Linear algebra - repetisjon

$A \in \mathbb{R}^{m \times n}$: $m \times n$-matrise

$$A = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

m rader

n kolonner

$\boxed{\text{Addisjon :}}$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$$

Oppgave:

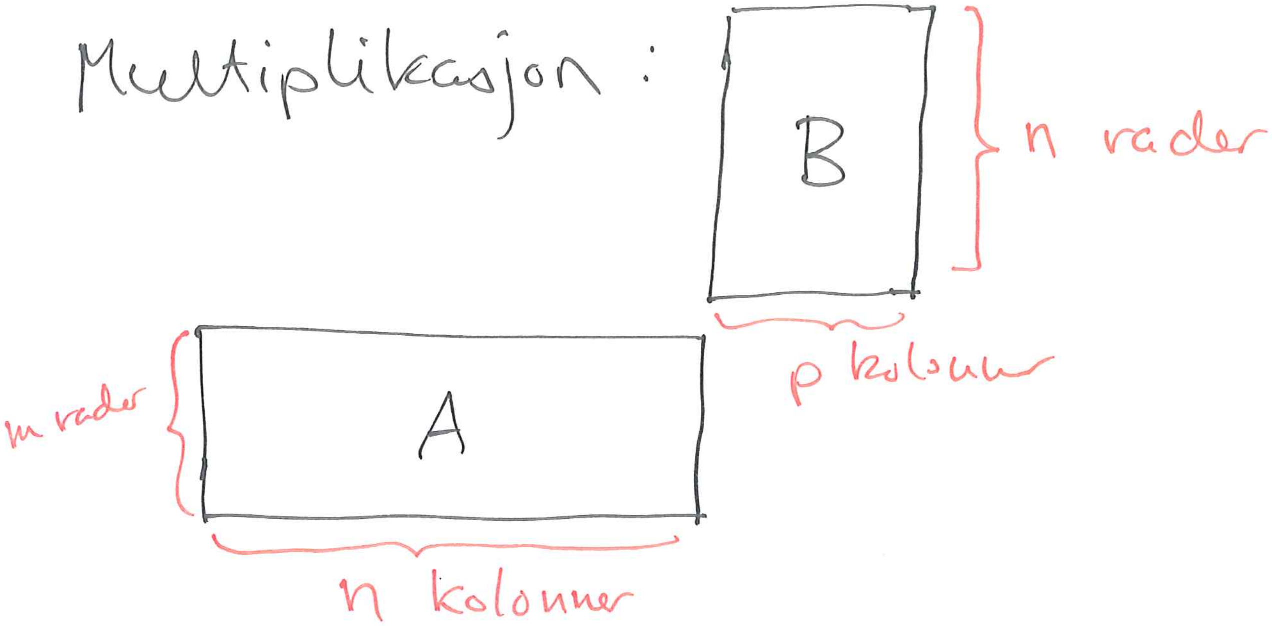
Regn ut $A + B$ og $A + C$, hvis mulig.

$$A + B = \begin{bmatrix} 1+3 & 0+2 \\ 2+1 & 3+1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 4 & 2 \\ 3 & 4 \end{bmatrix}}}$$

2x2-matrise 3x2-matrise

$A + C$: Ikke mulig!

Multiplikasjon :



AB kan beregnes!

$m \times n \qquad n \times p$

Like!

Må kreve :

Antall kolonner i A

= Antall rader i B.

Resultatet AB blir

$m \times p$ - matrise!

