COMPARISON THEOREMS FOR DEFORMATION FUNCTORS VIA INVARIANT THEORY

JAN ARTHUR CHRISTOPHERSEN AND JAN O. KLEPPE

Abstract. We compare deformations of algebras to deformations of schemes in the setting of invariant theory. Our results generalize comparison theorems of Schlessinger and the second author for projective schemes. We consider deformations (abstract and embedded) of a scheme X which is a good quotient of a quasi-affine scheme X' by a linearly reductive group G and compare them to invariant deformations of an affine G-scheme containing X' as an open invariant subset. The main theorems give conditions for when the comparison morphisms are smooth or isomorphisms.

1. Introduction

Given a projective scheme X defined by equations $f_1, \dots, f_m \in k[x_0, \dots, x_n]$, perturbing the equations in a flat manner so that they remain homogeneous induce deformations of X. In practice this is often the only way to construct examples of deformations. In more stringent terms we have a map between the degree 0 embedded deformations of the affine cone $C(X)$ and deformations of X in $\mathbb{P}^n$. If we take into account trivial deformations we get a map to the deformations of X as scheme. The question is, when do we get all deformations this way?

To see precisely what is going on we should compare deformation functors on Artin rings. If $R = k[x_0, \dots, x_n]$ and $S = R/(f_1, \dots, f_m)$ then the above describes maps $\mathrm{Def}^0_{S/R} \to \mathrm{Hilb}_{X/\mathbb{P}^n}$ where $\mathrm{Def}^0_{S/R}$ is the functor of degree 0 deformations of S as R-algebra and $\mathrm{Def}^0_S \to \mathrm{Def}_X$ where Def^0_S is the functor of degree 0 deformations of S as k-algebra. Generalizing comparisons theorems of Schlessinger ([Sch71], [Sch73]) the second author gave in [Kle79] exact conditions for when these maps are isomorphisms. The object of this paper is to further generalize these to other situations where one can compare deformations of algebras to deformations of schemes.

The comparison map for projective schemes factors through deformations of the open subset of $C(X)$ where the vertex $\{0\}$ is removed. Thereafter one compares deformations to $X = (C(X) \setminus \{0\})/k^*$ via the quotient map. A natural question is if this can be generalized to closed subschemes of toric varieties corresponding to ideals in the Cox ring. It turns out one can go even further, i.e. the quotient need not be by a quasi-torus.

In this paper we consider schemes X that are good quotients of a quasi-affine scheme X' by a linearly reductive group G. We assume that $X' \subseteq \operatorname{Spec} S$, where S is a finitely generated k-algebra and that G acts on S inducing the action on X'. We can then compare Def^G_S to Def_X where Def^G_S is the functor of invariant deformations of S. The precise definitions of these settings are formulated with what we call G-quadruples - see Definition 3.2. Given such a situation we define in Definition 3.10 a G-subquadruple induced by a G-invariant ideal in S. This give us a setting to compare local Hilbert functors with the deformations functor of an invariant surjection of k-algebras.

Linearly reductive groups have many properties coming from the Reynolds operator which make it possible to prove things, e.g. taking invariants is exact. Another reason to work with them is that the functor of invariant deformations is well defined and has the usual nice properties of a good deformation theory. This was proven by Rim.

Our main result on the local Hilbert functor is Theorem 4.3. The conditions are depth conditions along the complement of X' in $\operatorname{Spec} S$ and along the locus where the quotient map fails to be geometric and smooth. We state also corollaries for subschemes of toric varieties and weighted projective space.

For the abstract deformation functor Def_X the results are not so exact due to the presence of infinitesimal automorphisms. It is not clear what the correct assumptions should be but we found it useful to use results of Altmann regarding rigidity of $\mathbb{Q}$-Gorenstein toric singularities as a guide. If $\pi : X' \to X$ is the quotient map set $\mathcal{S} = \pi_* \mathcal{O}_{X'}$. As in the Hilbert functor case we get conditions involving the depth of $\mathcal{S}$ along the locus in X where the quotient map fails to be geometric and smooth, but also where the isotropy groups are not finite.

A new ingredient is what we call a set of Euler derivations coming from the Lie algebra $\mathfrak{g}$ of G. These are a generator set for the sheaf of derivations of $\mathcal{S}$ over $\mathcal{O}_X$ which we prove is free. They define an equivariant map $E : \Omega^1_{\mathcal{S}/\mathcal{O}_X} \to \mathcal{S} \otimes \mathfrak{g}^*$. The cokernel $\mathcal{Q}$ plays an important role for the obstructions to comparing Def^G_S and Def_X. The support of $\mathcal{Q}$ is contained in the set of points $x \in X$ where $\pi^{-1}(x)$ fails to have finite isotropy. In particular for toric varieties it is contained in the non-simplicial locus.

Our most general result for Def_X is Theorem 5.19. The results are more easily presented when G is a quasitorus, i.e. the product of a torus and a finite abelian group (Theorem 5.27) and even better when $\mathcal{Q} = 0$ (Theorem 5.20). If S is a regular ring then we get a rigidity statement as corollary.

As a by-product of our generalization of the Euler derivation we are able to give criteria for when there exists a generalized Euler sequence for the scheme X. See Definition 5.9 and Theorem 5.12.

We conclude with examples of how our results can be used to study deformations of toric varieties and subschemes of these. In particular we consider Calabi-Yau hypersurfaces in simplicial toric Fano varieties, first order deformations of toric singularities and reprove rigidity results of Altmann ([Alt95, 6.5]) and of Totaro ([Tot12, Theorem 5.1]).

Throughout this paper k is an algebraically closed field, in Section 5 we assume characteristic 0 and in Section 6 that $k = \mathbb{C}$.

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2. Preliminaries

2.1. **Cotangent cohomology.** To fix notation we give a short description of the cotangent modules and sheaves. Given a ring R and an R-algebra S there is a complex of free S modules; the *cotangent complex* $\mathbb{L}^{S/R}_{\bullet}$. See e.g. [And74, p. 34] for a definition. For an S module M we get the *cotangent cohomology* modules $T^i(S/R; M) = H^i(\operatorname{Hom}_S(\mathbb{L}^{S/R}_{\bullet}, M))$. If R is the ground field we abbreviate $T^i(S/R; M) = T^i_S(M)$ and $T^i_S(S) = T^i_S = T^i_X$ if $X = \operatorname{Spec} S$.

If X is a scheme we may globalise these as follows. If $\mathcal{R}$ is a sheaf of rings on X and $\mathcal{S}$ an $\mathcal{R}$ algebra we set $\mathcal{L}_\bullet^{\mathcal{S}/\mathcal{R}}$ to be the complex of sheaves associated with the presheaves $U \mapsto \mathbb{L}_\bullet^{\mathcal{S}(U)/\mathcal{R}(U)}$. Let $\mathcal{F}$ be an $\mathcal{S}$ module. We get the cotangent cohomology sheaves $\mathcal{T}^i(\mathcal{S}/\mathcal{R};\mathcal{F})$ as the cohomology sheaves of $\mathcal{H}om_{\mathcal{S}}(\mathcal{L}_\bullet^{\mathcal{S}/\mathcal{R}},\mathcal{F})$ and the cotangent cohomology groups $T^i(\mathcal{S}/\mathcal{R};\mathcal{F})$ as the cohomology of $\operatorname{Hom}_{\mathcal{S}}(\mathcal{L}_\bullet^{\mathcal{S}/\mathcal{R}},\mathcal{F})$.

Because of the functoriality of these constructions we have $\mathcal{T}^i(\mathcal{S}/\mathcal{R};\mathcal{F})$ as the sheaf associated to the presheaf $U \mapsto T^i(\mathcal{S}(U)/\mathcal{R}(U);\mathcal{F}(U))$ and $T^\bullet(\mathcal{S}/\mathcal{R};\mathcal{F})$ as the hyper-cohomology of $\mathcal{H}om_{\mathcal{S}}(\mathcal{L}_\bullet^{\mathcal{S}/\mathcal{R}},\mathcal{F})$. In particular there is a "local-global" spectral sequence

$$H^p(X,\mathcal{T}^q(\mathcal{S}/\mathcal{R};\mathcal{F})) \Rightarrow T^n(\mathcal{S}/\mathcal{R};\mathcal{F})\,. \tag{2.1}$$

If $\mathcal{S}$ is the structure sheaf $\mathcal{O}_X$ and $\mathcal{R}$ corresponds to the ground field, then we abbreviate as above to $T^i_X(\mathcal{F})$.

The properties of the cotangent cohomology we use, e.g. the Zariski-Jacobi sequence and flat base change results, may be found in [And74]. We include here one result that does not seem to be well known. Let $Z \subseteq X$ be a closed subscheme, then following Laudal [Lau79, 3.2.10] one may define cotangent cohomology with support in Z denoted $T^i_Z(\mathcal{O}_X/\mathcal{R};\mathcal{F})$. If $Z \subseteq X = \operatorname{Spec} S$ is given by $V(I)$, we write $T^i_I(S/R;S) = T^i_Z(\mathcal{O}_X/\mathcal{R};\mathcal{O}_X)$.

Theorem 2.1. [Lau79, Theorem 3.2.11] *There is a long exact sequence*

$$\cdots \to T^i_Z(\mathcal{O}_X/\mathcal{R};\mathcal{F}) \to T^i(\mathcal{O}_X/\mathcal{R};\mathcal{F}) \to T^i(\mathcal{O}_{X\setminus Z}/\mathcal{R};\mathcal{F}) \to T^{i+1}_Z(\mathcal{O}_X/\mathcal{R};\mathcal{F}) \to \dots$$

and a spectral sequence yielding

$$T^p(\mathcal{O}_X/\mathcal{R};\mathcal{H}^q_Z(\mathcal{F})) \Rightarrow T^n_Z(\mathcal{O}_X/\mathcal{R};\mathcal{F})\,.$$

Here $\mathcal{H}^q_Z(\mathcal{F})$ are the local cohomolgy sheaves, see Section 3.4.

2.2. **Deformation functors.** The deformation theory we use in this paper is described in various degrees of generality and readability in [LS67], [Sch68], [Ill71], [Lau79], [Ser06] and [Har10]. For a slightly different, but applicable, newer approach see e.g. [FM98]. For the functor of invariant deformations see [Rim80].

Let $\mathbf{C}$ be the category of Artin local k-algebras with residue field k. We list here the deformation functors on $\mathbf{C}$ of interest to us. We denote the functor of deformations of a scheme X by Def_X. The functor of embedded deformations of a subscheme $X \subset Y$ is denoted $\operatorname{Hilb}_{X/Y}$ and called the local Hilbert functor. The deformations of an R-algebra S is denoted $\operatorname{Def}_{S/R}$ and if $R = k$ we simply write Def_S.

If G is an algebraic group acting on a scheme X then it also acts on the set of deformations over A. If $\sigma \in G$ and $f : \mathcal{X} \to \operatorname{Spec} A$ is a deformation then σ acts by

$$\begin{array}{ccc} X & \xhookrightarrow{\;i\;} & \mathcal{X} \\ \downarrow & & \downarrow f \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} A \end{array} \quad \mapsto \quad \begin{array}{ccc} X & \xhookrightarrow{\;i\circ\sigma^{-1}\;} & \mathcal{X} \\ \downarrow & & \downarrow f \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} A \end{array}\,.$$

If G is linearly reductive then there is a well behaved sub-deformation functor of invariant deformations Def^G_X. Similarly if $R \to S$ is equivariant for a linearly reductive group G then there is a subfunctor $\operatorname{Def}^G_{S/R}$ of invariant deformations. If G is a quasi-torus so that the action corresponds to a grading by an abelian group C then we often write $\operatorname{Def}^0_{S/R}$ for the degree $0 \in C$ deformations instead of $\operatorname{Def}^G_{S/R}$.

The tangent and obstruction spaces for Def_X are $T^i_X = T^i(\mathcal{O}_X/k;\mathcal{O}_X)$ for $i=1$ and 2. If $f : X \to Y$ is a closed embedding then the tangent and obstruction spaces for $\mathrm{Hilb}_{X/Y}$ are $T^i(\mathcal{O}_X/f^{-1}\mathcal{O}_Y;\mathcal{O}_X)$ for $i=1$ and 2. The tangent and obstruction spaces for $\mathrm{Def}_{S/R}$ are $T^i(S/R;S)$ for $i=1$ and 2 and finally the tangent and obstruction spaces for $\mathrm{Def}^G_{S/R}$ are $T^i(S/R;S)^G$ for $i=1$ and 2. If D is one of these deformation functors let T^i_D, $i=1,2$, denote the corresponding tangent and obstruction space.

Recall that a morphism $F \to G$ of functors is *smooth* if for any surjection $B \to A$ in $\mathbf{C}$, the morphism

$$F(B) \to F(A) \times_{G(A)} G(B)$$

is surjective. A functorial map of deformation functors $D \to D'$ induces maps $T^i_D \to T^i_{D'}$. We use throughout the standard criteria for smoothness, namely that $T^1_D \to T^1_{D'}$ is surjective and $T^2_D \to T^2_{D'}$ is injective. See [Ser06, Proposition 2.3.6] for a more general statement and proof.

If $T^1_D \to T^1_{D'}$ is an isomorphism and $T^2_D \to T^2_{D'}$ is injective it is not necessarily true that $D \to D'$ is an isomorphism. This is true if D and D' satisfy Schlessinger's condition H_4, see [Sch68, 2.11 and 2.15]. If $X \subset Y$ is a closed subscheme then $\mathrm{Hilb}_{X/Y}$ satisfies H_4. In general Def_X does not, but in our case we will not only have an isomorphism at the tangent level but also surjectivity of infinitesimal automorphisms allowing us to state that the functors we compare are isomorphic (Lemma 5.1).

3. G-quadruples

3.1. **Definitions.** We consider now a standard situation in invariant theory. Definitions of different types of quotients in algebraic geometry vary slightly in the literature. For us the best one is the original notion of good (and geometric) quotient due to Seshadri.

Definition 3.1 ([Ses72] Definition 1.5)**.** Let G be an affine algebraic group acting on an algebraic scheme Y and $\pi : Y \to X$ a morphism. Then we say that π is a *good quotient* if the following properties hold:

(i) π is a surjective, affine G-invariant morphism
(ii) $(\pi_*\mathcal{O}_Y)^G = \mathcal{O}_Y$
(iii) if $W \subseteq Y$ is closed and G-invariant then $\pi(W)$ is closed in X
(iv) if W_1 and W_2 are closed, disjoint and G-invariant in Y, then $\pi(W_1)$ and $\pi(W_2)$ are disjoint in X.

In this case one writes $X = Y//G$. If G is reductive acting algebraically on $Y = \mathrm{Spec}(A)$ then $Y \to \mathrm{Spec}(A^G)$ is a good quotient. (See e.g. [Ses72, Theorem 1.1].) If $\pi : Y \to Y//G$ is a good quotient and all G-orbits are closed then π is called a *geometric* quotient ([Ses72, Definition 1.6]). The condition closed orbits is equivalent to that π induces a bijection between G-orbits and $Y//G$.

Remark. It will be essential for us that the good quotient map is affine. In some definitions of good this is not included. That would allow e.g. the non-separated quotient $(\mathbb{C}^2 \setminus \{(0,0\})//C^*$ where C^* acts by $\mathrm{diag}(\lambda,\lambda^{-1})$.

It is convenient to give the situation we study a name.

Definition 3.2. Let G be a linearly reductive algebraic group. Let X be a noetherian k-scheme, $Z \subsetneq X$ a (possibly empty) closed subset, S a finitely generated k-algebra on which G acts and $J \subseteq S$ an ideal such that the G action restricts to an action on $\mathrm{Spec}\, S \setminus V(J)$. We call (X, Z, S, J) a *G-quadruple* if the following properties are satisfied: If $X' = \operatorname{Spec} S \setminus V(J)$ and $U = X \setminus Z$ then there is a commutative diagram

$$\begin{array}{ccccc} U' & \hookrightarrow & X' & \hookrightarrow & \operatorname{Spec} S \\ \downarrow{\scriptstyle \pi_{|U'}} & & \downarrow{\scriptstyle \pi} & & \\ U & \hookrightarrow & X & & \end{array}$$

where $U' = \pi^{-1}(U)$ and

(i) π is a good quotient of X' by the action G
(ii) $\pi_{|U'}$ is a geometric quotient and smooth of relative dimension $\dim G$
(iii) $\operatorname{depth}_J S \geq 1$.

The sheaf of algebras $\mathcal{S} = \pi_* \mathcal{O}_{X'}$ is called *the associated sheaf of algebras* of the G-quadruple.

We will throughout the rest of this paper use the above notation to refer to the various objects in the definition.

Remark. A G-quadruple may be constructed using Geometric Invariant Theory. (See [MFK94], although the presentation in [Muk03] applies more directly to our situation.) Starting with the action of G on a reduced and irreducible affine scheme $Y = \operatorname{Spec} S$ and a character χ of G, let Y^{ss} and Y^s be the semistable and stable points of Y with respect to χ. This gives a G-quadruple by letting $X' = Y^{ss}$, $J = I(Y \setminus Y^{ss})$, $X = Y^{ss}//G$, U' equals the open subset of Y^s where the quotient map is smooth, $U = U'/G$ and $Z = X \setminus U$.

