Liaison invariants and the Hilbert scheme of codimension 2 subschemes in $\mathbb{P}^{n+2}$

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Abstract

In this paper we study the Hilbert scheme $\mathrm{Hilb}^{p(v)}(\mathbb{P})$ of equidimensional locally Cohen-Macaulay codimension 2 subschemes, with a special look to surfaces in $\mathbb{P}^4$ and 3-folds in $\mathbb{P}^5$, and the Hilbert scheme stratification $\mathrm{H}_{\gamma,\rho}$ of constant cohomology. For every $(X) \in \mathrm{Hilb}^{p(v)}(\mathbb{P})$ we define a number δ_X in terms of the graded Betti numbers of the homogeneous ideal of X and we prove that $1 + \delta_X - \dim_{(X)} \mathrm{H}_{\gamma,\rho}$ and $1 + \delta_X - \dim T_{\gamma,\rho}$ are CI-biliaison invariants where $T_{\gamma,\rho}$ is the tangent space of $\mathrm{H}_{\gamma,\rho}$ at (X). As a corollary we get a formula for the dimension of any generically smooth component of $\mathrm{Hilb}^{p(v)}(\mathbb{P})$ in terms of δ_X and the CI-biliaison invariant. Both invariants are equal in this case.

Recall that, for space curves C, Martin-Deschamps and Perrin have proved the smoothness of the "morphism" $\phi : \mathrm{H}_{\gamma,\rho} \to E_\rho :=$ isomorphism classes of graded modules M satisfying $\dim M_v = \rho(v)$, given by sending C onto its Rao module. For surfaces X in $\mathbb{P}^4$ we have two Rao modules $M_i \simeq \oplus H^i(\mathcal{I}_X(v))$ of dimension $\rho_i(v)$, $\rho := (\rho_1, \rho_2)$ and an induced extension $b \in {}_0\mathrm{Ext}^2(M_2, M_1)$ and a result of Horrocks and Rao saying that a triple $D := (M_1, M_2, b)$ of modules M_i of finite length and an extension b as above determine a surface X up to biliaison. We prove that the corresponding "morphism" $\varphi : \mathrm{H}_{\gamma,\rho} \to \mathrm{V}_\rho =$ isomorphism classes of graded modules M_i satisfying $\dim(M_i)_v = \rho_i(v)$ and commuting with b, is smooth, and we get a smoothness criterion for $\mathrm{H}_{\gamma,\rho}$, i.e. for the equality of the two biliaison invariants. Moreover we get some smoothness results for $\mathrm{Hilb}^{p(v)}(\mathbb{P})$, valid also for 3-folds, and we give examples of obstructed surfaces and 3-folds. The linkage result we prove in this paper turns out to be useful in determining the structure and dimension of $\mathrm{H}_{\gamma,\rho}$, and for proving the main biliaison theorem above.

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Contents

1 Introduction.

A main object of this paper is to find the dimension of the Hilbert scheme, $\mathrm{Hilb}^{p(v)}(\mathbb{P})$, of equidimensional locally Cohen-Macaulay (lCM) codimension 2 subschemes of $\mathbb{P} := \mathbb{P}^{n+2}$. As an initial ambitious goal we look for a formula for the dimension of any reduced component V of the Hilbert scheme $\mathrm{Hilb}^{p(v)}(\mathbb{P})$ in terms of the graded Betti numbers of the homogeneous ideal I_X of a general element (X) of V. Since we expect the matrices in the minimal resolution of I_X to play a role, it seems natural to modify our goal by introducing a biliaison invariant in the dimension formula. Indeed in this paper we explicitly define an invariant $\delta_X^{n+1}(-n-3)$ in terms of the graded Betti numbers of I_X and $H_*^n(\mathcal{O}_X)$ and we prove that

$$\dim V = 1 + \delta_X^{n+1}(-n-3) - \mathrm{sumext}(X)$$

where $\mathrm{sumext}(X)$ is a CI-biliaison invariant (Corollary 9.4). In the case X is a curve ($n = 1$) with Hartshorne-Rao module M, we use results of [38] to prove

$$\mathrm{sumext}(X) = \sum_{i=0}^{1} {}_0\mathrm{ext}_R^i(M, M) \ ,$$

(Theorem 3.7 and Remark 3.9) and there is a similar, but much more complicated formula in the surface case (which we may deduce from Remark 6.4).

Let $\mathrm{H}_{\gamma,\rho} \subseteq \mathrm{Hilb}^{p(v)}(\mathbb{P})$ be the Hilbert scheme whose k-points (X) corresponds to equidimensional lCM codimension 2 subschemes X of $\mathbb{P}^{n+2}$ with constant cohomology (see [38] for the curve case). If X is *any* equidimensional lCM codimension 2 subscheme of $\mathbb{P}$, we define $\mathrm{obsumext}(X)$ in the following way,

$$\mathrm{obsumext}(X) = 1 + \delta_X^{n+1}(-n-3) - \dim_{(X)} \mathrm{H}_{\gamma,\rho} \, .$$

We define $\mathrm{sumext}(X)$ by the same expression provided we have replaced $\mathrm{H}_{\gamma,\rho}$ by its tangent space, $T_{\gamma,\rho}$, at (X). Then we prove that $\mathrm{sumext}(X)$ and $\mathrm{obsumext}(X)$ are CI-biliaison invariants (Theorem 9.1). Since every arithmetically Cohen-Macaulay codimension 2 subscheme is in the liaison class of a complete intersection (CI) by Gaeta's theorem, it follows that $\mathrm{sumext}(X) = \mathrm{obsumext}(X) = 0$ and that $\dim_{(X)} \mathrm{Hilb}^{p(v)}(\mathbb{P}) = 1 + \delta_X^{n+1}(-n-3)$ for $n > 0$ if X is arithmetically Cohen-Macaulay (Corollary 9.6). Even though we do not prove the explicit expression of $\mathrm{sumext}(X)$ in terms the Rao modules of X in general, the theorem is motivated from the fact that the Rao modules are invariant under biliaison up to shift. In fact it seems more effective to compute $\mathrm{sumext}(X)$ and $\mathrm{obsumext}(X)$ by considering a nice representative X' in its even liaison class, e.g. the minimal element, and to compute $\delta_{X'}^{n+1}(-n-3)$, $\dim_{(X')} \mathrm{H}_{\gamma,\rho}$, and $\dim T_{\gamma,\rho}$ for X'.

Since the curve case of the results above is rather well understood ([38], [33]), we will in the present paper mostly concentrate on the study of the Hilbert scheme $\mathrm{H}(d,p,\pi)$ of surfaces of degree d and arithmetic (resp. sectional) genus p (resp. π). Recall that, for space curves C, Martin-Deschamps and Perrin proved the smoothness of the "morphism" $\phi : \mathrm{H}_{\gamma,\rho} \to E_\rho$: $=$ isomorphism classes of graded R-modules M satisfying $\dim M_v = \rho(v)$, given by sending C onto its Rao module. Earlier Rao proved that any graded R-module M of finite length determines the liaison class of a curve, up to dual and shift in the grading ([47]). Note that Rao's result is related to the surjectivity of ϕ, while the smoothness of ϕ implies infinitesimal surjectivity. For surfaces in $\mathbb{P}^4$ there is a result in Bolondi's paper [6], stating that a triple $D := (M_1, M_2, b)$ of graded modules M_i of finite length and an extension $b \in {}_0\mathrm{Ext}^2(M_2, M_1)$ determine the biliaison class of a surface X such that $M_i \simeq \oplus H^i(\mathcal{I}_X(v))$ modulo some shift in the grading. The result is a consequence of the main theorem of [48] and Horrocks' classification of stable vector bundles ([24]), as mentioned by Rao in

[48]. Therefore it is natural to consider the stratification $\mathrm{H}_{\gamma,\rho}$ of $\mathrm{H}(d,p,\pi)$ where now $\rho := (\rho_1, \rho_2)$ and $\rho_i(v) = \dim H^i(\mathcal{I}_X(v))$, and to ask for the smoothness of the corresponding "morphism" $\varphi : \mathrm{H}_{\gamma,\rho} \to \mathrm{V}_\rho :=$ isomorphism classes of triples (M_1, M_2, b) where M_i are graded R-modules which satisfy $\dim(M_i)_v = \rho_i(v)$ and where an isomorhpism between triples is an isomorphism between the corresponding modules which commutes with the extensions. We prove in section 5 that the answer is yes (Theorem 5.3). As a corollary we get a smoothness criterion for $\mathrm{H}_{\gamma,\rho}$ (Corollary 5.4, Remark 6.3), i.e. for the equality $\mathrm{sumext}(X) = \mathrm{obsumext}(X)$ to hold. Note that we do not prove that φ extends to a morphism of schemes; we only prove that the corresponding morphism of the local deformation functors is formally smooth. This, however, takes fully care of what we want.

In section 6 we determine the tangent space of $\mathrm{H}_{\gamma,\rho}$ at (X), and we prove a local isomorphism $\mathrm{H}_{\gamma,\rho} \simeq \mathrm{H}(d,p,\pi)$ at (X) under some conditions (Proposition 6.1, Remark 6.2). Note, however, that if X has seminatural cohomology, we know that $\mathrm{H}_{\gamma,\rho} \simeq \mathrm{H}(d,p,\pi)$ at (X) by the semicontinuity of $\dim H^i(\mathcal{I}_X(v))$ and this observation mostly suffices for our applications. In section 7 we prove a useful linkage result (Theorem 7.1) which we apply to determine the structure and the dimension of $\mathrm{H}_{\gamma,\rho}$ and to prove our main theorem on the biliaison invariants. In this section we also give conditions for a linked surface to be e.g. non-generic, thus proving the existence of surfaces with "smaller" cohomology in some cases (Proposition 7.4).

Since the technical problems in describing well the stratification of $\mathrm{H}(d,p,\pi)$ and the morphism φ are quite complicated (see [31]), we don't follow up this trace for equidimensional lCM codimension 2 subschemes $X \subseteq \mathbb{P}^{n+2}$ of dimension $n \geq 3$. Instead we only use our main theorem on the biliaison invariance of $\mathrm{sumext}(X)$ and $\mathrm{obsumext}(X)$ together with some new results on the smoothness and the dimension of $\mathrm{Hilb}^{p(v)}(\mathbb{P})$ in our study of the Hilbert schemes of e.g. 3-folds in section 9. We also give a vanishing criterion for $H^1(\mathcal{N}_X)$, but unfortunately, as in [33], the results we get require that the Hartshorne-Rao modules are rather "small". When the conditions of these vanishing criteria do not hold, we give examples of obstructed surfaces and 3-folds.

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2 Notations and terminology.

A surface (resp. curve) X is an equidimensional, locally Cohen-Macaulay subscheme (lCM) of $\mathbb{P}^4$ (resp. $\mathbb{P}^3$) of dimension 2 (resp. 1) with sheaf ideal $\mathcal{I}_X$ and normal sheaf $\mathcal{N}_X = \mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_X, \mathcal{O}_X)$. If $\mathcal{F}$ is a coherent $\mathcal{O}_\mathbb{P}$-Module, we let $H^i(\mathcal{F}) = H^i(\mathbb{P}, \mathcal{F})$, $H^i_*(\mathcal{F}) = \oplus_v H^i(\mathcal{F}(v))$ and $h^i(\mathcal{F}) = \dim H^i(\mathcal{F})$, and we denote by $\chi(\mathcal{F}) = \Sigma(-1)^i h^i(\mathcal{F})$ the Euler-Poincaré characteristic. Then $p(v) = \chi(\mathcal{O}_X(v))$ is the Hilbert polynomial of X. Put $n = \dim X$ and

$$s(X) = \min\{v | h^0(\mathcal{I}_X(v)) \neq 0\},$$
$$e(X) = \max\{v | h^n(\mathcal{O}_X(v)) \neq 0\}.$$

Let $I = I_X = H^0_*(\mathcal{I}_X)$ be the homogeneous ideal. I is a graded module over the polynomial ring $R = k[X_0, X_1, .., X_{n+2}]$, where k is supposed to be algebraically closed (and of characteristic zero in section 5, 6 and in all examples since we there may use results and methods of papers relying on this assumption). The postulation γ of X is the function defined over the integers by $\gamma(v) = \gamma_X(v) = h^0(\mathcal{I}_X(v))$.

Let $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ denote the Hilbert scheme of equidimensional lCM codimension 2 subschemes of $\mathbb{P}^{n+2}$ with Hilbert polynomial p (cf. [21]). X is called *unobstructed* if $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ is smooth at the corresponding point (X), otherwise X is obstructed. A subscheme of $\mathbb{P}^{n+2}$ belonging to a sufficiently small open irreducible subset U of $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ (small enough so that any (X) of U satisfies all the openness properties which we want it to have) is called a *generic* subscheme of $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$, and accordingly, if we state that a generic subscheme has a certain property, then there is a non-empty open irreducible subset of $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ of subschemes having this property.

In the case of curves we put $\mathrm{H}(d,g) = \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ provided $p(v) = dv+1-g$. Moreover we let $M = M(C) := H^1_*(\mathcal{I}_C)$ be the deficiency or Hartshorne-Rao module of the curve C. The deficiency function ρ is defined by $\rho(v) = h^1(\mathcal{I}_C(v))$. Let $\mathrm{H}(d,g)_{\gamma,\rho}$ (resp. $\mathrm{H}(d,g)_\gamma$) denote the subscheme of $\mathrm{H}(d,g)$ of curves with constant cohomology given by γ and ρ, (resp. constant postulation γ) where "constant" means flat deformations of the corresponding modules, see [38]. Let Def_M be the local deformation functor consisting of graded deformations M_S of M to $\mathbb{P}^3 \times \mathrm{Spec}(S)$ modulo graded isomorphisms of M_S over M, where S is a local artinian k-algebra with residue field k, i.e. such that M_S is S-flat and $M_S \otimes k = M$.

For a surface X we define the arithmetic genus p by $p = \chi(\mathcal{O}_X) - 1$, while the sectional genus π is given by $\chi(\mathcal{O}_X(1)) = d - \pi + 1 + \chi(\mathcal{O}_X)$. By Riemann-Roch's theorem we have

$$p(v) = \chi(\mathcal{O}_X(v)) = \frac{1}{2}dv^2 - (\pi - 1 - \frac{1}{2}d)v + \chi(\mathcal{O}_X). \tag{1}$$

Put $\mathrm{H}(d,p,\pi) = \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ in this case. Moreover let $M_i = M_i(X)$ be the deficiency modules $H^i_*(\mathcal{I}_X)$ for i = 1,2. The deficiency $\rho = (\rho_1, \rho_2)$ of X is the function defined over the integers by $\rho(v) = \rho_X(v) = (\rho_1(v), \rho_2(v))$ where $\rho_i(v) = h^i(\mathcal{I}_X(v))$ for $i = 1,2$. Let $\mathrm{H}_{\gamma,\rho} = \mathrm{H}(d,p,\pi)_{\gamma,\rho}$ (resp. $\mathrm{H}_\gamma = \mathrm{H}(d,p,\pi)_\gamma$) denote the subscheme of $\mathrm{H}(d,p,\pi)$ of surfaces with constant cohomology given by γ and ρ, (resp. constant postulation γ) where again "constant" means flat deformations of the corresponding modules.

For the notion of linkage, we refer to [39]. Note that liaison (resp. even liaison or biliaison) is the equivalence relation generated by linkage (resp. direct linkages in an even number of steps).

For any graded R-module N, we have the right derived functors $H^i_{\mathfrak{m}}(N)$ and ${}_v\mathrm{Ext}^i_{\mathfrak{m}}(N,-)$ of $\Gamma_{\mathfrak{m}}(N) := \oplus_v \ker(N_v \to \Gamma(\mathbb{P}, \tilde{N}(v)))$ and $\Gamma_{\mathfrak{m}}(\mathrm{Hom}_R(N,-))_v$ respectively (cf. [20], exp. VI or [22]) where $\mathfrak{m} = (X_0, .., X_{n+2})$. We use small letters for the k-dimension and subscript v for the homogeneous part of degree v, e.g. ${}_v\mathrm{ext}^i_{\mathfrak{m}}(N_1,N_2) = \dim {}_v\mathrm{Ext}^i_{\mathfrak{m}}(N_1,N_2)$.

Let N_1 and N_2 be graded R-modules of finite type. As in [33] we need the spectral sequence

$$E_2^{p,q} = {}_v\mathrm{Ext}^p_R(N_1, H^q_{\mathfrak{m}}(N_2)) \Rightarrow {}_v\mathrm{Ext}^{p+q}_{\mathfrak{m}}(N_1,N_2) \tag{2}$$

([20], exp. VI) and the duality isomorphism

$${}_v\mathrm{Ext}^i_{\mathfrak{m}}(N_2,N_1) \cong {}_{-v-n-3}\mathrm{Ext}^{n+3-i}_R(N_1,N_2)^\vee, \quad i,v \in \mathbb{Z} \tag{3}$$

where $(-)^\vee = \mathrm{Hom}_k(-,k)$ (cf. [30], Thm. 1.1, see [28], Thm. 2.1.4 for a full proof). Moreover there is a long exact sequence

$$\to {}_v\mathrm{Ext}^i_{\mathfrak{m}}(N_1,N_2) \to {}_v\mathrm{Ext}^i_R(N_1,N_2) \to \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}}}(\tilde{N}_1, \tilde{N}_2(v)) \to {}_v\mathrm{Ext}^{i+1}_{\mathfrak{m}}(N_1,N_2) \to \tag{4}$$

([20], exp. VI) which at least for equidimensional, lCM subschemes of codimension 2 (with $n > 0$) relate the deformation theory of X, described by $H^{i-1}(\mathcal{N}_X) \simeq \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}}}(\tilde{I}, \tilde{I})$ for $i = 1,2$ (cf. [28], Rem. 2.2.6), to the deformation theory of the homogeneous ideal $I = I_X$, described by ${}_0\mathrm{Ext}^i_R(I,I)$, in the following exact sequence

$$\begin{aligned} 0 \to {}_v\mathrm{Ext}^1_R(I,I) \to H^0(\mathcal{N}_X(v)) \to {}_v\mathrm{Ext}^2_{\mathfrak{m}}(I,I) \\ \xrightarrow{\alpha} {}_v\mathrm{Ext}^2_R(I,I) \to H^1(\mathcal{N}_X(v)) \to {}_v\mathrm{Ext}^3_{\mathfrak{m}}(I,I) \to \end{aligned} \tag{5}$$

see [49] or [17] for related works on such deformation functors.

3 The dimension of $\mathrm{H}(d,g)$ and biliaison invariants.

In this section we consider the Hilbert scheme, $\mathrm{H}(d,g)$, of curves in $\mathbb{P}^3$ and results which we would like to generalize to surfaces in $\mathbb{P}^4$. We will focus on the dimension of the Hilbert schemes and some biliaison invariants which we naturally detect from this point of view.

