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SINGULARITIES IN CODIMENSION 1 OF THE HILBERT SCHEME. AN EXAMPLE.

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A well known example of Mumford shows that in the Hilbert scheme $H(14,24)$ of space curves of degree $d = 14$ and arithmetic genus $g = 24$, there is a non-reduced component consisting generically of smooth connected curves sitting on smooth surfaces of degree 3. In fact for any $d \geq 14$ the Hilbert scheme $H(d,g)$, where g is the largest number satisfying $g \leq \frac{1}{8}(d^2-4)$, contains a non-reduced component, the general member of which is a smooth curve on a smooth cubic surface. See (K1).

In this paper we will, however, give an example of a codimension 1 singularity of the Hilbert scheme $H = H(16,29)$ which is not an intersection of irreducible non-embedded components[1]. In fact we described in (K1,3.3) an irreducible closed subset $Z \subseteq H$ where $\dim Z = d+g+18 = 63$ and $\operatorname{codim}_Z H = 1$ and proved that H was singular along Z. The general member X of Z is, in this case too, smooth and sits on a smooth cubic surface Y. And $L = O_Y(X)$ equals $(12,4,4,4,4,2,2)$ via the usual isomorphism $\operatorname{Pic} Y \cong \mathbb{Z}^{\oplus 7}$. See (H,chV,4.8).
In the following we will partially prove all this and in particular prove that Z is contained in a unique irreducible non-embedded component. The uniqueness was not treated in (K1).

By a curve X we will mean an equidimensional, (locally) Cohen Macaulay subscheme of $P = \mathbb{P}^3_k$ of dimension 1, and $H(d,g)_{CM}$ denotes the Hilbert scheme of such curves. Now let $R = k[X_0, X_1, X_2, X_3]$ be a polynomial ring over an algebraically closed field k, and $A = R/I$ the minimal cone of $X \subseteq P$. If $I_X = \ker(O_P \to O_X)$, then $I_X = \tilde{I}$, and I has a minimal resolution of the form

1) Z is not an embedded component either.

1) $0 \longrightarrow \overset{h_3}{\underset{i}{\oplus}} R(-n_{3i}) \longrightarrow \overset{h_2}{\underset{i}{\oplus}} R(-n_{2i}) \longrightarrow \overset{h_1}{\underset{i}{\oplus}} R(-n_{1i}) \longrightarrow I \longrightarrow 0$

Define $s(X)$, $e = e(X)$ and $c = c(X)$ by

$$s(X) = \min_{1 \le i \le h_1} n_{1i} ,$$

2) $H^1(X, O_X(e)) \neq 0$ and $H^1(O_X(l)) = 0$ for $l > e$,
$H^1(P, I_X(c)) \neq 0$ and $H^1(I_X(l)) = 0$ for $l > c$.

Put $c = -\infty$ in the arithmetically Cohen Macaulay case, i.e. where $H^1(I_X(l)) = 0$ for all $l \in \mathbb{Z}$. Morever splitting 1) into two short exact sequenses we get

3) $$c(X) = \max n_{3i} - 4,$$

as in 24) and 25). By the fact $\max n_{1i} < \max n_{2i}$, one proves

$$e(X) \le \max n_{2i} - 4$$

with equality provided $\max n_{3i} \le \max n_{2i}$. Thus

4) $$e(X) < c(X) \text{ implies } \max n_{2i} < \max n_{3i}.$$

Since the resolution is minimal,

5) $$\min n_{1i} < \min n_{2i} < \min n_{3i}$$

For details, see (Kl, 2.2.7).

To begin with we recall two results from (Kl) and the new proposition 3. Since (Kl) has only appeared in my thesis and as a preprint we indicate the proofs of theorem 1 and proposition 6.

Liaison

Let $Y = V(F_1, F_2) \subseteq P$ be a global complete intersection of two surfaces of degree $f_i = \deg F_i$ for $i = 1,2$, in which case we say Y is of type (f_1, f_2), and let $X \subseteq Y$ be an inclusion of curves. In this situation there is a linked curve $X' \subseteq Y$ whose sheaf ideal $I_{X'/Y}$ in O_Y is

6) $$I_{X'/Y} = \underline{\mathrm{Hom}}_{O_Y}(O_X, O_Y)$$

by the definition of X'. By (PS, 1.3) X' is a curve, i.e. equidimensional and Cohen Macaulay, and the linked curve $X'' \subseteq Y$ of $X' \subseteq Y$ is just $X \subseteq Y$.