3.2. **Examples.** Here are some examples.

Example 3.3. *The* Spec *construction.* If $X = \operatorname{Spec} S$ and G is trivial then $(X, \emptyset, S, (1))$ is a G-quadruple.

Example 3.4. *The usual* Proj *construction.* If $X = \operatorname{Proj} S$ for S finitely generated $\mathbb{Z}$-graded k-algebra generated in degree 1 with irrelevant ideal $\mathfrak{m}$ and $G = k^*$ then $(X, \emptyset, S, \mathfrak{m})$ is a G-quadruple.

Example 3.5. *The* Proj *construction in general.* Let S be a finitely generated $\mathbb{Z}_+$ graded k-algebra with $S_0 = k$ and set $X = \operatorname{Proj} S$. Assume $S = R/I$ where $R = k[x_0, \ldots, x_n]$ is graded with $\deg x_i = q_i \in \mathbb{N}$. Let $\mathfrak{m} = (x_0, \ldots, x_n)$ and suppose $\operatorname{depth}_{\mathfrak{m}} S \geq 1$. This defines an embedding of X into the weighted projective space $\mathbb{P}(\mathbf{q}) = \mathbb{P}(q_0, \ldots, q_n)$.

Following Miles Reid we say that $\mathbb{P}(\mathbf{q})$ is *well formed* if no n of the $q_0, \ldots, q_n$ have a common factor. Similarly we will say that the corresponding grading on S is well formed. For every $\mathbf{q} \in \mathbb{N}^{n+1}$ there exists a well formed grading $\mathbf{q}'$ such that $\mathbb{P}(\mathbf{q}') \simeq \mathbb{P}(\mathbf{q})$, see e.g. [Del75, Proposition 1.3] or [Dol82, 1.3.1]. Let J'_k be the ideal of R generated by $\{x_i : k \nmid q_i\}$ and set

$$J' = \bigcap_{k \geq 2} J'_k = \bigcap_{\substack{p \text{ prime} \\ p | \operatorname{lcm}(q_0, \ldots, q_n)}} J'_p \,.$$

The singular locus of the well formed $\mathbb{P}(\mathbf{q})$ is $Z = V(J') \subseteq \mathbb{P}(\mathbf{q})$ and satisfies $\operatorname{codim}_Z \mathbb{P}(\mathbf{q}) \geq 2$.

Then $(X, X \cap Z, S, \mathfrak{m})$ is a G-quadruple.

Example 3.6. *The Cox construction for complex toric varieties.* This is our main example and it includes the previous ones, see e.g. [Cox95] and [CLS11, Chapter 5]. To fix notation for the rest of this paper we recall the construction. Using the standard notation for toric geometry let $X = X_\Sigma$ be an n-dimensional toric variety given by a fan Σ in $N_{\mathbb{R}}$. We assume here for simplicity that X_Σ has no torus factors, but this is not necessary for applying our results (see [CLS11, 5.1.11]). Let $\Sigma(1) = \{\rho_1, \dots, \rho_N\}$ be the set of rays of Σ and let v_i be the primitive generator of $\rho_i \cap N$. The divisor class group of X is given by the exact sequence

$$0 \to \mathbb{Z}^n \xrightarrow{b} \mathbb{Z}^N \to \mathrm{Cl}(X) \to 0$$

where $b(u) = (\langle u, v_1\rangle, \dots, \langle u, v_N\rangle)$. The ring $S = \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$ is naturally graded by the abelian group $\mathrm{Cl}(X)$ and with this grading it is called the *Cox ring* or *total homogeneous coordinate ring* of X.

For each cone σ in Σ there is a monomial

$$\prod_{\rho_i \notin \sigma} x_i \in S$$

and define $B(\Sigma)$ to be the ideal of S generated by these. It is called the *irrelevant ideal* of the Cox ring. Let $Z(\Sigma) = V(B(\Sigma)) \subseteq \operatorname{Spec} S$. Let G be the quasi-torus $\operatorname{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*)$. The theorem of Cox states that X is an almost geometric quotient for the action of G on $\operatorname{Spec} S \setminus Z(\Sigma)$.

Let $\operatorname{Sing}(X_\Sigma)$ be the singular locus, i.e.

$$\operatorname{Sing}(X_\Sigma) = \bigcup_{\substack{\sigma \in \Sigma \\ \sigma \text{ not smooth}}} V(\sigma)$$

where $V(\sigma)$ is the closure of the torus orbit corresponding to σ. The smooth locus is given by the subfan consisting of smooth cones in Σ. Then it follows from the construction (see e.g. [CLS11, Exercise 5.1.10]) that $(X_\Sigma, \operatorname{Sing}(X_\Sigma), S, B(\Sigma))$ is a G-quadruple. Note that a G invariant S-module M is the same thing as a $\mathrm{Cl}(X)$ graded S-module and that $M^G = M_0$, the degree $0 \in \mathrm{Cl}(X)$ part of M. Moreover if $\alpha = [D] \in \mathrm{Cl}(X)$ then $S_\alpha = \Gamma(X, \mathcal{O}_X(D))$. In particular the associated sheaf of algebras of the G-quadruple is in this case

$$\mathcal{S} = \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{O}_X(D)\,.$$

Example 3.7. *Quotient singularities.* Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite group without pseudo-reflections. Then $(\mathbb{C}^n/G, \operatorname{Sing}(\mathbb{C}^n/G), \mathbb{C}[x_1, \dots, x_n], (1))$ is a G-quadruple.

Example 3.8. *Grassmannians.* Consider the Grassmannian $\mathbb{G}(d, n)$ and let S be the polynomial ring on variables x_{ij} for $i = 1, \dots, d$ and $j = 1, \dots, n$. Thus $G = \mathrm{GL}_d$ acts on S by viewing the variables as entries in a $d \times n$ matrix. Let J be the ideal generated by the $\binom{n}{d}$ maximal minors in such a matrix. Then $(\mathbb{G}(d, n), \emptyset, S, J)$ is a G-quadruple.

Example 3.9. *Moduli spaces.* Compactifications of moduli spaces constructed using GIT on affine schemes give G-quadruples as explained above. Among these are the moduli of smooth hypersurfaces in $\mathbb{P}^n$, vector bundles and quiver representations.

3.3. **G-subquadruples.** We may also define G-subquadruples.

Definition 3.10. Given a G-quadruple (Y, W, R, J) and a G-invariant ideal $I \subseteq R$ we may construct a new G-quadruple as follows. Let $S = R/I$, set $\bar{J} = (I + J)/I$ and assume $\operatorname{depth}_{\bar{J}} S \geq 1$. Let $\pi : Y' \to Y$ be the good quotient as in Definition 3.2. The quasi-affine scheme $X' = \operatorname{Spec} S \setminus V(\bar{J})$ is a closed G-invariant subset of Y'. Thus $\pi : X' \to X = \pi(X')$ is a good quotient where X has structure sheaf $(\pi_* \mathcal{O}_{X'})^G$. It follows that $(X, X \cap W, S, \bar{J})$ is also a G-quadruple. We call it the *G-subquadruple induced by I.*

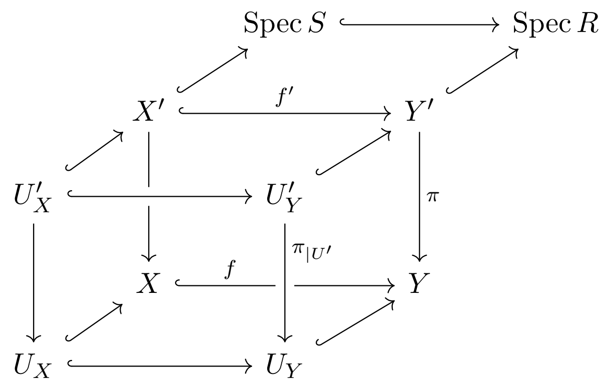

Figure 1. The diagram for G-subquadruples.

A G-subquadruple determines a diagram as in Figure 1 where $Z = X \cap W$, $U_X = X \cap U_Y$ and by definition $U'_X = \pi^{-1}(U_X)$. Let $\mathcal{S} = \pi_* \mathcal{O}_{X'}$ be the associated sheaf of algebras of $(X, Z, S, \bar{J})$. Let $f : X \to Y$ be the closed embedding. Let I_W be the radical ideal of $\operatorname{Spec} R \setminus U'_Y$ and set $I_Z = (I_W + I)/I$. We will use this notation and the notation in the diagram throughout.

Now $\pi_{|U'_Y}$ is a geometric quotient, so in particular inverse images of points are G-orbits. Thus for any G-invariant subset $V \subseteq U'_Y$ we have $\pi^{-1}(\pi(V)) = V$. Thus $X' \cap U'_Y = \pi^{-1}(\pi(X' \cap U'_Y)) = \pi^{-1}(\pi(X') \cap \pi(U'_Y))$ since $x, y \in U'_Y$ and $\pi(x) = \pi(y)$ implies x and y are in the same G-orbit. This shows that $U'_X = X' \cap U'_Y = \operatorname{Spec} S \setminus V(I_Z)$, so the above diagram is commutative. Note that the vertical square involving U'_X, U'_Y, U_X, U_Y is Cartesian, but the one involving X', Y', X, Y need not be.

3.4. **Depth and local cohomology.** We recall the connection between depth and local cohomology for rings and schemes which we use through out. See [Gro67, Section 3] and [Gro05, Exp. 1-3] for proofs and details. If S is a Noetherian ring, I an ideal of S, and M an S-module (not necessarily finitely generated) we define $\operatorname{depth}_I(M)$ by

$$\operatorname{depth}_I(M) = \max\{j \in \mathbb{Z} \cup \{\infty\} : H_I^{j-1}(M) = 0\} .$$

where $H_I^i(-)$ is the right derived functor of $\Gamma_I(-)$ and $\Gamma_I(M) = \ker(M \to \Gamma(\operatorname{Spec} S \setminus V(I), M))$. By Proposition 2.4 of [Gro05, Exp. 3], if reg denotes the length of a maximal M-regular sequence in I (letting $\text{reg} = \infty$ if $IM = M$), then $H_I^j(M) = 0$ for every $j < \text{reg}$, and if M is finitely generated then $\text{reg} = \operatorname{depth}_I(M)$.

For a quasi-coherent sheaf $\mathcal{F}$ on a noetherian scheme X we have the local cohomology groups $H_Z^i(X, \mathcal{F})$ and the local cohomology sheaves $\mathcal{H}_Z^i(\mathcal{F})$ with the spectral sequence

$$H^p(X, \mathcal{H}_Z^q(\mathcal{F})) \Rightarrow H_Z^n(X, \mathcal{F}) .$$

We define depth as in the ring case by

$$\operatorname{depth}_Z(\mathcal{F}) = \inf\{i \in \mathbb{Z} \cup \{\infty\} : \mathcal{H}_Z^i(\mathcal{F}) \neq 0\}$$

(see [Gro67, Proposition 2.2] and compare with [Gro67, Theorem 3.8]). Note that if $\mathcal{F}$ is coherent, we have $\operatorname{depth}_Z \mathcal{F} = \inf_{x \in Z} \operatorname{depth} \mathcal{F}_x$ where $\operatorname{depth} \mathcal{F}_x$ is the depth of $\mathcal{F}_x$ as $\mathcal{O}_{X,x}$ module (see [Gro67, Corollary 3.6]).

If $U = X \setminus Z$ and $j : U \to X$ is the inclusion then there are exact sequences

$$\cdots \to H^i_Z(X, \mathcal{F}) \to H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \to H^{i+1}_Z(X, \mathcal{F}) \to \cdots$$

and

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_*\mathcal{F}_{|U} \to \mathcal{H}^1_Z(\mathcal{F}) \to 0$$
$$\mathcal{H}^{i+1}_Z(\mathcal{F}) \simeq R^i j_* \mathcal{F}_{|U} \quad \text{for } i > 0\,.$$

The condition $\operatorname{depth}_Z(\mathcal{F}) \geq 2$ will appear many times and we see from the above that it is equivalent to that the natural map $\mathcal{F} \to j_*\mathcal{F}_{|U}$ is an isomorphism.

Let $(X, Z, S, \bar{J})$ be a G-quadruple with associated sheaf of algebras $\mathcal{S}$, for instance a G-subquadruple of (Y, W, R, J) induced by $I \subseteq R$, let $Z' = V(I_Z) \cap X'$ and note that $X' = \operatorname{Spec} S \setminus V(\bar{J})$ and $U'_X = X' \setminus Z'$ are open sets of $\operatorname{Spec} S$. Since $\operatorname{depth}_Z \mathcal{S}$ is so often used in this paper and $\mathcal{S}$ is in general not coherent, we want to relate it to $\operatorname{depth}_{Z'} \mathcal{O}_{X'}$. Note that we have

$$H^i(X, \mathcal{S}) \simeq H^i(X', \mathcal{O}_{X'}) \quad \text{and} \quad H^i(U_X, \mathcal{S}_{|U_X}) \simeq H^i(X' \setminus Z', \mathcal{O}_{X'})$$

for $i \geq 0$ because π is affine. Moreover using [Gro67, Corollary 5.6] and π affine we get that

$$H^i_{Z\cap V}(V, \pi_*\mathcal{O}_{X'}) \simeq H^i_{Z'\cap V'}(V', \mathcal{O}_{X'})$$

for every open affine $V \subseteq X$ and $V' = \pi^{-1}(V)$ open affine of $\operatorname{Spec} S$. It follows that $\mathcal{H}^i_Z(\mathcal{S}) = \pi_*(\mathcal{H}^i_{Z'}(\mathcal{O}_{X'}))$ and

$$\operatorname{depth}_Z \mathcal{S} = \operatorname{depth}_{Z'} \mathcal{O}_{X'}\,.$$

Notice that the latter is defined by a coherent sheaf, thus characterized by the length of maximal regular sequences. We also have

$$\mathcal{H}^i_{Z'}(\mathcal{O}_{X'}) \simeq H^i_{Z'}\widetilde{(X', \mathcal{O}_{X'})}_{|X'} \simeq \widetilde{H^i_{I_Z}(S)}_{|X'} \tag{3.1}$$

because $\widetilde{H^i_{\bar{J}}(S)}_{|X'} = 0$. More generally since $\operatorname{Spec} S$ is affine there is a diagram

$$\begin{array}{ccc} H^i(X', \mathcal{O}_{X'}) & \xrightarrow{\sim} & H^{i+1}_{\bar{J}}(S) \\ \downarrow & & \\ H^i(X' \setminus Z', \mathcal{O}_{X'}) & \xrightarrow{\sim} & H^{i+1}_{I_Z}(S) \end{array}$$

for $i > 0$; for $i = 0$ the horisontal maps are surjective with kernels S and $\operatorname{coker}(H^0_{I_Z}(S) \to S)$ respectively. Since the vertical map fits into a long exact sequence involving the local cohomology group $H^i_{Z'}(X', \mathcal{O}_{X'})$ we get an exact sequence of S-modules

$$\cdots \longrightarrow H^i_{Z'}(X', \mathcal{O}_{X'}) \longrightarrow H^{i+1}_{\bar{J}}(S) \longrightarrow H^{i+1}_{I_Z}(S) \longrightarrow H^{i+1}_{Z'}(X', \mathcal{O}_{X'}) \longrightarrow \cdots\,. \tag{3.2}$$

Remark. We may relate $\operatorname{depth}_{I_Z} S$, $\operatorname{depth}_{\bar{J}} S$ and $\operatorname{depth}_Z \mathcal{S} = \operatorname{depth}_{Z'} \mathcal{O}_{X'}$. We get

$$\operatorname{depth}_Z \mathcal{S} \geq d \quad \text{and} \quad \operatorname{depth}_{\bar{J}} S \geq d \iff \operatorname{depth}_{I_Z} S \geq d\,.$$

Indeed, the implication $\Leftarrow$ follows from $\operatorname{depth}_{I_Z} S \leq \operatorname{depth}_{\bar{J}} S$ (due to $I_Z \subseteq \bar{J}$) and (3.1). The implication $\Rightarrow$ follows from (3.2) and the spectral sequence above.

4. Deformations of the embedded scheme - $\mathrm{Hilb}_{X/Y}$

We begin with a general lemma.

Lemma 4.1. *If in a Cartesian square of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & Y \end{array}$$

the morphism π is flat and affine and f is a closed immersion then

$$T^i(\mathcal{O}_X/f^{-1}\mathcal{O}_Y;\pi_*\mathcal{O}_{X'}) \simeq T^i(\mathcal{O}_{X'}/{f'}^{-1}\mathcal{O}_{Y'};\mathcal{O}_{X'})$$

for all $i \geq 0$.

Proof. We first compare the corresponding sheaves. The diagram is Cartesian so f' is also a closed immersion. Closed immersions are affine and the base change of an affine morphism is affine, so all morphisms in the diagram are affine.