Recall that $\chi(\mathcal{N}_C(v)) = 2dv + 4d$ and that $\chi(\mathcal{N}_C) = 4d$ is a lower bound for $\dim_{(C)} \mathrm{H}(d,g)$. For this reason the number $4d$ is often called the expected dimension of $\mathrm{H}(d,g)$ even though it often does not give the correct dimension of $\mathrm{H}(d,g)$ at (C). E.g. at ACM, generically comlete intersection curves the dimension is never $4d$ if $e(C) \geq s(C)$.

To give a more reliable estimate for the dimension of the components of $\mathrm{H}(d,g)$, we have found it convenient to introduce the following invariant, defined in terms of the numbers $n_{j,i}$ appearing in a minimal resolution of the homogeneous ideal I_C of C:

$$0 \to \bigoplus_{i=1}^{r_3} R(-n_{3,i}) \to \bigoplus_{i=1}^{r_2} R(-n_{2,i}) \to \bigoplus_{i=1}^{r_1} R(-n_{1,i}) \to I_C \to 0 \ . \tag{6}$$

Note that we can define the graded Betti numbers, $\beta_{j,k}$, of I_C by just putting $\oplus_{k=1}^{\infty} R(-k)^{\beta_{j,k}} := \oplus_{i=1}^{r_j} R(-n_{j,i})$.

Definition 3.1. *If C is a curve in $\mathbb{P}^3$, we let*

$$\delta^j_C(v) := \sum_i h^j(\mathcal{I}_C(n_{1,i}+v)) - \sum_i h^j(\mathcal{I}_C(n_{2,i}+v)) + \sum_i h^j(\mathcal{I}_C(n_{3,i}+v)).$$

Put $\delta^j(v) = \delta^j_C(v)$. In [33] we proved the following result (Lem. 2.2 of [33])

Lemma 3.2. *Let C be any curve of degree d in $\mathbb{P}^3$. Then the following expressions are equal*

$${}_0\mathrm{ext}^1_R(I_C,I_C) - {}_0\mathrm{ext}^2_R(I_C,I_C) = 1 - \delta^0(0) = 4d + \delta^2(0) - \delta^1(0) = 1 + \delta^2(-4) - \delta^1(-4).$$

Remark 3.3. *Comparing with the results and notations of [38] we recognize $1 - \delta^0(0)$ as δ_γ and $\delta^1(-4)$ as $\epsilon_{\gamma,\delta}$ in their terminology. By Lemma 3.2 it follows that the dimension of the Hilbert scheme $\mathrm{H}_{\gamma,M}$ of constant postulation and Rao module, which they show is $\delta_\gamma + \epsilon_{\gamma,\delta} - {}_0\mathrm{hom}(M,M)$ (Thm. 3.8, page 171), is also equal to $1 + \delta^2(-4) - {}_0\mathrm{hom}(M,M)$.*

Note that the difference of the ext-numbers in Lemma 3.2 is a lower bound for $\dim O_{\mathrm{H}(d,g)_\gamma,(C)}$ ([33], proof of Thm. 2.6 (i)). Mainly since $\mathrm{H}(d,g)_\gamma$ is a subscheme of $\mathrm{H}(d,g)$, we used this lower bound in [35], Thm. 24, to prove the following result

Theorem 3.4. *Let C be a curve in $\mathbb{P}^3$ and let $\delta^j(v) = \delta^j_C(v)$ for any j and v. Then the dimension of $\mathrm{H}(d,g)$ at (C) satisfies*

$$\dim_{(C)} \mathrm{H}(d,g) \geq 1 - \delta^0(0) = 4d + \delta^2(0) - \delta^1(0).$$

Moreover if C is a generic curve of a generically smooth component V of $\mathrm{H}(d,g)$ *and* $M = H^1_*(\mathcal{I}_C)$, *then*

$$\dim V = 4d + \delta^2(0) - \delta^1(0) + \ {}_{-4}\mathrm{hom}_R(I_C, M)$$

where ${}_{-4}\mathrm{Hom}_R(I_C, M)$ *is the kernel of the map*

$$\bigoplus_i H^1(\mathcal{I}_C(n_{1,i}-4)) \to \bigoplus_i H^1(\mathcal{I}_C(n_{2,i}-4))$$

induced by the corresponding map in (6).

Remark 3.5. *Let C be any curve in $\mathbb{P}^3$ and suppose*

$${}_{-4}\mathrm{Hom}_R(I_C, M) = {}_0\mathrm{Hom}_R(I_C, M) = 0.$$

Then C is unobstructed and the lower bound of the inequality of Theorem 3.4 is equal to $\dim_{(C)} \mathrm{H}(d,g)$ *by Thm. 2.6 of [33].*

Remark 3.6. *Let C be any curve in $\mathbb{P}^3$.*
(i) *If $M = 0$, then $\delta^1(0) = 0$ and we can use Remark 3.5 to see that C is unobstructed and that the lower bound of Theorem 3.4 is equal to* $\dim_{(C)} \mathrm{H}(d,g)$. *This coincides with [13].*
(ii) *If* $\operatorname{diam} M = 1$, $\dim M = r$ *and C is a generic curve, then C is unobstructed by [33] Cor. 1.6 and the lower bound is equal to $4d + \delta^2(0) + r\beta_{2,c}$. Indeed $r\beta_{1,c} = 0$ for a generic curve by [33], Cor. 4.4. Moreover in this case the "correction" number ${}_{-4}\mathrm{hom}_R(I_C, M)$ is equal to $r\beta_{1,c+4}$. Hence we get*

$$\dim V = 4d + \delta^2(0) + r(\beta_{2,c} + \beta_{1,c+4}).$$

This coincides with the dimension formula of [33], Thm. 3.4.

Theorem 3.4 is a consequence of the inclusion $\mathrm{H}(d,g)_\gamma \hookrightarrow \mathrm{H}(d,g)$ of schemes. One may try the same argument for the inclusion $\mathrm{H}(d,g)_{\gamma,\rho} \hookrightarrow \mathrm{H}(d,g)$ since we also for these schemes know tangent and obstruction spaces. This leads to

Theorem 3.7. *Let C be a curve in $\mathbb{P}^3$ and $M = H^1_*(\mathcal{I}_C)$. Then the dimension of* $\mathrm{H}(d,g)$ *at* (C) *satisfies*

$$\dim_{(C)} \mathrm{H}(d,g) \geq 1 + \delta^2(-4) - \sum_{i=0}^{2} {}_0\mathrm{ext}^i_R(M, M).$$

Moreover if C is a generic curve of a generically smooth component V of $\mathrm{H}(d,g)$, *then*

$$\begin{aligned}\dim V &= 4d + \delta^2(0) - \delta^1(0) + \delta^1(-4) - \sum_{i=0}^{1} {}_0\mathrm{ext}^i_R(M, M) \\ &= 1 + \delta^2(-4) - \sum_{i=0}^{1} {}_0\mathrm{ext}^i_R(M, M).\end{aligned}$$

Proof. We consider the stratification $\mathrm{H}(d,g)_{\gamma,\rho}$ of the Hilbert scheme $\mathrm{H}(d,g)$ and the "morphism" $\phi :$ $\mathrm{H}(d,g)_{\gamma,\rho} \to E_\rho$: $=$ isomorphism classes of R-modules M given by mapping (C) onto $M(C)$. By [38], Thm. 1.5, ϕ is smooth, and $\mathrm{H}(d,g)_{\gamma,M} := \phi^{-1}(M)$ is a scheme of dimension $1+\delta^2(-4)-{}_0\mathrm{hom}(M,M)$ (see Remark 3.3). If we ignore the scheme structures, we may still, for each curve C, consider the corresponding local deformation functor, ϕ_C, of ϕ at (C), defined on the category of local artinian k-algebras with residue field k. ϕ_C is smooth of fiber dimension as above by the results of [38], see also [33], Rem. 2.12 for the curve case and Theorem 5.3 of this paper for the corresponding result for surfaces.

It is well known that ${}_0\mathrm{Ext}^i_R(M,M)$ for $i = 1,2$, determine the local graded deformation functor, Def_M, of the R-module $M := M(C)$, e.g.

$$ {}_0\mathrm{ext}^1(M,M) - {}_0\mathrm{ext}^2(M,M) \le \dim E_{\rho,M} \le {}_0\mathrm{ext}^1(M,M), $$

where $E_{\rho,M}$ is the hull of Def_M ([37], Thm. 4.2.4). Moreover we have equality to the right if and only if Def_M is formally smooth. Combining with the smoothness of ϕ_C and its fiber dimension we get

$$ \begin{aligned} 1+\delta^2(-4) - \sum_{i=0}^{2} {}_0\mathrm{ext}^i(M,M) &\le \dim_{(C)} \mathrm{H}(d,g)_{\gamma,\rho} \\ &\le 1+\delta^2(-4) - {}_0\mathrm{hom}(M,M) + {}_0\mathrm{ext}^1(M,M) \end{aligned} \tag{7} $$

with equality to the right if and only if $\mathrm{H}(d,g)_{\gamma,\rho}$ is smooth at (C). This proves the inequality of the theorem since $\dim_{(C)} \mathrm{H}(d,g) \ge \dim_{(C)} \mathrm{H}(d,g)_{\gamma,\rho}$. We also get the final statement because, at a generic curve C with postulation γ and deficiency ρ, $\mathrm{H}(d,g)_{\gamma,\rho} \cong \mathrm{H}(d,g)$ around (C)! Indeed if we have $\dim_{(C)} \mathrm{H}(d,g)_{\gamma,\rho} < \dim_{(C)} \mathrm{H}(d,g)$, then a small neighborhood of (C) in $\mathrm{H}(d,g)_{\gamma,\rho}$ is not *open* in $\mathrm{H}(d,g)$, contradicting the assumption that C is generic in $\mathrm{H}(d,g)$. Hence we have equality in dimensions and in fact a local isomorphism (e.g. by generic flatness) since $\mathrm{H}(d,g)$ is smooth at (C). It follows that $\mathrm{H}(d,g)_{\gamma,\rho}$ is smooth at (C) and the inequality of (7) to the right is an equality. □

Remark 3.8. *Let $T_{\gamma,\rho}$ be the tangent space of $\mathrm{H}(d,g)_{\gamma,\rho}$ at (C). Then we easily see from the proof that the upper bound in (7) is equal to $\dim T_{\gamma,\rho}$.*

If we want to generalize Theorem 3.7 to codimension 2 subschemes in $\mathbb{P}^{n+2}$, the explicit replacements of $\sum_{i=0}^{1} {}_0\mathrm{ext}^i(M,M)$ in the generalized statements seem to be very complicated. However observing that $\sum_{i=0}^{1} {}_0\mathrm{ext}^i(M,M)$ is a biliaison invariant (since M is, up to a twist), it seems to be the following weaker form of Theorem 3.7 and (7) which is natural to generalize:

Remark 3.9. *If we define* $\mathrm{sumext}(C)$ *and* $\mathrm{obsumext}(C)$ *by* $\mathrm{sumext}(C) = 1+\delta^2(-4) - \dim T_{\gamma,\rho}$ *and* $\mathrm{obsumext}(C) = 1+\delta^2(-4) - \dim_{(C)} \mathrm{H}(d,g)_{\gamma,\rho}$, *then* $\mathrm{sumext}(C)$ *and* $\mathrm{obsumext}(C)$ *are biliaison invariants. We have*

$$ \mathrm{sumext}(C) \le \mathrm{obsumext}(C) $$

and equality holds if and only if $\mathrm{H}(d,g)_{\gamma,\rho}$ is smooth at (C). Furthermore if C is unobstructed and generic in $\mathrm{H}(d,g)$, then

$$ \dim_{(C)} \mathrm{H}(d,g) = 1+\delta^2(-4) - \mathrm{sumext}(C) \ . $$

We have not yet proved that $\mathrm{obsumext}(C)$ is a biliaison invariant, but it will follow from later results, or from [38], Thm. 1.5 and Remark 3.3.

For curves we have

$$ \mathrm{sumext}(C) = \sum_{i=0}^{1} {}_0\mathrm{ext}^i_R(M,M) \ , \quad \text{and} \tag{8} $$

$$ \sum_{i=0}^{1} {}_0\mathrm{ext}^i_R(M,M) \le \mathrm{obsumext}(C) \le \sum_{i=0}^{2} {}_0\mathrm{ext}^i_R(M,M) \tag{9} $$

which we may use to compute $\mathrm{sumext}(C)$ and estimate $\mathrm{obsumext}(C)$. We may also compute these invariants somewhere in the even liaison class, e.g. by letting C be the minimal curve and computing

$\dim_{(C)} \mathrm{H}(d,g)_{\gamma,\rho}$, $\dim T_{\gamma,\rho}$ and $\delta^2(-4)$ in this case. If D is in the even liaison class of C, $D \in \mathrm{H}_{\gamma',\rho'}$, and if we can compute $\delta^2_D(-4)$, then we get the dimensions of $\mathrm{H}_{\gamma',\rho'}$ and $T_{\gamma',\rho'}$, from the biliaison invariants.

4 The dimension and the smoothness of $\mathrm{H}(d,p,\pi)$.

In this section we consider the Hilbert scheme, $\mathrm{H}(d,p,\pi)$, of surfaces in $\mathbb{P}^4$. Our goal is to see how far we can generalize the results of the preceding section to surfaces. We will focus on the dimension and the smoothness of the Hilbert scheme.

To compute the dimension of the components of $\mathrm{H}(d,p,\pi)$, we consider the minimal resolution of $I = I_X$:

$$0 \to \bigoplus_{i=1}^{r_4} R(-n_{4,i}) \to \bigoplus_{i=1}^{r_3} R(-n_{3,i}) \to \bigoplus_{i=1}^{r_2} R(-n_{2,i}) \to \bigoplus_{i=1}^{r_1} R(-n_{1,i}) \to I \to 0, \tag{10}$$

and the invariant $\delta^j(v) = \delta^j_X(v)$ defined by

$$\begin{aligned} \delta^j_X(v) = \sum_i h^j(\mathcal{I}_X(n_{1,i}+v)) - \sum_i h^j(\mathcal{I}_X(n_{2,i}+v)) \\ + \sum_i h^j(\mathcal{I}_X(n_{3,i}+v)) - \sum_i h^j(\mathcal{I}_X(n_{4,i}+v)). \end{aligned} \tag{11}$$

Proposition 4.1. *Let X be any surface in $\mathbb{P}^4$ of degree d and sectional genus π. Then the following expressions are equal*

$$\begin{aligned} {}_0\mathrm{ext}^1_R(I,I) - {}_0\mathrm{ext}^2_R(I,I) + {}_0\mathrm{ext}^3_R(I,I) = 1 - \delta^0(0) = \chi(\mathcal{N}_X) - \delta^0(-5) \\ = \chi(\mathcal{N}_X) - \delta^3(0) + \delta^2(0) - \delta^1(0) = 1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5). \end{aligned} \tag{12}$$

Moreover

$$\chi(\mathcal{N}_X(v)) = dv^2 + 5dv + 5(2d+\pi-1) - d^2 + 2\chi(\mathcal{O}_X). \tag{13}$$

Proof. The first upper equality follows easily by applying ${}_v\mathrm{Hom}_R(-,I)$ (for $v=0$) to the resolution (10) because $\mathrm{Hom}_R(I,I) \simeq R$ and because the alternating sum of the dimension of the terms in a complex equals the alternating sum of the dimension of its homology groups. Similarly we compute $\delta^0(-5)$ which through the duality (3) leads to the alternating sum of ${}_0\mathrm{ext}^i_{\mathfrak{m}}(I,I)$. Combining with (5), recalling $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_X,\mathcal{I}_X) \cong \mathcal{O}_{\mathbb{P}}$ and $\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_X,\mathcal{I}_X) \cong \mathcal{N}_X$, we get the next equality in the first line. The other equalities involving $\delta^j(v)$ follow from (2), (3) and (4) as outlined in [33], Lem 2.2 in the curve case. The surface case is technically more complicated because the spectral sequence of the proof, $E_2^{p,q} = {}_v\mathrm{Ext}^p_R(I, H^q_{\mathfrak{m}}(I))$, contains one more non-vanishing term. The principal parts of the proof are, however, the same, and we leave this part to the reader. Similarly the arguments of [33], Rem 2.4, lead to the formula

$$\chi(\mathcal{N}_X(v)) = \chi(\mathcal{O}_X(v)) + \chi(\mathcal{O}_X(-v-5)) - d^2 \tag{14}$$

for any surface X, from which (13) of Proposition 4.1 easily follows provided we combine with (1). Since we do not have a reference of (13) in the generality of an *arbitrary surface* (i.e. locally Cohen-Macaulay and equidimensional, see Remark below) and since the arguments of [33], Rem 2.4 was only sketched, we will include a proof of (14).

Firstly, we compute $\chi(\mathcal{O}_X(v)) = \chi(\mathcal{O}_{\mathbb{P}}(v)) - \chi(\mathcal{I}_X(v))$, $\chi(\mathcal{O}_{\mathbb{P}}(v)) = \binom{v+4}{4}$, directly from (10) as a large sum of binomials. Recalling that $\chi(\mathcal{O}_X(v))$ is the polynomial (1) of degree 2, we get

$$\sum_{j=1}^{4}(-1)^{j-1}r_j = 1 \ , \ \sum_{j=1}^{4}(-1)^{j-1}\sum_i n_{j,i} = 0 \ \text{ and } \ \sum_{j=1}^{4}(-1)^{j-1}\sum_i n_{j,i}^2 = -2d \ . \qquad (15)$$

Now as in the very first part of the proof, we apply ${}_v\mathrm{Hom}_R(-,I)$ to (10). Since we get ${}_v\mathrm{Ext}^i_R(I,I) \cong H^{i-1}(\mathcal{N}_X(v))$ for $v >> 0$ and $i \geq 1$ directly from (2), (3) and (4) and we have $\mathrm{Hom}_R(I,I) \simeq R$, we find

$$\dim R_v \ - \ \chi(\mathcal{N}_X(v)) \ = \ \delta^0(v) \ = \ \sum_{j=1}^{4}(-1)^{j-1}\sum_i \chi(\mathcal{I}_X(n_{j,i}+v)) \ , \quad v \ >> \ 0 \ . \qquad (16)$$

By (10),

$$\chi(\mathcal{I}_X(-v-5)) = \sum_{j=1}^{4}(-1)^{j-1}\sum_i \chi(\mathcal{O}_{\mathbb{P}}(-n_{j,i}-v-5)) = \sum_{j=1}^{4}(-1)^{j-1}\sum_i \chi(\mathcal{O}_{\mathbb{P}}(n_{j,i}+v)).$$

The right hand side of (16) is therefore equal to

$$\chi(\mathcal{I}_X(-v-5)) - \sum_{j=1}^{4}(-1)^{j-1}\sum_i \chi(\mathcal{O}_X(n_{j,i}+v)).$$

Then we compute $\sum_{j=1}^{4}(-1)^{j-1}\sum_i \chi(\mathcal{O}_X(n_{j,i}+v))$ by just using (1) and (15). We get exactly

$$\sum_{j=1}^{4}(-1)^{j-1}\sum_i \chi(\mathcal{O}_X(n_{j,i}+v)) = \chi(\mathcal{O}_X(v)) - d^2,$$

and (16) translates to $\dim R_v - \chi(\mathcal{N}_X(v)) = \chi(\mathcal{I}_X(-v-5)) - \chi(\mathcal{O}_X(v)) + d^2$ and we get (14). $\square$

Remark 4.2. *Note that the formula* (13) *of Proposition 4.1 is certainly straightforward to prove for* smooth *surfaces by combining the well known formula*

$$\chi(\mathcal{N}_X(v)) = dv^2 + 5dv + 5(d-\pi+1) - 2K^2 + 14\chi(\mathcal{O}_X)$$

with the double point formula $d^2 - 10d - 5H.K - 2K^2 + 12\chi(\mathcal{O}_X) = 0$.

