M_{iS} *further to T. Then* ${}_0Ext_R^3(M_2, M_1)$, *(resp. coker* $\bar{e}_1$*) contains the obstruction of the existence of an element* b_T *(resp. a deformation* (M_{1T}, M_{2T}, b_T)*),* $b_T \in {}_0Ext_R^2(M_{2T}, M_{1T})$, *which maps to* $b_S \in {}_0Ext_R^2(M_{2S}, M_{1S})$ *(resp. to* (M_{1S}, M_{2S}, b_S)*) via* $(-)\otimes_T S$ *where* M_{iT} *are given (resp. some) deformations of* M_{iS}.

Proof (sketch) i) To prove the existence of the maps e_i and the commutative diagram of i), we enlarge the diagram (13) of the proof of proposition 2.1, say to a diagram (13*), by including more Ext-groups. Letting morphisms in what follows be the obvious (compositions of) maps from this enlarged diagram (13*), we see that

(22) $\qquad {}_0Ext_R^1(I, I)_{\rho 1} = \ker ({}_0Ext_R^1(I, I) \longrightarrow {}_0Ext_R^2(K_1', L_3'))$

mainly follows from

(23) ${}_0Ext_R^2(K_1', L_3') \cong {}_{-5}Ext_m^3(L_3', K_1')^\vee \cong {}_{-5}Ext_R^1(L_3', M_1)^\vee \cong {}_0Ext_m^4(M_1, L_3') \cong {}_0Hom_R(M_1, M_2)$

and the definition of ${}_0Ext_R^1(I,I)_{\rho 1}$. Moreover since we as in (23) can see that ${}_0Ext_R^1(K_1', K_1')$

$\to {}_0Ext^1(K_1',I)$ and ${}_0Ext_R^1(M_1, M_1) \to {}_{-5}Ext_R^2(I, M_1)^\vee$ are isomorphic, it follows easily that the composition ${}_0Ext_R^1(I,I)_{\rho 1} \hookrightarrow {}_0Ext_R^1(I,I) \to {}_0Ext^1(K_1',I)$ admits a factorization

$$\beta : {}_0Ext_R^1(I, I)_{\rho 1} \longrightarrow {}_0Ext_R^1(K_1', K_1')$$

which is Ψ_1 if we identify ${}_0Ext_R^1(K_1',K_1')$ with ${}_0Ext_R^1(M_1, M_1)$. Correspondingly (13*), (22) and ${}_0Ext_R^1(K_1',L_3') = 0$ (cf. the proof of (2.1)) imply that the composition ${}_0Ext_R^1(I,I)_{\rho 1} \hookrightarrow {}_0Ext_R^1(I,I) \to {}_0Ext^2(I,L_3')$ admits a factorization α fitting into a commutative diagram

$$(24) \qquad \begin{array}{ccc} {}_0Ext_R^1(I, I)_{\rho 1} & \xrightarrow{\alpha} & {}_0Ext_R^1(L_3', L_3') \\ \downarrow \beta & \circ & \downarrow \\ {}_0Ext_R^1(K_1', K_1') & \xrightarrow{f} & {}_0Ext_R^1(L_3', K_1') \end{array}$$

We shall deduce the morphisms e_i and the commutative diagram of i) as a "factorization" of (24). Indeed the duality (8) used twice and their spectral sequences, cf. (23), lead to

$$\begin{array}{ccc} {}_0Ext_R^1(L_3', L_3') & \longrightarrow & {}_0Ext_R^1(L_3', K_1') \\ \wr\| & \circ & \wr\| \\ {}_0Ext_m^5(M_2, L_3') & \longrightarrow & {}_0Ext_m^5(M_2, K_1') \end{array}$$

Using the spectral sequence converging to ${}_0Ext_m^5(M_2,-)$, we get a commutative diagram

$$\begin{array}{ccccccc} & & & & {}_0Ext_R^1(M_2, M_2) & \cdots\cdots\to & {}_0Ext_R^3(M_2, M_1) \\ & & & & \downarrow & & \downarrow \\ & & {}_0Ext_R^1(I, I)_{\rho 1} & \xrightarrow{\alpha} & {}_0Ext_m^5(M_2, L_3') & \longrightarrow & {}_0Ext_m^5(M_2, K_1') \\ & & \downarrow \varphi_2 & \circ & \downarrow & \circ & \downarrow \\ 0 & \to & {}_0Hom_R(M_2, E) & \to & {}_0Hom_R(M_2, H_m^5(L_3')) & \to & {}_0Hom_R(M_2, H_m^5(K_1')) \end{array}$$

where the dotted arrow is e_2 and where the lower horizontal sequence is exact by (18). Hence there exists a map e: ${}_0Ext_R^1(I, I)_{\rho 1} \to {}_0Ext_R^3(M_2, M_1)$ whose restriction to ${}_0Ext_R^1(I, I)_\rho = \ker \varphi_2$ factorizes via e_2. To get the commutative diagram of i), it remains to define e_1 such that $e_1 \cdot \Psi_1 = e$. Considering (24) and the last diagram above, it suffices to prove that the composition of f with ${}_0Ext_m^5(M_2,K_1') \to {}_0Hom_R(M_2,H_m^5(K_1'))$ vanishes. To see this we look to the proof of proposition 0.1 and the exact sequence $0 \to K_1' \to P_1' \to K_0 \to 0$ occurring there. We get that the composition of f with ${}_0Ext_R^1(L_3',K_1') \to {}_0Ext_R^1(L_3',P_1')$ vanishes because the composition naturally factorizes via ${}_0Ext_R^1(K_1',P_1')$ which is zero. We conclude mainly by ${}_0Ext_R^1(L_3',P_1') \cong {}_0Ext_m^5(M_2,P_1') \cong {}_0Hom_R(M_2,H_m^5(P_1'))$ and $H_m^5(K_1') \subseteq H_m^5(P_1')$. The claimed surjectivity of i) will be proved in iii) below.