Let $V = \operatorname{Spec} B \subseteq Y$ be an open affine subset, $U = \operatorname{Spec} A = f^{-1}(V)$, $\pi^{-1}(V) = \operatorname{Spec} B'$ and $\pi^{-1}(U) = \operatorname{Spec} A'$. The diagram locally corresponds to a cocartesian diagram of rings

$$\begin{array}{ccc} A' & \longleftarrow & B' \\ \uparrow & & \uparrow \pi^{\#} \\ A & \longleftarrow & B \end{array}$$

with $\pi^{\#}$ flat, so $T^q(A/B;A') \simeq T^q(A'/B';A')$ for all $q \geq 0$ ([And74, Appendice, Proposition 76]). Applying the corresponding cotangent cohomology sheaves we have

$$\mathcal{T}^q(\mathcal{O}_X/f^{-1}\mathcal{O}_Y;\pi_*\mathcal{O}_{X'})(U) = T^q(A/B;A')$$
$$\mathcal{T}^q(\mathcal{O}_{X'}/{f'}^{-1}\mathcal{O}_{Y'};\mathcal{O}_{X'})(\pi^{-1}(U)) = T^q(A'/B';A')$$

thus

$$\mathcal{T}^q(\mathcal{O}_X/f^{-1}\mathcal{O}_Y;\pi_*\mathcal{O}_{X'}) \simeq \pi_*\mathcal{T}^q(\mathcal{O}_{X'}/{f'}^{-1}\mathcal{O}_{Y'};\mathcal{O}_{X'})$$

for all $q \geq 0$.

Again because the maps are affine, $H^p(X,\pi_*\mathcal{F}) \simeq H^p(X',\mathcal{F})$ for any quasi-coherent $\mathcal{F}$. Thus all terms in the two spectral sequences (2.1) for the two cohomology groups are isomorphic and the result follows. □

Lemma 4.2. *Let $(X,Z,S,\bar{J})$ be a G-subquadruple of (Y,W,R,J) induced by $I \subseteq R$ with associated sheaf of algebras $\mathcal{S}$. If $\operatorname{depth}_Z \mathcal{S} \geq 1$ and the natural map $H^0(X,\mathcal{S}) \to H^0(U_X,\mathcal{S}_{|U_X})$ is surjective, equivalently $H^0(X,\mathcal{S}) \simeq H^0(U_X,\mathcal{S}_{|U_X})$, then*

(i) $\operatorname{depth}_Z \mathcal{O}_X \geq 1$
(ii) $H^0_{\bar{J}}(S) = H^0_{I_Z}(S) = 0$
(iii) *there is an isomorphism $H^1_{\bar{J}}(S) \simeq H^1_{I_Z}(S)$.*

Proof. Suppose $\operatorname{depth}_Z \mathcal{S} \geq 1$ and $H^0(X,\mathcal{S}) \to H^0(U_X,\mathcal{S}_{|U_X})$ is surjective. Since taking invariants is exact (i) follows from $\mathcal{H}^i_Z(\mathcal{S})^G = \mathcal{H}^i_Z(\mathcal{O}_X)$. We have $\operatorname{depth}_{\bar{J}} S \geq 1$ and $I_Z \subseteq \bar{J}$

by the assumption that $(X, Z, S, \bar{J})$ is a G-subquadruple, so we get (ii) and (iii) by considering the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & S & \longrightarrow & H^0(X,\mathcal{S}) & \longrightarrow & H^1_{\bar{J}}(S) & \longrightarrow & 0 \\ & & & & \| & & \downarrow & & & & \\ 0 & \longrightarrow & H^0_{I_Z}(S) & \longrightarrow & S & \longrightarrow & H^0(U_X,\mathcal{S}_{|U_X}) & \longrightarrow & H^1_{I_Z}(S) & \longrightarrow & 0 \end{array}$$

where the middle vertical map is an isomorphism by $H^0_Z(X,\mathcal{S}) \simeq H^0(X,\mathcal{H}^0_Z(\mathcal{S})) = 0$ and assumption.

Finally, to show that $\operatorname{depth}_Z \mathcal{S} \geq 1$ and $H^0(X,\mathcal{S}) \to H^0(U_X,\mathcal{S}_{|U_X})$ surjective is equivalent to $H^0(X,\mathcal{S}) \simeq H^0(U_X,\mathcal{S}_{|U_X})$, suppose the vertical map is an isomorphism. Then the diagram above (or the remark in Section 3.4) implies $H^0_{I_Z}(S) = 0$, hence $\operatorname{depth}_Z \mathcal{S} \geq 1$ by (3.1). □

Theorem 4.3. *If $(X, Z, S, \bar{J})$ is a G-subquadruple of (Y, W, R, J) induced by $I \subseteq R$ with associated sheaf of algebras $\mathcal{S}$ and*

(i) $H^0(X,\mathcal{S}) \simeq H^0(U_X,\mathcal{S}_{|U_X})$ *and* $\mathcal{H}^1_Z(\mathcal{O}_X) = 0$ (e.g. $\operatorname{depth}_Z \mathcal{S} \geq 2$),
(ii) $\operatorname{Hom}_R(I, H^1_{\bar{J}}(S))^G = 0$

then $\operatorname{Def}^G_{S/R}$ *and* $\operatorname{Hilb}_{X/Y}$ *are isomorphic deformation functors.*

Proof. To see that $\operatorname{depth}_Z \mathcal{S} \geq 2$ implies that the assumptions in (i) hold use $\mathcal{H}^i_Z(\mathcal{S})^G = \mathcal{H}^i_Z(\mathcal{O}_X)$, the spectral sequence of Section 3.4 which implies $H^i_Z(X,\mathcal{S}) = 0$ for $i \leq 1$, and the diagram in the proof of Lemma 4.2.

We will prove that both deformation functors are isomorphic to $\operatorname{Hilb}_{U_X/U_Y}$. Note first that if $f : X \to Y$ is any closed embedding then $T^0(\mathcal{O}_X/f^{-1}\mathcal{O}_Y;\mathcal{F}) = 0$ for any $\mathcal{O}_X$-module $\mathcal{F}$. Thus the spectral sequence in Theorem 2.1 yields $T^1_{I_Z}(S/R;S) \simeq \operatorname{Hom}_R(I, H^0_{I_Z}(S)) = 0$ by Lemma 4.2. Furthermore this and Lemma 4.2 (iii) show that $T^2_{I_Z}(S/R;S) \simeq \operatorname{Hom}_R(I, H^1_{I_Z}(S)) \simeq \operatorname{Hom}_R(I, H^1_{\bar{J}}(S))$.

We start with applying the long exact sequence in Theorem 2.1 and get

$$0 \to T^1(S/R;S) \to T^1(\mathcal{O}_{U'_X}/\mathcal{R};\mathcal{O}_{U'_X}) \to T^2_{I_Z}(S/R;S) \to T^2(S/R;S) \to T^2(\mathcal{O}_{U'_X}/\mathcal{R};\mathcal{O}_{U'_X})$$

where we set $i_Y : U'_Y \hookrightarrow \operatorname{Spec} R$ and $\mathcal{R} = (f')^{-1} i_Y^{-1}\mathcal{O}_{\operatorname{Spec} R}$. Now i_Y is an open immersion so $\mathcal{R} \simeq (f')^{-1}\mathcal{O}_{U'_Y}$. By assumption $\pi_{|U'_Y}$ is smooth, in particular flat so by Lemma 4.1

$$T^i(\mathcal{O}_{U'_X}/\mathcal{R};\mathcal{O}_{U'_X}) \simeq T^i(\mathcal{O}_{U_X}/f^{-1}\mathcal{O}_{U_Y};\pi_*\mathcal{O}_{U'_X})\,.$$

Therefore after taking invariants and using the conditions we see that

$$T^i(S/R;S)^G \to T^i(\mathcal{O}_{U_X}/f^{-1}\mathcal{O}_{U_Y};\mathcal{O}_{U_X})$$

is an isomorphism for $i = 1$ and injective for $i = 2$. Thus $\operatorname{Def}^G_{S/R} \simeq \operatorname{Hilb}_{U_X/U_Y}$.

We have $\mathcal{H}^i_Z(\mathcal{O}_X) = 0$ for $i \leq 1$ by Lemma 4.2 and assumption, so again the spectral sequence and vanishing of $T^0(\mathcal{O}_X/f^{-1}\mathcal{O}_Y;\mathcal{O}_X)$ implies $T^i_Z(\mathcal{O}_X/f^{-1}\mathcal{O}_Y;\mathcal{O}_X) = 0$ for $i \leq 2$. We apply the long exact sequence in Theorem 2.1 again to get $T^1(\mathcal{O}_X/f^{-1}\mathcal{O}_Y;\mathcal{O}_X) \simeq T^1(\mathcal{O}_{U_X}/f^{-1}\mathcal{O}_{U_Y};\mathcal{O}_{U_X})$ and $T^2(\mathcal{O}_X/f^{-1}\mathcal{O}_Y;\mathcal{O}_X) \to T^2(\mathcal{O}_{U_X}/f^{-1}\mathcal{O}_{U_Y};\mathcal{O}_{U_X})$ injective. Thus $\operatorname{Hilb}_{X/Y} \simeq \operatorname{Hilb}_{U_X/U_Y}$. □

Remark. (i) From the proof we see that the slightly weaker assumptions $\operatorname{depth}_Z \mathcal{S} \geq 1$, $\operatorname{Hom}_R(I, H^1_{I_Z}(S))^G = 0$ and $\operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{I}, \mathcal{H}^1_Z(\mathcal{O}_X)) = 0$ also imply the result in the theorem.

(ii) If the theorem applies and $T^2(S/R; S)^G = 0$ then $\operatorname{Hilb}_{X/Y}$ is unobstructed even though $H^1(X, \mathcal{N}_{X/Y})$ or $H^2(X, \mathcal{T}^2_{X/Y})$ do not vanish. See Example 4.8.

Corollary 4.4. *If $(X, Z, S, \bar{J})$ is a G-subquadruple of (Y, W, R, J) induced by $I \subseteq R$ and $\operatorname{depth}_{I_Z} S \geq 2$, then $\operatorname{Def}^G_{S/R}$ and $\operatorname{Hilb}_{X/Y}$ are isomorphic deformation functors.*

Proof. Both assumptions in Theorem 4.3 are satisfied because $\operatorname{depth}_{I_Z} S \geq 2$ implies $\operatorname{depth}_{\bar{J}} S \geq 2$, whence $H^1_{\bar{J}}(S) = 0$ and $\operatorname{depth}_Z \mathcal{S} \geq 2$ by remark of subsection 3.4. □

In the case of the Cox construction for toric varieties (Example 3.6) we get a corollary which is a generalization of the comparison theorem as stated in [PS85], cf. [Kle79, Theorem 3.6 and Remark 3.7]. Actually [Kle79, Remark 3.7] implies the comparison theorem in [PS85]. Proofs and full statements of the results we use here may be found in [CLS11, Chapter 5 and 6].

Let Y be a toric variety with Cox ring R. Every closed subscheme X of Y corresponds to a homogeneous, with respect to the $\operatorname{Cl}(Y)$ grading, ideal $I \subseteq R$, [CLS11, Proposition 6.A.6]. Moreover there is a sheafification construction taking any graded R-module M to a sheaf $\widetilde{M}$ on Y. In particular if $S = R/I$ then $\widetilde{S} = \bigoplus_{\alpha \in \operatorname{Cl}(Y)} \mathcal{O}_X(\alpha)$. Also, as in the case of projective space, one may compute sheaf cohomology from local cohomology at the irrelevant ideal ([EMS00, Proposition 2.3]). In particular there is an exact sequence

$$0 \to H^0_B(S) \to S \to \bigoplus_{\alpha \in \operatorname{Cl}(Y)} H^0(X, \mathcal{O}_X(\alpha)) \to H^1_B(S) \to 0\,. \tag{4.1}$$

We therefore get the following result for subschemes of toric varieties.

Corollary 4.5. *Let X be a subscheme of a toric variety Y corresponding to a homogeneous ideal I in the Cox ring R of Y. Set $S = R/I$ and let Z be the intersection of the singular locus of Y with X and $U = X \setminus Z$. Assume I is generated by homogeneous polynomials of degrees $\alpha_1, \dots, \alpha_m \in Cl(Y)$. If*

(i) $\operatorname{depth}_Z \mathcal{O}_X \geq 2$ *and* $H^0(X, \mathcal{O}_X(\alpha)) \simeq H^0(U, \mathcal{O}_{X|U}(\alpha))$ *for every* $\alpha \in Cl(Y)$ (e.g. $\operatorname{depth}_Z \mathcal{O}_X(\alpha) \geq 2$ for all $\alpha \in \operatorname{Cl}(Y)$) ,

(ii) $S_{\alpha_i} \simeq H^0(X, \mathcal{O}_X(\alpha_i))$ for all $i = 1, \dots m$

then $\operatorname{Def}^0_{S/R}$ *and* $\operatorname{Hilb}_{X/Y}$ *are isomorphic deformation functors.*

Proof. The first statement is just a rewrite of Theorem 4.3 (i) using $\mathcal{S} = \bigoplus_{\alpha \in \operatorname{Cl}(Y)} \mathcal{O}_X(\alpha)$. What is left is to show that $\operatorname{Hom}_R(I, H^1_B(S))_0 = 0$, where we mean degree 0 in the $\operatorname{Cl}(Y)$ grading. Clearly $\varphi \in \operatorname{Hom}_R(I, H^1_B(S))$ is determined by its values on generators of I, so if $H^1_B(S)_{\alpha_i} = 0$ for all $i = 1, \dots m$ we are done. But this follows from the statement in (ii) and the exact sequence (4.1). □

Corollary 4.6. *Let $X = \operatorname{Proj} S$ be a subscheme of a well formed weighted projective space $\mathbb{P}(\mathbf{q}) = \operatorname{Proj} R$ defined by the homogeneous ideal I. Let $\mathfrak{m}$ be the irrelevant maximal ideal of R and Z the intersection of the singular locus of $\mathbb{P}(\mathbf{q})$ with X. If*

(i) $\operatorname{depth}_Z \mathcal{O}_X \geq 2$ *and* $H^0(X, \mathcal{O}_X(m)) \simeq H^0(U_X, \mathcal{O}_{X|U_X}(m))$ *for every* $m \in \mathbb{Z}$, (e.g. $\operatorname{depth}_Z \mathcal{O}_X(m) \geq 2$ for all $m \in \mathbb{Z}$)

(ii) $\operatorname{Hom}_R(I, H^1_{\mathfrak{m}}(S))_0 = 0$

then $\mathrm{Def}^0_{S/R}$ *and* $\mathrm{Hilb}_{X/Y}$ *are isomorphic deformation functors.*

We give some examples to illustrate the conditions in Theorem 4.3 and Corollary 4.5. Note that in the (multi)-graded case we may think of $\mathrm{Def}^0_{S/R}$ as deformations that preserve the Hilbert function and that they correspond to Hilbert function strata of the Hilbert scheme, see [Kle98, Theorem 1.1] and [HS04].

Example 4.7. *Points.* If $X = \operatorname{Proj} S$ is s points in general enough position in $\mathbb{P}^n$, then the Hilbert function of X is

$$h_X(\nu) = \dim S_\nu = \inf\{s, \binom{\nu+n}{n}\}$$

by e.g [GMR83]. Let ν_0 be the smallest integer with

$$s \le \binom{\nu_0+n}{n}.$$

The exact sequence

$$0 \to S_\nu \to H^0(X, \mathcal{O}_X(\nu)) \to H^1(\mathcal{I}_X(\nu)) \to 0$$

and the fact that $h^0(\mathcal{O}_X(\nu)) = s$ yield $I_\nu = 0$ for $\nu < \nu_0$ and $H^1(\mathcal{I}_X(\nu)) \simeq H^1_{\mathfrak{m}}(S)_\nu = 0$ for $\nu \ge \nu_0$ so by Corollary 4.6, $\mathrm{Def}^0_{S/R} \simeq \mathrm{Hilb}_{X/\mathbb{P}^n}$ for points in general enough position.

Six general points in $\mathbb{P}^2$ will have Hilbert function $(1, 3, 6, 6, \dots)$ and hence $\mathrm{Def}^0_{S/R} \simeq \mathrm{Hilb}_{X/\mathbb{P}^n}$. On the other hand the complete intersection of a quadric and a cubic will have $h_X = (1, 3, 5, 6, 6, \dots)$. Thus $\dim I_2 = h^1(\mathcal{I}_X(2)) = 1$, so $\mathrm{Hom}(I, H^1_{\mathfrak{m}}(S))_0 \simeq k$ and the functors need not be isomorphic, cf. Corollary 4.6 (ii). Indeed since both obstruction spaces vanish (S is a complete intersection) the map $\mathrm{Def}^0_{S/R} \to \mathrm{Hilb}_{X/\mathbb{P}^n}$ corresponds dually to a surjection of formally smooth complete k-algebras. The above two cases show that the Hilbert function stratum given by $(1, 3, 6, 6, \dots)$ is an open subscheme of $\mathrm{Hilb}^6(\mathbb{P}^2)$ while $(1, 3, 5, 6, \dots)$ gives a smooth codimension 1 stratum.