Now we come to the analogue of Theorem 3.4. Also in this case ${}_0\mathrm{ext}^1_R(I,I) - {}_0\mathrm{ext}^2_R(I,I)$ is a lower bound of $\mathrm{H}(d,p,\pi)_\gamma$. Since the basic part of the proof of the Theorem below is similar to the proof of Theorem 3.4, we will only sketch the proof. Note that in the surface case, we do not succeed so nicely as in the curve case because the lower bound above is not directly given by the first equality of Proposition 4.1, due to the term ${}_0\mathrm{ext}^3_R(I,I)$. Since we have ${}_0\mathrm{Ext}^3_R(I,I) \cong {}_{-5}\mathrm{Ext}^2_{\mathfrak{m}}(I,I)^\vee \cong {}_{-5}\mathrm{Hom}_R(I,M_1)^\vee$ by (2) and (3) and $M_1 \cong H^2_{\mathfrak{m}}(I)$ we get at least

Proposition 4.3. *Let X be a surface in $\mathbb{P}^4$, let $M_i = H^i_*(\mathcal{I}_X)$ for $i = 1,2$ and put $I = I_X$ and $\delta^j(v) = \delta^j_X(v)$ for any j and v. Then the dimension of* $\mathrm{H}(d,p,\pi)$ *at (X) satisfies*

$$\dim_{(X)} \mathrm{H}(d,p,\pi) \geq 1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5) - \sum_i h^1(\mathcal{I}_X(n_{1,i}-5)).$$

Moreover let X be a generic surface of a generically smooth component V of $\mathrm{H}(d,p,\pi)$ *and suppose* ${}_{-5}\mathrm{Hom}_R(I,M_2) = 0$. *Then*

$$\dim V = 1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5) - \sum_{i=0}^{1} {}_{-5}\mathrm{ext}^i_R(I,M_1).$$

Proof. For the inequality, we remark that

$$ {}_0\mathrm{ext}^3_R(I,I) = {}_{-5}\mathrm{hom}_R(I,M_1) \le \sum_i h^1(\mathcal{I}_X(n_{1,i}-5)) $$

because ${}_{-5}\mathrm{Hom}_R(I,M_1)$ is the kernel of the map $\oplus_i H^1(\mathcal{I}_X(n_{1,i}-5)) \longrightarrow \oplus_i H^1(\mathcal{I}_X(n_{2,i}-5))$ induced by the corresponding map in (10). We conclude by Proposition 4.1.

To find $\dim V$ we proceed as in the proof of Theorem 3.4 (see the last part of the proof of Theorem 3.7 for a close idea), and we get $\dim V = {}_0\mathrm{ext}^1_R(I,I)$, i.e.

$$ \dim V = 1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5) + {}_0\mathrm{ext}^2_R(I,I) - {}_0\mathrm{ext}^3_R(I,I). $$

By (3) we have ${}_0\mathrm{ext}^2_R(I,I) = {}_{-5}\mathrm{ext}^3_{\mathfrak{m}}(I,I)$ and we conclude by the exact sequence associated to (2),

$$ 0 \to {}_{-5}\mathrm{Ext}^1_R(I,H^2_{\mathfrak{m}}(I)) \to {}_{-5}\mathrm{Ext}^3_{\mathfrak{m}}(I,I) \to {}_{-5}\mathrm{Hom}_R(I,H^3_{\mathfrak{m}}(I)) \to {}_{-5}\mathrm{Ext}^2_R(I,H^2_{\mathfrak{m}}(I)) \to \quad . \tag{17} $$

$\square$

Under more specific assumptions we are able to prove,

Proposition 4.4. *Let X be any surface in $\mathbb{P}^4$ and suppose*

$$ {}_0\mathrm{Hom}_R(I,M_1) = {}_{-5}\mathrm{Ext}^1_R(I,M_1) = {}_{-5}\mathrm{Hom}_R(I,M_2) = 0. $$

Then X is unobstructed and

$$ \dim_{(X)} \mathrm{H}(d,p,\pi) = 1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5) - {}_{-5}\mathrm{hom}_R(I,M_1). $$

Proof. Due to [27], Rem. 3.7 (cf. [49], Thm. 2.1), $\mathrm{H}(d,p,\pi)_\gamma \cong \mathrm{H}(d,p,\pi)$ at (X) provided ${}_0\mathrm{Hom}_R(I,M_1) = 0$. Then we see by the arguments of (17) that ${}_0\mathrm{Ext}^2_R(I,I) = 0$. It follows that $\mathrm{H}(d,p,\pi)_\gamma$ is smooth at (X) of dimension ${}_0\mathrm{ext}^1_R(I,I)$. Then we conclude by Proposition 4.1. $\square$

Remark 4.5. (i) *Proposition 4.4 is mainly proved in [30], sect. 1. In [30] we moreover use* (2) *and* (3) *to prove a vanishing result for $H^1(\mathcal{N}_X)$. Indeed we show that $H^1(\mathcal{N}_X) = 0$ provided*

$$ H^1(\mathcal{I}_X(n_{2,i})) = H^1(\mathcal{I}_X(n_{2,i}-5)) = 0 \text{ and } H^2(\mathcal{I}_X(n_{1,i})) = H^2(\mathcal{I}_X(n_{1,i}-5)) = 0 $$

for every i.

(ii) *Let X be an arithmetically Cohen-Macaulay surface in $\mathbb{P}^4$. Then $M_1 = M_2 = 0$ and $\delta^1(v) = \delta^2(v) = 0$ for every v and we can use Proposition 4.4 to see that X is unobstructed and $\dim_{(X)} \mathrm{H}(d,p,\pi) = 1 + \delta^3(-5) = 1 - \delta^0(0)$. This coincides with [13].*

We will illustrate the results of this section by an example. If the assumptions of Proposition 4.4 or Remark 4.5 are not satisfied, then the surface may be obstructed, and we refer to section 8 for such examples.

Example 4.6. *Let X be the smooth rational surface with invariants $d = 11$, $\pi = 11$ (no 6-secant) and $K^2 = -11$ (cf. [43] or [11], B1.17, see also [16]). In this case the graded modules $M_i \simeq \oplus H^i(\mathcal{I}_X(v))$ are supported at two consecutive degrees and satisfy*

$$ \begin{aligned} &\dim H^1(\mathcal{I}_X(3)) = 2, \qquad &&\dim H^2(\mathcal{I}_X(1)) = 3, \\ &\dim H^1(\mathcal{I}_X(4)) = 1, \qquad &&\dim H^2(\mathcal{I}_X(2)) = 1. \end{aligned} $$

Moreover $I = I_X$ admits a minimal resolution (cf. [11])

$$0 \to R(-9) \to R(-8)^{\oplus 3} \oplus R(-7)^{\oplus 3} \to R(-7)^{\oplus 2} \oplus R(-6)^{\oplus 12} \to R(-5)^{\oplus 10} \to I \to 0.$$

It follows that ${}_{-5}\mathrm{Hom}_R(I, M_2) = 0$ and ${}_{-5}\mathrm{Ext}^i_R(I, M_1) = 0$ for $i = 0, 1$. By Proposition 4.4, $\mathrm{H}(d, p, \pi)$ is smooth at (X) and

$$\begin{aligned}\dim_{(X)} \mathrm{H}(d, p, \pi) &= 1 + \delta^3(-5) - \delta^3(-5) + \delta^1(-5) \\ &= 1 + 12h^2(\mathcal{I}_X(1)) - h^2(\mathcal{I}_X(2)) + 3h^1(\mathcal{I}_X(3)) - h^1(\mathcal{I}_X(4)) = 41.\end{aligned}$$

In this example it is, however, easier to use Proposition 4.1 to get

$$\begin{aligned}1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5) &= \chi(\mathcal{N}_X) - \delta^3(0) + \delta^2(0) - \delta^1(0) \\ &= 5(2d + \pi - 1) - d^2 + 2\chi(\mathcal{O}_X) = 41\end{aligned}$$

because $\delta^i(0)$ for $i > 0$ is easily seen to be zero. We may also use Remark 4.5 to see $H^1(\mathcal{N}_X) = 0$. Since any smooth surface satisfies

$$H^2(\mathcal{N}_X) = 0 \quad \text{provided} \quad H^2(\mathcal{O}_X(1)) = 0$$

(due to the existence of the natural surjection $\mathcal{O}_X(1)^5 \to \mathcal{N}_X$), we may conclude as above directly from $\dim H^0(\mathcal{N}_X) = \chi(\mathcal{N}_X) = 41$.

One may hope that a generalization of Theorem 3.7 to surfaces will contain a more complete result. To do it we need to generalize some of the theorems in [38] to surfaces. This will be done in the next two sections. The biliaison statements of Remark 3.9 will be generalized to any codimension 2 lCM equidimensional subscheme of $\mathbb{P}^{n+2}$ and carried out in later sections.

5 The smoothness of the "morphism" $\varphi : \mathrm{H}_{\gamma,\rho} \to \mathrm{V}_\rho$.

In this section we prove the local smoothness of the "morphism" $\varphi : \mathrm{H}_{\gamma,\rho} \to \mathrm{V}_\rho :=$ isomorphism classes of graded R-modules M_1 and M_2 satisfying $\dim(M_i)_v = \rho_i(v)$ and commuting with b, given by sending the surface X onto the class of the triple (M_1, M_2, b) where $M_i = H^i_*(\mathcal{I}_X)$ and $b \in {}_0\mathrm{Ext}^2_R(M_2, M_1)$ is the extension determined by X (cf. Remark 5.2 (ii)). To prove our theorem we first take in Proposition 5.1 a close look to Bolondi's short exact "resolution" of the homogeneous ideal of a surface X ([6]) and how we can define the extension b given in Horrock's paper [24]. As in [11] the ideal is the cokernel of some syzygy modules of M_1 and M_2, up to direct free factors. The proposition somehow uses and extends a result of Rao for a curve C, namely that the minimal resolution of I_C can be put in the following form

$$0 \to L_4 \xrightarrow{\sigma \oplus 0} L_3 \oplus F_2 \to F_1 \to I_C \to 0 \tag{18}$$

where $0 \to L_4 \xrightarrow{\sigma} L_3 \to ... \to M \to 0$ is a minimal resolution of M and F_i are free modules ([47]). Moreover we use local flatness criteria to generalize Bolondi's construction in [6] so that it works for flat resolutions over a local ring, rather than over a field. This is also the approach of [23] in the curve case.

Let X be a surface in $\mathbb{P}^4$ and let

$$\begin{aligned}&0 \to P_5 \xrightarrow{\sigma_5} P_4 \xrightarrow{\sigma_4} P_3 \xrightarrow{\sigma_3} \ldots \longrightarrow P_0 \xrightarrow{\sigma_0} M_1 \to 0, \\ &0 \to Q_5 \xrightarrow{\tau_5} Q_4 \xrightarrow{\tau_4} Q_3 \xrightarrow{\tau_3} \ldots \longrightarrow Q_0 \xrightarrow{\tau_0} M_2 \to 0\end{aligned} \tag{19}$$

(for short $\sigma_\bullet : P_\bullet \to M_1 \to 0$ and $\tau_\bullet : Q_\bullet \to M_2 \to 0$) be minimal free resolutions over R. Let $K_\bullet$ and $L_\bullet$ be the syzygies of M_1 and M_2 respectively, i.e. $K_i = \ker \sigma_i$ and $L_i = \ker \tau_i$. Recall that syzygies have nice cohomological properties ([11], [6]), for instance

$$\begin{aligned} M_1 = H^1_*(\tilde{K}_1) \quad \text{and} \quad H^2_*(\tilde{K}_1) = H^3_*(\tilde{K}_1) = 0, \\ M_2 = H^3_*(\tilde{L}_3) \quad \text{and} \quad H^1_*(\tilde{L}_3) = H^2_*(\tilde{L}_3) = 0. \end{aligned} \tag{20}$$

There is a strong connection between the resolutions (19), the minimal resolution (10) of $I = I_X$ and the following minimal resolutions of $A = H^0_*(\mathcal{O}_X)$;

$$0 \to P_3' \xrightarrow{\sigma_3'} P_2' \xrightarrow{\sigma_2'} P_1' \xrightarrow{\sigma_1'} P_0 \oplus R \to A \to 0 \tag{21}$$

where the morphism $P_0 \oplus R \to A$ of (21) is naturally deduced from $P_0 \to M_1$ of (19) and the exact sequence $R \to A \to M_1 \to 0$ and where $\sigma'_\bullet : P'_\bullet \to \ker(P_0 \oplus R \to A) \to 0$ is a minimal R-free resolution (cf. [38], p. 46). The connection we have in mind can be formulated and proved for a *family* of surfaces with constant cohomology, at least locally, e.g. we can replace the field k by a local k-algebra S with residue field k. Now, in [6], Bolondi uses some ideas of Horrocks [24] to define an element $b \in {}_0\mathrm{Ext}^2_R(M_2, M_1)$ and the "Horrocks triple" $D =: (M_1, M_2, b)$ associated to X such that, conversely given $D = (M_1, M_2, b)$ where M_i are R-modules of finite length, there is a surface X whose homogeneous ideal I is defined in the following way. For some integer $h \in \mathbb{Z}$ there is an exact sequence $0 \to L_3' \to K_1' \to I(h) \to 0$ where L_3' (resp. K_1') is isomorphic to the syzygy L_3 (resp. K_1) up to some R-free module F_L (resp. F_K). Up to biliaison this construction is the inverse to the first approach which defines (M_1, M_2, b) from a given X. To prove the main smoothness theorem of this section, we need to adapt the approach above by determining F_L and F_K more explicitly and such that it works over (at least an artinian) S. Using also ideas of Rao's paper [47], we can prove

Proposition 5.1. *Let X be a surface in $\mathbb{P}^4_S$, flat over a local noetherian k-algebra S with residue field k, and suppose that $M_1 = H^1_*(\mathcal{I}_X)$, $M_2 = H^2_*(\mathcal{I}_X)$ and $I = I_X$ are flat S-modules. Then there exist minimal R-free resolutions of M_i, I and $A = H^0_*(\mathcal{O}_X)$ as in* (19)*,* (10) *and* (21)*, with $R = S[X_0, X_1, .., X_4]$. Moreover let $L_3' = \ker \sigma_1'$ and let K_1' be the kernel of the composition of σ_1' and the natural projection $P_0 \oplus R \to P_0$, cf.* (21)*. Then there is an exact sequence*

$$0 \longrightarrow L_3' \xrightarrow{b'} K_1' \longrightarrow I \longrightarrow 0 \tag{22}$$

of flat graded S-modules and a surjective morphism $d : {}_0\mathrm{Hom}_R(L_3', K_1') \longrightarrow {}_0\mathrm{Ext}^2_R(M_2, M_1)$, defining a triple (M_1, M_2, b) where $b = d(b')$, coinciding with the uniquely defined "Horrocks triple" of [24] or [6]. Moreover L_3' (resp. K_1') is the direct sum of a 3rd syzygy of M_2 (resp. 1st syzygy of M_1) up to a direct free factor, i.e. there exist R-free modules F_L and F_K such that the horizontal exact sequences in the diagram

$$\begin{array}{ccccc} 0 \longrightarrow K_1' & \longrightarrow & P_1' & \longrightarrow & P_0 \\ \downarrow & \circ & \downarrow & \circ & \| \\ 0 \to K_1 \oplus F_K & \to & P_1 \oplus F_K & \xrightarrow{\sigma_1 \oplus 0} & P_0 \end{array}$$

are isomorphic (i.e., the downarrows are isomorphisms). Similarly, the exact sequences $0 \to Q_5 \xrightarrow{(\tau_5,0)} Q_4 \oplus F_L \to L_3 \oplus F_L \to 0$ and $0 \to P_3' \to P_2' \to L_3' \to 0$ are isomorphic as well.

Remark 5.2. (i) *By a surface $X \subseteq \mathbb{P}^4_S$ in Proposition 5.1 we actually mean that $X \times_{\mathrm{Spec}(S)} \mathrm{Spec}(k)$ is a surface (i.e. locally Cohen-Macaulay and equidimensional of dimension 2).*

(ii) *The proposition above, defining the "Horrocks triple"* (M_1, M_2, b) *from a given* $X \subseteq \mathbb{P}^4_S$*, can be regarded as our definition of the "morphism"* $\varphi : \mathrm{H}_{\gamma,\rho} \to \mathrm{V}_\rho$ = *isomorphism classes of graded* R*-modules* M_1 *and* M_2 *satisfying* $\dim(M_i)_v = \rho_i(v)$ *and commuting with* b.

Proof. We obviously have minimal resolutions of $M_i \otimes_S k$, $I_X \otimes_S k$ and $A \otimes_S k$ as described above with $R = k[X_0, X_1, .., X_4]$, cf. (19), (10) and (21). These resolutions can easily be lifted to the minimal resolution of the proposition by cutting into short exact sequences and using the flatness of the modules involved.

By the definition of L_3' and K_1' there is a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & \longrightarrow & R & \longrightarrow & R \longrightarrow 0 \\ & & & & & & \downarrow & \circ & \downarrow \\ 0 \longrightarrow L_3' & \longrightarrow & P_1' & \longrightarrow & P_0 \oplus R & \longrightarrow & A \longrightarrow 0 \\ & & \| & \circ & \downarrow & \circ & \downarrow \\ 0 \longrightarrow K_1' & \longrightarrow & P_1' & \longrightarrow & P_0 & \longrightarrow & M_1 \longrightarrow 0 \end{array}$$

and we get the exact sequence (22) by the snake lemma. Comparing the lower exact sequence in the last diagram with the following part of the minimal resolution of M_1; $\to P_1 \to P_0 \to M_1 \to 0$, we get the commutative diagram of the proposition because K_1 is the 1st syzygy of M_1.