ii) A part of the enlarged diagram (13*) is

$$\begin{array}{ccc} {}_0Ext_R^1(K_1', K_1') & \longrightarrow & {}_0Ext_R^1(K_1', I) \\ \downarrow f & \circ & \downarrow \\ {}_0Ext_R^1(L_3', K_1') & \longrightarrow & {}_0Ext_R^1(L_3', I) \end{array}$$

where the vertical map to the right identifies with ${}_{-5}Ext_m^4(I,K_1')^\vee \cong {}_{-5}Ext_R^2(I,M_1)^\vee \to {}_0Ext_m^4(I,L_3') \cong {}_{-5}Hom_R(I,M_2)^\vee$, i.e. the dual of the connecting morphism of the spectral sequence ${}_{-5}E_2^{p,q}(I)$ of (3.2), and the commutative diagram of ii) is established.

To prove the claim $im\ \Psi_1 = ker\ e_1'$, we consider the spectral sequence ${}_0E_2^{p,3-p}(M_1) = {}_0Ext_R^p(M_1,H_m^{3-p}(I)) \Rightarrow {}_0Ext_m^3(M_1,I) \cong {}_{-5}Ext_R^2(I,M_1)^v$ and ${}_{-5}E_2^{p,4-p}(I) \Rightarrow {}_0Ext_R^1(I,I)^v$, and we get a diagram of exact sequences

$$
(25) \qquad
\begin{array}{ccccc}
 & & {}_0Ext_R^1(M_1, M_1) & & \\
 & & \uparrow & \searrow^{e_1'} & \\
{}_0Ext_R^1(I, I) & \longrightarrow & {}_{-5}Ext_R^2(I, M_1)^v & \longrightarrow & {}_{-5}Hom_R(I, M_2)^v \\
 & \searrow_{\varphi_1} & \downarrow & & \\
 & & {}_0Hom_R(M_1, M_2) & &
\end{array}
$$

where we recognize the dotted compositions as e_1' and φ_1. Since ${}_0Ext_R^1(I,I)_{\rho 1} = \ker \varphi_1$, we get the claim. Moreover note that the surjectivity of i) leads easily to $\Psi_1({}_0Ext^1(I,I)_\rho) = \ker \tilde{e}_1$. Finally we get coker $e_2 \subseteq {}_{-5}Hom_R(I, M_2)^v$ from the spectral sequence $E_2^{p,q}(M_2) = {}_0Ext_R^p(M_2,H_m^q(I))$. Indeed the map e_2 appears as the connecting homomorphism e_2: $E_2^{1,3}(M_2) \to E_2^{3,2}(M_2)$, i.e. coker $e_2 = E_3^{3,2}(M_2)$, and the injectivity assumption of ii) is precisely the injectivity of $E_2^{0,4}(M_2) \to E_2^{2,3}(M_2)$, i.e. we have $E_3^{0,4}(M_2) = 0$. Hence coker $e_2 = E_\infty^{3,2}(M_2) \subseteq {}_0Ext_m^5(M_2,I) \cong {}_{-5}Hom_R(I,M_2)^v$, as required.

iii) Now we sketch the proof of the obstruction statements. Indeed the commutative diagram of the proposition is a "factorization" of (24) and the corresponding obstruction statements for deforming b': $L_3' \to K_1'$ follows essentially from Laudal's book [L], cf. section 2.3 and theorem 2.3.3 (which treats the more difficult case of morphisms of algebras). In our case ${}_0Ext_R^1(L_3',K_1')$ contains the obstruction of deforming b_S': $L_{3S}' \to K_{1S}'$ further to T (cf. our section 1 for notations). The obstruction must sit in the subgroup Ex $= {}_0Ext_R^3(M_2,M_1)$ because it is the image of b_S via ${}_0Ext_R^2(M_{2S},M_{1S}) \cong {}_0Ext_R^2(M_{2T},M_{1S}) \to {}_0Ext_R^3(M_2,M_1)$, where the last morphism is the connecting homomorphism of the long exact sequence obtained by applying ${}_0Ext_R^2(M_{2T},-)$ to the sequence $0 \to M_1 \to M_{1T} \to M_{1S} \to 0$ associated to the given deformation M_{1T} of M_{1S}. We get in this way the obstruction statements of iii) as a consequence of the corresponding, more well-known, statements involving L_{3S}', K_{1S}' and b_S'.