Example 4.8. *Curves.* Consider a smooth curve C sitting on a hypersurface V of degree $s < \sum_{i=0}^3 q_i$ in $\mathbb{P}(q_0, q_1, q_2, q_3)$. Let $C = \operatorname{Proj} S \subset \mathbb{P}(\mathbf{q}) = \operatorname{Proj} R$ with $S = R/I$ and suppose that $\operatorname{Spec} R/I_V \setminus \{0\}$ is smooth, i.e. V quasi-smooth. Applying $\mathcal{H}om(-, \mathcal{O}_C)$ to the sequence $0 \to \mathcal{I}_{V/\mathbb{P}(\mathbf{q})} \to \mathcal{I}_{C/\mathbb{P}(\mathbf{q})} \to \mathcal{I}_{C/V} \to 0$ we get an exact sequence of normal bundles

$$0 \to \mathcal{N}_{C/V} \to \mathcal{N}_{C/\mathbb{P}(\mathbf{q})} \to \mathcal{O}_C(s) \to 0$$

(cf. [Har10, p. 93]) and $\mathcal{N}_{C/V} \simeq (\omega_C^{-1} \otimes \omega_V)^{-1}$. Thus by Serre duality $H^1(\mathcal{N}_{C/V}) = 0$ and $H^1(\mathcal{N}_{C/\mathbb{P}(\mathbf{q})}) \simeq H^1(\mathcal{O}_C(s))$ while $H^0(\mathcal{N}_{C/\mathbb{P}(\mathbf{q})}) \to H^0(\mathcal{O}_C(s))$ is surjective.

Applying this to the long exact sequence in Theorem 2.1 we get

$$0 \to \mathrm{Hom}_R(I, S)_0 \to H^0(\mathcal{N}_{C/\mathbb{P}(\mathbf{q})}) \to \mathrm{Hom}_R(I, H^1_{\mathfrak{m}}(S))_0 \to T^2(S/R; S)_0 \to 0\,, \tag{4.2}$$

because $T^2_{\mathfrak{m}}(S/R; S)_0 \simeq \mathrm{Hom}_R(I, H^1_{\mathfrak{m}}(S))_0$ and the composition

$$H^1(\mathcal{N}_{C/\mathbb{P}(\mathbf{q})}) \to T^3_{\mathfrak{m}}(S/R; S)_0 \to \mathrm{Hom}_R(I, H^2_{\mathfrak{m}}(S))_0 \to H^1(\mathcal{O}_C(s))$$

is injective. Similarly there is a commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{N}_{C/\mathbb{P}(\mathbf{q})}) & \longrightarrow & \mathrm{Hom}_R(I, H^1_{\mathfrak{m}}(S)) \\ \downarrow\!\!\!\downarrow & & \downarrow \\ H^0(\mathcal{O}_C(s)) & \longrightarrow\!\!\!\!\!\rightarrow & H^1(\mathcal{I}_C(s)) \end{array}$$

so if $\mathrm{Hom}_R(I, H^1_{\mathfrak{m}}(S))_0 \simeq H^1(\mathcal{I}_C(s))$ then $T^2(S/R;S)_0 = 0$ and $\mathrm{Def}^0_{S/R}$ is unobstructed. If $d_0, d_1, \ldots, d_m$ are the degrees of the minimal generators of I and $s = d_0$, then it is easy to see that there is such an isomorphism if $H^1(\mathcal{I}_C(d_i)) = 0$ for all $i = 1, \ldots, m$.

If $\mathbb{P}(\mathbf{q}) = \mathbb{P}^3$ and $s = 2$ then $V \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and we can use the Künneth formula to compute these cohomology groups. The outcome is that the above vanishing holds and we get $\mathrm{Hom}_R(I, H^1_{\mathfrak{m}}(S))_0 \simeq H^1(\mathcal{I}_C(2))$, so $T^2(S/R;S)_0 = 0$. Moreover if C has bidegree (p,q) with $2 \le p \le q$ then

$$h^1(\mathcal{I}_C(2)) = \begin{cases} 0 & \text{if } p > 2 \\ \max\{0, q-3\} & \text{if } p = 2 \end{cases}.$$

Thus $\mathrm{Def}^0_{S/R} \simeq \mathrm{Hilb}_{C/\mathbb{P}^3}$ are smooth if $p > 2$ or $q = 3$ while $\mathrm{Def}^0_{S/R}$ corresponds to a smooth stratum of codimension $q-3$ in $\mathrm{Hilb}^{d,g}(\mathbb{P}^3)$ otherwise, see [Tan80].

If $s = 3$ then among curves on a cubic surface in $\mathbb{P}^3$ we find the curve that gives rise to Mumford's example of an irreducible component W of $\mathrm{Hilb}^{14,24}(\mathbb{P}^3)$ which is not reduced at its generic point ([Mum62]). Let H be a hyperplane section of V and E a line on V. Take C to be a generic element of the complete linear system $|4H + 2E|$ on V. Such a curve may be constructed as the linked curve to the curve consisting of two disjoint conics in a general $(3,6)$ complete intersection, and there is a resolution

$$0 \to R(-9) \to R(-8)^2 \oplus R(-7)^2 \to R(-6)^3 \oplus R(-3) \to I \to 0\,,$$

cf. [Cu81]. One computes that $h^1(\mathcal{I}_C(d)) = 0$ when $d \ge 6$ and $h^1(\mathcal{I}_C(3)) = h^1(\mathcal{O}_C(3)) = 1$, so $\mathrm{Hom}_R(I, H^1_{\mathfrak{m}}(S)) \simeq H^1(\mathcal{I}_C(3)) \simeq k$ and $T^2(S/R;S)_0 = 0$ in (4.2). Thus $\mathrm{Def}^0_{S/R}$ is represented by a formally smooth complete k-algebra which by (4.2) corresponds to the reduced subscheme of the component W at its generic point.

5. Deformations of the scheme - Def_X

In this section we must assume that the characteristic of the ground field is 0. Fix for the whole of this section a G-quadruple (X, Z, S, J) with associated sheaf of algebras $\mathcal{S}$. Moreover fix M, a finitely generated SG-module, and set $\mathcal{F} = \pi_*(\widetilde{M_{|X'}})$ to be the corresponding sheaf of $\mathcal{S}G$-module on X. This section is mostly concerned with computing cotangent groups with values in $\mathcal{S}$ or $\mathcal{O}_X$, but because of future applications it is better to be a little more general and consider values in M, $\mathcal{F}$ and $\mathcal{F}^G$.

5.1. Conditions for $\mathrm{Def}^G_S \to \mathrm{Def}_X$ to be smooth or an isomorphism.

Lemma 5.1. *If $T^2(\mathcal{S}/\mathcal{O}_X;\mathcal{S})^G = 0$ and $T^2_J(S/k;S)^G = 0$ then $\mathrm{Def}^G_S \to \mathrm{Def}_X$ is smooth. If moreover $T^1(\mathcal{S}/\mathcal{O}_X;\mathcal{S})^G = 0$ and $T^1_J(S/k;S)^G = 0$ then $\mathrm{Def}^G_S \to \mathrm{Def}_X$ is an isomorphism.*

Proof. First note that since taking G invariants is exact we have $T^i_X(\mathcal{S})^G \simeq T^i_X$. Secondly [And74, Appendice, Proposition 56] and the fact that π is affine imply $T^i_X(\mathcal{S}) \simeq T^i(\pi^{-1}\mathcal{O}_X/k;\mathcal{O}_{X'})$. The Zariski-Jacobi sequence for $k \to \pi^{-1}\mathcal{O}_X \to \mathcal{O}_{X'}$ reads

$$\to T^i(\mathcal{O}_{X'}/\pi^{-1}\mathcal{O}_X;\mathcal{O}_{X'}) \to T^i_{X'} \to T^i(\pi^{-1}\mathcal{O}_X/k;\mathcal{O}_{X'}) \to T^{i+1}(\mathcal{O}_{X'}/\pi^{-1}\mathcal{O}_X;\mathcal{O}_{X'}) \to$$

so the first condition yields $(T^i_{X'})^G \to T^i_X$ surjective for $i = 1$ and injective for $i = 2$. The exact sequence in Theorem 2.1 for $X' \subseteq \operatorname{Spec} S$ is

$$\cdots \to T^i_J(S/k;S) \to T^i_S \to T^i_{X'} \to T^{i+1}_J(S/k;S) \to \cdots$$

so the second condition implies the same for $(T^i_S)^G \to (T^i_{X'})^G$ and we have proven the first statement.

To prove the second statement we need to show that the conditions imply $\mathrm{Def}^G_S(A) \to \mathrm{Def}_X(A)$ injective for all objects A of $\mathbf{C}$. More explicitly we must show that if S^1_A and S^2_A are invariant deformations over A mapping to X^1_A and X^2_A, then for every isomorphism of deformations $\varphi_{X,A} : X^1_A \to X^2_A$ there exists an equivariant isomorphism $\varphi_{S,A} : S^1_A \to S^2_A$ inducing $\varphi_{X,A}$.

If $T^1(\mathcal{S}/\mathcal{O}_X;\mathcal{S})^G = 0$ and $T^1_J(S/k;S)^G = 0$ then the same argument as above shows that $(T^i_S)^G \to T^i_X$ is surjective for $i = 0$ and injective for $i = 1$. We will prove the statement by induction on the length of A. Let $\rho : B \to A$ be a small extension and assume the statement is true for A. Let S^1_B and S^2_B be invariant deformations of S over B mapping to X^1_B and X^2_B. Assume $\varphi_{X,B} : X^1_B \to X^2_B$ is an isomorphism of deformations. Let $\varphi_{X,A} : X^1_A \to X^2_A$ be the induced isomorphism where $X^i_A = X^i_B \times_{\mathrm{Spec}\, B} \mathrm{Spec}\, A$. By induction there exists an isomorphism $\varphi_{S,A} : S^1_A \to S^2_A$ inducing $\varphi_{X,A}$ and where $S^i_A = S^i_B \otimes_B A$.

The obstruction to lifting $\varphi_{S,A}$ to an isomorphism $S^1_B \to S^2_B$ is in $(T^1_S)^G \otimes \ker\rho$ and maps to 0 in $T^1_X \otimes \ker\rho$ because a lifting of $\varphi_{X,A}$ exists (namely $\varphi_{X,B}$). Thus by injectivity at the T^1 level the obstruction vanishes and there exists a lifting $\varphi'_{S,B} : S^1_B \to S^2_B$. It maps to an isomorphism $\varphi'_{X,B}$ which may differ from our given $\varphi_{X,B}$. However $\varphi_{X,B} - \varphi'_{X,B}$ defines an element of $T^0_X \otimes \ker\rho$ and the surjectivity at the T^0 level yields an element $D_{S,B} \in (T^0_S)^G \otimes \ker\rho$ mapping to $\varphi_{X,B} - \varphi'_{X,B}$. Thus $\varphi_{S,B} := \varphi'_{S,B} + D_{S,B}$ is an isomorphism inducing $\varphi_{X,B}$. □

We will later need a statement with more general values. The proof is the same as the first part above.

Lemma 5.2. *If* $T^2(\mathcal{S}/\mathcal{O}_X;\mathcal{F})^G = 0$ *and* $T^2_J(S/k;M)^G = 0$ *then there is a surjective morphism* $T^1_S(M)^G \twoheadrightarrow T^1_X(\mathcal{F}^G)$.

The modules $T^i_J(S/k;M)$ are easily described in terms of local cohomology.

Lemma 5.3. *If* $\mathrm{depth}_J M \geq 1$ *then there is an isomorphism* $T^1_J(S/k;M) \simeq \mathrm{Der}_k(S, H^1_J(M))$ *and an exact sequence*

$$0 \to T^1(S/k;H^1_J(M)) \to T^2_J(S/k;M) \to T^0(S/k;H^2_J(M)) \to T^2(S/k;H^1_J(M))\,.$$

In particular if $\mathrm{depth}_J M \geq 2$ *or* S *is regular then* $T^2_J(S/k;M) \simeq \mathrm{Der}_k(S, H^2_J(M))$ *as SG-modules.*

Proof. Since $H^0_J(M) = 0$ the exact sequence comes from the edge exact sequence for the spectral sequence in Theorem 2.1. □

5.2. **Euler derivations.** We show that if $\mathrm{depth}_Z \mathcal{S} \geq 2$ then the relative tangent sheaf $\Theta_{X'/X}$ is globally free of rank equal to the dimension of G. This will allow us to give finer criteria for the equivalence of deformation functors. The result follows from general considerations and we start with some lemmas.

Let G be a reductive group acting on an affine scheme $X = \mathrm{Spec}\, A$ over a field k of characteristic 0. Let $\mathfrak{g} = T_e$ be the Lie algebra of G (where e is the identity element in G). Consider the corresponding representation of Lie algebras $\phi : A \otimes \mathfrak{g} \to \mathrm{Der}_{A^G}(A, A)$. Recall that if $\xi \in \mathfrak{g} = \mathrm{Der}_k(k[G], k(e))$ then $\phi(1 \otimes \xi)$ is the composition

$$A \xrightarrow{\mu} A \otimes k[G] \xrightarrow{1\otimes\xi} A \otimes k(e) \simeq A \tag{5.1}$$

where μ is the comultiplication of the group action. Note that if f is invariant then $\mu(f) = f \otimes 1$ so the composition is in $\mathrm{Der}_{A^G}(A, A)$.

If $D \in \mathrm{Der}_{A^G}(A, A)$ and x is a closed point of $\mathrm{Spec}\, A$, let D_x be the value in $\mathrm{Der}_{A^G}(A, k(x))$. For a fixed closed point x let $p_x : G \to Gx$ be the orbit map. Then $\phi(1 \otimes \xi)_x(f) = \xi(p_x^{\#}(f))$ so $\phi(\xi)_x$ equals the image of ξ via the map of Zariski tangent spaces $dp_x : T_e \to T_{Gx,x}$. Note that $T_{Gx,x} = \mathrm{Der}_k(\mathcal{O}_{Gx}, k(x)) \subseteq \mathrm{Der}_{A^G}(A, k(x)) \subseteq \mathrm{Der}_k(A, k(x)) = T_{X,x}$.

Lemma 5.4. *If there is an open dense subset of* $\mathrm{Spec}\, A$ *where all isotropy groups are finite, then* $\phi : A \otimes_k \mathfrak{g} \to \mathrm{Der}_{A^G}(A, A)$ *is injective.*

Proof. Choose a basis $\xi_1, \ldots, \xi_r$ for $\mathfrak{g}$ and assume $\phi(\sum a_i \otimes \xi_i) = 0$. Then $\sum a_i(x) dp_x(\xi_i) = 0$ in $T_{Gx,x}$ for all x. But by assumption dp_x is an isomorphism on an open dense subset U. Thus for all $i = 1, \ldots, r$ we have $a_i = 0$ on U so $a_i = 0$ in A. □

Lemma 5.5. *If the quotient map* $\mathrm{Spec}\, A \to \mathrm{Spec}\, A^G$ *is smooth of relative dimension* $\dim G$ *and a geometric quotient then* $\phi : A \otimes_k \mathfrak{g} \to \mathrm{Der}_{A^G}(A, A)$ *is an isomorphism.*

Proof. Injectivity follows from Lemma 5.4. We will show surjectivity for the maps of stalks $A_x \otimes \mathfrak{g} \to \mathrm{Der}_{A^G}(A, A)_x = \mathrm{Der}_{A^G_y}(A_x, A_x)$ where $x \mapsto y$ via the quotient map. By assumption $\mathcal{O}_{Gx} \simeq A \otimes_{A^G} k(y)$ and so $\mathrm{Der}_{A^G_y}(A_x, k(x)) \simeq \mathrm{Der}_k(\mathcal{O}_{Gx}, k(x)) = T_{Gx,x} \simeq \mathfrak{g}$. On the other hand since $A^G \to A$ is smooth we have $\mathrm{Der}_{A^G_y}(A_x, k(x)) \simeq \mathrm{Der}_{A^G_y}(A_x, A_x) \otimes_{A_x} k(x)$ and so the surjectivity follows from Nakayama's Lemma. □

We write $M^\vee = \mathrm{Hom}_A(M, A)$ for the dual module. The above construction yields also a map $\psi : \Omega_{A/A^G} \to \mathrm{Hom}_k(\mathfrak{g}, A)$ namely the composition

$$\Omega_{A/A^G} \to \Omega^{\vee\vee}_{A/A^G} = \mathrm{Hom}_A(\mathrm{Der}_{A^G}(A, A), A) \xrightarrow{\phi^\vee} \mathrm{Hom}_k(\mathfrak{g}, A)\,.$$

Thus $\psi(da)(\xi) = \phi(1 \otimes \xi)(a)$

Lemma 5.6. *If x is a closed point in* $\mathrm{Spec}\, A$ *mapping to* $y \in \mathrm{Spec}\, A^G$ *with finite isotropy group and Gx is closed in* $\mathrm{Spec}\, A$ *then* $\psi_x : \Omega_{A_x/A^G_y} \to \mathrm{Hom}_k(\mathfrak{g}, A_x)$ *is surjective.*

Proof. By Nakayama's Lemma it is enough to prove $\Omega_{A_x/A^G_y} \otimes_{A_x} k(x) \to \mathrm{Hom}_k(\mathfrak{g}, k(x))$ is surjective, which we can do by showing that the k-dual map is injective. But this is the map $\mathfrak{g} \otimes k(x) \to \mathrm{Der}_{A^G_y}(A_x, k(x))$ induced by ϕ. Thus it factors in the following way

$$\mathfrak{g} \otimes k(x) \simeq T_{Gx,x} \subseteq \mathrm{Der}_{A^G_y}(A_x, k(x))\,.$$

□

We now apply these local lemmas to our situation. By $\mathcal{F} \otimes_k \mathfrak{g}$ we mean the $\mathcal{S}G$-module that as $\mathcal{S}$-module is isomorphic to the sum of $\dim \mathfrak{g}$ copies of $\mathcal{F}$ and the G-action is given by $g(f \otimes \xi) = gf \otimes \mathrm{Ad}_g(\xi)$ where $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is the adjoint action. Similarly we can define $\mathcal{F} \otimes_k \mathfrak{g}^*$ for the dual of the adjoint action.