Moreover as the dualizing sheaf $\omega_Y \simeq O_Y(f_1+f_2-4)$ and the corresponding $\omega_X \simeq \underline{\mathrm{Hom}}_{O_Y}(O_X, \omega_Y)$,

6) yields

7) $I_{X'/Y} \simeq \omega_X(4-f_1-f_2)$ and $I_{X/Y} \simeq \omega_{X'}(4-f_1-f_2)$.
Hence there is an exact sequence

8) $0 \longrightarrow \omega_X(4-f_1-f_2) \to O_Y \to O_{X'} \longrightarrow 0$
and a similar one interchanging X and X'.
Twisting by 1 and taking Euler - Poincaré characteristics we deduce by Riemann Roch's theorem that

9)
$$d + d' = f_1 f_2$$
$$g - g' = (d-d')\frac{f_1 + f_2 - 4}{2}$$

Letting $h^i(F) = \dim H^i(F)$ for any coherent F, we have by 7)

$$h^0(I_{X/Y}(1)) = h^1(O_{X'}(f_1+f_2-4-1))$$

10)
$$h^1(I_X(1)) = h^1(I_{X'}(f_1+f_2-4-1))$$
$$h^1(O_X(1)) = h^0(I_{X'/Y}(f_1+f_2-4-1))$$

See (Kl,2.3.3). Now a consequence of a main result in (Kl) is

Theorem 1 Let the numbers d,g,d',g',f_1 and f_2 satisfy 9).

i) The set of curves

$$U = \left\{ (X \subseteq P) \in H(d,g)_{CM} \;\middle|\; \begin{array}{l}\text{There is a } Y \text{ of type } (f_1,f_2) \text{ con-}\\ \text{taining } X \text{ and } H^1(I_X(f_i)) = 0 \text{ for } i=1,2\end{array} \right\}$$

is open in $H(d,g)_{CM}$. In particular

$$U_{\underline{f}}(d,g) = \{(X\subseteq P) \in U \mid H^1(I_X(f_i-4)) = 0 \text{ for } i = 1,2\}$$

is an open subscheme of $H(d,g)_{CM}$.

ii) There is a diagram of quasiprojective k-schemes

$$\begin{array}{ccc} U(d,g;f_1,f_2) & \simeq & U(d',g';f_1,f_2) \\ \downarrow p & & \downarrow p' \\ U_{\underline{f}}(d,g) & & U_{\underline{f}}(d',g') \end{array}$$

where p and p' are smooth surjective morphisms of geometrically irreducible fibers of fiber dimension $\sum_{i=1}^{2} h^0(I_{X/Y}(f_i))$ for any $(X\subseteq P)$ of $U_{\underline{f}}(d,g)$ and Y of type (f_1,f_2) containing X, and $\sum h^0(I_{X'/Y}(f_i))$ respectively.

In particular the irreducible , resp embedded components of $U_{\underline{f}}(d,g)$ and $U_{\underline{f}}(d',g')$ are in a one-to-one correspondence.

<u>Main lines of proof</u> In fact (Kl,1.1) implies the existence of a quasiprojective scheme $D(d,g;f_1,f_2)$ called the Hilbert-flag scheme, whose k-points are $D = D(d,g;f_1,f_2) = \{(X\subseteq Y\subseteq P) \mid (X\subseteq P) \in H(d,g)_{CM}$ and Y of type $(f_1,f_2)\}$, representing a correspondingly defined functor. And if $X_S \subseteq Y_S$ is an inclusion of flat curves in PxS over S whose closed fibers belong to D, we can still define the linked curve $X'_S \subseteq Y_S$ over S as in 6). One may see that X'_S is S-flat because $\mathrm{Ext}^1_{O_{Y_S}}(O_{X_S}, O_{Y_S} \otimes k(s)) = 0$ for any $s \in S$ and for the same reason liaison and specialization commutes, and we therefore get an isomorphism of <u>schemes</u>

11) $$D(d,g;\ f_1,f_2) \simeq D(d',g';f_1,f_2)$$

defined by liason (Kl,2.3.4) . Furthermore the natural projection $p : D \to H = H(d,g)$, given by $t = (X \subseteq Y \subseteq P) \to (X \subseteq P)$, is smooth at t provided $H^1(I_X(f_i)) = 0$ for $i = 1,2$. In fact let $S \hookrightarrow S'$ be a morphism of affine k-schemes whose k-algebras are local artinian rings with residue fields k, and let a diagram of deformations, hence of flat schemes

$$\begin{array}{ccccc} X_{S'} & \hookrightarrow & & \longrightarrow & PxS' \\ \uparrow & & o & & \uparrow \\ X_S & \hookrightarrow & Y_S & \hookrightarrow & PxS \end{array}$$

of $X\subseteq Y\subseteq P$ and $X\subseteq P$ be given. By $H^1(I_X(f_i)) = 0$ for $i = 1,2$, we have a surjective map

12) $$H^0(I_{X_{S'}}(f_i)) \twoheadrightarrow H^0(I_{X_S}(f_i))$$

where $I_{X_S} = \ker(O_{PxS} \to O_{X_S})$, and this gives readily the existence of an S'-flat $Y_{S'} \subseteq PxS'$ containing $X_{S'}$ such that $Y_{S'} \times_{S'} S = Y_S$. This proves precisely the smoothness of p, see (Kl,1.3.14). Smooth morphisms are open, and finally if $t = (X \subseteq V(F_1,F_2) \subseteq P)$ and $t' = (X \subseteq V(G_1,G_2) \subseteq P)$, there is an open $U \subseteq \mathbb{A}^1_k = \mathrm{Spec}\ k[T]$ containing $T = 0$ and $T = 1$ over which

$Y_U = V(F_1+T(G_1-F_1), F_2+T(G_2-F_2)) \subseteq PxU$

is a flat complete intersection. See (M1, page 57) Clearly $Y_U \supseteq XxU$ and this proves that the fibers of p are connected. Moreover the fiber dimension is easily found (Kl,1.3.12). Putting this together, recalling $h^1(I_{X'}(f_i)) = h^1(I_X(f_{3-i}-4))$ for i = 1,2 from 10), we get the theorem.