To prove the corresponding commutative diagram for L_3' and L_3, we sheafify (22), and we get $M_2 \simeq H^3_*(\tilde{L}_3')$. Recalling the definition of L_3', we get the exact sequence

$$H^4_*(\tilde{P}_2')^\vee \to H^4_*(\tilde{P}_3')^\vee \to M_2^\vee \simeq \mathrm{Ext}^5_R(M_2, R(-5)) \to 0$$

which we compare to the *minimal* resolution

$$Q_4^\vee \to Q_5^\vee \to \mathrm{Ext}^5_R(M_2, R) \to 0$$

obtained by applying $\mathrm{Hom}_R(-, R)$ to the resolution $Q_\bullet \to M_2$. Recalling $H^4_*(\tilde{P}_i')^\vee(5) \simeq P_i'^\vee$, we get the conclusion, as in the proof of Thm. 2.5 of [47].

Finally to define the morphism d and to see that the defined triple (M_1, M_2, b) corresponds to the one given by Horrocks' construction (seen to be unique by [24]), one may consult [6] for the case $S = k$ which, however, generalizes to a local ring S. The important part is as follows. The definition of K_1' and K_0 implies

$$\mathrm{Ext}^2(M_2, M_1) \simeq \mathrm{Ext}^3(M_2, K_0) \simeq \mathrm{Ext}^4(M_2, K_1').$$

Next, by Gorenstein duality, we know $\mathrm{Ext}^i_R(M_2, R) = 0$ for $i \neq 5$. Hence the definition of the syzygies L_i leads to $\mathrm{Ext}^4(M_2, K_1') \simeq \mathrm{Ext}^3(L_0, K_1') \simeq \mathrm{Ext}^1(L_2, K_1')$ and to a diagram

$$\begin{array}{ccccccc} {}_0\mathrm{Hom}_R(Q_3, K_1') & \to & {}_0\mathrm{Hom}(L_3, K_1') & \to & {}_0\mathrm{Ext}^1(L_2, K_1') & \to & 0 \\ & & \downarrow & & \downarrow & & \\ & & {}_0\mathrm{Hom}(L_3', K_1') & & {}_0\mathrm{Ext}^2_R(M_2, M_1) & & \end{array} \tag{23}$$

where the horizontal sequence is exact and the first (resp. second) vertical map is injective and split (resp. an isomorphism). We let $d : {}_0\mathrm{Hom}_R(L_3', K_1') \to {}_0\mathrm{Ext}^2_R(M_2, M_1)$ be the obvious composition, first using the "inverse" of the split map, and we get the conclusions of the proposition. □

Now we will show the smoothness of φ. Indeed using Proposition 5.1 for S artinian, we get a rather easy proof of

Theorem 5.3. *The "morphism" $\varphi : \mathrm{H}_{\gamma,\rho} \to \mathrm{V}_\rho$ = isomorphism classes of graded R-modules M_1 and M_2 satisfying $\dim(M_i)_v = \rho_i(v)$ and commuting with b, is smooth (i.e. for any surface X in $\mathbb{P}^4_k$, the corresponding local deformation functor of φ, given by $(X_S \subseteq \mathbb{P}^4_S) \mapsto$ class of (M_{1S}, M_{2S}, b_S), see right below, is formally smooth).*

Proof. Let $T \to S \to k$ be surjections of local artinian k-algebras with residue fields k such that $\ker(T \to S)$ is a k-module via $T \to k$. Let $X_S \subseteq \mathbb{P}^4_S$ be a deformation of $X \subseteq \mathbb{P}^4$ to S with constant postulation γ and constant deficiency $\rho = (\rho_1, \rho_2)$. Let (M_{1S}, M_{2S}, b_S) be the "Horrocks triple" defined by X_S (cf. Proposition 5.1). Note that M_{iS} for $i = 1,2$ are S-flat by the definition of $\mathrm{H}_{\gamma,\rho}$. Let (M_{1T}, M_{2T}, b_T) be a given deformation of (M_{1S}, M_{2S}, b_S) to T. To prove the smoothness at (X), we must show the existence of a deformation $X_T \subseteq \mathbb{P}^4_T$ of $X_S \subseteq \mathbb{P}^4_S$, whose corresponding "Horrocks triple" is precisely (M_{1T}, M_{2T}, b_T), modulo graded isomorphisms of (M_{1T}, M_{2T}) commuting with b_T.

We have by Proposition 5.1 minimal resolutions of M_{iS}, I_{X_S} and A_S over $R_S := S[X_0, X_1, .., X_4]$ as in (10), (19)-(21) and flat S-modules L_{iS}, K_{iS}, L'_{3S}, K'_{1S} fitting into the exact sequence (22) and a surjection d defined as the composition (cf. (23))

$$\begin{array}{ccccccc}
{}_0\mathrm{Hom}_{R_S}(L'_{3S}, K'_{1S}) & \to & {}_0\mathrm{Hom}_{R_S}(L_{3S}, K'_{1S}) & \to & {}_0\mathrm{Ext}^1_{R_S}(L_{2S}, K'_{1S}) \simeq & {}_0\mathrm{Ext}^2_{R_S}(M_{2S}, M_{1S}) \\
\Cup & & \Cup & & \Cup & \Cup \\
b'_S & \longrightarrow & \beta_S & \longrightarrow & b_S \quad \longrightarrow & b_S
\end{array} \tag{24}$$

"on the S-level" (β_S is simply the image of b'_S via the map of (24)) which lifts the corresponding resolutions/modules/sequences on the "k-level". Since M_{iT} are given deformations of M_{iS}, we can lift the minimal resolutions $\sigma_{\bullet S} : P_{\bullet S} \to M_{1S}$ and $\tau_{\bullet S} : Q_{\bullet S} \to M_{2S}$ further to T, thus proving the existence of deformations L_{iT}, K_{iT}, L'_{3T}, K'_{1T} of L_{iS}, K_{iS}, L'_{3S}, K'_{1S} resp. (the free submodules F_{LS} and F_{KS} of L'_{3S} and K'_{1S} are lifted trivially). So we have a diagram (23) and hence a sequence (24) "on the T-level" where the elements b'_T and β_T are not yet defined. The element $b_T \in {}_0\mathrm{Ext}^1(L_{2T}, K'_{1T}) \simeq {}_0\mathrm{Ext}^2_{R_T}(M_{2T}, M_{1T})$ is, however, given and if we consider the diagram (cf. (23))

$$\begin{array}{ccccccc}
{}_0\mathrm{Hom}_{R_T}(Q_{3T}, K'_{1T}) & \to & {}_0\mathrm{Hom}_{R_T}(L_{3T}, K'_{1T}) & \to & {}_0\mathrm{Ext}^1_{R_T}(L_{2T}, K'_{1T}) & \to & 0 \\
\downarrow & \circ & \downarrow \alpha & \circ & \downarrow & & \\
{}_0\mathrm{Hom}_{R_S}(Q_{3S}, K'_{1S}) & \to & {}_0\mathrm{Hom}_{R_S}(L_{3S}, K'_{1S}) & \to & {}_0\mathrm{Ext}^1_{R_S}(L_{2S}, K'_{1S}) & \to & 0
\end{array}$$

of exact horizontal sequences and surjective vertical maps deduced from $0 \to L_{3T} \to Q_{3T} \to L_{2T} \to 0$, we easily get a morphism $\beta_T \in {}_0\mathrm{Hom}(L_{3T}, K'_{1T})$ such that $\alpha(\beta_T) = \beta_S$, i.e., $\beta_T \otimes_T S = \beta_S$. Since $L'_{3S} \simeq L_{3S} \oplus F_{LS}$ we can decompose the map b'_S as $(\beta_S, \gamma_S) \in {}_0\mathrm{Hom}(L'_{3S}, K'_{1S})$, and taking any lifting $\gamma_T : F_{LT} \to K'_{1T}$ of γ_S, we get a map $b'_T = (\beta_T, \gamma_T) \in {}_0\mathrm{Hom}(L'_{3T}, K'_{1T})$ fitting into a commutative diagram

$$\begin{array}{ccc}
L_{3T} \oplus F_{LT} \simeq L'_{3T} & \xrightarrow{b'_T} & K'_{1T} \\
\downarrow & \circ & \downarrow \\
L_{3S} \oplus F_{LS} \simeq L'_{3S} & \xrightarrow{b'_S} & K'_{1S}\ .
\end{array}$$

Once having proved the existence of such a commutative diagram, we can define a surface X_T of $\mathbb{P}^4_T$ with the desired properties, thus proving the claimed smoothness. Indeed it is straightforward to see that $\operatorname{coker} b'_T$ is a (flat) deformation of $\operatorname{coker} b'_S = I_{X_S}$ to T. Moreover one knows that an $R_T := T[X_0, X_1, .., X_4]$-*module* $\operatorname{coker} b'_T$ which lifts a graded ideal I_{X_S} is again a *graded ideal* I_T (we can deduce this information by interpreting the isomorphisms $H^{i-1}(\mathcal{N}_X) \simeq \mathrm{Ext}^i_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_X, \mathcal{I}_X)$ for $i = 1,2$ in terms of their deformation theories from which we see that $\widetilde{\operatorname{coker} b'_T}$ is a sheaf ideal, and we conclude by taking global sections, cf. [49] or [33], Lem. 4.8 for further details). Hence we have proved the existence of a surface $X_T = \mathrm{Proj}(R_T/I_T)$, flat over T which via $T \to S$ reduces to X_S. By the construction above the corresponding "Horrocks triple" is precisely the given triple (M_{1T}, M_{2T}, b_T), and we are done. $\square$

Corollary 5.4. *Let X be a surface in $\mathbb{P}^4$. If the local deformation functors $Def(M_i)$ of M_i are formally smooth (for instance if ${}_0\mathrm{Ext}^2_R(M_i, M_i) = 0$) for $i = 1, 2$, and if*

$$ {}_0\mathrm{Ext}^3_R(M_2, M_1) = 0, $$

then $\mathrm{H}_{\gamma,\rho}$ is smooth at (X).

Proof. With notations as in the very first part of the proof of Theorem 5.3, it suffices to prove that there always exists a deformation (M_{1T}, M_{2T}, b_T) of (M_{1S}, M_{2S}, b_S) since then the proof above shows the existence of a deformation $X_T = \mathrm{Proj}(R_T/I_T)$ which reduces to X_S via $T \to S$. Since $Def(M_i)$ are formally smooth, it suffices to show the existence of b_T which maps to $b_S \in {}_0\mathrm{Ext}^2_{R_S}(M_{2S}, M_{1S})$. Let $\mathfrak{a} = \ker(T \to S)$. If we apply ${}_0\mathrm{Hom}_{R_T}(M_{2T}, -)$ to the exact sequence

$$ 0 \to \mathfrak{a} \otimes_T M_{1T} \cong \mathfrak{a} \otimes_k M_1 \to M_{1T} \to M_{1S} \to 0 $$

and use ${}_0\mathrm{Ext}^3_R(M_2, M_1) = 0$, we see that ${}_0\mathrm{Ext}^2_{R_T}(M_{2T}, M_{1T}) \to {}_0\mathrm{Ext}^2_{R_T}(M_{2T}, M_{1S})$ is surjective. Hence we get a surjective map

$$ {}_0\mathrm{Ext}^1_{R_T}(L_{3T}, K'_{1T}) \simeq {}_0\mathrm{Ext}^2_{R_T}(M_{2T}, M_{1T}) \to {}_0\mathrm{Ext}^1_{R_S}(L_{2S}, K'_{1S}) \simeq {}_0\mathrm{Ext}^2_{R_S}(M_{2S}, M_{1S}) $$

and we are done. $\square$

Remark 5.5. *If we, as in [38] for curves, had proven the existence of the "fiber" $\mathrm{H}_{\gamma,D}$, $D = (M_1, M_2, b)$, of φ as a scheme, then Theorem 5.3 must imply the smoothness of $\mathrm{H}_{\gamma,D}$ while [8] implies its irreducibility. Indeed [8], cor. 3.2 tells that the family of surfaces in $\mathbb{P}^4$ belonging to the same shift of the same liaison class, with fixed postulation, form an irreducible family, from which we see that $\mathrm{H}_{\gamma,D}$ is irreducible. Note that we can work with $\mathrm{H}_{\gamma,D}$ as a locally closed subset of $\mathrm{H}_{\gamma,\rho}$ (cf. the arguments of [4], cor. 2.2, and combine with Proposition 5.1), even though we have not proved that φ extends to a morphism of representable functors.*

6 The tangent space of $\mathrm{H}_{\gamma,\rho}$.

In this section we determine the tangent space of $\mathrm{H}_{\gamma,\rho}$ at (X) and we give a criterion for $\mathrm{H}_{\gamma,\rho} \cong \mathrm{H}(d, p, \pi)$ to be isomorphic as schemes at (X). We end this section by considering an example.

Let X be a surface in $\mathbb{P}^4$ with graded ideal $I = I_X$ and let $D = (M_1, M_2, b)$, $M_i = H^i_*(\tilde{I})$, be its "Horrocks triple". Recall that ${}_0\mathrm{Ext}^1_R(I, I)$ is the *tangent space* of H_γ at (X) because a deformation in H_γ keeps the postulation constant, i.e. it corresponds precisely to a graded deformation of I [38]. Moreover there exist maps

$$ \varphi_i : {}_0\mathrm{Ext}^1_R(I, I) \to {}_0\mathrm{Hom}_R(H^i_*(\tilde{I}), H^{i+1}_*(\tilde{I})) $$

taking an extension $0 \to I \to E \to I \to 0$ of ${}_0\mathrm{Ext}^1_R(I, I)$ onto the connecting homomorphism δ^i in the exact sequence

$$ H^i_*(\tilde{E}) \to H^i_*(\tilde{I}) \xrightarrow{\delta^i} H^{i+1}_*(\tilde{I}) \to H^{i+1}_*(\tilde{E}). $$

For saturated homogeneous ideals we have $I = H^0_*(\tilde{I})$, and it follows that the composition $E \to H^0_*(\tilde{E}) \to H^0_*(\tilde{I})$ is surjective, i.e. we get $\varphi_0 = 0$. Moreover note that if δ^{i-1} and δ^i are both zero for some i, then the exact sequence $0 \to I \to E \to I \to 0$ above defines an extension

$$0 \to H^i_*(\tilde{I}) \to H^i_*(\tilde{E}) \to H^i_*(\tilde{I}) \to 0.$$

Since $M_i = H^i_*(\tilde{I})$ for $i = 1, 2$ and $E = H^3_*(\tilde{I})$, there are well-defined morphisms

$$\psi_i : \ker(\varphi_1, \varphi_2) \to {}_0\mathrm{Ext}^1_R(M_i, M_i) \quad \text{for} \quad \mathrm{i} = 1, 2$$

where $(\varphi_1, \varphi_2) : {}_0\mathrm{Ext}^1_R(I, I) \to {}_0\mathrm{Hom}(M_1, M_2) \times {}_0\mathrm{Hom}(M_2, E)$ and φ_i are defined above. Recalling $\rho = (\rho_1, \rho_2)$ we put

$$ {}_0\mathrm{Ext}^1_R(I, I)_\rho := \ker(\varphi_1, \varphi_2). \tag{25}$$

Using base change theorems, as in [38], we easily show that $\ker(\varphi_1, \varphi_2)$ is the tangent space of $\mathrm{H}_{\gamma,\rho}$ at (X), i.e. we get

Proposition 6.1. *${}_0\mathrm{Ext}^1_R(I, I)_\rho$ is the tangent space of $\mathrm{H}_{\gamma,\rho}$ at (X). In particular if*

$$ {}_0\mathrm{Hom}_R(I, M_1) = 0, \quad {}_0\mathrm{Hom}_R(M_1, M_2) = 0 \quad \textit{and} \quad {}_0\mathrm{Hom}(M_2, E) = 0, \tag{26}$$

then the tangent spaces of $\mathrm{H}_{\gamma,\rho}, \mathrm{H}_\gamma$ and $\mathrm{H}(d, p, \pi)$ are isomorphic at (X). Indeed $\mathrm{H}_\gamma \cong \mathrm{H}(d, p, \pi)$ as schemes at (X), and if $\mathrm{H}_{\gamma,\rho}$ is smooth at (X), then $\mathrm{H}_{\gamma,\rho} \cong \mathrm{H}_\gamma$ are isomorphic as schemes at (X) as well.

Proof. As earlier remarked, cf. (2) and (5), ${}_0\mathrm{Ext}^1_R(I, I) \cong \mathrm{Ext}^1(\mathcal{I}_X, \mathcal{I}_X) \cong H^0(\mathcal{N}_X)$ provided ${}_0\mathrm{Hom}_R(I, M_1) = 0$. Moreover ${}_0\mathrm{Ext}^1_R(I, I)_\rho \cong {}_0\mathrm{Ext}^1_R(I, I)$ since $\varphi_i = 0$ for $i = 1, 2$.

For the isomorphism as schemes we remark that $\mathrm{H}_\gamma \simeq \mathrm{H}(d, p, \pi)$ follows from [27], Thm. 3.6 and Rem. 3.7 (see [49] and [33], proof of Thm. 2.6 (i) for details). Finally if $\mathrm{H}_{\gamma,\rho}$ is smooth at (X), then the embedding $\mathrm{H}_{\gamma,\rho} \hookrightarrow \mathrm{H}_\gamma$ is smooth at (X) (since the tangent map is surjective), hence etale, hence an isomorphism at (X) since the embedding is universally injective. $\square$

Remark 6.2. *If we suppose (26), then $\mathrm{H}_{\gamma,\rho} \cong \mathrm{H}_\gamma$ are isomorphic as schemes at (X) by [31], Thm. 3.7 without requiring the smoothness of $\mathrm{H}_{\gamma,\rho}$ at (X). See also Remark 9.3.*

In [31] we also gave almost complete proofs of Remark 6.2 and of the following two non-trivial results (cf. [31], Prop. 3.4 and Prop. 3.6). Note that Remark 6.3 generalizes Corollary 5.4.

Remark 6.3. *Let X be a surface in $\mathbb{P}^4$. Then for $i = 1, 2$ there exist morphisms $e_i : {}_0\mathrm{Ext}^1_R(M_i, M_i) \to {}_0\mathrm{Ext}^3_R(M_2, M_1)$ and an induced morphism*

$$\bar{e}_1 : {}_0\mathrm{Ext}^1_R(M_1, M_1) \to {}_0\mathrm{Ext}^3_R(M_2, M_1)/e_2({}_0\mathrm{Ext}^1_R(M_2, M_2))$$

such that if the local deformation functors $Def(M_i)$ of M_i are formally smooth (for instance if ${}_0\mathrm{Ext}^2_R(M_i, M_i) = 0$) for $i = 1, 2$, and if the morphism $\bar{e}_1$ is surjective, then V_ρ is smooth at $D = (M_1, M_2, b)$ (i.e. the local deformation functor of D is formally smooth).