Lemma 5.7. *If* $\mathrm{depth}_Z\, \mathcal{S} \geq 2$ *then there is an $\mathcal{S}G$ isomorphism*

$$\mathcal{T}^0(\mathcal{S}/\mathcal{O}_X; \mathcal{S}) \simeq \mathcal{S} \otimes_k \mathfrak{g}\,.$$

More generally if $\mathrm{depth}_Z\, \mathcal{F} \geq 2$ *then* $\mathcal{T}^0(\mathcal{S}/\mathcal{O}_X; \mathcal{F}) \simeq \mathcal{F} \otimes_k \mathfrak{g}$ *as $\mathcal{S}G$-modules.*

Proof. By definition $\pi_{|U'}$ satisfies locally the conditions in Lemma 5.5 . This globalizes to an isomorphism $\Theta_{U'/U} \simeq \mathcal{O}_{U'} \otimes_k \mathfrak{g}$. Since π is smooth on U' we get $\Omega^1_{U'/U} \simeq \mathcal{O}_{U'} \otimes \mathfrak{g}^*$. Since $\mathcal{T}^0(\mathcal{S}/\mathcal{O}_X;\mathcal{F}) = \mathcal{H}om(\Omega^1_{\mathcal{S}/\mathcal{O}_X},\mathcal{F})$ it also has depth ≥ 2 in Z, cf. Theorem 2.1. It is therefore isomorphic to $j_*\mathcal{H}om(\pi_*\Omega^1_{U'/U},\mathcal{F}_{|U}) \simeq \mathcal{F} \otimes_k \mathfrak{g}$ where j is the inclusion $U \subseteq X$. □

For example let $G = \mathbb{G}_m$ and A be a finitely generated algebra, or a localization of such, with algebra generators $x_1,\dots,x_n$. Then $\mu : A \to A \otimes k[t,t^{-1}]$ is of the form $f(x_1,\dots,x_n) \mapsto f(t^{a_1}x_1,\dots,t^{a_n}x_n)$ for integers $a_1,\dots,a_n$. We get

$$\phi(1 \otimes \frac{d}{dt}_{|t=1})(f) = \frac{d}{dt} f(t^{a_1}x_1,\dots,t^{a_n}x_n)_{|t=1} = \sum_{i=1}^n a_i x_i \frac{\partial f}{\partial x_i}$$

by the chain rule. Thus the generator of $\mathfrak{g}$ maps to the Euler derivation of the $\mathbb{Z}$-graded algebra A. This motivates the name in the following definition.

Definition 5.8. Assuming $\operatorname{depth}_Z \mathcal{S} \geq 2$, let $E_1,\dots,E_r$ be a set of global sections that generate the free sheaf $\mathcal{T}^0(\mathcal{S}/\mathcal{O}_X;\mathcal{S})$. We call such a set a *set of Euler derivations* for the G-quadruple. A set of Euler derivations defines an $\mathcal{S}G$-map $E : \Omega^1_{\mathcal{S}/\mathcal{O}_X} \to \mathcal{S} \otimes \mathfrak{g}^*$. Let $\mathcal{Q} = \operatorname{coker}(E)$ and $Z(\mathcal{Q})$ be the support of $\mathcal{Q}$.

The morphism E is just the natural map $\Omega^1_{\mathcal{S}/\mathcal{O}_X} \to (\Omega^1_{\mathcal{S}/\mathcal{O}_X})^{\vee\vee}$ after choosing a basis for the free sheaf $(\Omega^1_{\mathcal{S}/\mathcal{O}_X})^{\vee\vee}$ thus we have an exact sequence

$$0 \to \mathcal{T}or(\Omega^1_{\mathcal{S}/\mathcal{O}_X}) \to \Omega^1_{\mathcal{S}/\mathcal{O}_X} \xrightarrow{E} \mathcal{S} \otimes \mathfrak{g}^* \to \mathcal{Q} \to 0 \tag{5.2}$$

where $\mathcal{T}or(\Omega^1_{\mathcal{S}/\mathcal{O}_X})$ is the torsion submodule of $\Omega^1_{\mathcal{S}/\mathcal{O}_X}$.

If $\operatorname{depth}_Z \mathcal{S} \geq 2$ and the sheaf $\mathcal{T}^1(\mathcal{S}/\mathcal{O}_X;\mathcal{S})^G = 0$ then the Zariski-Jacobi sequence for $k \to \mathcal{O}_X \to \mathcal{S}$ leads to a short exact sequence

$$0 \to \mathcal{T}^0(\mathcal{S}/\mathcal{O}_X;\mathcal{S})^G \to (\mathcal{T}^0_{\mathcal{S}})^G \to \mathcal{T}^0(\mathcal{O}_X/k;\mathcal{S})^G \to 0\,.$$

Therefore by Lemma 5.7 we can make the following definition.

Definition 5.9. If $\operatorname{depth}_Z \mathcal{S} \geq 2$ and the sheaf $\mathcal{T}^1(\mathcal{S}/\mathcal{O}_X;\mathcal{S})^G = 0$ then we call the short exact sequence

$$0 \to (\mathcal{S} \otimes \mathfrak{g})^G \to \Theta^G_{\mathcal{S}} \to \Theta_X \to 0$$

the *Euler sequence* associated to the G-quadruple. Here $\Theta_X = \mathcal{T}^0(\mathcal{O}_X/k;\mathcal{O}_X)$.

Remark. If $G \to \mathrm{GL}(V)$ is a representation and $S = \operatorname{Sym}(V^*)$ then $\operatorname{Der}_k(S) \simeq S \otimes V$ as SG-module. From the above we have an SG-map $S \otimes \mathfrak{g} \to S \otimes V$. In this case it is induced from the natural map $\mathfrak{g} \to \operatorname{Hom}(V,V) \simeq \operatorname{Sym}^1(V^*) \otimes V$ given by the representation.

Lemma 5.10. *If* $\operatorname{depth}_Z \mathcal{F} \geq 2$ *then*

$$\mathcal{T}^i(\mathcal{S}/\mathcal{O}_X;\mathcal{F}) \simeq \mathcal{E}xt^i_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X},\mathcal{F}) \text{ for } i = 0,1,2.$$

Proof. Consider the spectral sequence $\mathcal{E}xt^p_{\mathcal{S}}(\mathcal{T}_q(\mathcal{S}/\mathcal{O}_X;\mathcal{S}),\mathcal{F}) \Rightarrow \mathcal{T}^{p+q}(\mathcal{S}/\mathcal{O}_X;\mathcal{F})$. Since the $\mathcal{T}_q(\mathcal{S}/\mathcal{O}_X;\mathcal{S})$ are supported on Z when $q \geq 1$, the depth condition yields the result. □

Assume $\mathcal{S}$ is a sheaf of $\mathcal{O}_X$-algebras on a scheme X and $Z \subseteq X$ is locally closed. If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{S}$-modules, denote by $\operatorname{Ext}^i_{\mathcal{S},Z}(\mathcal{F},\mathcal{G})$, respectively $\mathcal{E}xt^i_{\mathcal{S},Z}(\mathcal{F},\mathcal{G})$, the higher derived functors of $\mathcal{G} \mapsto H^0_Z(X,\mathcal{H}om_{\mathcal{S}}(\mathcal{F},\mathcal{G}))$, respectively $\mathcal{G} \mapsto \mathcal{H}^0_Z(X,\mathcal{H}om_{\mathcal{S}}(\mathcal{F},\mathcal{G}))$. We refer to SGA 2 Exposè VI ([Gro05]) for details and the results we will use.

Lemma 5.11. *Assume* $\operatorname{depth}_Z \mathcal{F} \geq 2$.

(i) *There is an isomorphism* $\mathcal{E}xt^1_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) \simeq \mathcal{H}om_{\mathcal{S}}(\mathcal{Q}, \mathcal{H}^2_Z(\mathcal{F}))$.

(ii) *If* $\operatorname{depth}_Z(\mathcal{F} \otimes \mathfrak{g})^G \geq 3$ *or* $\mathcal{Q} = 0$ *then*

$$\mathcal{T}^1(\mathcal{S}/\mathcal{O}_X; \mathcal{F})^G = \mathcal{E}xt^1_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G = \mathcal{H}om_{\mathcal{S}}(\mathcal{Q}, \mathcal{H}^2_Z(\mathcal{F}))^G = 0\,.$$

Proof. By Lemma 5.7, $\mathcal{H}om(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) \simeq \mathcal{F} \otimes \mathfrak{g}$ so we get an exact sequence

$$\cdots \to \mathcal{E}xt^1_{\mathcal{S},Z}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) \to \mathcal{E}xt^1_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) \to \mathcal{R}^1 j_*(\mathcal{F} \otimes \mathfrak{g})_{|U} \xrightarrow{\varepsilon} \mathcal{E}xt^2_{\mathcal{S},Z}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) \to \cdots$$

from the long exact sequence for $\mathcal{E}xt^i_{\mathcal{S},Z}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})$. Here j is the inclusion of U in X. Now since $\operatorname{depth}_Z \mathcal{F} \geq 2$ the left sheaf vanishes. The sheaf $\mathcal{R}^1 j_*(\mathcal{F} \otimes \mathfrak{g})_{|U}$ is isomorphic to $\mathcal{H}^2_Z(\mathcal{F} \otimes \mathfrak{g}) \simeq \mathcal{H}^2_Z(\mathcal{F}) \otimes \mathfrak{g}$. Using the spectral sequence $\mathcal{E}xt^p_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{H}^q_Z(\mathcal{F})) \Rightarrow \mathcal{E}xt^{p+q}_{\mathcal{S},Z}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})$ we get

$$\mathcal{E}xt^2_{\mathcal{S},Z}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) \simeq \mathcal{H}om_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{H}^2_Z(\mathcal{F}))\,.$$

Thus our exact sequence becomes

$$0 \to \mathcal{E}xt^1_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) \to \mathcal{H}^2_Z(\mathcal{F}) \otimes \mathfrak{g} \xrightarrow{\varepsilon} \mathcal{H}om_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{H}^2_Z(\mathcal{F})) \to \cdots$$

where ε is the map induced by E in the sequence (5.2). It follows that

$$\mathcal{E}xt^1_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) = \ker \varepsilon \simeq \mathcal{H}om_{\mathcal{S}}(\mathcal{Q}, \mathcal{H}^2_Z(\mathcal{F}))$$

proving the first statement.

Clearly if $\mathcal{Q} = 0$ then $\mathcal{E}xt^1_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) = 0$ and the inclusion

$$\mathcal{E}xt^1_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G \hookrightarrow \mathcal{H}^2_Z(\mathcal{F} \otimes \mathfrak{g})^G \simeq \mathcal{H}^2_Z((\mathcal{F} \otimes \mathfrak{g})^G)$$

yields the second statement. □

We now have conditions for the existence of an Euler sequence.

Theorem 5.12. *If* $\operatorname{depth}_Z \mathcal{S} \geq 2$ *and* $\mathcal{H}om_{\mathcal{S}}(\mathcal{Q}, \mathcal{H}^2_Z(\mathcal{S}))^G = 0$ *then there is an Euler sequence*

$$0 \to (\mathcal{S} \otimes \mathfrak{g})^G \to \Theta^G_{\mathcal{S}} \to \Theta_X \to 0\,.$$

Proposition 5.13. *The closed subset* $Z(\mathcal{Q}) \subset X$ *is contained in the set of points* $x \in X$ *where* $\pi^{-1}(x)$ *contains a point with positive dimensional isotropy group.*

Proof. We may prove the result locally on an affine chart, i.e. for $\pi : \operatorname{Spec} A \to \operatorname{Spec} A^G$. A standard result in invariant theory says that the stable points are the complement of $\pi^{-1}(\pi(L))$ where L is the locus of points with positive dimensional isotropy. The result follows now from Lemma 5.6. □

Example 5.14. For toric varieties Proposition 5.13 says that $Z(\mathcal{Q})$ is contained in the non-simplicial locus. In fact one can prove directly using Euler derivations that these loci are equal. In particular for simplicial toric varieties $\mathcal{Q} = 0$ and we recover the Euler sequence of [BC94, Theorem 12.1]

Example 5.15. For Grassmannians (Example 3.8) even Z is empty, Proposition 5.13 applies. If $\mathcal{L}$ is the universal subbundle and $\mathcal{P}$ is the universal quotient bundle then the tangent sheaf is $\mathcal{P} \otimes \mathcal{L}^\vee$. Our Euler sequence is just the usual sequence induced by the tautological exact sequence.

5.3. **The groups $T^i(\mathcal{S}/\mathcal{O}_X;\mathcal{F})$.** As we shall see in the computations here and the results in the next section the behavior of the sheaf $\mathcal{Q}$, the cokernel of the Euler derivations (Definition 5.8) plays an important role when comparing deformation functors.

Lemma 5.16. *If* $\operatorname{depth}_Z \mathcal{F} \geq 2$ *and* $\mathcal{H}om_{\mathcal{S}}(\mathcal{Q}, \mathcal{H}^2_Z(\mathcal{F}))^G = 0$ *then*

$$T^i(\mathcal{S}/\mathcal{O}_X;\mathcal{F})^G \simeq \operatorname{Ext}^i_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G \text{ for } i = 0,1,2\,.$$

Proof. Lemma 5.11 says that under the conditions above, $\mathcal{T}^1(\mathcal{S}/\mathcal{O}_X;\mathcal{F})^G = \mathcal{E}xt^1_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G = 0$. We have local global spectral sequences which at the E_2 level are

$$H^p(X, \mathcal{T}^q(\mathcal{S}/\mathcal{O}_X;\mathcal{F})) \Rightarrow T^{p+q}(\mathcal{S}/\mathcal{O}_X;\mathcal{F})$$
$$H^p(X, \mathcal{E}xt^q_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})\,.$$

Write (just for this proof) $\mathcal{T}^i = \mathcal{T}^i(\mathcal{S}/\mathcal{O}_X;\mathcal{F})$ and $\mathcal{E}^i = \mathcal{E}xt^i_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})$. The edge homomorphisms for the spectral sequence $\mathcal{E}xt^p_{\mathcal{S}}(\mathcal{T}_q(\mathcal{S}/\mathcal{O}_X;\mathcal{S}), \mathcal{F}) \Rightarrow \mathcal{T}^{p+q}(\mathcal{S}/\mathcal{O}_X;\mathcal{F})$ yield natural maps $\mathcal{E}^i \to \mathcal{T}^i$. Lemma 5.10 implies $T^i(\mathcal{S}/\mathcal{O}_X;\mathcal{F})^G \simeq \operatorname{Ext}^i_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G$ for $i = 0, 1$. Since $\mathcal{E}xt^1_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G = 0$ the spectral sequences yield a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(X,\mathcal{E}^0)^G & \longrightarrow & \operatorname{Ext}^2_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G & \longrightarrow & H^0(X,\mathcal{E}^2)^G & \longrightarrow & H^3(X,\mathcal{E}^0)^G \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^2(X,\mathcal{T}^0)^G & \longrightarrow & T^2(\mathcal{S}/\mathcal{O}_X;\mathcal{F})^G & \longrightarrow & H^0(X,\mathcal{T}^2)^G & \longrightarrow & H^3(X,\mathcal{T}^0)^G
\end{array}$$

so the isomorphism for $i = 2$ follows from Lemma 5.10 and the Five Lemma. □

Proposition 5.17. *Assume* $\operatorname{depth}_Z \mathcal{F} \geq 2$

(i) *If* $\mathcal{H}om_{\mathcal{S}}(\mathcal{Q}, \mathcal{H}^2_Z(\mathcal{F}))^G = 0$ *then* $T^1(\mathcal{S}/\mathcal{O}_X;\mathcal{F})^G \simeq H^1(X, (\mathcal{F}\otimes\mathfrak{g})^G)$ *and there is an exact sequence*

$$0 \to H^2(X, (\mathcal{F}\otimes\mathfrak{g})^G) \to T^2(\mathcal{S}/\mathcal{O}_X;\mathcal{F})^G \to H^0(X, \mathcal{E}xt^2_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G) \to H^3(X, (\mathcal{F}\otimes\mathfrak{g})^G)$$

(ii) *If* $\operatorname{depth}_Z(\mathcal{F}\otimes\mathfrak{g})^G \geq 3$ *then there is an exact sequence*

$$0 \to \operatorname{Hom}_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{H}^2_Z(\mathcal{F}))^G \to H^0(X, \mathcal{E}xt^2_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G) \to \operatorname{Hom}_{\mathcal{S}}(\mathcal{Q}, \mathcal{H}^3_Z(\mathcal{F}))^G\,.$$

(iii) *If* $\mathcal{Q} = 0$ *then* $\mathcal{E}xt^2_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) \simeq \mathcal{H}om_{\mathcal{S}}(\mathcal{T}or(\Omega^1_{\mathcal{S}/\mathcal{O}_X}), \mathcal{H}^2_Z(\mathcal{F}))$.