The details of this obstruction calculus also show the surjectivity of ${}_0Ext_R^1(I, I)_\rho \to {}_0Ext_R^1(M_1, M_1) \times_{Ex} {}_0Ext_R^1(M_2, M_2)$. Indeed if (r_1,r_2) is an element of this product and r_i corresponds to a deformation M_{iS}' of M_i to the dual numbers $S = k[\epsilon]$, then the image of r_2 (resp. r_1) in Ex is the difference of the obstruction $o(b;M_{1S}',M_{2S}')$ - $o(b;M_{1S}',M_{2S})$, resp. $o(b;M_{1S},M_{2S})$ - $o(b;M_{1S}',M_{2S})$ where M_{iS} is the trivial deformation of M_i and $o(b;M_{1S}',M_{2S}')$ denotes the obstruction of the existence of an element $b_S \in {}_0Ext_R^2(M_{2S}',M_{1S}')$ mapping to b $\in {}_0Ext_R^2(M_2, M_1)$. Since r_1 and r_2 map to the same element in Ex and $o(b;M_{1S},M_{2S}) = 0$, we get $o(b;M_{1S}',M_{2S}') = 0$. Hence there exists a deformation of b': $L_3' \to K_1'$ whose cokernel is a deformation of ideal I, i.e. the deformation defines an element of ${}_0Ext_R^1(I, I)_\rho$ mapping to (r_1,r_2), and the surjectivity is proved. This completes the proof (modulo some details).

One may from the proof of proposition 3.3iii) (where the obstruction calculus was considered) see that the surjectivity onto the fiber product of (3.3i) is closely related to the surjectivity of the tangent map of $\varphi : H_{\gamma,\rho} \to V_\rho$ and hence to the smoothness of φ. We expect indeed this obstruction calculus idea to lead to an independent proof of the smoothness of $\varphi : H_{\gamma,\rho} \to V_\rho$. We will, however, concentrate on other aspects of proposition 3.3, such as the following criterion for V_ρ to be smooth at (M_1,M_2,b). The result is an immediate consequence of (3.3iii) and we state it as

Proposition 3.4 *Let X be a surface in* $\mathbf{P}^4$. *If the local deformation functors* $Def(M_i)$ *are formally smooth at* M_i *(for instance if* ${}_0Ext_R^2(M_i, M_i) = 0$*) for* $i = 1,2$, *and if the morphism*

$$\bar{e}_1 : {}_0Ext_R^1(M_1, M_1) \longrightarrow {}_0Ext_R^3(M_2, M_1)/e_2({}_0Ext_R^1(M_2, M_2))$$

of proposition 3.3 *is surjective, then* V_ρ *is smooth at* $D = (M_1, M_2, b)$ *(i.e. the local deformation functor* $Def(D)$ *is formally smooth at* D*).*

We now come to the main theorem of this section

Theorem 3.5 *Let X be a surface in* $\mathbf{P}^4$, *let*

$$d_1 : {}_0Hom_R(M_1, M_1) \longrightarrow coker\ d_2 = {}_0Ext_R^2(M_2, M_1)/d_2({}_0Hom_R(M_2, M_2))$$

be the map of theorem 2.4, *and let* $T_{\gamma,D}$ *be the tangent space of the fiber of* $\varphi : H_{\gamma,\rho} \to V_\rho$ *at* $(X \subseteq \mathbf{P}^4)$ *determined in that theorem. Then the tangent space* $T_{V\rho,D} \cong {}_0Ext_R^1(I,I)_\rho/T_{\gamma,D}$ *of* V_ρ *at* $D = (M_2, M_1, b)$ *is given by the following exact sequence*

$$0 \longrightarrow coker\ d_1 \longrightarrow T_{V\rho,D} \longrightarrow {}_0Ext_R^1(M_1, M_1) \times_{Ex} {}_0Ext_R^1(M_2, M_2) \longrightarrow 0$$

where $Ex = {}_0Ext_R^3(M_2, M_1)$ *and where* ${}_0Ext_R^1(I, I)_\rho$ *is the tangent space of* $H_{\gamma,\rho}$ *at* $(X \subseteq \mathbf{P}^4)$.

We will prove the theorem above as an easy consequence of

Proposition 3.6 *Let X be a surface in* $\mathbf{P}^4$, *let*

$$\begin{aligned} d_1 &: {}_0Hom_R(M_1, M_1) \longrightarrow coker\ d_2 = {}_0Ext_R^2(M_2, M_1)/d_2({}_0Hom_R(M_2, M_2)) \\ e_2 &: {}_0Ext_R^1(M_2, M_2) \longrightarrow {}_0Ext_R^3(M_2, M_1) \\ \bar{e}_1 &: {}_0Ext_R^1(M_1, M_1) \longrightarrow coker\ e_2 \end{aligned}$$

be the maps of theorem 2.4 *and proposition* 3.3, *and let* $E_2^{p,q} = {}_{-5}Ext_R^p(I, H_m^q(I))$ *be the spectral sequence converging to* ${}_{-5}Ext_m^{p+q}(I, I) \cong {}_0Ext_R^{5-p-q}(I, I)^\vee$. *Then the tangent space* ${}_0Ext_R^1(I, I)_\rho$ *(resp.* $T_{V\rho,D}$*) of* $H_{\gamma,\rho}$ *at* $(X \subseteq \mathbf{P}^4)$ *(resp. of* V_ρ *at* $D = (M_2, M_1, b)$*) is determined by the three exact sequences below;*