For a similar result, see (Bu).

Remark 2 In the applications it is often necessary to make a sequence of liaisons, and it is therefore desireable to use the set U of the theorem for such f_1 and f_2 for which there always exists a Y of type (f_1,f_2) containing X. By (PS,3.7) there exists a Y of type (f_1,f_2) where $f_1 = s(X)$ and $f_2 \leq \max n_{1i}$, the n_{ji}'s belong to 1). In particular if $e(X) < c(X)$ we get by 4) that $\max n_{1i} \leq \max n_{3i} -2 = c(X)+2$, i.e. there exists a global complete intersection Y of type $(s(X), c(X)+2)$ containing X.

We will now state a proposition, the proof of which is a consequence of

13) $c(X') = f_1+f_2-4-c(X)$

14) $e(X') = f_1+f_2-4-s(X/Y)$

15) $s(X'/Y) = f_1+f_2-4-e(X)$

where $n = s(X/Y)$ is the least integer satisfying $H^0(I_{X/Y}(n)) \neq 0$. These formulas results immediately from 10) provided $H^1(I_X(l)) = 0$ for $l \neq c(X)$.

Proposition 3 i) Let $(X \subseteq P) \in H(d,g)_{CM}$ and suppose $e(X)<c(X)<s(X)$ and $H^1(I_X(l)) = 0$ for $l \neq c(X)$. If we make liaison with a Y of type (f_1,f_2) where $f_1 = s(X)$ and $f_2 \leq c(X)+2$, then the linked curve $(X' \subseteq P)$ of $H(d,g)_{CM}$ satisfies

$$e(X')<c(X')<s(X') \text{ and } H^1(I_{X'}(l)) = 0 \text{ for } l \neq c(X')$$

ii) Moreover the linked curve satisfies either

1) $s(X') < s(X)$, or

2) $s(X') = s(X)$ and $c(X') < c(X)$

or we have the following situation 3) :

$$s(X') = s(X)\ ,\ c(X') = c(X)\ ,\ e(X') = e(X),$$

and the minimal resolutions of the sheaf ideals I_X and $I_{X'}$ are both of the form

$$0 \to O_P(-2r-2)^{\oplus r} \to O_P(-2r-1)^{\oplus 4r} \to O_P(-2r)^{\oplus 3r+1} \to I_{X(')} \to 0$$

In this case, called the stationary case, we have

$$c(X)-2 = e(X),\ s(X) = c(X)+2 \text{ and } h^1(I_X(c(X))) = r$$

and for the degree and arithmetic genus of X and X',

$$d = \tfrac{1}{2}s^2,\ g = \tfrac{1}{3}(s-3)(s^2-1) \text{ where } s = s(X) = 2r$$

<u>Proof</u> i) Since the sequence of sheaf ideals

$$0 \to I_Y \to I_{X'} \to I_{X'/Y} \to 0$$

is exact and $H^1(I_Y(l)) = 0$ for any l, we get

16) $s(X') = \min(s(X), s(X'/Y))$

because $s(Y) = s(X)$. Correspondingly $s(X) \leq s(X/Y)$.
Observe also that

17) $e(X') < c(X') \Longleftrightarrow s(X/Y) > c(X)$

by 13) and 14) and since by assumption $c(X) < s(X) \leq s(X/Y)$, we get

$$e(X') < c(X').$$

To prove $c(X') < s(X')$ we see as in 17) that

$$c(X') < s(X'/Y)$$

because $e(X) < c(X)$. To finish prop 3i) it remains to prove

$$c(X') < s(X) = f_1,$$

see 16). However by 13) this is equivalent to

$$f_1+f_2-4-c(X) < f_1, \text{ i.e. to } f_2-4 < c(X)$$

which is true by the assumption on f_2.