Remark 6.4. *Let X be a surface in $\mathbb{P}^4$ and let $\epsilon = \dim \operatorname{coker} \bar{e}_1$. Then*

$$\dim {}_0\mathrm{Ext}^1_R(I, I)_\rho = 1 + \delta^3(-5) + \sum_{i=0}^{3} (-1)^i \; {}_0\mathrm{ext}^i_R(M_2, M_1)$$
$$- \sum_{i=0}^{1} (-1)^i \; {}_0\mathrm{ext}^i_R(M_1, M_1) - \sum_{i=0}^{1} (-1)^i \; {}_0\mathrm{ext}^i_R(M_2, M_2) + \epsilon.$$

To illustrate the results we have proved, we consider an example of a surface X of $\mathbb{P}^4$ where actually V_ρ is smooth and non-trivial at the corresponding (M_1, M_2, b), cf. Corollary 5.4. Moreover all conditions of Proposition 6.1 are satisfied, and it follows that $\mathrm{H}_{\gamma,\rho}$ and $\mathrm{H}(d,p,\pi)$ are isomorphic and smooth at (X).

Example 6.5. *Let X be the smooth elliptic surface with invariants $d = 11$, $\pi = 12$ and $K^2 = -4$ (cf. [43] or [11], B7.6). Then the graded modules $M_i \simeq \oplus H^i(\mathcal{I}_X(v))$ for $i = 1, 2$ vanish for every v except in the following cases*

$$h^1(\mathcal{I}_X(3)) = 1, \quad h^2(\mathcal{I}_X(1)) = 2, \quad h^2(\mathcal{I}_X(2)) = 1.$$

Moreover $I = I_X$ admits a minimal resolution (cf. [11])

$$0 \to R(-8) \to R(-7)^{\oplus 6} \to R(-6)^{\oplus 13} \to R(-5)^{\oplus 8} \oplus R(-4) \to I \to 0.$$

It follows that ${}_0\mathrm{Ext}^i(M_j, M_j) = 0$ for $i \geq 2$ and $j = 1, 2$ and that ${}_0\mathrm{Ext}^3(M_2, M_1) = 0$. By Corollary 5.4 and Proposition 6.1 we get that $\mathrm{H}(d,p,\pi) \cong \mathrm{H}_{\gamma,\rho}$ are smooth at (X). If we, however, want to compute the dimension of $\mathrm{H}(d,p,\pi)$ at (X) and will avoid Remark 6.4 which we have not proved, we still have to use the results of section 4. Let us only use the two "most general" results there, Proposition 4.1 and Propositions 4.3, to illustrate the principle of semicontinuity a little extended (to include the semicontinuity of the graded Betti numbers). Let V be the generically smooth component of $\mathrm{H}(d,p,\pi)$ to which (X) belongs. Since $\mathrm{H}(d,p,\pi) \cong \mathrm{H}_{\gamma,\rho}$ at (X), then a generic surface $\tilde{X}$ of V also belongs to $\mathrm{H}_{\gamma,\rho}$. Inside H_γ, hence inside $\mathrm{H}_{\gamma,\rho}$, the graded Betti numbers of the homogeneous ideal of the surfaces obey semicontinuity by Remark 7(b) of [34]!! Since we from the minimal resolution of I_X can see that, for every i, $\beta_{j,i} \neq 0$ for at most one j and since the Hilbert functions of X and $\tilde{X}$ are the same, they have exactly the same graded Betti numbers. Moreover note that $h^i(\mathcal{I}_{\tilde{X}}(v)) = h^i(\mathcal{I}_X(v))$ for any i, v since X has seminatural cohomology. It follows that

$$\dim V = 1 + \delta^3(-5) - \delta^3(-5) + \delta^1(-5) =$$

$$1 + h^3(\mathcal{I}_X(-1)) + 8h^3(\mathcal{I}_X) + 13h^2(\mathcal{I}_X(1)) - 6h^2(\mathcal{I}_X(2)) - h^1(\mathcal{I}_X(3)) = 50.$$

Since we have proved $\dim V = 1 + \delta^3(-5) - \delta^3(-5) + \delta^1(-5)$ it is easier to use Proposition 4.1 to get

$$\dim V = \chi(\mathcal{N}_X) - \delta^3(0) + \delta^2(0) - \delta^1(0) = 5(2d + \pi - 1) - d^2 + 2\chi(\mathcal{O}_X) = 50$$

because $\delta^i(0)$ for $i > 0$ is easily seen to be zero.

7 Linkage of surfaces.

The main result of this section shows how to compute the dimension of $\mathrm{H}_{\gamma,\rho}$ and the dimension of its tangent space at (X) provided we know how to solve the corresponding problem for a linked surface X' (Theorem 7.1). In another related result (Proposition 7.4 with $c > 0$) we give conditions on e.g. a generic surface of $\mathrm{H}(d,p,\pi)$ such that corresponding linked surface X' is non-generic in the sense $\dim_{(X')} \mathrm{H}_{\gamma',\rho'} < \dim_{(X')} \mathrm{H}(d',p',\pi')$. It follows that a new surface, the generic one with "smaller" cohomology, has to exist! Indeed recall that linkage is a well known method for proving existence of surfaces with certain properties, e.g. see [42], [26], [40], [46], [12], [44], [1] to mention a few papers which use linkage in this way. In these and similar papers we see that the linked surface X' is usually generic if X is generic. In Remark 7.2 we notice that if certain cohomological assumptions, cf. (30), are satisfied, then X' is generic if and only if X is generic. Using Proposition 7.4 with $c > 0$, however, then some of the cohomology groups of (30) are non-zero, and under some assumptions we get the existence of a non-generic surface $X' \in \mathrm{H}_{\gamma',\rho'}$ and hence a generic one $\notin \mathrm{H}_{\gamma',\rho'}$ as well. In proving the results of this section we substantially need the theory of linkage of families developed in [29].

Since the main even liaison result of this paper, which we prove in the final section, requires that the linkage theorem of this section is proven for equidimensional locally Cohen-Macaulay codimension 2 subschemes of $\mathbb{P}^{n+2}$, we prove Theorem 7.1 in this generality. The other results and examples of this section deal, however, with surfaces.

Now, if the surfaces X and X' are (algebraically) linked by a complete intersection (a CI) Y of type (f,g), then the dualizing sheaf $\omega_{X'}$ satisfies $\omega_{X'} = \mathcal{I}_{X/Y}(f+g-5)$ where $\mathcal{I}_{X/Y} = \ker(\mathcal{O}_Y \to \mathcal{O}_X)$ ([45], [39]). Moreover $\omega_X = \mathcal{I}_{X'/Y}(f+g-5)$ and we get

$$\begin{aligned} \chi(\mathcal{O}_X(v)) + \chi(\mathcal{O}_{X'}(f+g-5-v)) &= \chi(\mathcal{O}_Y(v)) \\ h^i(\mathcal{I}_{X'}(v)) &= h^{3-i}(\mathcal{I}_X(f+g-5-v)), \quad \text{for } i = 1 \text{ and } 2 \\ h^i(\mathcal{I}_{X'/Y}(v)) &= h^{2-i}(\mathcal{O}_X(f+g-5-v)), \quad \text{for } i = 0 \text{ and } 2 \\ h^i(\mathcal{O}_{X'}(v)) &= h^{2-i}(\mathcal{I}_{X/Y}(f+g-5-v)), \quad \text{for } i = 0 \text{ and } 2 \end{aligned} \tag{27}$$

from which we deduce $d + d' = fg$ and $\pi' - \pi = (d' - d)(f+g-4)/2$.

The generalization of (27) to equidimensional lCM codimension 2 subschemes of $\mathbb{P}^{n+2}$ is clear, e.g. we have

$$h^i(\mathcal{I}_{X'/Y}(v)) = h^{n-i}(\mathcal{O}_X(f+g-n-3-v)), \quad \text{for } i = 0 \text{ and } n. \tag{28}$$

Note that we now have n deficiency modules, whose dimensions $\rho_i(v) = h^i(\mathcal{I}_X(v))$, $i = 1,2,...,n$ determine the vector function $\rho = (\rho_1,...,\rho_n)$. Using this vector function, we easily generalize (25) in such a way that we get the tangent space ${}_0\mathrm{Ext}^1_R(I_X, I_X)_\rho$ of the Hilbert scheme $\mathrm{H}_{\gamma,\rho} \subseteq \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ of constant cohomology in this case. We allow $n = 0$ in which case there is no ρ and $\mathrm{H}_{\gamma,\rho} \subseteq \mathrm{Hilb}^{p(v)}(\mathbb{P}^2)$ should be taken as the Hilbert scheme of constant postulation ("the postulation Hilbert scheme") and ${}_0\mathrm{Ext}^1_R(I_X, I_X)_\rho$ as ${}_0\mathrm{Ext}^1_R(I_X, I_X)$. We have (cf. [38] for the curve case of the theorem),

Theorem 7.1. *Let X and X' be two equidimensional locally Cohen-Macaulay codimension 2 subschemes of $\mathbb{P}^{n+2}$, linked by a complete intersection $Y \subseteq \mathbb{P}^{n+2}$ of type (f,g), and suppose that (X) (resp. (X')) belongs to the Hilbert scheme $\mathrm{H}_{\gamma,\rho}$ (resp. $\mathrm{H}_{\gamma',\rho'}$) of constant cohomology. Then*

$$\text{i)} \quad \begin{aligned} &\dim_{(X)} \mathrm{H}_{\gamma,\rho} + h^0(\mathcal{I}_X(f)) + h^0(\mathcal{I}_X(g)) \\ &\qquad = \dim_{(X')} \mathrm{H}_{\gamma',\rho'} + h^0(\mathcal{I}_{X'}(f)) + h^0(\mathcal{I}_{X'}(g)) \end{aligned}$$

or equivalently,

$$\begin{aligned} \dim_{(X')} \mathrm{H}_{\gamma',\rho'} = \dim_{(X)} \mathrm{H}_{\gamma,\rho} &+ h^0(\mathcal{I}_{X/Y}(f)) + h^0(\mathcal{I}_{X/Y}(g)) \\ &- h^n(\mathcal{O}_X(f-n-3)) - h^n(\mathcal{O}_X(g-n-3)). \end{aligned}$$

ii) *The dimension formulas of i) remain true if we replace $\dim_{(X)} \mathrm{H}_{\gamma,\rho}$ and $\dim_{(X')} \mathrm{H}_{\gamma',\rho'}$ by the dimensions of their tangent spaces ${}_0\mathrm{Ext}^1_R(I_X, I_X)_\rho$ and ${}_0\mathrm{Ext}^1_R(I_{X'}, I_{X'})_{\rho'}$ respectively.*

iii) $\mathrm{H}_{\gamma,\rho}$ *is smooth at (X) if and only if $\mathrm{H}_{\gamma',\rho'}$ is smooth at (X')*

Proof. Let $D(p(v); f, g)$ be the Hilbert flag scheme parametrizing pairs (X, Y) of equidimensional lCM codimension 2 subschemes of $\mathbb{P}^{n+2}$ such that Y is a CI of type (f, g) containing X. By [29], Thm. 2.6, there is an isomorphism of schemes,

$$D(p(v); f, g) \simeq D(p'(v); f, g), \tag{29}$$

given by sending (X, Y) onto (X', Y) where X' is linked to X by Y (cf. [25], Prop. (A.1)). We may suppose $n \geq 1$ in Theorem 7.1 since the case $n = 0$ is completely solved by Prop. 1.7 of [32]. Then the projection morphism $p : D(p(v); f, g) \to \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$, given by $(X, Y) \mapsto (X)$, is smooth at (X, Y) provided $H^1(\mathcal{I}_X(f)) = H^1(\mathcal{I}_X(g)) = 0$ ([29], Thm. 1.16 (b)). By [29], Lem. 1.17 and Rem. 1.20, this smoothness holds if we replace the vanishing above with the claim that the set of global sections of the corresponding twisted ideal sheaves over the local ring of $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ at (X) are locally free and commute with base change. Hence the following restriction of p to $p^{-1}(\mathrm{H}_{\gamma,\rho})$, $p^{-1}(\mathrm{H}_{\gamma,\rho}) \to \mathrm{H}_{\gamma,\rho}$, is smooth, (or see [38] for related arguments). Since the fiber dimension of p at (X, Y) is precisely

$$h^0(\mathcal{I}_{X/Y}(f)) + h^0(\mathcal{I}_{X/Y}(g)) = h^0(\mathcal{I}_X(f)) + h^0(\mathcal{I}_X(g)) - h^0(\mathcal{I}_Y(f)) - h^0(\mathcal{I}_Y(g))$$

by [29], Thm. 1.16 (a), we get any conclusion of the theorem if we combine with (28). $\square$

Remark 7.2. *Let X and X' be two surfaces in $\mathbb{P}^4$, linked by a CI of type (f, g). Then the arguments of the proof above show that we can, under the assumptions*

$$H^1(\mathcal{I}_X(f)) = H^1(\mathcal{I}_X(g)) = 0 \quad \text{and} \quad H^1(\mathcal{I}_{X'}(f)) = H^1(\mathcal{I}_{X'}(g)) = 0 \tag{30}$$

replace $\mathrm{H}_{\gamma,\rho}$ and $\mathrm{H}_{\gamma',\rho'}$ in Theorem 7.1 (i) (resp. their tangent spaces in Theorem 7.1 (ii)) by $\mathrm{H}(d, p, \pi)$ and $\mathrm{H}(d', p', \pi')$ (resp. by $H^1(\mathcal{N}_X)$ and $H^1(\mathcal{N}_{X'})$) and get valid dimension formulas involving the whole Hilbert schemes (resp. their tangent spaces). Hence assuming (30), *it follows that X is unobstructed if and only if X' is unobstructed, see [29], Prop. 3.12 for a generalization. Note also that we from the proof above (i.e. from [29], Thm. 1.16 (b)) and* (30) *get that X is generic if and only if X' is generic, see [29], Prop. 3.8 for a related general result.*

Example 7.3. *Let X be the smooth rational surface of $\mathrm{H}(11, 0, 11)$ of Example 4.6, let Y be a CI of type* $(5, 5)$ *containing X, and let X' be the linked surface. Using* (27) *we deduce $\chi(\mathcal{O}_{X'}(v)) = 7v^2 - 12v + 9$ from $\chi(\mathcal{O}_X(v)) = (11v^2 - 9v + 2)/2$, i.e. (X') belongs to $\mathrm{H}(d', p', \pi') = \mathrm{H}(14, 8, 20)$ by* (1). *Moreover $\omega_{X'} = \mathcal{I}_{X/Y}(5)$ is globally generated (cf. the resolution of I of Example 4.6) and the graded modules $M'_i \simeq \oplus H^i(\mathcal{I}_{X'}(v))$ are supported at two consecutive degrees and satisfy*

$$\begin{aligned} \dim H^1(\mathcal{I}_{X'}(3)) &= 1, & \qquad \dim H^2(\mathcal{I}_{X'}(1)) &= 1, \\ \dim H^1(\mathcal{I}_{X'}(4)) &= 3, & \qquad \dim H^2(\mathcal{I}_{X'}(2)) &= 2. \end{aligned}$$

From these informations we find the minimal resolution of $I' = I_{X'}$ to be

$$0 \to R(-9)^{\oplus 3} \to R(-8)^{\oplus 14} \to R(-7)^{\oplus 23} \to R(-6)^{\oplus 11} \oplus R(-5)^{\oplus 2} \to I' \to 0.$$

Combining Example 4.6 and Remark 6.2 we see that $\mathrm{H}_{\gamma,\rho}$ is smooth at (X) and $\dim_{(X)} \mathrm{H}_{\gamma,\rho} = 41$. Thanks to Theorem 7.1, we get that $\mathrm{H}_{\gamma',\rho'}$ is smooth at (X') and that

$$\dim_{(X')} \mathrm{H}_{\gamma',\rho'} = \dim_{(X)} \mathrm{H}_{\gamma,\rho} + 2h^0(\mathcal{I}_{X/Y}(5)) - 2h^2(\mathcal{O}_X(0)) = 57.$$

Moreover by Remark 7.2 or Proposition 6.1, $\mathrm{H}(d', p', \pi') \simeq \mathrm{H}_{\gamma',\rho'}$ is smooth at (X') and $\dim_{(X')} \mathrm{H}(d', p', \pi') = 57$. Note that in this case we neither have ${}_0\mathrm{Ext}^3(M_2, M_1) = 0$ nor ${}_{-5}\mathrm{Hom}_R(I, M_2) = 0$, i.e. we can not use Corollary 5.4 or Proposition 4.4 to conclude that $\mathrm{H}_{\gamma',\rho'}$ is smooth at (X'). But, as we have seen, the linkage result above takes care of the smoothness and the dimension.

If a surface X of $\mathbb{P}^4$ is contained in a CI Y of type (f,g), then there is an inclusion map $I_Y \to I_X$ which induces a morphism $l^{i+1}_{X/Y} : H^i(\mathcal{N}_X) \to H^i(\mathcal{O}_X(f)) \oplus H^i(\mathcal{O}_X(g))$ for every i. We let $\beta_{X/Y}$ be the composition of $l^1_{X/Y}$ with the natural map $H^0(\mathcal{O}_X(f)) \oplus H^0(\mathcal{O}_X(g)) \to H^1(\mathcal{I}_X(f)) \oplus H^1(\mathcal{I}_X(g))$.