$$0 \longrightarrow T_{\gamma,D} \cong (E_2^{0,4})^\vee \longrightarrow {}_0Ext_R^1(I, I)_\rho \longrightarrow T_{V\rho,D} \longrightarrow 0$$

$$0 \longrightarrow coker\ d_1 \longrightarrow ker[(E_3^{1,3})^\vee \to {}_0Hom_R(M_2, E)] \longrightarrow ker\ e_2 \longrightarrow 0$$

$$0 \longrightarrow ker[(E_3^{1,3})^\vee \to {}_0Hom_R(M_2, E)] \longrightarrow T_{V\rho,D} \longrightarrow ker\ \bar{e}_1 \longrightarrow 0$$

and $E_3^{1,3} \subseteq E_2^{1,3} = {}_{-5}Ext_R^1(I, M_2)$. *In particular* $\dim {}_0Ext_R^1(I, I)_\rho =$

$$1 + \delta^3(-5) + \sum_{i=0}^{3} (-1)^i \cdot {}_0ext_R^i(M_2, M_1) - \sum_{i=1}^{1} (-1)^i \cdot {}_0ext_R^i(M_1, M_1) - \sum_{i=1}^{1} (-1)^i \cdot {}_0ext_R^i(M_2, M_2) + \varepsilon$$

where ϵ *is defined by* $\epsilon = \dim coker\ \bar{e}_1$.

Proof The first exact sequence follows immediately from proposition 3.1 and remark 2.6ii).

To prove the exactness of the third sequence, we use the spectral sequence $E_2^{p,q} = {}_{-5}Ext_R^p(I, H_m^q(I))$ and the arguments of (25) to get the long exact horizontal sequence in

$$\begin{array}{ccccccc} {}_{-5}Ext_R^3(I, M_1)^{\vee} & \xrightarrow{d^{\vee}_{2,-1}} & {}_{-5}Ext_R^1(I, M_2)^{\vee} & \longrightarrow & {}_0Ext_R^1(I, I)_{\rho 1}/(E_4^{0,4})^{\vee} & \rightarrow & {}_0Ext_R^1(M_1, M_1) \xrightarrow{e_1'} \\ & & \downarrow & \circ & \downarrow \varphi_2 & & \\ & & {}_0Hom_R(M_2, E) & \longrightarrow & {}_0Hom_R(M_2, E) & & \end{array}$$

Note that coker $d_{2,-1}^{\vee} = (E_3^{1,3})^{\vee} = (E_\infty^{1,3})^{\vee}$ and that $d_{2,-1}^{\vee}$ (cf. remark 2.6ii) is the composition of $d_1 : {}_{-5}Ext_R^3(I, M_1)^{\vee} \cong {}_0Hom_R(M_1, M_1) \rightarrow$ coker d_2 with the edge morphism coker $d_2 = E_3^{2,2}(M_2) \hookrightarrow {}_{-5}Ext_R^1(I, M_2)^{\vee}$ of the spectral sequence $E_2^{p,q}(M_2) = {}_0Ext_R^p(M_2, H_m^q(I))$. Since the composition coker $d_2 = E_3^{2,2}(M_2) \rightarrow {}_{-5}Ext_R^1(I, M_2)^{\vee} \rightarrow E_2^{0,4}(M_2) = {}_0Hom_R(M_2, E)$ is zero, we get a well-defined morphism coker $d_{2,-1}^{\vee} = (E_3^{1,3})^{\vee} \rightarrow {}_0Hom_R(M_2, E)$ which commutes with φ_2 in the diagram above. Combining with $\Psi_1({}_0Ext_R^1(I,I)_\rho)$ = ker $\bar{e}_1$, we deduce the 3. exact sequence.

To prove the exactness of the 2. sequence, we consider once more the spectral sequence $E_2^{p,4-p}(M_2)$ above. There are three non-vanishing terms, leading to the exact sequence

$$(26) \quad 0 \longrightarrow E_\infty^{2,2}(M_2) \longrightarrow ker\ [{}_{-5}Ext_R^1(I, M_2)^{\vee} \rightarrow E_\infty^{0,4}(M_2)] \longrightarrow E_\infty^{1,3}(M_2) \longrightarrow 0$$

Observe that coker $d_2 = E_3^{2,2}(M_2) = E_\infty^{2,2}(M_2)$ and ker $e_2 = E_3^{1,3}(M_2) = E_\infty^{1,3}(M_2)$ and since $E_\infty^{0,4}(M_2) \subseteq E_2^{0,4}(M_2) = {}_0Hom(M_2, E)$, we see that (26) is isomorphic to

$$(27) \quad 0 \longrightarrow coker\ d_2 \longrightarrow ker\ [{}_{-5}Ext_R^1(I, M_2)^{\vee} \rightarrow {}_0Hom(M_2, E)] \longrightarrow ker\ e_2 \longrightarrow 0$$

This leads indeed to the 2. exact sequence if we recall the factorization of $d_{2,-1}^{\vee}$ via d_1 which implies that we in (27) can replace coker d_2 by coker d_1 and ~~by~~ ${}_{-5}Ext_R^1(I, M_2)^{\vee}$ by coker $d_{2,-1}^{\vee}$ and still preserve exactness.