ii) Since $s(X') = \min(s(X), s(X'/Y)) \leq s(X)$ is always true we get either 1) of prop 3ii) or the case $s(X')=s(X)$ which we consider in the following. As in remark 2,

$$s(X') \leq c(X')+2,$$

and the inequality $f_2 \leq c(X)+2$ leads therefore to

$$c(X') = f_1+f_2-4-c(X) \leq f_1-2 = s(X')-2.$$

Combining we get

$$s(X') = c(X') +2,$$

and since $s(X') = s(X) \leq f_2 \leq c(X)+2$ we have either

2) of prop 3ii) or the case

$$s(X') = s(X) \text{ and } s(X) = f_2 = c(X)+2$$

which we now consider. By 3), 4) and 5),

$$s(X) = \min n_{1i} \leq \min n_{2i}-1 \leq \min n_{3i}-2$$

18) $$\begin{matrix} \wedge\!\mid & & & \wedge\!\mid \end{matrix}$$

$$\max n_{1i} \leq \max n_{2i}-1 \leq \max n_{3i}-2 = c(X)+2,$$ and $s(X)=c(X)+2$ leads to equalities everywhere in 18). The resolution of I_X must therefore be of the form

$$0 \longrightarrow O_P(-s-2)^{\oplus l} \longrightarrow O_P(-s-1)^{\oplus(l+t-1)} \longrightarrow O_P(-s)^{\oplus t} \rightarrow I \rightarrow 0$$

where $s = s(X)$. Since $s = s(X') = c(X')+2$ and $e(X') < c(X')$, the resolution of $I_{X'}$ is of the same form replacing t by t' and l by l'. As $t \geq 3$ and $t' \geq 3$ we have $s(X) = s(X/Y)$ and $s(X') = s(X'/Y)$ and 14), 15) gives readily

$$e(X) = f_2-4 = e(X') \text{ and } e(X) = c(X)-2.$$

Moreover subtracting

$$\binom{s+2}{3} -d(s-1)-1+g = \chi(I_X(s-1)) = 0$$

$$\binom{s}{3}-d(s-3)-1+g = \chi(I_X(s-3)) =0$$

where $\chi(I_X(l))= \chi(O_P(l))-\chi(O_X(l)) = \binom{l+3}{3}-(dl+1-g)$ is the Hilbert polynomial of the sheaf ideal I_X, we get

$$2d = \binom{s+2}{3}-\binom{s}{3} = s^2$$

The genus of X is therefore

$$g = d(s-1)+1-\binom{s+2}{3} = \tfrac{1}{3}(s^2-1)(s-3)$$

Finally

$$\chi(I_X(s-2)) = -h^1(I_X(s-2)) = \binom{s+1}{3} - d(s-2) - 1 + g = -\tfrac{1}{2}s$$

and

$$\chi(I_X(s)) = h^0(I_X(s)) = \binom{s+3}{3} - ds - 1 + g = \tfrac{1}{2}(3s+2)$$

The same arguments hold for the linked curve as well, and we are done.

Example. We will illustrate by considering the example $H = H(16,29)_{CM}$ in question. Start with a curve $X \subseteq P$ of H satisfying $H^1(I_X(l)) = 0$ for $l \neq 4$ and $h^1(I_X(4)) = 1$. Since $\chi(I_X(l)) = \binom{l+3}{3} - (dl+1-g)$, we get $\chi(I_X(3)) = 0$, $\chi(I_X(4)) = -1$, $\chi(I_X(5)) = 4$. Hence $e(X) = 2$, $c(X) = 4$ and $s(X) = 5$.

Now if we make a sequence of liaisons and each time use a complete intersection $Y \supseteq X$ of type $(s(X), c(X)+2)$ we get in succession linked curves with datas $(d,g,e(X),c(X),s(X))$ as follows (16,29,2,4,5), (14,22,2,3,5), (11,13,1,3,4),
8 (9,7,1,2,4), (7,4,0,2,3), (5,1,0,1,3), (4,0,-1,1,2), (2,-1,-2,0,2), (2,-1,-2,0,2)

These datas are immediately found by using 13), 14), 15), together with computing $\chi(I_X(s(X)))$. Moreover the reader may readily check that $X \subseteq P$ and the linked curves of this sequence belong to the corresponding sets $U_{\underline{f}}$ of theorem 1i), thus giving more generally

Corollary 4 For any numbers d,g and r, let

$$U_r(d,g) = \left\{ (X \subseteq P) \in H(d,g)_{CM} \;\middle|\; \begin{array}{l} h^1(I_X(c(X))) = r \text{ and } H^1(I_X(l)) = 0 \text{ for} \\ l \neq c(X) \text{ and } e(X) < c(X) < s(X) \end{array} \right\}$$

Then the functions $e(-)$, $c(-)$ and $s(-)$ defined on $U_r(d,g)$ are constant and $U_r(d,g)$ is open in $H(d,g)_{CM}$. Moreover $U_r(d,g)$ is smooth, resp irreducible if and only if the corresponding set of the stationary case

$$U_r(2r^2, \tfrac{1}{3}(2r-3)(4r^2-1))$$

is smooth, resp irreducible.