Proposition 7.4. *Let X and X' be surfaces in $\mathbb{P}^4$, geometrically linked by a complete intersection $Y \subseteq \mathbb{P}^4$ of type (f,g), let $(X) \in \mathrm{H}_{\gamma,\rho}$ and $(X') \in \mathrm{H}_{\gamma',\rho'}$ and suppose $\dim_{(X)} \mathrm{H}_{\gamma,\rho} = \dim_{(X)} \mathrm{H}(d,p,\pi)$. Let $c := \dim_{(X')} \mathrm{H}(d',p',\pi') - \dim_{(X')} \mathrm{H}_{\gamma',\rho'}$ and suppose $H^1(\mathcal{I}_X(f)) = H^1(\mathcal{I}_X(g)) = 0$ and that $l^2_{X/Y}$ is injective. Then*

$$h^1(\mathcal{I}_{X'}(f)) + h^1(\mathcal{I}_{X'}(g)) - h^2(\mathcal{I}_{X'}(f)) - h^2(\mathcal{I}_{X'}(g)) \le c \le h^1(\mathcal{I}_{X'}(f)) + h^1(\mathcal{I}_{X'}(g)) \tag{31}$$

and we have equality on the right hand side if and only if $\mathrm{H}(d',p',\pi')$ is smooth at (X'). Furthermore, if $h^1(\mathcal{I}_{X'}(v)) \cdot h^2(\mathcal{I}_{X'}(v)) = 0$ for $v = f$ and $v = g$, then

$$c = h^1(\mathcal{I}_{X'}(f)) + h^1(\mathcal{I}_{X'}(g)).$$

Proof. Since X and X' are generically complete intersections (due to geometric linkage) of codimension 2 in $\mathbb{P}^4$, it follows that the cotangent sheaves $\mathcal{A}^2_X$ and $\mathcal{A}^2_{X'}$ are zero (cf. [10]). The vanishing of the obstruction group, $A^2(X \subseteq Y)$, of the Hilbert flag scheme $D(p(v);f,g)$ at (X,Y) is therefore equivalent to $\beta_{X/Y}$ being surjective and $l^2_{X/Y}$ being injective by (1.11) of [29], so $A^2(X \subseteq Y) = 0$ by assumption. Moreover since the linkage is geometric, we get $A^2(X' \subseteq Y) = 0$ by Cor. 2.14 of [29], i.e. $\beta_{X'/Y}$ is surjective, $l^2_{X'/Y}$ is injective and $D(p'(v);f,g)$ is smooth at (X',Y). Hence [29], Thm. 1.27 applies (to a component V satisfying $\dim V = \dim_{(X')} \mathrm{H}(d',p',\pi')$) to get the bounds of the codimension c above provided we can show that $\mathrm{H}_{\gamma',\rho'}$, in a neighborhood of (X'), is dense in an (f,g)-maximal subset of $\mathrm{H}(d',p',\pi')$ (i.e. dense in the image under the first projection of some non-embedded component of $D(p'(v);f,g)$). By the proof of Theorem 7.1 we see that the restriction of the first projection p' to $p'^{-1}(\mathrm{H}_{\gamma',\rho'})$, $p'^{-1}(\mathrm{H}_{\gamma',\rho'}) \to \mathrm{H}_{\gamma',\rho'}$, is smooth. It follows that $\mathrm{H}_{\gamma',\rho'}$ is, locally at (X'), (f,g)-maximal provided we can show

$$\dim_{(X',Y)} p'^{-1}(\mathrm{H}_{\gamma',\rho'}) = \dim_{(X',Y)} D(p'(v);f,g).$$

Thanks to (29) it suffices to show $\dim_{(X,Y)} p^{-1}(\mathrm{H}_{\gamma,\rho}) = \dim_{(X,Y)} D(p(v);f,g)$ which readily follows from the assumptions $\dim_{(X)} \mathrm{H}_{\gamma,\rho} = \dim_{(X)} \mathrm{H}(d,p,\pi)$ and $H^1(\mathcal{I}_X(f)) = H^1(\mathcal{I}_X(g)) = 0$ because the first projection, $p : D(p(v);f,g) \to \mathrm{Hilb}^{p(v)}(\mathbb{P}^4)$, as well as its restriction to $p^{-1}(\mathrm{H}_{\gamma,\rho})$, are smooth at (X,Y) by Remark 7.2. Then we get the final conclusion from [29], Cor. 1.29, which states that $h^1(\mathcal{I}_{X'}(v)) \cdot h^2(\mathcal{I}_{X'}(v)) = 0$ for $v = f$ and g implies that $\mathrm{H}(d',p',\pi')$ is smooth at (X') and we are done. $\square$

Example 7.5. *Let Z be the surface which is linked to the surface $(X') \in \mathrm{H}(14,8,20)$ of Example 7.3 via a complete intersection of type $(5,6)$ containing X'. Then (Z) belongs to $\mathrm{H}(16,15,27)$, $\omega_Z = \mathcal{I}_{X'/Y}(6)$ is globally generated, and $M_i(Z) = \oplus H^i(\mathcal{I}_Z(v))$, $i = 1,2$, are supported at two consecutive degrees. Moreover;*

$$\begin{aligned} &h^0(\mathcal{I}_Z(5)) = 1, \quad h^1(\mathcal{I}_Z(4)) = 2 \quad \textit{and} \quad h^1(\mathcal{I}_Z(5)) = 1 \\ &h^2(\mathcal{O}_Z(1)) = 1, \quad h^2(\mathcal{I}_Z(2)) = 3 \quad \textit{and} \quad h^2(\mathcal{I}_Z(3)) = 1\,. \end{aligned} \tag{32}$$

By Proposition 4.1, we know $\chi(\mathcal{N}_{X'}) = 5(2d' + \pi' - 1) - d'^2 + 2\chi(\mathcal{O}_{X'}) = 57$ and since we obviously have $h^2(\mathcal{N}_{X'}) = 0$ (from $h^2(\mathcal{O}_{X'}(1)) = 0$) and we get $h^0(\mathcal{N}_{X'}) = 57$ from Example 7.3, we conclude that $h^1(\mathcal{N}_{X'}) = 0$. The conditions of Proposition 7.4 are therefore satisfied (replacing X by X' there).

Hence, at (Z), we get that $\mathrm{H}(16,15,27)_{\gamma,\rho}$ *is smooth of codimension 1 in* $\mathrm{H}(16,15,27)$. *Moreover* $\mathrm{H}(16,15,27)$ *is smooth at (Z), and*

$$\dim_{(Z)} \mathrm{H}(16,15,27)_{\gamma,\rho} =$$
$$\dim_{(X')} \mathrm{H}_{\gamma',\rho'} + h^0(\mathcal{I}_{X'/Y}(5)) + h^0(\mathcal{I}_{X'/Y}(6)) - h^2(\mathcal{O}_{X'}) - h^2(\mathcal{O}_{X'}(1)) = 65.$$

Hence Z belongs to a unique generically smooth component V of $\mathrm{H}(16,15,27)$ *of dimension 66, and since the generic surface $\tilde{Z}$ of V do not have the same cohomology as Z (since $\tilde{Z} \notin \mathrm{H}(16,15,27)_{\gamma,\rho}$), we must get*

$$\dim H^0(\mathcal{I}_{\tilde{Z}}(5)) = \dim H^1(\mathcal{I}_{\tilde{Z}}(5)) = 0$$

while elsewhere the dimension of the cohomology groups is unchanged, i.e. it is as in (32).

8 Obstructed surfaces in $\mathbb{P}^4$.

In this section we explicitly prove the existence of obstructed surfaces. Our examples are as close as they can be to the arithmetically Cohen-Macaulay case. Indeed, in the examples, one of the Rao modules in the pair (M_1, M_2) vanishes, the other is 1-dimensional. Moreover in Proposition 4.4 and Remark 4.5 we gave conditions which imply unobstructedness. Our Example 8.3 is minimal with respect to the mentioned conditions in the sense that only one of the many cohomology groups, claimed in Remark 4.5 (i) to vanish, is non-zero. It also shows that we in Remark 7.2 can not skip the assumption (30) since we in Example 8.3 link an unobstructed surface to an obstructed surface where one of the cohomology groups of (30) is non-zero. Moreover, note that once having constructed one obstructed surface we can find infinitely many by linking under the assumption (30).

In the following proposition we consider a codimension 2 subscheme X of $\mathbb{P}^{n+2}$, containing a CI Y of type (f_1, f_2), in order to find obstructed codimension 2 subschemes of $\mathbb{P}^{n+2}$ for $n \geq 1$. In this situation we recall that the inclusion map $I_Y \to I_X$ induces a morphism $H^0(\mathcal{N}_X) \to \oplus_{i=1}^2 H^0(\mathcal{O}_X(f_i))$ whose composition with $\oplus_{i=1}^2 H^0(\mathcal{O}_X(f_i)) \to \oplus_{i=1}^2 H^1(\mathcal{I}_X(f_i))$ we denote $\beta_{X/Y}$. Note that we below do not need the cotangent sheaves to vanish since we work only with tangent (and not obstruction) spaces of $D(p(v); f_1, f_2)$.

Proposition 8.1. *Let X be an equidimensional locally Cohen-Macaulay codimension 2 subscheme of $\mathbb{P}^{n+2}$, and let Y and Y_0 be two complete intersections containing X, both of type (f_1, f_2) such that*

i) $\beta_{X/Y}$ *is surjective* and β_{X/Y_0} *is not surjective,*
ii) $H^n(\mathcal{I}_X(f_i - n - 3)) = 0$ *for $i = 1$ and $i = 2$.*

Let X' (resp. X_0') be linked to X by Y (resp. Y_0). Then X_0 is obstructed. Moreover if X is unobstructed, then so is X'.

Proof. If $A^1(X \subseteq Y)$ is the tangent space of the Hilbert flag scheme $D(p(v); f_1, f_2)$ at (X, Y), then it is shown in [29], (1.11) that there is an exact sequence

$$0 \to \oplus_{i=1}^2 H^0(\mathcal{I}_{X/Y}(f_i)) \to A^1(X \subseteq Y) \to H^0(\mathcal{N}_X) \to \oplus_{i=1}^2 H^1(\mathcal{I}_X(f_i))$$

where the rightmost map is $\beta_{X/Y}$. The corresponding exact sequence for $(X \subseteq Y_0)$ together with the assumption (i) show that

$$\dim A^1(X \subseteq Y) < \dim A^1(X \subseteq Y_0)$$

because it is easy to see $h^0(\mathcal{I}_{X/Y}(v)) = h^0(\mathcal{I}_{X/Y_0}(v))$ for every v. We *claim* that $D(p(v); f_1, f_2)$ is not smooth at (X, Y_0). Suppose the converse. Since it is shown in [29], Thm. 1.16 (a) that the fibers of the first projection $p : D(p(v); f_1, f_2) \to \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ are irreducible, it follows that there exists an irreducible component W of $D(p(v); f_1, f_2)$ which contains both points, (X, Y) and (X, Y_0). Hence if $D(p(v); f_1, f_2)$ is smooth at (X, Y_0), we get

$$\dim A^1(X \subseteq Y_0) = \dim W \leq \dim_{(X,Y)} D(p(v); f_1, f_2) \leq \dim A^1(X \subseteq Y),$$

i.e. a contradiction.

Thanks to (29) we get that $D(p'(v); f_1, f_2)$ is not smooth at (X'_0, Y_0). Since $h^1(\mathcal{I}_{X'_0}(f_i - n - 3)) = h^n(\mathcal{I}_X(f_{3-i} - n - 3)) = 0$ for $i = 1, 2$, cf. (27), and since the vanishing of $H^1(\mathcal{I}_{X'_0}(f_i - n - 3))$ implies that the first projection $p' : D(p'(v); f_1, f_2) \to \mathrm{Hilb}^{p'(v)}(\mathbb{P}^{n+2})$ is smooth at (X'_0, Y_0) by [29], Thm. 1.16 (b), we conclude that X'_0 is obstructed. Finally, for the last conclusion, if we have the surjectivity of $\beta_{X/Y}$ and assume the unobstructedness of X, we get that $D(p(v); f_1, f_2)$ is smooth at (X, Y) by [29], Prop. 3.12. Using (29) and (27) once more we conclude that X' is unobstructed, and we are done. □

We think the surjectivity of $\beta_{X/Y}$ may often hold, provided the generators of I_Y are among the minimal generators of I_X, but this is difficult to prove. In the Buchsbaum case, however, it is easy to see the surjectivity, as observed in [7] for curves. Indeed even though the statement of Proposition 8.1 and the remark below generalizes [7], Prop. 2.1 by far, the ideas of the proof are quite close to the idea in Prop. 2.1 of [7].

Remark 8.2. *In this remark we consider surfaces in $\mathbb{P}^4$ with minimal resolution given as in* (10).

(i) *Using* (5) *and the spectral sequence* (2) *we get an exact sequence*

$$\to H^0(\mathcal{N}_X) \to {}_0\mathrm{Hom}_R(I_X, H^2_{\mathfrak{m}}(I_X)) \xrightarrow{\alpha} {}_0\mathrm{Ext}^2_R(I_X, I_X) \to$$

where ${}_0\mathrm{Hom}_R(I_X, H^2_{\mathfrak{m}}(I_X)) \simeq \oplus_i H^1(\mathcal{I}_X(n_{1,i}))$ *provided* $H^1(\mathcal{I}_X(n_{2,i})) = 0$ *for any* i. *The natural map* $H^0(\mathcal{N}_X) \to {}_0\mathrm{Hom}_R(I_X, H^2_{\mathfrak{m}}(I_X)) \simeq \oplus_i H^1(\mathcal{I}_X(n_{1,i}))$, *which we denote* β_X, *is correspondingly defined as* $\beta_{X/Y}$ *above, but with the difference that a set of all minimal generators of* I_X *is used. In particular if the generators of* I_Y *are among the minimal generators of* I_X, *then the composition of* β_X *with the projection* $\oplus_i H^1(\mathcal{I}_X(n_{1,i})) \to \oplus_{i=1}^2 H^1(\mathcal{I}_X(f_i))$ *is* $\beta_{X/Y}$. *It follows that if*

$${}_0\mathrm{Ext}^2_R(I_X, I_X) = 0 \quad \text{and} \quad H^1(\mathcal{I}_X(n_{2,i})) = 0 \text{ for any i },$$

then $\beta_{X/Y}$ *is surjective. Note that, by* (3) *and* (2) *(cf. the proof of Proposition 4.4),* ${}_0\mathrm{Ext}^2_R(I_X, I_X) = 0$ *provided* ${}_{-5}\mathrm{Ext}^1_R(I, M_1) = {}_{-5}\mathrm{Hom}_R(I, M_2) = 0$, *i.e. provided*

$$H^1(\mathcal{I}_X(n_{2,i} - 5)) = 0 \text{ and } H^2(\mathcal{I}_X(n_{1,i} - 5)) = 0 \text{ for every } i.$$

(ii) *If, however, the minimal generators* $\{F_1, F_2\}$ *of* I_Y *do not belong to a set of minimal generators of* I_X, *say* $F_i = H_i \cdot G_i$ *for some* $G_i \in I_X$, $i = 1, 2$, *then* $\beta_{X/Y}$ *is easily seen to be non-surjective under a manageable assumption. Indeed let* g_i *be the degree of the form* G_i, *let* Y_0 *be the CI with homogeneous ideal* $I_{Y_0} = (G_1, G_2)$ *and suppose the obvious map*

$$h \ : \ \oplus_{i=1}^2 H^1(\mathcal{I}_X(g_i)) \xrightarrow{(H_1,H_2)} \oplus_{i=1}^2 H^1(\mathcal{I}_X(f_i)) \text{ is not surjective.}$$

Then $\beta_{X/Y}$ *can not be surjective because it factors via* h, *i.e.* $\beta_{X/Y} = h \circ \beta_{X/Y_0}$*!*

Example 8.3. *If we link the smooth quintic scroll* Z *of* $\mathrm{H}(5, -1, 1)$ *with Rao modules* $H^1_*(\mathcal{I}_Z) = 0$, $H^2_*(\mathcal{I}_Z) \simeq k$ *and minimal resolution (cf. [11], B.2.1),*

$$0 \to R(-5) \to R(-4)^{\oplus 5} \to R(-3)^{\oplus 5} \to I_Z \to 0, \tag{33}$$

using a CI of type $(5,6)$ *containing* Z*, then the ideal of the linked surface* X *has a minimal resolution*

$$0 \to R(-11) \to R(-10)^{\oplus 5} \to R(-9)^{\oplus 10} \to R(-8)^{\oplus 5} \oplus R(-6) \oplus R(-5) \to I_X \to 0$$

and Rao modules given by $H^2_*(\mathcal{I}_X) = 0$*,* $h^1(\mathcal{I}_X(6)) = 1$ *and* $H^1(\mathcal{I}_X(v)) = 0$ *for* $v \neq 6$*. Using* (27) *we see that* (X) *belongs to* $\mathrm{H}(d,p,\pi) = \mathrm{H}(25,99,71)$*. This surface* X *has invariants such that Proposition 8.1 and Remark 8.2 apply. Indeed we can link* X *to two different surfaces* X' *and* X'_0 *using CI's* Y *and* Y_0 *containing* X*, both of type* $(6,8)$*, generated in the following way. Let* F_5*, resp.* F_6*, be the minimal generator of* I_X *of degree 5, resp. 6, and let* G *be a general element of* $H^0(\mathcal{I}_X(8))$*. Then we take* Y*, resp.* Y_0*, to be given by* $I_Y = (F_6, G)$*, resp.* $I_{Y_0} = (H \cdot F_5, G)$ *where* H *is a linear form. We may check that all assumptions of Remark 8.2 are satisfied. Hence we get that* X' *and* X'_0 *belong to a common irreducible component of* $\mathrm{H}(d',p',\pi') = \mathrm{H}(23,80,61)$*, that* X'_0 *is obstructed with minimal resolution*

$$0 \to R(-8) \to R(-7)^{\oplus 5} \oplus R(-8) \oplus R(-9) \to R(-6)^{\oplus 6} \oplus R(-8) \to I_{X'_0} \to 0,$$

while X' *is unobstructed with minimal resolution*

$$0 \to R(-8) \to R(-7)^{\oplus 5} \oplus R(-9) \to R(-6)^{\oplus 6} \to I_{X'} \to 0.$$

Note that it is straightforward to find these resolutions since X' *and* X'_0 *are bilinked to* Z *and we know the minimal resolution of* I_Z*, see [39] or the sequence* (39) *appearing later in this paper. We observe that common direct free factors ("ghost terms") are present in the minimal resolution, similar to what happens for obstructed curve with "small Rao module", cf. [33]. Moreover since the assumptions of Proposition 4.4 are satisfied for* X'*, we also get the unobstructedness of* X' *from that Proposition and the dimension,*

$$\dim_{(X')} \mathrm{H}(23,80,61) = 1 + \delta^3(-5) - \delta^2(-5) + \delta^1(-5) = 163.$$

However, since the conditions of Remark 4.5 (i) also hold, we get $H^1(\mathcal{N}_{X'}) = 0$ *and hence it is easier to compute* $\dim_{(X')} \mathrm{H}(23,80,61)$ *by using Proposition 4.1. We get*

$$\dim_{(X')} \mathrm{H}(23,80,61) = \chi(\mathcal{N}_{X'}) = 5(2d' + \pi' - 1) - d'^2 + 2\chi(\mathcal{O}_{X'}) = 163.$$

Note that neither the assumptions of Proposition 4.4, nor the assumptions of Remark 4.5 (i), are satisfied for X'_0*. Indeed Remark 4.5 (i) a little extended will show* $h^1(\mathcal{N}_{X'_0}) = 1$ *(i.e. just compute the dimension using* (17)*). The surface* X'_0 *is easily seen to be reducible, as pointed out to me by H. Nasu.*

Example 8.4. *If we link the surface* X'_0 *of Example 8.3 using a general CI of type* $(9,9)$ *containing* X'_0*, we get a smooth obstructed surface* S *of degree 58. Indeed the assumptions of Remark 7.2 are satisfied. So* S *is obstructed, and we have used Macaulay 2 ([19]) to verify that* S *is smooth provided the CI's used in the linkages of Example 8.3 are general enough under the specified restrictions. The surface* S *is in the biliaison class of the Veronese surface in* $\mathbb{P}^4$*.*

Finally if we link S *via a general CI of type* $(9,12)$ *containing* S*, we get an obstructed surface* S' *of degree 50 by Remark 7.2. We have used Macaulay 2 to verify that the surface is smooth. The surface* S' *is in the biliaison class of the quintic elliptic scroll in* $\mathbb{P}^4$*. Since* S' *is bilinked to the surface* X'_0 *of Example 8.3 we easily find the minimal resolution of* $I_{S'}$ *to be*

$$0 \to R(-11) \to R(-10)^{\oplus 5} \oplus R(-11) \oplus R(-12)^{\oplus 2} \to R(-9)^{\oplus 7} \oplus R(-11) \to I_{S'} \to 0.$$

Note that we again have "ghost terms" in the minimal resolution in degree $c+5$ *where* $h^2(\mathcal{I}_{S'}(c)) \neq 0$*. This feature seems to be related to obstructedness, as in the curve case, cf. [33].*

9 Even liaison of codimension 2 subschemes of $\mathbb{P}^{n+2}$.

In this section we prove the main even liaison theorem of this paper, which holds for any equidimensional lCM codimension 2 subscheme X of $\mathbb{P}^{n+2}$. We also generalize Proposition 4.4 and the vanishing result for $h^1(\mathcal{N}_X)$ of Remark 4.5 to schemes X of dimension $n > 2$ and we give an example of an obstructed 3-fold.