To see the dimension of ${}_0Ext_R^1(I, I)_\rho$, we have by the 2. and the 3. exact sequence that

$$dim\ T_{V_\rho,D} = dim\ coker\ d_1 + dim\ ker\ e_2 + dim\ ker\ \bar{e}_1 = dim\ coker\ d_1 + {}_0ext^1(M_2,M_2) -$$

$${}_0ext^3(M_2,M_1) + dim\ coker\ e_2 + {}_0ext^1(M_1,M_1) - dim\ coker\ e_2 + dim\ coker\ \bar{e}_1$$

and we conclude by the 1. exact sequence and corollary 2.7.

Proof of theorem 3.5. The theorem follows from the exact sequences of proposition 3.6 and the exact sequence

$$0 \longrightarrow ker\ e_2 \longrightarrow {}_0Ext_R^1(M_1, M_1) \times_{Ex} {}_0Ext_R^1(M_2, M_2) \longrightarrow ker\ \bar{e}_1 \longrightarrow 0$$

see also proposition 3.3i), and we get easily the theorem.

If we in addition to the tangent spaces locate their obstruction spaces, we can prove the following result for which we also have a direct proof in the spirit of proposition 3.3iii) available. We have, however, chosen to sketch the proof by determining (or rather indicating) their obstruction spaces.

Theorem 3.7 *Let X be a surface in $\mathbf{P}^4$, and suppose*

$${}_0Hom_R(I, M_1) = 0, \quad {}_0Hom_R(M_1, M_2) = 0 \quad \text{and} \quad {}_0Hom(M_2, E) = 0$$

Then the Hilbert schemes $H_{\gamma,\rho} \cong H_{\gamma,\rho 1} \cong H_\gamma \cong H(d,p,\pi)$ are isomorphic at $(X \subseteq \mathbf{P}^4)$.

Proof (sketch) It is straightforward to see that their tangent spaces are isomorphic by proposition 3.1. Moreover the isomorphism $H_\gamma \cong H(d,p,\pi)$ is proven in [K1], th. 3.6 and rem. 3.7. To get the theorem it suffices to prove that the obstruction $o_{\gamma,\rho}(I)$ of deforming a graded ideal I (i.e. the surface X) in $H_{\gamma,\rho}$ is vanishing provided the corresponding obstruction $o(I) \in {}_0Ext_R^2(I, I)$ of deforming I in H_γ is zero. By theorem 1.1 the vanishing of $o_{\gamma,\rho}(I)$ is equivalent to the vanishing of the corresponding object of V_ρ which one may write as

$$(o(M_1), o(M_2), o(b)) \in {}_0Ext_R^2(M_1, M_1) \times {}_0Ext_R^2(M_2, M_2) \times coker\ \bar{e}_1$$

If we continue the horizontal exact sequence of (25) we get a diagram

$$\text{(28)} \qquad \begin{array}{ccccccccc} & {}_0Ext_R^1(M_1, M_1) & \longrightarrow & coker\ e_2 & \longrightarrow & coker\ \bar{e}_1 \to 0 & & {}_0Ext_R^2(M_1, M_1) & \\ & \wr\| & \circ & \downarrow & \circ & \downarrow i & & \downarrow j & \\ \to & {}_{-5}Ext_R^2(I, M_1)^\vee & \longrightarrow & {}_{-5}Hom_R(I, M_2)^\vee & \longrightarrow & {}_0Ext_R^2(I, I) & \longrightarrow & {}_{-5}Ext_R^1(I, M_1)^\vee & \longrightarrow 0 \end{array}$$

where the obstructions involved are mapped to each other or are mapped to the same element over a common image. In particular $o(I) = 0$ implies $o(M_1) = 0$ because j is injective. Moreover one may also see $o(M_2) = 0$ while the injectivity of i implies $o(b) = 0$ as well (i.e. o(I) is the image some element of ${}_0Ext_R^1(M_1, M_1)$ which, mainly via the map $\bar{e}_1$, maps to the couple $(o(M_2), o(b))$ of vanishing obstructions) and we are done.

Remark 3.8 Consulting proposition 3.3ii) we see that ${}_{-5}Hom_R(I, M_2) = 0$ and the injectivity of the natural map ${}_0Hom_R(M_2, E) \to {}_0Ext_R^2(M_2, M_2)$ lead to a surjective map

$${}_0Ext_R^1(I, I)_\rho \longrightarrow {}_0Ext_R^1(M_1, M_1)$$

Using an easy part of the obstruction argument in the proof of theorem 3.7, we can therefore prove that the "morphism" $\tau : H_{\gamma,\rho} \to E_{\varrho}$ = isomorphism classes of R-modules M_1, defined by sending $(X \subseteq \mathbf{P}^4)$ onto $M_1(X) = H_*^1(I_X)$, is *smooth* at $(X \subseteq \mathbf{P}^4)$. Indeed the surjective map of the Ext-groups is the tangent map of τ, and, under the given assumptions above, we see that the obstructions o(I) and $o(M_1)$ of the proof of theorem 3.7 vanish simultaneously, whence the conclusion.