Proof. The function $\chi(I_{X_1}(l))$ on $H(d,g)$, i.e. with l fixed and varying $(X_1 \subseteq P) \in H(d,g)$, is constant.
Since for $(X \subseteq P) \in U_r(d,g)$,

$$\chi(I_X(l)) = \begin{cases} h^0(I_X(l)) & \text{for } l > c(X) \\ -h^1(I_X(l)) & \text{for } l = c(X) \\ h^1(O_X(l)) & \text{for } -4 < l < c(X) \end{cases}$$

we find that $e(-)$, $c(-)$ and $s(-)$ are constant on $U_r(d,g)$. By the semicontinuity of $h^i(I_X(1))$ they are also constant in some $H(d,g)$-neighbourhood of an arbitrary $(X \subseteq P) \in U_r(d,g)$. Thus $U_r(d,g)$ is open in $H(d,g)$.

Now let $(f_1,f_2) = (s(X),c(X)+2)$ where $(X \subseteq P) \in U_r(d,g)$. If the $U_{\underline{f}}$'s are as in theorem 1, then we claim that $U_r(d,g) \subseteq U_{\underline{f}}(d,g)$ and $U_r(d',g') \subseteq U_{\underline{f}}(d',g')$. In fact remark 2 and $c(X) < s(X) = f_1 \leq f_2 = c(X)+2$ gives easily $H^1(I_X(f_i)) = 0 = H^1(I_X(f_i-4))$ for $i = 1,2$, thus proving $U_r(d,g) \subseteq U_{\underline{f}}(d,g)$. And moreover by proposition 3,

$$s(X') \leq s(X) = f_1$$

and by 13) and $f_2 = c(X)+2$,

$$c(X')+2 = f_1 \leq f_2$$

Hence for any $(X' \subseteq P) \in U_r(d',g')$ there exists a $Y \subseteq P$ of type (f_1,f_2) containing X', and again

$$H^1(I_{X'}(f_i)) = 0 = H^1(I_{X'}(f_i-4))$$

for $i = 1,2$ for instance by 10). It follows that $U_r(d',g') \subseteq U_{\underline{f}}(d',g')$.

Finally consider the diagram of theorem 1ii) and let

$U_r(d,g;f_1,f_2) = p^{-1}(U_r(d,g))$ and $U_r(d',g';f_1 f_2) = p'^{-1}(U_r(d',g'))$.

We get easily a diagram

$$\begin{array}{ccc} U_r(d,g;f_1,f_2) & \simeq & U(d',g';f_1,f_2) \\ \downarrow p & & \downarrow p' \\ U_r(d,g) & & U_r(d',g') \end{array}$$

where p and p' are smooth surjective morphisms of geometrically irreducible fibers. This diagram covers each step in the liaison sequence ending with the stationary case. In fact we proceed from the diagram above letting $(f_1',f_2') = (s(X'), c(X')+2)$ where $(X' \subseteq P) \in U_r(d',g')$ etc. This proves the corollary.

Remark 5 i) By proposition 6, $U_r(d,g)$ is smooth of dimension 4d if it is non-empty.

ii) It is well known that the set $U_1(2,-1)$ of corollary 4 is smooth and irreducible. For a reference we get by (B) that the moduli scheme M(0,1) of stable rank 2 vector bundles is a smooth connected scheme and so is $U_1(2,-1)$ by (Kl 1). It follows that the set $U_1(16,29)$ is smooth and irreducible.

On the cohomology of the normal bundle.

Let $N_X = Hom_{O_P}(I_X, O_X)$ be the normal bundle of $X \subseteq P$ and consider the minimal resolution 1).

Proposition 6. Let $X \subseteq P$ be a curve.

i) If $c(X) < \min n_{2i}$ and

$$H^1(I_X(n_{1i}-4)) = 0 \quad \text{for } 1 \le i \le r_1$$

then

$$h^1(N_X) = \sum_{i=1}^{r_1} h^1(O_X(n_{1i})) - \sum_{i=2}^{r_2} h^1(O_X(n_{2i})) + \sum_{i=3}^{r_3} h^1(O_X(n_{3i}))$$

In particular if $e(X) < s(X)$, then

$$H^1(N_X) = 0$$

ii) Moreover let W(s), $s = n_{1i}$ for some i, be the closed subset of H(d,g) given by

$$W(s) = \{ (X \subseteq P) \in H(d,g) \mid h^0(I_X(s)) > 0 \}$$

Then

$$\operatorname{codim}_{W(s)} H(d,g) = h^1(I_X(s)) \quad \text{at } (X \subseteq P)$$

provided the three conditions of i) are satisfied.

Main lines of proof We will only need the vanishing result of $H^1(N_X)$ together with ii) of which we concentrate. If M and $N = \oplus N_l$ are graded R-modules, let ${}_l\mathrm{Ext}^i_m(M,-)$ be the right derived functor of the covariant left-exact $\Gamma_m(\mathrm{Hom}_R(M,-))_l$ where $\Gamma_m(N)_l = \ker(N_l \longrightarrow \Gamma(P,N(l)))$, and consider the exact sequence (SGA2,expVI)

19) $\longrightarrow {}_0\mathrm{Ext}^2_m(I,I) \longrightarrow {}_0\mathrm{Ext}^2_R(I,I) \longrightarrow \mathrm{Ext}^2_{O_P}(\tilde{I},\tilde{I}) \longrightarrow {}_0\mathrm{Ext}^3_m(I,I) \longrightarrow 0$