Proof. Since the condition in (i) implies $\mathcal{T}^1(\mathcal{S}/\mathcal{O}_X;\mathcal{F})^G = 0$ by Lemma 5.10 and Lemma 5.11 (i) follows from the local-global spectral sequence and Lemma 5.7 and Lemma 5.16.

Let j be the inclusion of U in X. To prove (ii) we consider the long exact sequence for $\mathcal{E}xt^i_{\mathcal{S},Z}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})$ as in the proof of Lemma 5.11, and get

$$\mathcal{H}^2_Z(\mathcal{F}\otimes\mathfrak{g}) \to \mathcal{E}xt^2_{\mathcal{S},Z}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) \to \mathcal{E}xt^2_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) \to \mathcal{H}^3_Z(\mathcal{F}\otimes\mathfrak{g}) \xrightarrow{u} \mathcal{E}xt^3_{\mathcal{S},Z}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})\,. \tag{5.3}$$

Taking invariants and using the depth assumption this yields

$$0 \to \mathcal{E}xt^2_{\mathcal{S},Z}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G \to \mathcal{E}xt^2_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G \to \mathcal{H}^3_Z(\mathcal{F}\otimes\mathfrak{g})^G \xrightarrow{u} \mathcal{E}xt^3_{\mathcal{S},Z}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G\,.$$

Using the spectral sequence $\mathcal{E}xt^p_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{H}^q_Z(\mathcal{F})) \Rightarrow \mathcal{E}xt^{p+q}_{\mathcal{S},Z}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})$ we get

$$\mathcal{E}xt^2_{\mathcal{S},Z}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G \simeq \mathcal{E}xt^0_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{H}^2_Z(\mathcal{F}))^G .$$

Moreover there is a composite map $\varepsilon = v \circ u$

$$\mathcal{H}^3_Z(\mathcal{F} \otimes \mathfrak{g}) \xrightarrow{u} \mathcal{E}xt^3_{\mathcal{S},Z}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) \xrightarrow{v} \mathcal{H}om_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{H}^3_Z(\mathcal{F}))$$

where v is the map to $E_2^{0,3}$ in the above spectral sequence. Thus $\ker u \subseteq \ker \varepsilon$. On the other hand ε is the map induced by E in the exact sequence (5.2). Therefore $\ker \varepsilon = \mathcal{H}om_{\mathcal{S}}(\mathcal{Q}, \mathcal{H}^3_Z(\mathcal{F}))$ and we get the exact sequence in the statement.

To prove (iii) note that if $\mathcal{Q} = 0$ then even without the depth assumption the above argument shows that $\ker \varepsilon = 0$. Thus u is injective. Moreover by Lemma 5.11, $\mathcal{E}xt^1_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})$ vanishes so (5.3) becomes

$$0 \to \mathcal{H}^2_Z(\mathcal{F} \otimes \mathfrak{g}) \to \mathcal{H}om_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{H}^2_Z(\mathcal{F})) \to \mathcal{E}xt^2_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F}) \to 0 .$$

The sequence (5.2) is now short exact and we can apply $\mathcal{H}om_{\mathcal{S}}(-, \mathcal{H}^2_Z(\mathcal{F}))$ to get an exact sequence

$$0 \to \mathcal{H}^2_Z(\mathcal{F} \otimes \mathfrak{g}) \to \mathcal{H}om_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{H}^2_Z(\mathcal{F})) \to \mathcal{H}om_{\mathcal{S}}(\mathcal{T}or(\Omega^1_{\mathcal{S}/\mathcal{O}_X}), \mathcal{H}^2_Z(\mathcal{F})) \to 0 .$$

This proves (iii). □

Remark. If $\mathcal{S}$ is regular then $H^0(X, \mathcal{E}xt^2_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{F})^G) \to \operatorname{Hom}_{\mathcal{S}}(\mathcal{Q}, \mathcal{H}^3_Z(\mathcal{F}))^G$ is surjective. See the proof of Theorem 6.10 below.

5.4. **Results.**

Example 5.18. *Finite G.* We can apply the above to the situation in Example 3.7, i.e. $G \subset \mathrm{GL}_n(\mathbb{C})$ is finite without pseudo-reflections and

$$(X, Z, S, J) = (\mathbb{C}^n/G, \operatorname{Sing}(\mathbb{C}^n/G), \mathbb{C}[x_1, \ldots, x_n], (1)) .$$

Assume that $\operatorname{codim}_Z X \geq 3$ (the singularity need not be isolated). Then since X is Cohen-Macaulay, $\operatorname{depth}_Z \mathcal{O}_X \geq 3$ and since G is finite, $\operatorname{depth}_{I_Z} S \geq 3$ and $\mathcal{Q} = 0$. It follows from Proposition 5.17, Lemma 5.10 and Lemma 5.1 that X is rigid. This was first proven by Schlessinger, see [Sch71] and [Sch73].

In general consider an affine G-quadruple

$$(X, Z, S, J) = (\operatorname{Spec} S^G, Z, S, (1))$$

with finite G and Z the locus where $\pi : \operatorname{Spec} S \to X$ is not étale. Then by the above results we have $\mathrm{Def}^G_S \simeq \mathrm{Def}_X$ if $\operatorname{depth}_{I_Z} S \geq 3$. In particular if S is Cohen-Macaulay and equidimensional then it is enough to assume $\operatorname{codim} Z \geq 3$. See also [Ste88, Section 7].

Our main result in its most general form is

Theorem 5.19. *Let (X, Z, S, J) be a G-quadruple with associated sheaf of algebras $\mathcal{S}$ and assume* $\operatorname{depth}_Z \mathcal{S} \geq 2$. *If*

(i) $\mathcal{H}om_{\mathcal{S}}(\mathcal{Q}, \mathcal{H}^2_Z(\mathcal{S}))^G = 0$
(ii) $H^0(X, \mathcal{E}xt^2_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{S})^G) = 0$
(iii) $H^2(X, (\mathcal{S} \otimes \mathfrak{g})^G) = 0$
(iv) $T^1(S/k; H^1_J(S))^G = 0$ *and* $\operatorname{Der}_k(S, H^2_J(S))^G = 0$

then $\mathrm{Def}^G_S \to \mathrm{Def}_X$ *is smooth. If moreover*

(v) $H^1(X, (\mathcal{S} \otimes \mathfrak{g})^G) = 0$
(vi) $\operatorname{Der}_k(S, H^1_J(S))^G = 0$

then $\operatorname{Def}^G_S \to \operatorname{Def}_X$ *is an isomorphism.*

Proof. The result follows directly from Lemma 5.1 and Proposition 5.17. □

Using the same proposition and lemma we get a slight improvement if $\mathcal{Q} = 0$.

Theorem 5.20. *Let* (X, Z, S, J) *be a* G*-quadruple with associated sheaf of algebras* $\mathcal{S}$ *and assume* $\operatorname{depth}_Z \mathcal{S} \geq 2$*. If*

(i) $\mathcal{Q} = 0$
(ii) $\operatorname{Hom}_{\mathcal{S}}(\mathcal{T}or(\Omega^1_{\mathcal{S}/\mathcal{O}_X}), \mathcal{H}^2_Z(\mathcal{S}))^G = 0$

and (iii) *and* (iv) *of Theorem 5.19 hold, then* $\operatorname{Def}^G_S \to \operatorname{Def}_X$ *is smooth. If moreover* (v) *and* (vi) *of Theorem 5.19 hold then* $\operatorname{Def}^G_S \to \operatorname{Def}_X$ *is an isomorphism.*

We give two examples to illustrate how the statement fails if the assumptions are not met.

Example 5.21. For smooth curves in $\mathbb{P}^3$ we only need to verify (iv) of Theorem 5.20 to conclude that $\operatorname{Def}^0_S \to \operatorname{Def}_X$ is smooth since Z is empty and $(\mathcal{S} \otimes \mathfrak{g})^G = \mathcal{O}_X$. We claim that a general space curve $X = \operatorname{Proj} S$ in $\operatorname{Hilb}^{d,g}(\mathbb{P}^3)$ satisfying $g < d + 3$ also satisfy (iv). Indeed in this case J is the irrelevant maximal ideal $(x_0, .., x_3)$, thus $H^2_J(S) \simeq \bigoplus_\nu H^1(X, O_X(\nu))$. We get $\operatorname{Der}_k(S, H^2_J(S))_0 = 0$ provided $H^1(X, \mathcal{O}_X(1)) = 0$. It is well known that a general curve with $g < d + 3$ satisfies this property.

We need to show $T^1(S/k, H^1_J(S))_0 = 0$. Let I be the homogeneous ideal of X in $\mathbb{P}^3 = \operatorname{Proj} R$. Since $\operatorname{Hom}_R(I, H^1_J(S))_0 = T^1(S/R, H^1_J(S)) \to T^1(S/k, H^1_J(S))_0$ is surjective, it suffices to show $\operatorname{Hom}(I, H^1_J(S))_0 = 0$. Space curves of maximal rank (i.e. $H^1(\mathcal{I}_X(v)) = 0$ provided $H^0(\mathcal{I}_X(v)) \neq 0$) necessarily satisfy this property. The main theorem of Ellia and Ballico in [BE85] implies that the general curve in the range $g < d - 3$ has maximal rank. Thus the first four assumptions of the theorem are satisfied.

It follows that $\operatorname{Def}^0_S \to \operatorname{Def}_X$ is smooth in this range. But if $g > 0$ then $H^1(X, \mathcal{O}_X) \neq 0$, i.e. (v) does not hold and we do not expect $\operatorname{Def}^0_S \to \operatorname{Def}_X$ to be an isomorphism.

Example 5.22. Let $X = \operatorname{Spec} A$ be the affine cone over $\mathbb{G}(2, 4)$ in the Plücker embedding. It is a node in $\mathbb{A}^6$ so $\dim T^1_X = 1$. On the other hand $A = S^G$ where $S = k[x_{1,1}, \ldots, x_{2,4}]$ and $G = \mathrm{SL}_2$ corresponding to an affine G-quadruple $(X, \{0\}, S, (1))$. Of course $T^1_S = 0$. If $\mathfrak{m} \subset A$ is the ideal of $\{0\}$ in X, then the ideal $I_Z = \mathfrak{m}S \subset S$ is generated by the 2×2 minors of a general 2×4 matrix so $\operatorname{depth}_Z S = 3$. Thus conditions (i), (iii) and (iv) of Theorem 5.19 are satisfied.

We compute $\operatorname{Ext}^2_S(\Omega^1_{S/A}, S)$. Let f_{ij} be the minor with columns i and j. If $1 \leq i < j \leq 4$ write $\{k, l\} = \{1, 2, 3, 4\} \setminus \{i, j\}$. Then one checks that for each pair (α, β) with $1 \leq \alpha < \beta \leq 4$

$$\sum_{1 \leq i < j \leq 4} \epsilon_{ij} \frac{\partial f_{ij}}{x_{\alpha\beta}} \cdot f_{kl} = 0$$

for suitable signs ϵ_{ij}. We may use this to construct a free SG-resolution

$$0 \to S \xrightarrow{M} S^6 \to S \otimes_k V^* \to \Omega^1_{S/A} \to 0$$

where the entries of M generate I_Z and V is the vector space of 2×4 matrices. In particular

$$\operatorname{Ext}^2_S(\Omega^1_{S/A}, S)^G \simeq (S/I_Z)^G = A/\mathfrak{m} = k$$

and indeed condition (ii) of Theorem 5.19 fails.

We assume now that G is a *quasitorus* (also called a diagonalizable group). See e.g. [ADHL15, Section 1.2] for details and proofs of the statements below. We recall the definition

Definition 5.23. A quasitorus is an affine algebraic group G whose algebra of regular functions $\Gamma(G, \mathcal{O}_G)$ is generated as a k-vector space by the characters $\chi : G \to k^*$. A torus is a connected quasitorus.

One proves that a quasitorus is a direct product of a torus and a finite abelian group. It is also characterized by the fact that any rational representation of G splits into one-dimensional subrepresentations. In particular in our case the adjoint action is trivial so $\mathcal{F} \otimes \mathfrak{g} \simeq \mathcal{F}^r$ as $\mathcal{S}G$-modules.

Lemma 5.24. *If G is a quasitorus and $\mathcal{G}$ is an $\mathcal{S}G$-module then there are inclusions*

$$\operatorname{Hom}_{\mathcal{S}}(\mathcal{Q}, \mathcal{G})^G \subseteq \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{Q}^G, \mathcal{G}^G) \subseteq (\mathcal{G}^G)^r .$$

Proof. Since G is quasitorus $\mathcal{S} \otimes \mathfrak{g} \simeq \mathcal{S}^r$ as $\mathcal{S}G$-modules and the globally generated free sheaf $\mathcal{S}^r$ is generated by invariants. Thus $\mathcal{Q}$ is generated by G invariants, so

$$\operatorname{Hom}_{\mathcal{S}}(\mathcal{Q}, \mathcal{G})^G \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{Q}^G, \mathcal{G}^G)$$

is injective. The second inclusion follows from first taking invariants of (5.2) and then applying $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{G}^G)$. □

The results become better if the codimension of $Z(\mathcal{Q})$ is sufficiently large.

Proposition 5.25. *If G is a quasitorus and*

$$\operatorname{depth}_Z \mathcal{F} \geq 2, \quad \operatorname{depth}_Z \mathcal{F}^G \geq 3 \text{ and } \operatorname{depth}_{Z(\mathcal{Q})} \mathcal{F}^G \geq 4$$

then there is an exact sequence

$$0 \to rH^2(X, \mathcal{F}^G) \to T^2(\mathcal{S}/\mathcal{O}_X; \mathcal{F})^G \to \operatorname{Hom}_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{H}^2_Z(\mathcal{F}))^G \to rH^3(X, \mathcal{F}^G)$$

Proof. From Proposition 5.17 and Lemma 5.24 it is enough to prove that

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}^G, \mathcal{H}^3_Z(\mathcal{F}^G)) = 0$$

but this follows from the assumptions and the following lemma. □

Lemma 5.26. *Let A be a noetherian ring, M a finitely generated A-module, $I \subseteq J$ two ideals of A and suppose $\operatorname{depth}_I M \geq d$ and $\operatorname{depth}_J M \geq d+1$. If Q is an A-module with support contained in $V(J)$ then $\operatorname{Hom}_A(Q, H^d_I(M)) = 0$.*

Proof. Consider the two spectral sequences

$$E_2^{p,q} = H^p_I(\operatorname{Ext}^q_A(Q, M)) \quad \text{and} \quad {}'E_2^{p,q} = \operatorname{Ext}^p_A(Q, H^q_I(M))$$

which both converge to $\operatorname{Ext}^{p+q}_{A,I}(Q, M)$. We have $H^q_I(M) = 0$ for $q < d$ so the second spectral sequence yields $\operatorname{Hom}_A(Q, H^d_I(M)) \simeq \operatorname{Ext}^d_{A,I}(Q, M)$. On the other hand since $\operatorname{depth}_J M \geq d+1$ we have $\operatorname{Ext}^q_A(Q, M) = 0$ for $q < d+1$. Thus by the first spectral sequence $\operatorname{Ext}^d_{A,I}(Q, M) = 0$, cf. [Gro05, Exp. VII], Lemma 1.1. □

Theorem 5.27. *Assume G is a quasitorus and let (X, Z, S, J) be a G-quadruple with associated sheaf of algebras $\mathcal{S}$ and assume $\operatorname{depth}_Z \mathcal{S} \geq 2$. If*

(i) $\operatorname{depth}_Z \mathcal{O}_X \geq 3$ *and* $\operatorname{depth}_{Z(\mathcal{Q})} \mathcal{O}_X \geq 4$

(ii) $\operatorname{Hom}_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{H}^2_Z(\mathcal{S}))^G = 0$
(iii) *G is finite or $H^2(X, \mathcal{O}_X) = 0$*

and (iv) *of Theorem 5.19 hold, then $\operatorname{Def}^G_S \to \operatorname{Def}_X$ is smooth. If moreover*

(v) *G is finite or $H^1(X, \mathcal{O}_X) = 0$*

and (vi) *of Theorem 5.19 hold then $\operatorname{Def}^G_S \to \operatorname{Def}_X$ is an isomorphism.*

Proof. Apply Lemma 5.1, Proposition 5.17 and Proposition 5.25. □

If S is regular then we get a statement about the rigidity of X and more generally about rigidity of X along a sheaf $\mathcal{F}$, i.e. when $T^1_X(\mathcal{F}) = 0$. Note that if $(X, Z, S, \bar{J})$ is a G-subquadruple of (Y, W, R, J) as defined in Definition 3.10 then the vanishing of $T^1_Y(f_*\mathcal{O}_X)$ implies that the forgetful map $\operatorname{Hilb}_{X/Y} \to \operatorname{Def}_X$ is smooth. This is often useful for proving unobstructedness.