Corollary 3.9 (*Small M_2*) *Let X be a surface in $\mathbf{P}^4$, and suppose its modules E and $H_m^5(R)$ are supported to the left of M_2 (i.e $(M_2)_v \neq 0$ implies $E_v = 0$ and $v > -5$). Moreover suppose the local deformation functor $Def(M_1)$ is formally smooth at M_1 (e.g. suppose ${}_0Ext_R^2(M_1,M_1) = 0$), and that ${}_{-5}Hom_R(I, M_2) = 0$. Then $H_{\gamma,\rho}$ and V_ρ are smooth at $(X \subseteq \mathbf{P}^4)$ and $D = (M_1,M_2,b)$ respectively and the dimension of $H_{\gamma,\rho}$ at $(X \subseteq \mathbf{P}^4)$ is*

$$dim_X H_{\gamma,\rho} = 1 + \delta^3(-5) - \delta^2(-5) - \sum_{i=0}^{1} (-1)^i \cdot {}_0ext_R^i(M_1, M_1)$$

Proof $H_{\gamma,\rho}$ and V_ρ is smooth at ($X \subseteq \mathbf{P}^4$) and D = (M_1,M_2,b) respectively by remark 3.8 and theorem 1.1. Moreover proposition 3.3ii) and ${}_{-5}Hom_R(I, M_2) = 0$ imply that coker $\bar{e}_1 = 0$ and we get the dimension formula from proposition 3.6 and 0.5 (note that (0.5) implies that ${}_0Ext^2(M_2, M_2) \cong {}_0Ext^4(M_2, M_1)$), and we conclude easily.

To illustrate our results, we consider an example of a surface X of $\mathbf{P}^4$ where V_ρ is smooth and non-trivial at the corresponding (M_1,M_2,b) by proposition 3.4 or corollary 3.9. The surface X has also the weak variant of "natural cohomology" given by proposition 0.4, and the dimension formula of corollary 3.9 simplifies therefore further by introducing $\delta^1(-5)$. Moreover the conditions of theorem 3.7 will be satisfied, and it follows that $H_{\gamma,\rho}$ and $H(d,p,\pi)$ are isomorphic and smooth at ($X \subseteq \mathbf{P}^4$).

Example 3.10 Let X be the smooth rational surface with invariants d = 11, π = 11 (no 6-secant) and K^2 = -11 (cf. [P] or [DES]). In this case the graded modules $M_i \cong \oplus H^i(I_X(v))$ are supported at two consecutive degrees and satisfy

$$dim\ H^1(I_X(3)) = 2\ ,\quad dim\ H^1(I_X(4)) = 1$$
$$dim\ H^2(I_X(1)) = 3\ ,\quad dim\ H^2(I_X(2)) = 1$$

Moreover I = I(X) admits a minimal resolution (cf. [DES])

$$0 \to R(-9) \to R(-8)^{\oplus 3} \oplus R(-7)^{\oplus 3} \to R(-7)^{\oplus 2} \oplus R(-6)^{\oplus 12} \to R(-5)^{\oplus 10} \to I \to 0$$

It follows at once that ${}_{-5}Hom_R(I, M_2) = 0$ and ${}_0Ext^i(M_j, M_j) = 0$ for $i \geq 2$ and j = 1,2. By (3.9), (3.7) and (0.5) we get that $H(d,p,\pi) \cong H_{\gamma,\rho}$ is smooth at ($X \subseteq \mathbf{P}^4$) and $\dim_X H_{\gamma,\rho}$ =

$$1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5) = 1 + 12h^2(I_X(1)) - h^2(I_X(2)) + 3h^1(I_X(3)) - h^1(I_X(4)) = 41$$

In this example it is, however, easier to use proposition 0.3 to get

$$1+\delta^3(-5)-\delta^2(-5)+\delta^1(-5) = \chi(N_X)-\delta^3(0)+\delta^2(0)-\delta^1(0) = 5(2d+\pi-1)-d^2+2\chi(O_X) = 41$$

because $\delta^i(0)$ for $i > 0$ is easily seen to be zero. Finally, for the use of proposition 3.4, we remark that, in this example, ${}_0Ext^3(M_2, M_1)$ might be non-vanishing, but we still have a surjective map $\bar{e}_1$ by (3.3ii) and ${}_{-5}Hom_R(I, M_2) = 0$, i.e. (3.4) applies.

LIAISON OF SURFACES

In this section we will show how to compute the dimension of $H_{\gamma,\rho}$ and the dimension of its tangent space at ($X \subseteq \mathbf{P}^4$) provided we know how to solve the corresponding problem for a linked surface X', and visa versa. The result is particularly interesting when we start with a surface which is generic in $H(d,p,\pi)$ (e.g. $H_{\gamma,\rho} \cong H(d,p,\pi)$ at X) and it turns out that the corresponding linked surface is non-generic (e.g. $\dim_{X'} H_{\gamma',\rho'} < \dim_{X'} H(d',p',\pi')$). In that case a new surface (the generic one) with smaller cohomology has to exist. In remark 4.2 we give a criterion for the linked surface to be non-generic which can be useful for solving such problems.