Using that the projective dimension of $\tilde{I}$ is 1 (locally), one proves $N_X \simeq \underline{\mathrm{Ext}}^1_{O_P}(\tilde{I},\tilde{I})$. Hence

$$H^1(N_X) \simeq \mathrm{Ext}^2_{O_P}(\tilde{I},\tilde{I}).$$

Moreover the spectral sequence ${}_0\mathrm{Ext}^p_R(I,H^{3-p}_m(I)) \Longrightarrow {}_0\mathrm{Ext}^3_m(I,I)$ implies that

$${}_0\mathrm{Ext}^3_m(I,I) \simeq {}_0\mathrm{Hom}(I,H^3_m(I)) \simeq {}_0\mathrm{Hom}(I,\oplus H^1(O_X(1))).$$

Finally the right-exactness of ${}_0\mathrm{Ext}^2_R(I,-)$ applied to $\oplus R(-n_{1i}) \longrightarrow I \longrightarrow 0$ and the duality

$$H^1(\tilde{I}(n_{1i}-4))^\vee \simeq \mathrm{Ext}^2_{O_P}(\tilde{I}(n_{1i}-4), O_P(-4)) \simeq {}_0\mathrm{Ext}^2_R(I,R(-n_{1i}))$$

proves that ${}_0\mathrm{Ext}^2(I,I) = 0$. Combining we get

$$H^1(N_X) \simeq {}_0\mathrm{Hom}(I,\oplus H^1(O_X(1))) = \ker(\oplus H^1(O_X(n_{1i})) \xrightarrow{\Psi} \oplus H^1(O_X(n_{2i})))$$

(and since one may prove that $\mathrm{coker}\,\Psi \simeq \oplus H^1(O_X(n_{3i}))$ we have i)). Anyway $H^1(N_X) = 0$ provided $e(X) < s(X) = \min n_{1i}$. For details, see (Kl,2.2.9).

The conclusion of ii) follows easily from the theory of Hilbert-flag schemes developped in (Kl). In fact consider

$$D = \{(X \subseteq Y \subseteq P) \mid (X \subseteq P) \in H(d,g) \text{ and } Y \text{ a hypersurface of deg } s\}$$

Then $W(s) = \mathrm{Im}\,p$ via the natural projection $p : D \longrightarrow H(d,g)$. If A^1 and A^2 are the tangent space and "obstruction space" of D at $(X \subseteq Y \subseteq P)$ respectively, there is an exact sequence

20) $0 \longrightarrow H^0(I_X(s)) \longrightarrow A^1 \longrightarrow H^0(N_X) \xrightarrow{\gamma} H^1(I_X(s)) \longrightarrow A^2 \longrightarrow H^1(N_X) \longrightarrow H^1(O_X(s))$

see (Kl,1.3.). Therefore the conclusion of ii) follows provided γ is surjective and $H^1(N_X) = 0$

However continuing 19) to the left we get

$$\begin{array}{ccccccc} {}_0\mathrm{Ext}^1(I,I) & \to & \mathrm{Ext}^1_{O_P}(\tilde{I},\tilde{I}) & \xrightarrow{\beta} & {}_0\mathrm{Ext}^2_m(I,I) & \to & {}_0\mathrm{Ext}^2(I,I) \\ & & \wr\!\mid & & \wr\!\mid & & \| \\ & & H^0(N_X) & & \oplus H^1(I_X(n_{1i})) & & 0 \end{array}$$

observing that

$${}_0\mathrm{Ext}^2_m(I,I) = {}_0\mathrm{Hom}(I,H^2_m(I)) = \ker(\oplus H^1(I_X(n_{1i})) \to \oplus H^1(I_X(n_{2i})))$$

and recalling $c(X) < \min n_{2i}$. Now composing β with the projection $\oplus H^1(I_X(n_{1i})) \twoheadrightarrow H^1(I_X(s))$ we get the map γ which therefore is surjective.

<u>Remark 7</u> Let $(X \subseteq Y \subseteq P) \in D$ and suppose X is a <u>divisor</u> on Y. Then A^2 is seen to be the cokernel of some map α: $H^1(O_Y(s)) \to H^1(N_{X/Y})$ where $N_{X/Y} = \underline{\mathrm{Hom}}_{O_Y}(I_{X/Y}, O_X) \simeq \omega_X(4-s)$ by (Kl,1.3). Hence if $\underline{s \leq 3}$[1], then $A^2 = 0$ and we have under these conditions a result similar to proposition 6 where we in the proof use 20) instead of 19).
This gives

i) $H^1(N_X) \xrightarrow{\sim} H^1(O_X(s))$

and

ii) $h^1(I_X(s)) - h^1(O_X(s)) \leq \mathrm{codim}_{W(\lambda)} H(d,g) \leq h^1(I_X(s))$ at $(X \subseteq P)$

with equality to the right if and only if $H(d,g)$ is smooth at $(X \subseteq P)$. Moreover using i) we easily get

iii) $\dim D = h^0(N_X) + h^0(I_X(s)) - h^1(I_X(s)) = 4d + \chi(I_X(s)) = (4-s)d + g - 2 + \binom{s+3}{3}$ for the dimension of D at $(X \subseteq Y \subseteq P)$.