Corollary 5.28. *Assume G is a quasitorus and let (X, Z, S, J) be a G-quadruple with S a regular ring, M a finitely generated SG-module and $\mathcal{F} = \pi_*(\widetilde{M_{|X'}})$. If*

(i) $\operatorname{depth}_Z \mathcal{F} \geq 2$*,* $\operatorname{depth}_Z \mathcal{F}^G \geq 3$ *and* $\operatorname{depth}_{Z(\mathcal{Q})} \mathcal{F}^G \geq 4$
(ii) *G is finite or $H^2(X, \mathcal{F}^G) = 0$*
(iii) $\operatorname{Hom}_{\mathcal{S}}(\Omega^1_{\mathcal{S}/\mathcal{O}_X}, \mathcal{H}^2_Z(\mathcal{F}))^G = 0$
(iv) $\operatorname{Der}_k(S, H^2_J(M))^G = 0$

then $T^1_X(\mathcal{F}) = 0$. In particular if the conditions hold for $M = S$ then X is rigid.

Proof. This follows from Lemma 5.2, Lemma 5.3 and Proposition 5.25. □

Remark. By the Hochster-Roberts theorem X will be Cohen-Macaulay and equidimensional if S is a regular ring so we may exchange depth with codimension if $M = S$.

Example 5.29. *Weighted projective space.* Consider now $X = \mathbb{P}(\mathbf{q}) = \mathbb{P}(q_0, \dots, q_n)$ as described in Example 3.5. We use Corollary 5.28 to prove that *if no $n-1$ of the $q_0, \dots, q_n$ have a common factor then $\mathbb{P}(\mathbf{q})$ is rigid.*

The subscheme Z is the singular locus of X and the condition means that $\operatorname{codim} Z \geq 3$. Let $S = k[x_0, \dots, x_n]$ with $n \geq 2$. We have

$$J' = \bigcap_{\substack{p \text{ prime} \\ p \mid \operatorname{lcm}(q_0, \dots, q_n)}} (x_i : p \nmid q_i)$$

so in fact $\operatorname{codim} J' \geq 3$. The sheaf $\mathcal{Q}$ is trivial since the isotropy is finite everywhere. This takes care of condition (i).

The cohomology $H^2(X, \mathcal{O}_X) = 0$ and since $J = (x_0, \dots, x_n)$ clearly $H^2_J(S) = 0$. We are left with (iii) but as in Example 5.18 since locally the quotient is by a finite group on a smooth space, $\operatorname{depth}_Z \mathcal{S} = \operatorname{codim} Z$ and $\mathcal{H}^2_Z(\mathcal{S}) = 0$.

6. Applications to toric varieties

We collect here some results for toric varieties X_Σ over $\mathbb{C}$ that illustrate the various aspects of our comparison theorems. We consider the G-quadruple $(X_\Sigma, \operatorname{Sing}(X_\Sigma), S, B(\Sigma))$ described in Example 3.6 and will use the notation defined there. Note that toric varieties are Cohen Macaulay and S is a polynomial ring. Moreover the condition $\operatorname{depth}_Z \mathcal{S} \geq 2$ is always satisfied since $U' = \operatorname{Spec} S \setminus Z(\Sigma')$ where Σ' is the fan of smooth cones in Σ and $\operatorname{codim} Z(\Sigma') \geq 2$.

6.1. **Subschemes of simplicial toric varieties.** As explained before Corollary 4.5 a subscheme of a toric variety yields a G-subquadruple induced by a homogeneous I in the Cox ring R of the toric variety $Y = Y_\Sigma$. Let $X \subset Y$ be the subscheme. We have $Z = \operatorname{Sing}(Y) \cap X$, $S = R/I$ and $J = (I + B(\Sigma))/I$.

We assume for simplicity that Y is simplicial and all maximal cones have dimension d and that X is Cohen-Macaulay and equidimensional. This means that the quotient

$$\pi : Y' = \operatorname{Spec} R \setminus V(B(\Sigma)) \to Y$$

is a geometric quotient with all isotropy finite. In particular $\mathcal{Q} = 0$. Moreover locally on a chart $U_\sigma \subset Y$, corresponding to a maximal cone in Σ, the quotient map sits in a commutative diagram

$$\begin{array}{ccc} \mathbb{C}^d & \hookrightarrow & \mathbb{C}^d \times (\mathbb{C}^*)^r \\ \downarrow & & \downarrow \pi \\ \mathbb{C}^d/G_\sigma & \xrightarrow{\simeq} & U_\sigma \end{array}$$

where G_σ is finite abelian. See e.g. the proof of [BC94, Theorem 1.9]. If Z is closed in $X \subseteq Y$ then $\operatorname{codim} \pi^{-1}(Z) = \operatorname{codim} Z$.

Since $\mathcal{Q} = 0$ we may apply Theorem 5.20, but we still need to ensure that

$$\operatorname{Hom}_{\mathcal{S}}(\mathcal{T}or(\Omega^1_{\mathcal{S}/\mathcal{O}_X}), \mathcal{H}^2_Z(\mathcal{S}))^G = 0\,.$$

Because $\operatorname{codim} \pi^{-1}(Z) = \operatorname{codim} Z$ this will be the case if $\operatorname{codim} Z \geq 3$.

Theorem 6.1. *If X is an equidimensional Cohen-Macaulay subscheme of a simplicial toric variety Y and*

(i) $\operatorname{codim} Z \geq 3$
(ii) $H^2(X, \mathcal{O}_X) = 0$
(iii) $T^1(S/k; H^1_J(S))_0 = 0$ *and* $\operatorname{Der}_k(S, H^2_J(S))_0 = 0$

then $\operatorname{Def}^0_S \to \operatorname{Def}_X$ *is smooth. If moreover*

(i) $H^1(X, \mathcal{O}_X) = 0$
(ii) $\operatorname{Der}_k(S, H^1_J(S))_0 = 0$

then $\operatorname{Def}^G_S \to \operatorname{Def}_X$ *is an isomorphism.*

We give now two often studied situations where the theorem applies. First, let $X = \operatorname{Proj} S$ where S is a finitely generated $\mathbb{Z}_+$ graded algebra as in Example 3.5.

Corollary 6.2. *Let $X = \operatorname{Proj} S$ be an equidimensional Cohen-Macaulay subscheme of a well formed weighted projective space $\mathbb{P}(q_0, \dots, q_n)$ defined by the homogeneous ideal I. Let $\mathfrak{m}$ be the irrelevant maximal ideal of S and Z the intersection of the singular locus of $\mathbb{P}(\mathbf{q})$ with X. Assume $\operatorname{codim} Z \geq 3$ and $\operatorname{depth}_{\mathfrak{m}} S \geq 2$. If $H^2(X, \mathcal{O}_X) = 0$ and $H^1(X, \mathcal{O}_X(q_i)) = 0$ for all $i = 0, \dots, n$ then $\operatorname{Def}^0_S \to \operatorname{Def}_X$ is smooth. If moreover $H^1(X, \mathcal{O}_X) = 0$ then $\operatorname{Def}^G_S \to \operatorname{Def}_X$ is an isomorphism.*

Secondly let X be a Calabi-Yau hypersurface in a simplicial Gorenstein Fano toric variety Y. (See the survey book [CK99] or the original paper by Batyrev [Bat94] for details.) Thus Y is given by the normal fan of a simple reflexive polytope and X is a divisor in the class of $-K_Y$ and therefore ample and Cartier. In particular $\omega_X = \mathcal{O}_X$ and $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$. We refer to it as a Calabi-Yau hypersurface even though it may be highly singular.

Let D_i be the divisors corresponding to the rays ρ_i. The hypersurface X is defined by some $f \in R_\beta$ where $\beta = \sum D_i$. For our results it is not necessary to assume any generality for f. We will need the following lemma and are grateful to Benjamin Nill for supplying the proof. (We use standard toric geometry notation as found in e.g. [CLS11].)

Lemma 6.3. *Let P be a simple reflexive lattice polytope with inward normal fan Σ and let Y be the corresponding simplicial Gorenstein Fano toric variety. For every ray ρ_i in Σ the $\mathbb{Q}$-Cartier divisor $E_i = \sum_{j\neq i} D_j = -K_Y - D_i$ is nef and big.*

Proof. Let P^* be the dual reflexive polytope and v_i the primitive lattice points on ρ_i (i.e., the vertices of P^*). Let h be the piecewise linear function on $N_\mathbb{R}$ such that $h(v_i) = 0$ and $h(v_j) = -1$ for $j \neq i$. Then E_i is the divisor associated to h and $P_h = \{m \in M_\mathbb{R} : \langle m, v_i\rangle \geq 0 \text{ and } \langle m, v_j\rangle \geq -1 \text{ for } i \neq j\}$ is the corresponding polytope. Geometrically, P_h is P after moving the facet $F_i = (v_i)^*$ of P one step inwards (so that the origin now lies on the boundary of P_h). Thus $\dim P_h = \dim P$ and E_i is big.

Let σ be a maximal cone of Σ, and σ^* the corresponding vertex of P. Since Σ is simplicial, E_i is $\mathbb{Q}$-Cartier so there exists $l_\sigma \in M_\mathbb{Q}$ with $\langle l_\sigma, v\rangle = h(v)$ for $v \in \sigma$. It is well known that if l_σ is contained in P_h for all σ then E_i is nef.

There are two cases, $v_i \notin \sigma$ or $v_i \in \sigma$. If $v_i \notin \sigma$, then l_σ evaluates to -1 on any vertex of σ. In other words, $l_\sigma = \sigma^* \in P$. Thus l_σ evaluates ≥ -1 on any v_j. Moreover, since by duality $l_\sigma = \sigma^* \notin F_i$, we get that l_σ evaluates > -1 on v_i. However, P is a lattice polytope, hence l_σ as a vertex of P is a lattice point, so l_σ evaluates ≥ 0 on v_i.

Assume $v_i \in \sigma$. Since Σ is simplicial, the facet of P^* corresponding to the maximal cone σ is the convex hull of v_i and a $(d-2)$-dimensional face G of P^*. Note that by duality G^* is an edge of P, and σ^* is contained in G^* and the facet $F_i = (v_i)^*$. In particular, the intersection of the affine hull of G^* with the hyperplane H orthogonal to v_i (which is parallel to F_i) is precisely the point l_σ. Since G^* is an edge of P^*, it has a vertex w different from σ^*. Since w is not in F_i (otherwise, G^* would be contained in F_i, hence v_i would be contained in G), we have that v_i evaluates > -1 on w. Because P is a lattice polytope, w is a lattice point, so v_i evaluates ≥ 0 on w. In particular, l_σ (as it lies on the affine hull of G^* and evaluates to 0 with v_i) lies between σ^* and w. Therefore, $l_\sigma \in P$. This implies that not only l_σ evaluates to 0 on v_i, but it evaluates ≥ -1 with all other v_j's, as desired. □

Theorem 6.4. *Let X be a Calabi-Yau hypersurface in a simplicial Gorenstein Fano toric variety Y defined by $f \in R_\beta$ where $R = \mathbb{C}[x_1, \dots, x_N]$ is the Cox ring of Y and β is the anti-canonical class. If $\dim Y \geq 3$ and $\operatorname{codim}(X \cap \operatorname{Sing}(Y)) \geq 3$ in X then Def_X is smooth and its tangent space is isomorphic to the degree β part of the Jacobian algebra of f. That is*

$$T^1_X \simeq \left(\mathbb{C}[x_1, \dots, x_N]/(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N})\right)_\beta .$$

Proof. We try to apply Theorem 6.1. Since $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$ by the Calabi-Yau property, it suffices to show that $H^1_J(S)_\beta$, $H^2_J(S)_{D_i}$ and $H^1_J(S)_{D_i}$ all vanish. From the exact sequence

$$0 \to R(-\beta) \to R \to S \to 0$$

we see that this would follow from

(i) $H^1_J(R)_\beta = 0$ and $H^1_J(R)_{D_i} = 0$
(ii) $H^2_J(R)_0 \simeq H^1(Y, \mathcal{O}_Y) = 0$
(iii) $H^2_J(R)_{D_i} \simeq H^1(Y, \mathcal{O}_Y(D_i)) = 0$

(iv) $H_J^{j+1}(R)_{D_i-\beta} \simeq H^j(Y, \mathcal{O}_Y(K_Y + D_i)) = 0$ for $j = 1, 2$.

Now (i) is true because $\operatorname{depth}_J(R) \geq 2$ and (ii) is true for all complete toric varieties. To prove (iii) we have that $-K_Y$ is ample so we may e.g. apply a vanishing theorem of Mustaţă as stated in [CLS11, Theorem 9.3.7]. The last vanishing follows from the assumption that $\dim Y \geq 3$, Lemma 6.3 and the $\mathbb{Q}$-Cartier version of a vanishing result of Batyrev-Borisov as stated in [CLS11, Theorem 9.3.5]. □

6.2. **Local cohomology computations.** In the following it will be important to be able to compute with the modules $H_B^i(S)$ where S is the Cox ring of a toric variety and B the irrelevant ideal for some fan. There is a combinatorial method due to Mustaţă and we will make some simplifications in the case $i = 2$.

Let $\{m_1, \dots, m_s\}$ be monomial generators for any squarefree monomial ideal $B \subseteq S$. For $I \subseteq \{1, \dots, s\}$ let T_I be the simplicial complex on the vertex set $\{1, \dots, s\}$ where $\{j_1, \dots, j_k\}$ is a face if $x_i \nmid \operatorname{lcm}(m_{j_1}, \dots, m_{j_k})$ for some $i \in I$. If $p \in \mathbb{Z}^m$ define $\operatorname{neg}(p) \subseteq \{1, \dots, m\}$ to be the set $\{i \mid p_i \leq -1\}$. Let $\{e_1, \dots, e_m\}$ be the standard generators of $\mathbb{Z}^m$.

Theorem 6.5. ([Mus00, Theorem 2.1]) *If $p \in \mathbb{Z}^m$ then there are isomorphisms $H_B^i(S)_p \simeq \widetilde{H}^{i-2}(T_I; k)$ when $I = \operatorname{neg}(p)$. Moreover the map*

$$H_B^i(S)_p \xrightarrow{\cdot x_i} H_B^i(S)_{p+e_i}$$

corresponds to the map $H^{i-2}(T_{\operatorname{neg}(p)}; k) \to H^{i-2}(T_{\operatorname{neg}(p+e_i)}; k)$ induced in cohomology by the inclusion $T_{\operatorname{neg}(p+e_i)} \subseteq T_{\operatorname{neg}(p)}$. In particular, if $p_i \neq -1$, then $\cdot x_i$ is an isomorphism.

From now on assume S is the Cox ring and B the irrelevant ideal for a fan Σ.

Lemma 6.6. *The codimension 2 prime ideals of B are the (x_i, x_j) with ρ_i and ρ_j not in the same cone in Σ.*

Proof. This follows directly from the description of the prime components of B. See e.g. [CLS11, Proposition 5.1.6]. □

Let B_2 be the intersection of the codimension 2 primes of B. Let K be the simplicial complex which has B_2 as Stanley-Reisner ideal. Let Γ be the graph with vertices $\{0, \dots, m\}$ and edges $\{i, j\}$ when ρ_i and ρ_j are in the same cone in Σ. Define $C(\Gamma)$ to be the clique complex of Γ. Let Γ_I be the induced subgraph with vertices in I.