Now, if X and X' are linked by a complete intersection Y of type (f,g), we recall ([PS], [M]) that the dualizing sheaf $\omega_{X'}$ satisfies $\omega_{X'} = I_{X/Y}(f+g-5)$ where $I_{X/Y} = \ker(O_Y \to O_X)$, and moreover $\omega_X = I_{X'/Y}(f+g-5)$. We get

$$\begin{aligned} &\chi(O_X(v)) + \chi(O_{X'}(f+g-5-v)) = \chi(O_Y(v)) \\ (29) \qquad &h^i(I_{X'}(v)) = h^{3-i}(I_X(f+g-5-v)) \quad , \text{ for } i = 1 \text{ and } 2 \\ &h^i(I_{X'/Y}(v)) = h^{2-i}(O_X(f+g-5-v)) \text{ , for } i = 0 \text{ and } 2 \\ &h^i(O_{X'}(v)) = h^{2-i}(I_{X/Y}(f+g-5-v)) \text{ , for } i = 0 \text{ and } 2 \end{aligned}$$

from which we deduce d + d' = fg and $\pi' - \pi = (d'-d)(f+g-4)/2$. Our main result is

Theorem 4.1 *Let X and X' be two surfaces in* $\mathbf{P}^4$ *which are linked (algebraically) by a complete intersection* $Y \subseteq \mathbf{P}^4$ *of type (f,g), and suppose X (resp. X') belongs to the Hilbert scheme* $H_{\gamma,\rho} = H(d,p,\pi)_{\gamma,\rho}$ *(resp.* $H_{\gamma',\rho'} = H(d',p',\pi')_{\gamma',\rho'}$*) of constant cohomology. Then*

$$i) \ \dim_X H_{\gamma,\rho} + h^0(I_X(f)) + h^0(I_X(g)) = \dim_{X'} H_{\gamma',\rho'} + h^0(I_{X'}(f)) + h^0(I_{X'}(g))$$

or equivalently

$$\dim_{X'} H_{\gamma',\rho'} = \dim_X H_{\gamma,\rho} + h^0(I_{X/Y}(f)) + h^0(I_{X/Y}(g)) - h^2(O_X(f-5)) - h^2(O_X(g-5))$$

ii) The dimension formulas of i) remain true if we replace $\dim_X H_{\gamma,\rho}$ *and* $\dim_{X'} H_{\gamma',\rho'}$ *by the dimension of their tangent spaces* ${}_0\mathrm{Ext}_R^1(I(X), I(X))_\rho$ *and* ${}_0\mathrm{Ext}_R^1(I(X'), I(X'))_{\rho'}$ *respectively.*

iii) $H_{\gamma,\rho}$ *is smooth at* $(X \subseteq \mathbf{P}^4)$ *if and only if* $H_{\gamma',\rho'}$ *is smooth at* $(X' \subseteq \mathbf{P}^4)$

Proof Let $D(d,p,\pi;f,g)$ be the Hilbert flag scheme consisting of pairs (X,Y) of surfaces of $\mathbf{P}^4$ such that $(X \subseteq \mathbf{P}^4) \in H(d,p,\pi)$ and Y is a complete intersection of type (f,g) containing X. By [K4], theorem 2.6, there is a liaison isomorphism

$$(30) \qquad D(d,p,\pi;f,g) \xrightarrow{\simeq} D(d',p',\pi';f,g).$$

given by sending (X,Y) onto (X',Y) where X' is linked to X by Y. Morover the projection morphism p : $D(d,p,\pi;f,g) \to H(d,p,\pi)$, given by $(X,Y) \to X$, is *smooth* at (X,Y) provided $H^1(I_X(f)) = H^1(I_X(g)) = 0$ ([K4], theorem 1.16ii). By [K4], remark 1.20, this smoothness holds if we replace the vanishing above with the claim that the corresponding sheaves on $H(d,p,\pi)$ are locally free and commute with base change. It follows that the restriction of p to $p^{-1}(H_{\gamma,\rho})$ is smooth, and since the fiber dimension of p at (X,Y) is precisely $h^0(I_{X/Y}(f)) + h^0(I_{X/Y}(g)) = h^0(I_X(f)) + h^0(I_X(g)) - h^0(I_Y(f)) - h^0(I_Y(g))$ by [K4], theorem 1.16i), we get easily any conclusion of the theorem.

Remark 4.2 i) The arguments of the proof also shows that we can, under the assumptions

$$H^1(I_X(f)) = H^1(I_X(g)) = 0 \ , \quad H^1(I_{X'}(f)) = H^1(I_{X'}(g)) = 0$$

replace $H_{\gamma,\rho}$ and $H_{\gamma',\rho'}$ in (4.1i) (resp. their tangent spaces in (4.1ii)) by $H(d,p,\pi)$ and $H(d',p',\pi')$ (resp. by $H^1(N_X)$ and $H^1(N_{X'})$ in (4.1ii)) and get valid dimension formulas involving the whole Hilbert schemes (resp. their tangent spaces).

