Halphen Gaps and Good Space Curves

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Jan O. Kleppe

1 Main Results

Let $\mathrm{H}(d,g)_S$ be the Hilbert scheme of smooth connected space curves. In this paper we state the

Conjecture 1.1. *Let* $s \geq 4$, $d \geq \binom{s+2}{2}$ *and* $g \geq sd - \binom{s+3}{3} + 2$. *If* $g \leq G_0(d,s)$, *then there exists a* generically smooth irreducible component V *of* $\mathrm{H}(d,g)_S$ *of dimension* $\dim V = (4-s)d + g + \binom{s+3}{3} - 2$ *whose general curve has maximal corank and sits on a smooth surface of degree* s *in* $\mathbb{P}^3$.

Here a curve C (with sheaf ideal $\mathcal{I}_C$) has maximal corank if $H^1(\mathcal{I}_C(v)) = 0$ whenever $H^2(\mathcal{I}_C(v)) \neq 0$. Moreover $G_0(d,s)$ was introduced by C. Walter in [2] where he **conjectured** that there exists curves C in $\mathrm{H}(d,g)_S$ sitting on a smooth degree-s surface in $\mathbb{P}^3$ (whence (d,g,s) is not an Halphen gap) in the range

(1.2) $$(s-4)d + 1 - \binom{s-1}{3} \leq g \leq G_0(d,s)\ ,\ d \geq \binom{s-1}{2} \text{ and } s \geq 4.$$

Using [1] we easily get Conjecture 1.1 for $s = 4$ from [1, Prop. 2.1]. In this case $G_0(d,4)$ is 1 (for $d \not\equiv 0 \bmod 4$), otherwise 2 less that the maximum genus in range C. Note that Walter proves (1.2) for $s = 4$ with C of maximal corank by extending Mori's well known existence result for curves on a smooth quartic. In [1] we also show the existence of generically smooth irreducible components of $\mathrm{H}(d,g)_S$ of dimension $d+g+18$ whose general curve C sits on a smooth cubic surface in the range

$$3d - 18 + (d-12)(d-15)/18 \leq g \leq 1 + d(d-3)/6\ ,\quad (d,g) \notin \{(30,91),(33,103),(34,109)\},$$

and we characterize those C that are not of maximal corank, see [1, Prop. 3.1 and 3.5].

In this paper [K98] we show Conjectures 1.1 and (1.2) for every $s \geq 4$, $d \geq \binom{s}{2}$ and $g \geq (s-3)d + 1 - \binom{s}{3}$ provided $G_1'(d,s) < g \leq G_0(d,s)$ where $g = G_1'(d,s)$, for fixed s, is a piecewise linear curve in the (d,g)-plane defined in [K98, Rem. 3.5]. Using the upper bound of $G_1'(d,s)$ given in Rem. 3.5, we get

$$G_0(d,s) - G_1'(d,s) \geq s^2(s-4)/32.$$

Since one may similarly prove $G_0(d,s) - G_1'(d,s) \leq s(s+2)^2/32$, the range where we prove both conjectures is asymptotically $s^3/32$. This result is a consequence of a more general result (see [K98, Thm. 2.9]), showing that we may link curves of maximal rank ($H^1(\mathcal{I}_C(v)) = 0$ for $v \geq s(C) :=$ minimal degree of a surface $\supset C$) with $H^2(\mathcal{I}_C(s(C)-1)) = 0$ and $H^1(\mathcal{N}_C) = 0$ ($\mathcal{N}_C$ the normal sheaf) to curves belonging to a unique component V satisfying Conjecture 1.1 by using a complete intersection of e.g. bidegree $(s(C)+1,t)$ for $t \geq s(C)+4$. Such curves are supposed to exist for every (d,g) in the A' or FW-range (but complete proofs seem missing). If we, however, suppose that such curves exist everywhere in the FW-range, the Conjectures 1.1 and (1.2) will hold in an even larger range, see Theorem 3.3 and Cor. 3.4 of [K98].

References

[1] J.O. Kleppe. On the existence of Nice Components in the Hilbert Scheme $\mathrm{H}(d,g)$ of Smooth Connected Space Curves. *Boll. U.M.I.* (7) 8-B (1994), 305-326.

[2] C. Walter. Curves on Surfaces with a Multiple Line. *J. reine angew. Math.* 412 (1990), 48-62.