<u>Singularities of codimension 1 of H(16,29).</u>

In the introduction we described a family Z of $H = H(16,29)$ whose general curve $X \subseteq P$ was contained in a smooth surface Y of degree $s=3$ and $L = O_Y(X) = (12,4,4,4,4,2,2)$ via $\mathrm{Pic}\, Y \simeq \mathbb{Z}^{\oplus 7}$.

Since we by remark 7 have a surjective

$$L = \underline{\mathrm{Hom}}_{O_Y}(I_{X/Y}, O_Y) \twoheadrightarrow \underline{\mathrm{Hom}}_{O_Y}(I_{X/Y}, O_X) \simeq \omega_X(1)$$

with kernel O_Y and since $L(-4) = (0,0,0,0,0,-2,-2)$ we find

$$h^1(O_X(3)) = h^0(\omega_X(-3)) = h^0(L(-4)) = 1,$$

21) $$h^1(N_X) = h^1(O_X(3)) = 1,\quad h^1(O_X(l)) = 0 \text{ for } l \geq 4,$$

$$\dim Z = d+g+18 = 63$$

---

1) and for instance X is reduced

see remark 7i) and 7iii). Moreover by (Kl,3.1.3),

$$H^1(I_X(l)) = 0 \text{ for } l \notin \{3,4,5,6\}$$

See also (D).

Now if $V \subseteq H$ is any irreducible non-embedded component containing Z, then we claim that

22) V is a reduced i.e. a generically smooth component of dimension 4d = 64 and a sufficently general curve $(X_1 \subseteq P)$ of V satisfies

$$s(X_1) = 5 , \quad e(X_1) = 2 \text{ and}$$

$$h^1(I_{X_1}(l)) = \begin{cases} 1 & \text{for } l = 4 \\ 0 & \text{for } l \neq 4. \end{cases}$$

This is proved in (Kl,3.3) . And then remark 5ii) implies that the family Z is contained in only one component V of the form 22). Moreover H(16,29) is singular along Z because a general curve $(X \subseteq P)$ of Z satisfies $h^1(N_X) = 1$ by 21), and for a general curve $(X_1 \subseteq P)$ of V we have $h^0(N_{X_1}) = \dim V = 4d$, hence $H^1(N_{X_1}) = 0$ (or simply, $H^1(N_{X_1}) = 0$ by proposition 6). Finally by the structure theorem of (L,5.2.10) the completion $\hat{O}_{H,X}$ of the local ring of H(16,29) at the general $(X \subseteq P)$ of Z is a complete intersection (a power series k-algebra divided out by an element). It follows that Z is not an embedded component either.

Finally we briefly indicate a proof of 22). By the semicontinuity of $h^0(I_X(l))$ there are three possibilities for a general curve $(X_1 \subseteq P)$ of V

A) $s(X_1) = 5$, B) $s(X_1) = 4$ and C) $s(X_1) = 3$

because $\chi(I_{X_1}(5)) = 4$. The case C) is easily excluded because any such maximal irreducible family V of H has dimension $d+g+63<4d$ by remark 7iii), contradicting the assumption that V is a component. The case B) leads similarly to a contradiction as proved in (Kl,3.3) .

To motivate for this we will remark that, classically, maximal irreducible families of curves sitting on surfaces of degree 4 should have dimension $g+33 = 62<4d$
See (N) on (K1,p148) . Hence curves as in B) should not form a component.

In the remaining case A) we get

$$s(X_1)=5 \quad , \quad h^1(O_{X_1}(3)) = h^1(I_{X_1}(3)) \quad \text{and } h^1(I_{X_1}(4)) = 1$$

because $\chi(I_{X_1}(3)) = 0$ and $\chi(I_{X_1}(4)) = -1$.

Ai) First suppose $h^1(I_{X_1}(3)) = 0$, and let $Y_1 \supseteq X_1$ be of type (5,5). For the linked curve we get by 9) and 10), $d(X_1') = 9$, $g(X_1') = 8$, $s(X_1') = 4$, $c(X_1') = 2$ and $e(X_1') = 1$
Hence

$$s(X_1') = c(X_1')+2 \quad \text{and } c(X_1') > e(X_1').$$

By 4) and 5),

$$\begin{array}{ccccc} s(X_1') = \min n_{1i} & \leq & \min n_{2i}-1 & \leq & \min n_{3i}-2 \\ \wedge\!| & & & & \wedge\!| \\ \max n_{1i} & \leq & \max n_{2i}-1 & \leq & \max n_{3i}-2 = c(X_1')+2 \end{array}$$

So we have equalities everywhere, and the resolution of $I_1' = \oplus H^0(I_{X_1'}(l))$ is therefore