$\boxed{\text{Transponering:}}$

A^T hvor A er $m \times n$-matrise

er en $n \times m$-matrise:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1m} & \cdots & a_{nm} \end{bmatrix}, \quad A^T = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

Eksempel:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$$

Regn ut AC^T og CB^T.

$$C^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$A : 2 \times 2 \,, \quad C^T : 2 \times 3$$

Like

2×3

$$AC^T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 1 + 0 \cdot 2 & 1 \cdot 0 + 0 \cdot 3 & 1 \cdot 0 + 0 \cdot 1 \\ 2 \cdot 1 + 3 \cdot 2 & 2 \cdot 0 + 3 \cdot 3 & 2 \cdot 0 + 3 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 8 & 9 & 3 \end{bmatrix}$$

Ligningssystemer

$$2x + 1 \cdot y = b$$

$$1 \cdot x + a y = 1$$

$\odot$ = koeffisient~~matrisen~~er

$$A = \begin{bmatrix} 2 & 1 \\ 1 & a \end{bmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} b \\ 1 \end{pmatrix}$$

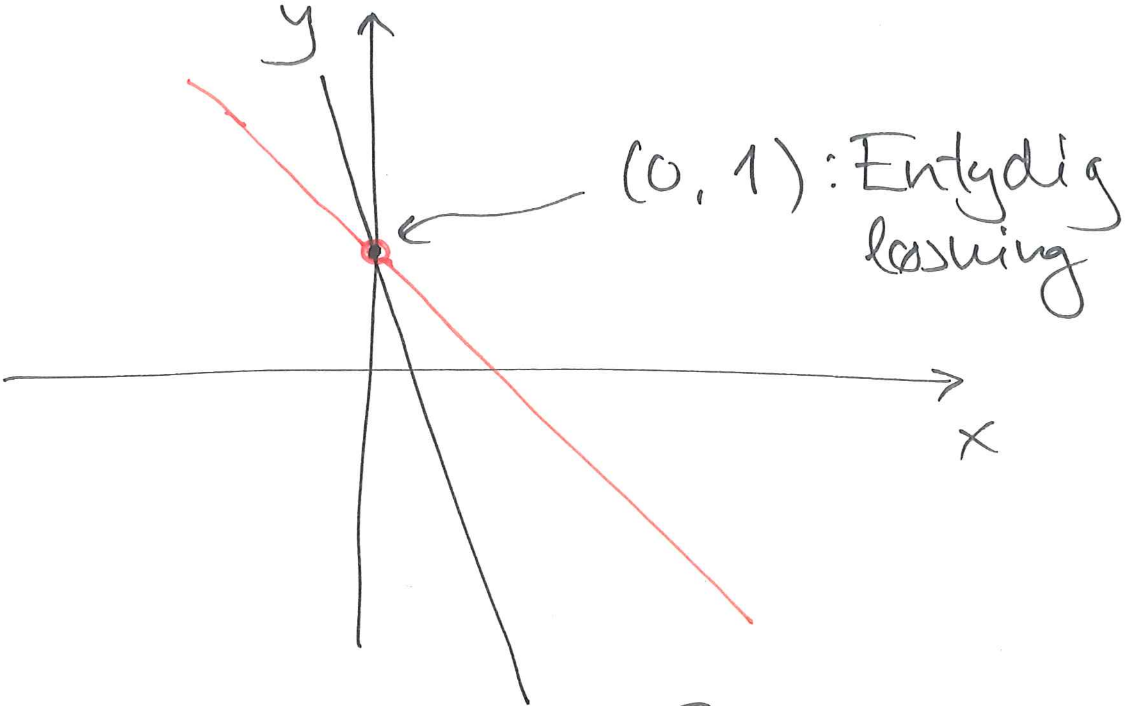
Total-matrise:

$$\left[\begin{array}{cc|c} 2 & 1 & b \\ 1 & a & 1 \end{array}\right]$$

Sett $a = b = 1$:

$$2x + y = 1 \Leftrightarrow y = 1 - 2x$$

$$x + y = 1 \Leftrightarrow y = 1 - x$$



$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] -\tfrac{1}{2} \sim \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} \end{array}\right] \cdot 2$$

$$\sim \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right] -1 \sim \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \cdot \tfrac{1}{2}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

Dvs $\boxed{\begin{array}{l} x = 0 \\ y = 1 \end{array}}$

$$\left[\begin{array}{cc|c} 2 & 1 & b \\ 1 & a & 1 \end{array}\right] \quad \left(\begin{array}{l} 2x+y=b \\ x+ay=1 \end{array}\right)$$