First we define $\delta_X^m(v)$. Let

$$0 \to \bigoplus_{i=1}^{r_{n+2}} R(-n_{n+2,i}) \to \bigoplus_{i=1}^{r_{n+1}} R(-n_{n+1,i}) \to ... \to \bigoplus_{i=1}^{r_2} R(-n_{2,i}) \to \bigoplus_{i=1}^{r_1} R(-n_{1,i}) \to I \to 0 \tag{34}$$

be a minimal resolution of $I = I_X$ and let the invariant $\delta^m(v) = \delta_X^m(v)$ be defined by

$$\delta_X^m(v) = \sum_{j=1}^{n+2} \sum_{i=1}^{r_j} (-1)^{j+1} h^m(\mathcal{I}_X(n_{j,i} + v)) \ . \tag{35}$$

Since adding common direct free factors in consecutive terms of (34) does not change $\delta_X^m(v)$, the resolution of I does not really need to be minimal in the definition of $\delta_X^m(v)$.

Theorem 9.1. *Let X and X' be two equidimensional locally Cohen-Macaulay codimension 2 subschemes of $\mathbb{P}^{n+2}$, linked to each other in two steps by two complete intersections, and suppose that (X) (resp. (X')) belongs to the Hilbert scheme $\mathrm{H}_{\gamma,\rho}$ (resp. $\mathrm{H}_{\gamma',\rho'}$) of constant cohomology. Then*

i) $$\delta_X^{n+1}(-n-3) - \dim_{(X)} \mathrm{H}_{\gamma,\rho} = \delta_{X'}^{n+1}(-n-3) - \dim_{(X')} \mathrm{H}_{\gamma',\rho'} \ .$$

In particular $\mathrm{obsumext}(X) := 1 + \delta_X^{n+1}(-n-3) - \dim_{(X)} \mathrm{H}_{\gamma,\rho}$ *is a biliaison invariant.*

ii) $$\delta_X^{n+1}(-n-3) - \dim {}_0\mathrm{Ext}^1_R(I_X, I_X)_\rho = \delta_{X'}^{n+1}(-n-3) - \dim {}_0\mathrm{Ext}^1_R(I_{X'}, I_{X'})_{\rho'}.$$

In particular $\mathrm{sumext}(X) := 1 + \delta_X^{n+1}(-n-3) - \dim {}_0\mathrm{Ext}^1_R(I_X, I_X)_\rho$ *is a biliaison invariant.*

iii) *We have* $\mathrm{sumext}(X) \le \mathrm{obsumext}(X)$, *with equality if and only if* $\mathrm{H}_{\gamma,\rho}$ *is smooth at* (X).

Remark 9.2. *This result is motivated by Remarks 3.9 and 6.4. Indeed we were quite convinced that Theorem 9.1 was true before starting proving it. Note that the dimension formula of Remark 6.4 was quite involved already for the case $n = \dim X = 2$ and we expect a very complicated formula for $n > 2$. So Theorem 9.1 may be a good practical approach to the problem of studying $\mathrm{H}_{\gamma,\rho}$ and $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ with respect to smoothness and dimension for $n > 1$. However, except for the other results of this paper, we have no better option for the use of Theorem 9.1 that to first compute* $\mathrm{sumext}(X)$ *and* $\mathrm{obsumext}(X)$ *through a nice representative in the even liaison class, e.g. for the minimal element of the class, before we use it for an arbitrary element in the even liaison class.*

Remark 9.3. *For the application of Theorem 9.1 there is one natural situation where $\mathrm{H}_{\gamma,\rho}$ is isomorphic to $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ at (X), namely in the case X has seminatural cohomology. We say a subscheme $X \subseteq \mathbb{P}^{n+2}$ has seminatural cohomology if for every $v \in \mathbb{Z}$, at most one of groups $H^0(\mathcal{I}_X(v)), H^1(\mathcal{I}_X(v)), ..., H^{n+1}(\mathcal{I}_X(v))$ are non-zero. In this case a generization (i.e. a deformation to more general element in $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$) of X is forced to have the same cohomology as X by the semicontinuity of $h^i(\mathcal{I}_X(v))$, i.e. $\mathrm{H}_{\gamma,\rho} \cong \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ as schemes at (X).*

Proof. Let X be linked to X_1 by a CI $Y \subseteq \mathbb{P}^{n+2}$ of type (f,g) and let X_1 be linked to X' by some CI $Y' \subseteq \mathbb{P}^{n+2}$ of type (f',g'). If (X_1) belongs to the Hilbert scheme $\mathrm{H}_1 := \mathrm{H}_{\gamma_1,\rho_1}$ of constant cohomology, then by Theorem 7.1,

$$\begin{aligned}\dim_{(X_1)} \mathrm{H}_1 = \dim_{(X)} \mathrm{H}_{\gamma,\rho} + h^0(\mathcal{I}_{X/Y}(f)) + h^0(\mathcal{I}_{X/Y}(g)) \\ - h^n(\mathcal{O}_X(f-n-3)) - h^n(\mathcal{O}_X(g-n-3)),\end{aligned}$$

$$\begin{aligned}\dim_{(X_1)} \mathrm{H}_1 = \dim_{(X')} \mathrm{H}_{\gamma',\rho'} + h^0(\mathcal{I}_{X'/Y'}(f')) + h^0(\mathcal{I}_{X'/Y'}(g')) \\ - h^n(\mathcal{O}_{X'}(f'-n-3)) - h^n(\mathcal{O}_{X'}(g'-n-3)).\end{aligned}$$

Let $h = f'+g'-f-g$. Using (28) twice we get $h^0(\mathcal{I}_{X'/Y'}(v)) = h^0(\mathcal{I}_{X/Y}(v-h))$. Hence

$$\begin{aligned}\dim_{(X')} \mathrm{H}_{\gamma',\rho'} = \dim_{(X)} \mathrm{H}_{\gamma,\rho} + h^0(\mathcal{I}_{X/Y}(f)) + h^0(\mathcal{I}_{X/Y}(g)) \\ - h^0(\mathcal{I}_{X/Y}(f'-h)) + h^0(\mathcal{I}_{X/Y}(g'-h)) + \eta\end{aligned} \tag{36}$$

where η is defined by

$$\eta := h^n(\mathcal{O}_{X'}(f'-n-3)) + h^n(\mathcal{O}_{X'}(g'-n-3)) - h^n(\mathcal{O}_X(f-n-3)) - h^n(\mathcal{O}_X(g-n-3)). \tag{37}$$

Next we need to find a free resolution of $I' = I_{X'}$ in terms of the minimal resolution of $I = I_X$ in (34). If we define E by the exact sequence

$$0 \to \oplus_{i=1}^{r_{n+2}} R(-n_{n+2,i}) \to \dots \to \oplus_{i=1}^{r_3} R(-n_{3,i}) \to \oplus_{i=1}^{r_2} R(-n_{2,i}) \to E \to 0, \tag{38}$$

we may put (34) in the form $0 \to E \to \oplus_{i=1}^{r_1} R(-n_{1,i}) \to I \to 0$. Then it is well known that there is an exact sequence

$$0 \to E(-h) \oplus R(-f-h) \oplus R(-g-h) \to \oplus_{i=1}^{r_1} R(-n_{1,i}-h) \oplus R(-f') \oplus R(-g') \to I' \to 0 \tag{39}$$

which combined with (38) yields a free resolution of I' (see [39]).

We will use this resolution of I' and (34) to see the connection between $\delta_X^{n+1}(-n-3)$ and $\delta_{X'}^{n+1}(-n-3)$. First we need to compute β defined by

$$\beta := \sum_{j=1}^{n+2} \sum_{i=1}^{r_j} (-1)^{j+1} \alpha(n_{j,i}-n-3) \quad \text{where} \quad \alpha(v) := h^n(\mathcal{O}_{X'}(v+h)) - h^n(\mathcal{O}_X(v))\ .$$

We *claim* that

$$\beta = h^0(\mathcal{I}_X(f)) + h^0(\mathcal{I}_X(g)) - h^0(\mathcal{I}_X(f'-h)) - h^0(\mathcal{I}_X(g'-h)) + h^0(\mathcal{I}_X(-h)). \tag{40}$$

Indeed by (28),

$$\alpha(v) = h^0(\mathcal{I}_{X_1/Y'}(f'+g'-n-3-v-h)) - h^0(\mathcal{I}_{X_1/Y}(f+g-n-3-v)).$$

Moreover since $0 \to \mathcal{I}_{Y'} \to \mathcal{I}_{X_1} \to \mathcal{I}_{X_1/Y'} \to 0$ and $0 \to \mathcal{I}_Y \to \mathcal{I}_{X_1} \to \mathcal{I}_{X_1/Y} \to 0$ are exact, we get

$$\alpha(v) = h^0(\mathcal{I}_Y(f+g-n-3-v)) - h^0(\mathcal{I}_{Y'}(f+g-n-3-v)). \tag{41}$$

Let $r(v) := \dim R_{(-n-3+v)}$. Combining with the minimal resolutions of I_Y and I'_Y, we get

$$\alpha(v) := r(f-v) + r(g-v) - r(-v) - r(f'-h-v) - r(g'-h-v) + r(-h-v).$$

Then we get the claim since (34) implies

$$h^0(\mathcal{I}_X(v)) = \sum_{j=1}^{n+2}\sum_{i=1}^{r_j}(-1)^{j+1}r(v - n_{j,i} + n + 3)$$

for any v and since $h^0(\mathcal{I}_X(0)) = 0$.

Using the resolution of I' deduced from (39) and the definition (35) we get

$$\delta_{X'}^{n+1}(-n-3) = \sum_{j=1}^{n+2}\sum_{i=1}^{r_j}(-1)^{j+1}h^n(\mathcal{O}_{X'}(n_{j,i} + h - n - 3)) + \epsilon$$

where ϵ is defined by

$$\begin{aligned}\epsilon := h^n(\mathcal{O}_{X'}(f' - n - 3)) + h^n(\mathcal{O}_{X'}(g' - n - 3)) \\ - h^n(\mathcal{O}_{X'}(f + h - n - 3)) - h^n(\mathcal{O}_{X'}(g + h - n - 3)).\end{aligned}$$

Comparing ϵ with η in (37) and recalling the definition of α, we have $\epsilon = \eta - \alpha(f-n-3) - \alpha(g-n-3)$. Moreover the definition of α, the proven claim and (35) lead to $\delta_{X'}^{n+1}(-n-3) = \delta_X^{n+1}(-n-3) + \beta + \epsilon$. Combining we get

$$\delta_{X'}^{n+1}(-n-3) = \delta_X^{n+1}(-n-3) + \beta + \eta - \alpha(f - n - 3) - \alpha(g - n - 3).$$

Comparing with (36) we get (i) of the Theorem provided we can show that

$$\begin{aligned}h^0(\mathcal{I}_{X/Y}(f)) + h^0(\mathcal{I}_{X/Y}(g)) - h^0(\mathcal{I}_{X/Y}(f' - h)) - h^0(\mathcal{I}_{X/Y}(g' - h)) \\ = \beta - \alpha(f - n - 3) - \alpha(g - n - 3).\end{aligned}$$

Suppose $h \geq 0$. Looking at (40), we see it suffices to show

$$-h^0(\mathcal{I}_Y(f)) - h^0(\mathcal{I}_Y(g)) + h^0(\mathcal{I}_Y(f' - h)) + h^0(\mathcal{I}_Y(g' - h)) = -\alpha(f - n - 3) - \alpha(g - n - 3).$$

Thanks to (41) it remains to show

$$h^0(\mathcal{I}_Y(f' - h)) + h^0(\mathcal{I}_Y(g' - h)) = h^0(\mathcal{I}_{Y'}(f)) + h^0(\mathcal{I}_{Y'}(g)).$$

Using the minimal resolutions of I_Y and $I_{Y'}$ and that $h = f' + g' - f - g \geq 0$, we easily show that both sides of the last equation is equal to $\dim R_{(f-f')} + \dim R_{(f-g')} + \dim R_{(g-f')} + \dim R_{(g-g')}$ and we get what we want, i.e.

$$\delta_X^{n+1}(-n-3) - \dim_{(X)} \mathrm{H}_{\gamma,\rho} = \delta_{X'}^{n+1}(-n-3) - \dim_{(X')} \mathrm{H}_{\gamma',\rho'} \tag{42}$$

provided $h \geq 0$. Suppose $h < 0$. Then we can start with X' and link in two steps back to X, i.e. we get an even liaison with $h' = f + g - f' - g' \geq 0$ in which case we know that (42) holds. Hence (42) is proved in general.

To show (ii) of the Theorem we only need to remark that, due to Theorem 7.1, (36) holds if we replace $\dim_{(X)} \mathrm{H}_{\gamma,\rho}$ and $\dim_{(X')} \mathrm{H}_{\gamma',\rho'}$ by the dimension of their tangent spaces ${}_0\mathrm{Ext}^1_R(I_X, I_X)_\rho$ and ${}_0\mathrm{Ext}^1_R(I_{X'}, I_{X'})_{\rho'}$ respectively. With the proof of Theorem 9.1 (i) above, we therefore get (42) with the mentioned replacements, i.e. we get Theorem 9.1 (ii).

Finally Theorem 9.1 (iii) follows by combining (i) and (ii) since e.g. the smoothness of $\mathrm{H}_{\gamma,\rho}$ at (X) is equivalent to $\dim_{(X)} \mathrm{H}_{\gamma,\rho} = \dim {}_0\mathrm{Ext}^1_R(I_X, I_X)_\rho$. $\square$

Corollary 9.4. *Let X be an equidimensional lCM codimension 2 subschemes of* $\mathbb{P}^{n+2}$*, and suppose* (X) *be a generic point of a generically smooth component* V *of* $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$*. Then* $\mathrm{sumext}(X) = \mathrm{obsumext}(X)$ *and*

$$\dim V = 1 + \delta_X^{n+1}(-n-3) - \mathrm{sumext}(X).$$

Proof. Arguing as the last part of the proof of Theorem 3.7, we get that $\mathrm{H}_{\gamma,\rho}$ is isomorphic to $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ at (X). Hence $\mathrm{H}_{\gamma,\rho}$ is smooth at (X). Then we conclude by Theorem 9.1. □

Corollary 9.5. *Let X be a surface in* $\mathbb{P}^4$*. If the local deformation functors* $Def(M_i)$ *of* M_i *are formally smooth (for instance if* ${}_0\mathrm{Ext}_R^2(M_i, M_i) = 0$*) for* $i = 1, 2$*, and if* ${}_0\mathrm{Ext}_R^3(M_2, M_1) = 0$*, then*

$$\mathrm{sumext}(X) = \mathrm{obsumext}(X).$$

Proof. By Corollary 5.4 we get that $\mathrm{H}_{\gamma,\rho}$ is smooth at (X) and we conclude by Theorem 9.1 (iii). □

Corollary 9.6. *Let X be an arithmetically Cohen-Macaulay codimension 2 subschemes of* $\mathbb{P}^{n+2}$*. Then* $\mathrm{sumext(X)} = \mathrm{obsumext}(X) = 0$*. Moreover,*

(i) *if $n > 0$, then X is unobstructed and*

$$\dim_{(X)} \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2}) = 1 + \delta_X^{n+1}(-n-3) = 1 - \delta_X^0(0) = \chi(\mathcal{N}_X) + (-1)^n \delta_X^0(-n-3),$$

(ii) *if $n = 0$, then* H_γ *is smooth at* (X) *and*

$$\dim_{(X)} \mathrm{H}_\gamma = 1 + \delta_X^1(-3) = 1 - \delta_X^0(0) = h^0(\mathcal{N}_X) + \delta_X^0(-3).$$

Proof. By Gaeta's theorem ([14], [15], cf. [2], [3]) X is in the liaison class of a complete intersection Y. Suppose $n > 0$. Then $\mathrm{H}_{\gamma,\rho} \cong \mathrm{H}_\gamma \cong \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ at (X) by [13] or [27], Rem. 3.7, (cf. [49], Thm. 2.1). Thanks to Theorem 9.1 it suffices to show that $\mathrm{sumext(Y)} = 0$, or equivalently that $\dim {}_0\mathrm{Ext}_R^1(I_Y, I_Y)_\rho = 1 + \delta_Y^{n+1}(-n-3)$. By definition, cf. (25), and (5), ${}_0\mathrm{Ext}_R^1(I_Y, I_Y)_\rho = {}_0\mathrm{Ext}_R^1(I_Y, I_Y) = h^0(\mathcal{N}_Y)$ and it is trivial to show $h^0(\mathcal{N}_Y) = 1 + \delta_Y^{n+1}(-n-3)$ by using duality and the minimal resolution of I_Y.