23) $0 \longrightarrow R(-6) \xrightarrow{N} R(-5)^{\oplus 6} \longrightarrow R(-4)^{\oplus 6} \longrightarrow I_1' \longrightarrow 0$

Splitting 23) into short exact sequences, one proves that

24) $0 \rightarrow H^1(I_{X_1'}(l)) \longrightarrow H^3(O_P(-6+l)) \longrightarrow H^3(O_P(-5+l))$

is exact. And dualizing 23) we get

$$R(5)^{\oplus 6} \xrightarrow{{}^tN} R(6) \longrightarrow C \longrightarrow 0$$

where by 24) the graded cokernel C satisfies

25) $C_{-l-4} \simeq \mathrm{Hom}_k(H^1(I_{X_1'}(l)),k)$.

C is therefore of finite length, hence supported at the maximal ideal of R. It follows that the radical of the ideal generated by the elements of the matrix $N = [L_1,L_1,\ldots,L_6]$ is

$$r((L_1,L_2,\ldots,L_6)) = (X_0,X_1,X_2,X_3).$$

Combining with the degree $\deg L_i = 1$, we have

$$(L_1,L_2,\ldots,L_6) = (X_0,X_1,X_2,X_3),$$

i.e. $C \simeq C_{-6}$ and by 25) and 10) we get

$$H^1(I_{X_1}(l)) = 0 \qquad \text{for } l \neq 4.$$

It follows from proposition 6 that

$$H^1(N_{X_1}) = 0$$

The corresponding component V is therefore as in 22).

Aii) The remaining case for a general curve $(X_1 \subseteq P)$ of V is

$$h^1(I_{X_1}(3)) = 1, \quad H^1(I_{X_1}(l)) = 0 \text{ for } l<3 \text{ and } s(X_1)=5.$$

If we link $X_1 \subseteq P$ by a complete intersection Y_1 of type $(f_1, f_2) = (5,5)$ and we consider the resolution of the sheaf ideal of the linked curve X_1' in P, one proves as in Ai) that

$$H^1(I_{X_1'}(l)) = 0 \quad \text{for } l \notin \{2,3\} \text{ , i.e. } H^1(I_{X_1}(l)) = 0 \quad \text{for } l \notin \{3,4\}$$

But then there is an open set U of the <u>component</u> V which is contained in the set $U_{\underline{f}}(16,29)$ of theorem 1 because $H^1(I_{X_1}(5)) = 0 = H^1(I_{X_1}(1))$.

The corresponding family $p'(p^{-1}(U))$ obtained by liaison is therefore open in H(9,8), hence form an irreducible <u>component</u> V' of H(9,8). The general curve $X_1' \subseteq P$ of V' satisfies, however, $s(X_1') = \min(5, f_1+f_2-4-e(X_1)) = 3$, and using proposition 6ii) we get the contradiction

$$\operatorname{codim}_{V'} H(9,8) = h^1(I_{X_1'}(3)) = 1.$$

<u>References</u>

(B) Barth, W.: Some properties of stable rank 2 vector bundles on $\mathbb{P}_n$. Math. Ann. 226, 125 - 150 (1977)

(Bu) Buchweitz. R.O: Contributions à la theorie des singularités. Univ. Paris VI (Thesis)

(D) Demazure M.: Surface de Del Pezzo II. Eclater n points dans $\mathbb{P}^2$. Sem. sur les sing. des surfaces. Centre de Math. Ecole Polytechnique (1976 - 1977)

(H) Hartshorne, R. Algebra Geometry. Springer Verlag (1977).

(Kl) Kleppe, J.O. The Hilbert-flag scheme, its properties and its connection with the Hilbert scheme. Applications to curves in 3-space. No 5-March 31-1981. Preprint Ser, Inst of Math., Univ. of Oslo (and part of thesis).

(Kl1) Kleppe, J.O. : Deformations of reflective sheaves of rank 2 on $\mathbb{P}^3$. No 4 - March 10 - 1982. Preprint Ser. Inst. of Math., Univ. of Oslo.

(L) Laudal, O.A. : Formal Moduli of Algebraic Structures Springer Lecture Notes 754 (1979).

(M) Mumford, D. Further Pathologies in Algebraic Geometry. Amer. J. Math. 84 642-648 (1962).

(M1) Mumford, D. : Lectures on Curves on an Algebraic Surface. Annals of Math. Studies 59, Princeton Univ. Press (1966).

(N) Noether, M. Zur Grundlegung der Theorie der algebraischen Raumcurven. J. Reine u. Angew. Math. 271 - 318 (1882) (or Steinerische Preisschrift, Verlag d. Kønig. Akad.d Wiss., Berlin 1883).

(PS) Peskine, C and Szpiro, L.: Liaison des variétés algébriques. Invent. Math. 26, 271 - 302 (1974)

(SGA2) Grothendieck, A.: Cohomologie locale des faisceaux coherents et Théorèmes de Lefshetz locaux et globaux. North-Holland, Amsterdam (1968).

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