ii) If, however, the linkage is geometric, and if we assume $H^1(I_X(f)) = H^1(I_X(g)) = 0$, the injectivity of the natural map $H^1(N_X) \to H^2(I_X(f)) \oplus H^2(I_X(g))$ and $\dim_X H_{\gamma,\rho} = \dim_X H(d,p,\pi)$, we can use [K4], theorem 1.27, to get bounds for the codimension $c = \dim_{X'} H(d',p',\pi') - \dim_{X'} H_{\gamma',\rho'}$. Indeed combining theorem 1.27 with corollary 2.14 of [K4], we get

$$(31) \qquad h^1(I_{X'}(f)) + h^1(I_{X'}(g)) - h^2(I_{X'}(f)) - h^2(I_{X'}(g)) \le c \le h^1(I_{X'}(f)) + h^1(I_{X'}(g))$$

and moreover, the right inequality is an equality if and only if $H(d',p',\pi')$ is smooth at X'. Finally by [K4], corollary 1.29, $H(d',p',\pi')$ is smooth at X' (i.e. $c = h^1(I_{X'}(f))+h^1(I_{X'}(g))$) provided $H^1(I_{X'}(v))\bullet H^2(I_{X'}(v)) = 0$ for $v = f$ *and* $v = g$.

Example 4.3 Let X be the smooth rational surface of $H(11,11,0)$ of example 3.10, let Y be a complete intersection of type (5,5) containing X, and let X' be the linked surface. Using (29) we deduce $\chi(O_{X'}(v)) = 7v^2-12v+9$ from $\chi(O_X(v)) = (11v^2-9v+2)/2$, i.e. X' belongs to $H(d',p',\pi') = H(14,20,8)$. Moreover $\omega_{X'} = I_{X/Y}(5)$ is globally generated (cf. the resolution of I of (3.10)) and the graded modules $M_i' \cong \oplus H^i(I_{X'}(v))$ are supported at two consecutive degrees and satisfy

$$\begin{aligned} &\dim H^1(I_{X'}(3)) = 1\ , \quad \dim H^1(I_{X'}(4)) = 3 \\ &\dim H^2(I_{X'}(1)) = 1\ , \quad \dim H^2(I_{X'}(2)) = 2 \end{aligned}$$

From these informations we find the minimal resolution of $I' = I(X')$ to be

$$0 \to R(-9)^{\oplus 3} \to R(-8)^{\oplus 14} \to R(-7)^{\oplus 23} \to R(-6)^{\oplus 11} \oplus R(-5)^{\oplus 2} \to I' \to 0$$

Thanks to theorem 4.1, we get that $H_{\gamma',\rho'}$ is smooth at $(X' \subseteq \mathbf{P}^4)$ and that

$$\dim_{X'} H_{\gamma',\rho'} = \dim_X H_{\gamma,\rho} + 2h^0(I_{X/Y}(5)) - 2h^2(O_X(0)) = 57$$

Moreover by remark 4.2i) or theorem 3.7, $H(d',p',\pi') \cong H_{\gamma',\rho'}$ is smooth at $(X' \subseteq \mathbf{P}^4)$ and $\dim_{X'} H(d',p',\pi') = 57$. Note that in this case we neither have ${}_0Ext^3(M_2, M_1) = 0$ nor ${}_{-5}Hom_R(I, M_2) = 0$, i.e we can not apply corollary 3.9, and in order to use proposition 3.4 we have to argue hardly for the surjectivity of $\bar{e}_1$. But, as we have seen, the linkage result above takes easily care of the smoothness and the dimension.

Example 4.4 Let Z be the surface which is linked to the surface $X' \in H(14,20,8)$ of example 4.3 via a complete intersection of type (5,6) containing X'. Then Z belongs to $H(16,27,15)$, $\omega_Z = I_{X'/Y}(6)$ is globally generated, and $M_i(Z) = \oplus H^i(I_Z(v))$, i = 1,2, are supported at two consecutive degrees, and moreover;

$$(31) \qquad \begin{aligned} &h^0(I_Z(5)) = 1\ , \quad h^1(I_Z(4)) = 2 \ \textit{and}\ h^1(I_Z(5)) = 1 \\ &h^2(O_Z(1)) = 1\ , \quad h^2(I_Z(2)) = 3 \ \textit{and}\ h^2(I_Z(3)) = 1 \end{aligned}$$

By proposition 0.3, we know $\chi(N_{X'}) = 5(2d'+\pi'-1)-d'^2+2\chi(O_{X'}) = 57$ and since we obviously have $h^2(N_{X'}) = 0$ (from $h^2(O_{X'}(1)) = 0$), we get $h^1(N_{X'}) = 0$ from (4.3). The conditions of remark 4.2ii) are therefore satisfied, and, at Z, we get that $H(16,27,15)_{\gamma,\rho}$ is smooth of codimension 1 in $H(16,27,15)$ (which is smooth as well). Moreover

$$dim_Z H(16,27,15)_{\gamma,\rho} = dim_X H_{\gamma',\rho'} + h^0(I_{X'/Y}(5)) + h^0(I_{X'/Y}(6)) - h^2(O_{X'}) - h^2(O_{X'}(1)) = 65$$

Hence Z belongs to a unique generically smooth component V of $H(16,27,15)$ of dimension 66, and the generic surface $\tilde{Z}$ of V satisfies dim $H^0(I_{\tilde{Z}}(5)) =$ dim $H^1(I_{\tilde{Z}}(5)) = 0$ while elsewhere the dimension is unchanged, i.e. it is as in (31).

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