Lemma 6.7. *The complex K is the Alexander dual of $C(\Gamma)$. In particular there is a one to one correspondence between facets of $C(\Gamma)$ and a minimal generating set for B_2 given by $F \mapsto x_{F^c}$. This correspondence identifies T_I with the complex with vertex set equal the set of facets of $C(\Gamma)$ containing an element of I and $\{F_{i_1}, \dots, F_{i_k}\}$ is a face if $i \in \bigcap F_{i_j}$ for some $i \in I$.*

Proof. The facets of K are the complements $\{i, j\}^c$ where $\{i, j\}$ is a non-edge of Γ. We have $f \in C(\Gamma)^\vee \Leftrightarrow f^c \notin C(\Gamma) \Leftrightarrow f^c$ contains a non-edge of $\Gamma \Leftrightarrow f \subseteq \{i, j\}^c$ for a non-edge $\{i, j\}$ $\Leftrightarrow f \in K$. A set $\{x_{F_{i_1}^c}, \dots, x_{F_{i_k}^c}\}$ is a face of T_I if there is an $i \in I$ with $i \notin \bigcup F_{i_k}^c = (\bigcap F_{i_k})^c$, that is if $i \in \bigcap F_{i_k}$. □

Lemma 6.8. *If $i \in I$ let f_i be the face of T_I given by $\{F : i \in F\}$. The map $H_0(\Gamma_I) \to H_0(T_I)$ defined by $[i] \mapsto [F]$ where F is any vertex in f_i is an isomorphism.*

Proof. First note that $\{i,j\}$ is an edge in Γ_I iff there is a maximal clique F of Γ containing $\{i,j\}$ which is iff $f_i \cap f_j \neq \emptyset$. To show that the map is well defined assume $[i]=[j]$. Then there is an edge path through the vertices $i=i_0, i_1, \dots, i_r = j$ in Γ_I. Then since the $f_{i_k} \cap f_{i_{k+1}} \neq \emptyset$ we may find a path from any vertex of f_i to any vertex in f_j.

The map is clearly surjective. To show injectivity assume $F \in f_i$ and $G \in f_j$ are connected by an edge path $F = F_0, F_1, \dots, F_r = G$. This means that the intersection of cliques $F_k \cap F_{k+1}$ contains an element $i_k \in I$. This implies that there are edges $\{i_k, i_{k+1}\} \in \Gamma$ and therefore in the induced subgraph Γ_I. Thus $[i]=[j]$. □

Proposition 6.9. *There is an isomorphism* $H^2_B(S)_p \simeq \widetilde{H}^0(\Gamma_I, k)$ *where* $I = \operatorname{neg}(p)$.

Proof. This follows now directly from Theorem 6.5, Lemma 6.7 and 6.8. □

6.3. T^1 **for toric singularities.** There has been much interest in computing deformations of affine toric varieties. In particular there are several combinatorial descriptions of these due to Altmann, see e.g. [Alt00] and the references therein. We give here an alternative description using the Cox ring and compute T^1 in some special cases. To simplify matters we will assume $\operatorname{codim}\operatorname{Sing}(X) \geq 3$.

Let X be an affine toric variety with cone σ and Cox ring S. In the grading defined by the abelian group $\operatorname{Cl}(X)$ we have $X = \operatorname{Spec} S_0$. Set $Z = \operatorname{Sing}(X)$. Thus $(X, Z, S, (1))$ is an affine G-quadruple where $G = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Cl}(X), \mathbb{C}^*)$. The associated sheaf of algebras is just S itself. If Σ is the fan consisting of smooth faces of σ and $B = B(\Sigma)$ the corresponding irrelevant ideal then the sheaves $\mathcal{H}^i_Z(\mathcal{S})$ correspond to the modules $H^i_B(S)$.

Let $n = \dim X = \operatorname{rk} N$, $m = |\Sigma(1)|$ and $r = \operatorname{rk}\operatorname{Cl}(X) = m - n$. Let $C \subseteq \operatorname{Cl}(X)$ be the free part of the abelian group. Fix an isomorphism $C \simeq \mathbb{Z}^r$ so that the map $\mathbb{Z}^m \to C$ has matrix $A = (a_{ij})$. The $\mathbb{Z}^r$ part of the $\operatorname{Cl}(X)$ grading on $S = \mathbb{C}[x_1, \dots, x_m]$ is given by the columns of A, i.e. $\deg x_j = (a_{1j}, \dots, a_{rj})$. A set of Euler derivations as in Definition 5.8 is

$$\{E_i = \sum_j a_{ij} x_j \frac{\partial}{\partial x_j} : i = 1, \dots, r\}\,.$$

Thus the module Q corresponding to the sheaf $\mathcal{Q}$ in Definition 5.8 has a graded presentation

$$\bigoplus_{j=1}^{m} S(-\deg x_j) \xrightarrow{\tilde{A}} S^r \to Q \to 0 \tag{6.1}$$

where $\tilde{A} = (a_{ij} x_j)$. The support of Q on X is the non-simplicial locus corresponding to the irrelevant ideal $B(Q) \subset S$ for the fan consisting of simplicial faces of σ.

Theorem 6.10. *Let X be an affine toric variety.*

(i) *If X is simplicial then* $\operatorname{Hom}_S(\Omega^1_{S/S_0}, H^2_B(S))_0 \simeq T^1_X$.

(ii) *If* $\operatorname{codim}\operatorname{Sing}(X) \geq 3$*, then there is an exact sequence*

$$0 \to \operatorname{Hom}_S(\Omega^1_{S/S_0}, H^2_B(S))_0 \to T^1_X \to \operatorname{Hom}_S(Q, H^3_B(S))_0 \to 0$$

and inclusions $\operatorname{Hom}_S(Q, H^3_B(S))_0 \subseteq \operatorname{Hom}_{S_0}(Q_0, H^3_{B_0}(S_0)) \subseteq H^3_{B_0}(S_0)^r$.

Proof. As in the proof of Lemma 4.1, $T^1_X = T^1_{S_0} = T^1_{S_0}(S)_0$. The Zariski-Jacobi sequence for $\mathbb{C} \to S_0 \to S$ and the fact that S is regular yields $T^1_{S_0}(S) \simeq T^2_{S/S_0}$. Furthermore by Lemma 5.10 we have $T^2_{S/S_0} \simeq \operatorname{Ext}^2_S(\Omega^1_{S/S_0}, S)$. The result, except for the 0 on the right in the exact sequence in (ii), now follows by applying the proof of Proposition 5.17 to the affine case.

Recall from the proof of Proposition 5.17 that the cokernel of $\mathrm{Hom}_S(\Omega^1_{S/S_0}, H^2_B(S)) \to \mathrm{Ext}^2_S(\Omega^1_{S/S_0}, S)$ is the kernel K of $H^3_B(S)^r \to \mathrm{Ext}^3_{S,B}(\Omega^1_{S/S_0}, S)$ in the long exact sequence for $\mathrm{Ext}^i_{S,B}(\Omega^1_{S/S_0}, S)$. We have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & & & & & \mathrm{Ext}^1_S(\Omega^1_{S/S_0}, H^2_B(S)) \\
 & & & & & & \downarrow \\
0 & \longrightarrow & K & \longrightarrow & H^3_B(S)^r & \xrightarrow{u} & \mathrm{Ext}^3_{S,B}(\Omega^1_{S/S_0}, S) \\
 & & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle v} \\
0 & \longrightarrow & \mathrm{Hom}_S(Q, H^3_B(S)) & \longrightarrow & H^3_B(S)^r & \longrightarrow & \mathrm{Hom}_S(\Omega^1_{S/S_0}, H^3_B(S))
\end{array}$$

where the right column comes from the spectral sequence for $\mathrm{Ext}^i_{S,B}(\Omega^1_{S/S_0}, S)$. By assumption $H^2_B(S)_0 = 0$ so by the below Lemma 6.11, $\mathrm{Ext}^1_S(\Omega^1_{S/S_0}, H^2_B(S))_0 = 0$ and $K \simeq \mathrm{Hom}_S(Q, H^3_B(S))$. □

Lemma 6.11. *Let G be a linearly reductive group acting on an n-dimensional k vector space V. Let $A = \mathrm{Sym}(V)$ and let N be an AG-module with $N^G = 0$. Then $\mathrm{Ext}^1_A(\Omega^1_{A/A^G}, N)^G = 0$.*

Proof. If

$$A^G = k[\varphi_1, \dots, \varphi_m] \subseteq A = k[x_1, \dots, x_n]$$

then the beginning of an AG projective resolution of Ω^1_{A/A^G} looks like

$$\cdots \to A^m \xrightarrow{\left(\frac{\partial \varphi_j}{\partial x_i}\right)} A \otimes_k V^* \to \Omega^1_{A/A^G} \to 0\,.$$

If M is the image of A^m in $A \otimes_k V^*$ then M is generated by invariants, so $\mathrm{Hom}_A(M, N)^G = 0$. Thus $\mathrm{Ext}^1_A(\Omega^1_{A/A^G}, N)^G = 0$ by the long exact sequence for Ext. □

We can ask when the contribution from $\mathrm{Hom}_S(\Omega^1_{S/S_0}, H^2_B(S))_0 = \mathrm{Der}_{S_0}(S, H^2_B(S))_0$ vanishes. This would happen if $H^2_B(S)_{\alpha_i} = 0$ for $i = 1, \dots, m$ where α_i is the degree of x_i. Lifting this to the $\mathbb{Z}^m$ grading we need $H^2_B(S)_p = 0$ when $p = q + e_i$ and $q = (\dots, \langle u, v_j \rangle, \dots)$ for some $u \in M$. If we assume $\mathrm{codim}\, \mathrm{Sing}(X) \geq 3$ then $H^2_B(S)_0 = H^2_{\mathrm{Sing}(X)}(S_0) = 0$. In the $\mathbb{Z}^m$ grading this means $H^2_B(S)_q = 0$ when q is as above. Thus by Theorem 6.5 it is enough to check when the coordinate $\langle u, v_i \rangle$ of q equals -1.

Let Γ^f be the graph with vertices $\{0, \dots, m\}$ and edges $\{i, j\}$ when $\{\rho_i, \rho_j\}$ span a 2-dimensional face of σ and Γ^f_I the induced subgraph as above. If $\mathrm{codim}\, \mathrm{Sing}(X) \geq 3$ then Γ^f_I is a subgraph of $\Gamma_I(\Sigma)$ with the same vertex set so the map $\widetilde{H}_0(\Gamma^f_I) \to \widetilde{H}_0(\Gamma_I)$ is surjective. If $u \in M$ and $\langle u, v_i \rangle = -1$ let

$$I_i(u) = \{j \in \{1, \dots, m\} : \langle u, v_j \rangle \leq -1\} \setminus \{i\}$$

and set $\Gamma_i(u) = \Gamma^f_{I_i(u)}$. (These graphs also appear in the study of deformations of smooth complete toric varieties in [Ilt11].) Thus we have

Proposition 6.12. *Let X be an affine toric variety that is smooth in codimension 2 and $\Gamma_i(u)$ is connected for all $u \in M$. Then $\mathrm{Hom}_S(\Omega^1_{S/S_0}, H^2_B(S))_0 = 0$.*

The connectivity of these graphs can be proven if there is a polytope we can use. For this we need the following lemma.

Lemma 6.13. *Let P be a polytope, H^* a closed half-space with bounding hyperplane H and v a vertex of P with $v \in H$. If $\Gamma_v(H)$ is the induced subgraph of the edge graph of P on the vertices in*

$$\{w \in \operatorname{vert} P : w \in H^*\} \setminus \{v\}$$

then $\Gamma_v(H)$ is connected.

Proof. If $H \cap \operatorname{int} P \neq \emptyset$ let H' be the supporting hyperplane in H^* that is parallell to H. By standard arguments every vertex in $\Gamma_v(H)$ is either on H' or has a neighbor vertex in H^* that is nearer to H'. Thus $\Gamma_v(H)$ is connected in this case.

If $H \cap \operatorname{int} P = \emptyset$ then H is a supporting hyperplane. If $P \subset H^*$ then $\Gamma_v(H)$ is the edge graph of P with one vertex removed. Since edge graphs of d-polytopes are d-connected $\Gamma_v(H)$ is connected (trivially also for $d = 0, 1$). If $P \cap H^* = F$ is a face then the same argument applied to F yields the result. □

Proposition 6.14. *A $\mathbb{Q}$-Gorenstein affine toric variety that is smooth in codimension 2 has $\operatorname{Hom}_S(\Omega^1_{S/S_0}, H^2_B(S))_0 = 0$.*

Proof. An affine toric variety is $\mathbb{Q}$-Gorenstein if and only if there is a primitive $u_0 \in M$ and a positive integer g such that $\langle u_0, v_i \rangle = g$ for all generators v_i of the rays of σ. This means that if P is the convex hull of the v_i in the hyperplane $\langle u_0, - \rangle = g$, then for all $u \in M$ the graph $\Gamma_i(u)$ is a $\Gamma_v(H)$ as in Lemma 6.13. □

Example 6.15. We compute T^1_X for a non-simplicial toric 3-dimensional Gorenstein isolated singularity. Note that it follows from Theorem 6.10 and Proposition 6.14 that a toric d-dimensional $\mathbb{Q}$-Gorenstein isolated singularity is rigid if $d \geq 4$. (For a complete description of T^1 for all toric $\mathbb{Q}$-Gorenstein singularities see [Alt95] and [Alt00].) From Theorem 6.10 and Proposition 6.14 it follows that $T^1_X \simeq \operatorname{Hom}(Q, H^3_B(S))_0$.

The cone σ has a special form as it comes from a plane lattice polygon P with smooth face fan (and more than 3 vertices). If $(\alpha_1, \beta_1), ..., (\alpha_m, \beta_m)$ are the vertices of such a P then the primitive generators of the rays of σ are $v_i = (\alpha_i, \beta_i, 1)$.

The fan Σ consists of all faces of σ except σ itself. Using the description of the prime components in [CLS11, Proposition 5.1.6] one sees that $B = B_2$ in the notation Section 6.2. Thus we may use the methods there to show that $H^3_B(S)_p \simeq H^1(\Gamma_I, k)$ where $I = \operatorname{neg}(p)$. Now it is easy to see that $H^3_B(S)_p \neq 0$ if and only if all $p_i \leq -1$. Note that the Gorenstein property can be seen through the one to one correspondence $p \mapsto -p - (1, \ldots, 1)$ between non-zero pieces of $H^3_B(S)_0$ and S_0.

In our situation $\operatorname{Cl}(X) \simeq \mathbb{Z}^r$ so there is an exact sequence

$$0 \to \mathbb{Z}^3 \xrightarrow{A'} \mathbb{Z}^m \xrightarrow{A} \mathbb{Z}^r \to 0$$

where the rows of A' are $[\alpha_i, \beta_i, 1]$ and $A = (a_{ij})$ gives the set of Euler derivations. Using the presentation of Q in (6.1) we may describe the $\mathbb{Z}^m$ graded pieces of $\operatorname{Hom}(Q, H^3_B(S))_0$ as follows. It is convenient to use the above one-to-one correspondence so set $q = -p - (1, \ldots, 1)$. If $\operatorname{Hom}(Q, H^3_B(S))_{0,p} \neq 0$ then $A \cdot q = 0$ and $\operatorname{neg}(q) = \emptyset$. If this is satisfied then $\operatorname{Hom}(Q, H^3_B(S))_{0,p}$ is isomorphic to the kernel of the submatrix A^t_q consisting of the rows A^t_i of A^t where $q_i > 0$.

From the smoothness of P it follows that *all* the 3×3 minors of A' are non-zero. Thus A has the special property:

(6.2) All $r \times r$ minors of A are non-zero.

Let k be the number of i with $q_i \neq 0$. If $k \geq r$ then (6.2) implies that A_q^t has trivial kernel. On the other hand if $1 \leq k < r$ let $q' = (q_{i_1}, \dots, q_{i_k})$ be the vector of non-zero components. Since $Aq = 0$ we must have $(A_q^t)^t q' = 0$, but this is impossible since by (6.2) the columns of $(A_q^t)^t$ are independent.

We conclude that the only non-zero $\operatorname{Hom}(Q, H_B^3(S))_{0,p}$ is when $p = (-1, \dots, -1)$ and that $\dim T_X^1 = \dim \operatorname{Hom}(Q, H_B^3(S))_0 = r = m - 3$.

6.4. **Rigidity results.** The above together with Corollary 5.28 yield rigidity results for toric singularities. First we reprove a theorem of Altmann in [Alt95].

Corollary 6.16. *A $\mathbb{Q}$-Gorenstein affine toric variety that is smooth in codimension 2 and simplicial in codimension 3 is rigid.*

Proof. This follows directly from Corollary 5.28 and Proposition 6.14. □

With the same type of arguments we can reprove Totaro's generalization, [Tot12, Theorem 5.1] of theorems of Bien-Brion and de Fernex-Hacon.

Corollary 6.17. *A toric Fano variety that is smooth in codimension 2 and simplicial in codimension 3 is rigid.*

Proof. Corollary 6.16 takes care of the local situation so what is left to prove is condition (iv) in Corollary 5.28. This follows from $H^2_{B(\Sigma)}(S)_{D_i} = H^1(X, \mathcal{O}_X(D_i) = 0$ since X is Fano. Alternatively, the Fano condition implies that the fan Σ is the face fan of a polytope. Thus the same arguments as above show that $H^2_{B(\Sigma)}(S)_{D_i} = 0$. □

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Matematisk institutt, Postboks 1053 Blindern, University of Oslo, N-0316 Oslo, Norway
E-mail address: christop@math.uio.no

Oslo Metropolitan University, Faculty of Technology, Art and Design, PB 4 St. Olavs plass, N-0130 Oslo, Norway
E-mail address: JanOddvar.Kleppe@oslomet.no
URL: https://www.cs.hioa.no/~jank/