Moreover note that for *any* equidimensional lCM codimension 2 subschemes X of $\mathbb{P}^{n+2}$, we easily show

$$\sum_{i=1}^{n+1} {}_0\mathrm{ext}_R^i(I_X, I_X) = 1 - \delta_X^0(0) = \chi(\mathcal{N}_X) + (-1)^n \delta_X^0(-n-3). \tag{43}$$

as in Proposition 4.1 (see the first sentence of the proof for the left equality and second and third sentence of the proof for the right equality). Hence if X is arithmetically Cohen-Macaulay we get ${}_0\mathrm{ext}_R^i(I_X, I_X) = 0$ for $i \geq 2$ and we are done in the case $n > 0$. The case $n = 0$ is similar and easier. □

Remark 9.7. *Corollary 9.6 coincides with [13] if $n > 0$, and with [18] and [36], Rem. 4.6 if $n = 0$.*

Example 9.8. *Let X be the smooth rational surface of* $\mathrm{H}(11, 0, 11)$ *of Example 4.6. Note that X has seminatural cohomology and hence we have* $\mathrm{H}_{\gamma,\rho} \cong \mathrm{H}(d, p, \pi)$ *at* (X) *by Remark 9.3. Moreover* $I = I_X$ *admits a minimal resolution*

$$0 \to R(-9) \to R(-8)^{\oplus 3} \oplus R(-7)^{\oplus 3} \to R(-7)^{\oplus 2} \oplus R(-6)^{\oplus 12} \to R(-5)^{\oplus 10} \to I \to 0. \tag{44}$$

By Example 4.6 we conclude that $\mathrm{H}_{\gamma,\rho} \cong \mathrm{H}(d, p, \pi)$ *is smooth at* (X) *and that* $\dim_{(X)} \mathrm{H}(d, p, \pi) = 41$*. However, since X is rational we obviously get* $1 + \delta_X^3(-5) = 1$ *from (44). By Theorem 9.1 we find* $\mathrm{sumext}(X) = \mathrm{obsumext}(X) = -40$*. Now we link twice to get X', first using a CI of type $(5, 5)$, then* *a CI of type* $(5,6)$*, both times using a common hypersurface of degree 5. Looking at* (39) *we find a free resolution of* $I' = I_{X'}$ *of the form*

$$\begin{aligned}0 \to R(-10) \to R(-9)^{\oplus 3} \oplus R(-8)^{\oplus 3} \to R(-8)^{\oplus 2} \oplus R(-7)^{\oplus 12} \oplus R(-6) \\ \to R(-6)^{\oplus 10} \oplus R(-5) \to I' \to 0.\end{aligned} \tag{45}$$

By (28)*,* $h^2(\mathcal{O}_{X'}) = 15$ *and* $h^2(\mathcal{O}_{X'}(1)) = 1$ *and we get* $1 + \delta^3_{X'}(-5) = 25$*. It follows from Theorem 9.1 and Proposition 6.1 that* $\mathrm{H}_{\gamma',\rho'} \cong \mathrm{H}(d',p',\pi')$ *is smooth at* (X') *of dimension* $1 + \delta^3_{X'}(-5) - \mathrm{sumext}(X) = 65$*. Compare with Examples 7.3 and 7.5.*

Before considering examples of 3-folds, we want to generalize some of the results of section 4. For recent papers on the Hilbert scheme of 3-folds, see [5] and its references. See also [12] for a long list of examples of 3-folds of non general type.

Proposition 9.9. *Let* X *be an equidimensional lCM codimension 2 subschemes of* $\mathbb{P}^{n+2}$*, let* $M_i = H^i_*(\mathcal{I}_X)$ *for* i = 1,...,n *and* $I = I_X$ *and suppose*

$${}_0\mathrm{Hom}_R(I, M_1) = 0 \quad \text{and} \quad {}_{-n-3}\mathrm{Ext}^{n-j}_R(I, M_j) = 0 \text{ for every } j,\ 1 \le j \le n.$$

Then ${}_0\mathrm{Ext}^2_R(I,I) = 0$*,* X *is unobstructed and*

$$\dim_{(X)} \mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2}) = {}_0\mathrm{ext}^1_R(I,I).$$

E.g. let $\dim X = 3$*. Then* X *is unobstructed and* $\dim_{(X)} \mathrm{Hilb}^{p(v)}(\mathbb{P}^5) = {}_0\mathrm{ext}^1_R(I,I)$ *if, for every* i*,*

$$H^1(\mathcal{I}_X(n_{1,i})) = H^3(\mathcal{I}_X(n_{1,i}-6)) = H^2(\mathcal{I}_X(n_{2,i}-6)) = H^1(\mathcal{I}_X(n_{3,i}-6)) = 0. \tag{46}$$

If in addition

$$H^2(\mathcal{I}_X(n_{1,i}-6)) = 0,\ H^1(\mathcal{I}_X(n_{2,i}-6)) = 0 \ \text{ and } \ H^1(\mathcal{I}_X(n_{1,i}-6)) = 0,$$

for every i*, then* $\dim_{(X)} \mathrm{Hilb}^{p(v)}(\mathbb{P}^5) = 1 - \delta^0_X(0) = \chi(\mathcal{N}_X) - \delta^0_X(-6)$.

Proof. Thanks to [27], Rem. 3.7 (cf. [49], Thm. 2.1), the Hilbert scheme H_γ of constant postulation is isomorphic to $\mathrm{Hilb}^{p(v)}(\mathbb{P}^{n+2})$ at (X) provided ${}_0\mathrm{Hom}_R(I, M_1) = 0$. By (3) we get ${}_0\mathrm{Ext}^2_R(I,I) = 0$ provided ${}_{-n-3}\mathrm{Ext}^{n+1}_{\mathfrak{m}}(I,I) = 0$. By (2) and $M_j \cong H^{j+1}_{\mathfrak{m}}(I)$ we deduce the vanishing of the latter from the assumptions of the proposition. It follows that H_γ is smooth at (X) of dimension ${}_0\mathrm{ext}^1_R(I,I)$.

Suppose $n = 3$. By the definition of ${}_v\mathrm{Ext}^\bullet_R(I,-)$ and (34) we easily prove the vanishing of all $\mathrm{Ext}^\bullet_R(I,-)$-groups of the first part of the proposition from the explicit vanishings in (46). Moreover due (43), to get the final formula it suffices to show ${}_0\mathrm{Ext}^j_R(I,I) = 0$ for $j = 3,4$. By (3) we must prove ${}_{-n-3}\mathrm{Ext}^{n-j}_{\mathfrak{m}}(I,I) = 0$ for $j = 0,1$. This is shown in exactly the same way as we did for ${}_{-n-3}\mathrm{Ext}^{n+1}_{\mathfrak{m}}(I,I) = 0$, i.e. by using (2) and (34) and we are done. □

Remark 9.10. (i) *We can also generalize Remark 4.5 to equidimensional lCM codimension 2 subschemes* $X \subseteq \mathbb{P}^{n+2}$ *of higher dimension. Indeed using* (5)*,* (2) *and* (3)*, see the proof above, we get* $H^1(\mathcal{N}_X) = 0$ *provided* ${}_0\mathrm{Ext}^3_{\mathfrak{m}}(I,I) = 0$ *and* ${}_{-n-3}\mathrm{Ext}^{n+1}_{\mathfrak{m}}(I,I) = 0$*, e.g. provided*

$${}_0\mathrm{Ext}^j_R(I, M_{2-j}) = 0 \text{ for } 0 \le j \le 1 \quad \text{and} \quad {}_{-n-3}\mathrm{Ext}^{n-j}_R(I, M_j) = 0 \text{ for } 1 \le j \le n.$$

Similarly $H^2(\mathcal{N}_X) = 0$ *provided* ${}_0\mathrm{Ext}^4_{\mathfrak{m}}(I,I) = 0$ *and* ${}_{-n-3}\mathrm{Ext}^n_{\mathfrak{m}}(I,I) = 0$*, e.g. provided*

$${}_0\mathrm{Ext}^j_R(I, M_{3-j}) = 0 \ \text{ for } 0 \le j \le 2 \ \text{ and } \ {}_{-n-3}\mathrm{Ext}^{n-j}_R(I, M_{j-1}) = 0 \ \text{ for } 2 \le j \le n.$$

We can in this way easily get a vanishing criteria for $H^q(\mathcal{N}_X)=0$ for every $q\geq 1$.
(ii) *Suppose for instance $n=\dim X=3$. Then $H^1(\mathcal{N}_X)=0$ if, for every i,*

$$\begin{aligned} &H^1(\mathcal{I}_X(n_{2,i}))=H^2(\mathcal{I}_X(n_{2,i}-6))=0,\\ &H^2(\mathcal{I}_X(n_{1,i}))=H^3(\mathcal{I}_X(n_{1,i}-6))=0 \ \text{ and } \ H^1(\mathcal{I}_X(n_{3,i}-6))=0. \end{aligned}$$

Moreover $H^2(\mathcal{N}_X)=0$ if, for every i,

$$\begin{aligned} H^1(\mathcal{I}_X(n_{3,i}))=0,\ H^2(\mathcal{I}_X(n_{2,i}))=H^1(\mathcal{I}_X(n_{2,i}-6))=0 \ \text{ and}\\ H^3(\mathcal{I}_X(n_{1,i}))=H^2(\mathcal{I}_X(n_{1,i}-6))=0. \end{aligned}$$

As in the surface case, if some of the assumptions of Proposition 9.9 or Remark 9.10 are not satisfied, we can find examples of obstructed 3-folds (e.g. X_0' in the example below). Note that all assumptions of Proposition 9.9 and Remark 9.10 (ii) are satisfied for X_0', except $H^3(\mathcal{I}_{X_0'}(n_{1,i}-6))=0$ for one i.

Example 9.11. *We start with the smooth 3-fold $Z\subseteq \mathbb{P}:=\mathbb{P}^5$ of [41] of degree 7 with Ω-resolution*

$$0\to \mathcal{O}_{\mathbb{P}}^{\oplus 4}\to \Omega_{\mathbb{P}}(2)\to \mathcal{I}_Z(4)\to 0,$$

where $\Omega_{\mathbb{P}}$ is the kernel of the map $\mathcal{O}_{\mathbb{P}}(-1)^6\to\mathcal{O}_{\mathbb{P}}$ induced by the multiplication with $(X_0,..,X_5)$. Note that $h^1(\mathcal{I}_Z(2))=1$. If we link Z, first using a CI of type $(4,4)$ to get a 3-fold Z', then a CI of type $(6,7)$ to link Z' to X, then X is a 3-fold with properties such that Proposition 8.1 applies. Indeed the ideas of Remark 8.2 also apply except for how we proved ${}_0\mathrm{Ext}^2_R(I,I)=0$. By the proof of Proposition 9.9, however, we have ${}_0\mathrm{Ext}^2_R(I,I)=0$ for 3-folds provided $H^3(\mathcal{I}_X(n_{1,i}-6))=H^2(\mathcal{I}_X(n_{2,i}-6))=H^1(\mathcal{I}_X(n_{3,i}-6))=0$ for all i. To see that all these $H^i(\mathcal{I}_X(j))$-groups vanish, we first find the minimal resolution of $I_{Z'}$. Combining the exact sequence $0\to\mathcal{O}_{\mathbb{P}}\to\mathcal{O}_{\mathbb{P}}(1)^6\to\Omega_{\mathbb{P}}^{\vee}\to 0$ with the mapping cone construction for how we get the resolution of $I_{Z'}$ from the resolution of I_Z, we find the minimal resolution

$$0\to R(-6)\to R(-5)^{\oplus 6}\to R(-4)^{\oplus 6}\to I_{Z'}\to 0.$$

Hence $H^1_*(\mathcal{I}_{Z'})=0$, $H^2_*(\mathcal{I}_{Z'})=0$ and we get $H^3_*(\mathcal{I}_X)=0$, $H^2_*(\mathcal{I}_X)=0$ and $H^1_*(\mathcal{I}_X)\simeq H^1(\mathcal{I}_X(7))\simeq k$, cf. (27). Now since the Koszul resolution induced by the regular sequence $\{X_0,..,X_5\}$ implies that

$$0\to\mathcal{O}_{\mathbb{P}}(-6)\to\mathcal{O}_{\mathbb{P}}(-5)^{\oplus 6}\to\mathcal{O}_{\mathbb{P}}(-4)^{\oplus 15}\to\mathcal{O}_{\mathbb{P}}(-3)^{\oplus 20}\to\mathcal{O}_{\mathbb{P}}(-2)^{\oplus 15}\to\Omega_{\mathbb{P}}\to 0$$

is exact, we can use the mapping cone construction to find the following Ω-resolution,

$$0\to\mathcal{O}_{\mathbb{P}}(-9)^{\oplus 6}\to\Omega_{\mathbb{P}}(-7)\oplus\mathcal{O}_{\mathbb{P}}(-7)\oplus\mathcal{O}_{\mathbb{P}}(-6)\to\mathcal{I}_X\to 0$$

of $\mathcal{I}_X$, leading to the minimal resolution

$$0\to R(-13)\to R(-12)^{\oplus 6}\to R(-11)^{\oplus 15}\to ...\to I_X\to 0.$$

It follows that all $n_{3,i}=11$ in the minimal resolution of I_X and hence we see that ${}_0\mathrm{Ext}^2_R(I,I)=0$.

Then we proceed exactly as in Example 8.3. Indeed we link X to two different 3-folds X' and X_0' using CI's Y and Y_0 containing X, both of type $(7,9)$, as follows. Let F_6, resp. F_7, be the minimal generator of I_X of degree 6, resp. 7, and let G be a general element of $H^0(\mathcal{I}_X(9))$. Then we take Y, resp. Y_0, to be given by $I_Y=(F_7,G)$, resp. $I_{Y_0}=(H\cdot F_6,G)$ where H is a linear form. We may *check that all assumptions of Proposition 8.1 are satisfied. Hence we get that X' and X_0' belong to a common irreducible component of* $\mathrm{Hilb}^{p(v)}(\mathbb{P}^5)$*, that X_0' is obstructed with minimal resolution*

$$0 \to R(-9) \to R(-8)^{\oplus 6} \oplus R(-9) \oplus R(-10) \to R(-7)^{\oplus 7} \oplus R(-9) \to I_{X_0'} \to 0,$$

cf. (39)*, while X' is unobstructed with minimal resolution*

$$0 \to R(-9) \to R(-8)^{\oplus 6} \oplus R(-10) \to R(-7)^{\oplus 7} \to I_{X'} \to 0.$$

Again we have "ghost terms" in the minimal resolution of $I_{X_0'}$. From the resolution we find X_0' to be of degree 30 and with Hilbert polynomial

$$p(v) = 5v^3 - \frac{67}{2}v^2 + \frac{247}{2}v - 153.$$

The 3-fold X_0' is reducible. Moreover since the assumptions of Proposition 9.9 are satisfied for X', we also get the unobstructedness of X' from that Proposition and the dimension, $\dim_{(X')} \mathrm{Hilb}^{p(v)}(\mathbb{P}^5) = 1 - \delta^0_{X'}(0) = 327$*. Note that the assumptions of Proposition 9.9 are not satisfied for X_0', due to the existence of a minimal generator of degree 9 of $I_{X_0'}$ and the fact $h^3(\mathcal{I}_{X_0'}(3)) = 1$.*

Finally since Remark 7.2 generalizes to 3-folds by [29], Prop. 3.12, one may by linkage obtain infinitely many obstructed 3-folds in the liaison class of X_0'.

We will finish this section by finding the Hilbert polynomials of $\mathcal{O}_X$ and $\mathcal{N}_X$ for any equidimensional lCM 3-fold in $\mathbb{P}^5$ of degree d and sectional genus π. If S is a general hyperplane section, we have an exact sequence

$$0 \to \mathcal{O}_X(v-1) \to \mathcal{O}_X(v) \to \mathcal{O}_S(v) \to 0,$$

and we easily deduce

$$p(v) := \chi(\mathcal{O}_X(v)) = \frac{1}{6}dv^3 + \frac{1}{2}(d+1-\pi)v^2 + (\chi(\mathcal{O}_S) + \frac{d}{3} + \frac{1-\pi}{2})v + \chi(\mathcal{O}_X) \tag{47}$$

from (1). Moreover

Proposition 9.12. *Let X be an equidimensional lCM 3-fold in $\mathbb{P}^5$ of degree d and sectional genus π and let S be a general hyperplane section. Then*

$$\chi(\mathcal{N}_X(v)) = \frac{1}{3}dv^3 + 3dv^2 + (2\chi(\mathcal{O}_S) + 5(\pi-1) + \frac{38}{3}d - d^2)v + (6\chi(\mathcal{O}_S) + 15(\pi-1) + 20d - 3d^2).$$

Proof. Since we have no reference for this formula in this generality we sketch a proof. Indeed we claim that

$$\chi(\mathcal{N}_X(v)) = \chi(\mathcal{O}_X(v)) - \chi(\mathcal{O}_X(-v-6)) - d^2(v+3). \tag{48}$$

Note that, using (48), we get Proposition 9.12 by combining with (47). To show (48), we follow the proof of Proposition 4.1. In addition to the formulas in (15) (where we only replace $\sum_{j=1}^4$ by $\sum_{j=1}^5$) we get

$$\sum_{j=1}^{5}(-1)^{j-1}\sum_i n_{j,i}^3 = 6(1-\pi-2d).$$

Then we proceed as in (16). We get $\delta^0(v) = -\chi(\mathcal{I}_X(-v-6)) - \chi(\mathcal{O}_X(v)) + (3+v)d^2$ for $v >> 0$ and then the claim. $\square$

Example 9.13. *Let X be the smooth Calabi-Yau 3-fold of [12], sect. 6, with invariants $d = 17$, $\pi = 32$, $\chi(\mathcal{O}_X) = 0$ and $\chi(\mathcal{O}_S) = 24$, and deficiency modules $M_1 = 0$, $M_2 = 0$ and M_3 given by*

$$h^3(\mathcal{I}_X(1)) = 4, \quad h^3(\mathcal{I}_X(2)) = 2, \quad h^3(\mathcal{I}_X(v)) = 0 \text{ for } v \notin \{1,2\}.$$

Following [12] we find that $I = I_X$ has the following minimal resolution

$$0 \to R(-8)^{\oplus 2} \to R(-7)^{\oplus 8} \to R(-6)^{\oplus 5} \oplus R(-5)^{\oplus 2} \to I \to 0. \tag{49}$$

All assumptions of Proposition 9.9 are satisfied and we get that $\mathrm{Hilb}^{p(v)}(\mathbb{P}^5)$ is smooth at (X) of dimension

$$\dim_{(X)} \mathrm{Hilb}^{p(v)}(\mathbb{P}^5) = 1 - \delta_X^0(0) = 82.$$

Let us compute $\mathrm{obsumext}(X)$. *Note that X has seminatural cohomology and hence we have $\mathrm{H}_{\gamma,\rho} \cong \mathrm{Hilb}^{p(v)}(\mathbb{P}^5)$ at (X) by Remark 9.3. Since $h^3(\mathcal{O}_X) = 1$ and $h^3(\mathcal{O}_X(-1)) = 24$, it follows that* $\mathrm{obsumext}(X) = 1 + \delta_X^4(-6) - 82 = -28$ *by Theorem 9.1. Now we link twice to get X', first using a CI of type $(5,6)$, then a CI of type $(5,5)$, both times using a common hypersurface of degree 5. This is possible, cf. [12]. Thanks to* (39) *we find a free resolution of $I' = I_{X'}$ of the form*

$$0 \to R(-7)^{\oplus 2} \to R(-6)^{\oplus 8} \to R(-5)^{\oplus 6} \oplus R(-4) \to I' \to 0. \tag{50}$$

By (28) *$h^1(\mathcal{O}_{X'}(-2)) = 19$ and $h^1(\mathcal{O}_{X'}(-1)) = 0$ and we get $1 + \delta_{X'}^4(-6) = 20$. It follows from Theorem 9.1 and Proposition 6.1 that $\mathrm{H}_{\gamma',\rho'} \cong \mathrm{Hilb}^{p'(v)}(\mathbb{P}^5)$ is smooth at (X') of dimension*

$$1 + \delta_{X'}^4(-6) - \mathrm{sumext}(X) = 48.$$

We can also use Proposition 9.9 and check that $1 - \delta_{X'}^0(0) = 48$.

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