Hvilke krav må vi stille til a og b for at vi skal ha

1. entydig løsning
2. ingen løsning
3. uendelig mange løsninger?

$$\left[\begin{array}{cc|c} 2 & 1 & b \\ 1 & a & 1 \end{array}\right] \xleftarrow{-\frac{1}{2}} \sim \left[\begin{array}{cc|c} 2 & 1 & b \\ 0 & a-\frac{1}{2} & 1-\frac{1}{2}b \end{array}\right]$$

$$\left(a-\frac{1}{2}\right)y = 1-\frac{1}{2}b$$

1. $a-\frac{1}{2} \neq 0$: $y = \dfrac{1-\frac{1}{2}b}{a-\frac{1}{2}}$

$$2x+y=b \Rightarrow \cancel{x=\tfrac{1}{2}(y-b)} = \frac{1}{2}\left(\frac{1-\frac{1}{2}b}{a-\frac{1}{2}} - b\right)$$

$$x = \frac{b-y}{2} = \frac{1}{2}\left(b - \frac{1-\frac{1}{2}b}{a-\frac{1}{2}}\right)$$

Løsningen er $\underline{\text{entydig når } a \neq \tfrac{1}{2}}$.

2. $a = \frac{1}{2}$:

$$\left(a - \tfrac{1}{2}\right)y = 1 - \tfrac{1}{2}b$$

$$\underbrace{0 \cdot y}_{=0} = \underbrace{1 - \tfrac{1}{2}b}_{\neq 0}$$

$$Vs \neq Hs$$

Dvs : Ingen løsning $\underbrace{}_{}$ $b \neq 2$

når $a = \frac{1}{2}$ og $1 - \frac{1}{2}b \neq 0$.

(Vi sier at systemet er <u>inkonsistent</u>)

3. $a = \frac{1}{2}$ og $1 - \frac{1}{2}b = 0$ $(b = 2)$:

$$\left(a - \tfrac{1}{2}\right)y = 1 - \tfrac{1}{2}b$$

$$\underbrace{0y}_{=0 \text{ uansett } y} = 0$$

$$Vs = Hs$$

for alle verdier av y.

Dvs: Uendelig mange løsninger.

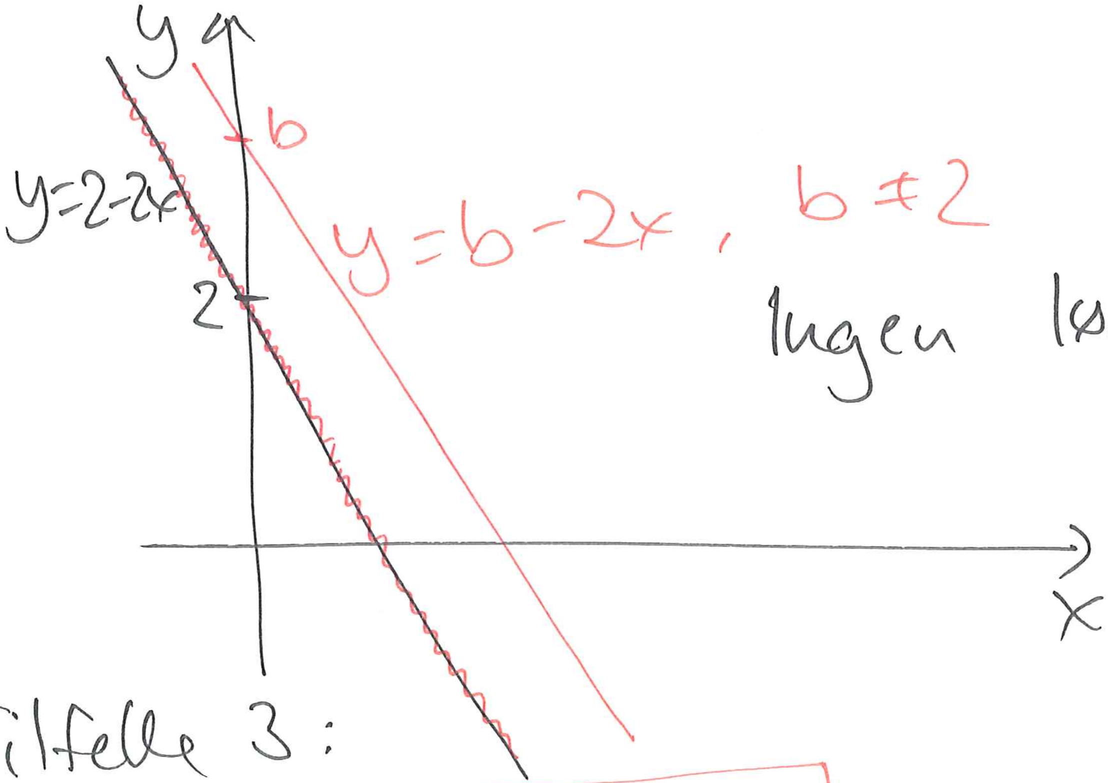
Tilfelle 2:

$$\begin{cases} \boxed{2x + y = b} , \quad b \neq 2 \\ x + \frac{1}{2}y = 1 \end{cases}$$

$$\boxed{2x + y = 2}$$

$$y = b - 2x$$

$$y = 2 - 2x \quad , b \neq 2$$



Ingen løsninger!

Tilfelle 3:

$b = 2$: $\boxed{y = 2 - 2x}$

∞ mange løsninger.

Identitets- og inversmatrisene

Når vi har $a \in \mathbb{R}$ er $a\frac{1}{a} = 1, a \neq 0$

$\Leftrightarrow \quad a a^{-1} = 1$

A en $\overbrace{\text{}}^{n \times n}$ matrise : $\left.\begin{aligned} AA^{-1} &= 1_n \\ A^{-1}A &= 1_n \end{aligned}\right\}$ Dersom A^{-1} eksisterer.

$$1_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{bmatrix}.$$

Vi kan løse $A\vec{x} = \vec{b}$ ved hjelp av A^{-1}:

$$\underbrace{A^{-1}A}_{1_n}\vec{x} = A^{-1}\vec{b} \qquad 1_n\vec{x} = \vec{x}$$

$$\underline{\vec{x} = A^{-1}\vec{b}}$$

$$[A \,\vdots\, 1_n] \sim [1_n \,\vdots\, A^{-1}]$$

Radoperasjoner

Eksempel

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Regn ut A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} -2 \\ \\ \\ \end{matrix} \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ -1 \\ \\ \end{matrix} \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] \begin{matrix} \\ -1 \\ \\ \end{matrix} \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

Inverterbarhet. Følgende utsagn er ekvivalente:

1. A er inverterbar (A^{-1} eksisterer)
2. A kan radreduseres til I_n
3. $A\vec{x} = \vec{b}$ har entydig løsning
4. $A\vec{x} = \vec{0}$ har kun løsningen $\vec{x} = \vec{0}$.
5. Kolonnevektorene til A er lineært uavhengige.
6. $\det A \neq 0$.

Determinanter

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Radreduksjon av A til øvre triangulær matrise B:

$$\det A = (-1)^{\text{ant. radbytter}} \underbrace{\det B}_{}$$

Produktet av diagonal-elementer

Eksempel 1

Vis at $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

er inverterbar.

Dersom $\det A \neq 0$, så er den inverterbar.

$$\det A = 1 \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} + 0$$

$$= 1(3 \cdot 1 - 0 \cdot 1) - 1(2 \cdot 1 - 0 \cdot 0)$$

$$= 3 - 2 = 1 \neq 0.$$ Inverterbar!

Eksempel 2

Vis at $B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ ikke

er inverterbar. Uttrykk kolonne-vektorene ved hverandre.

$$\det B = 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + 0$$

$$= 1(1-(-1)) - 1(2\cdot 1 - 0(-1))$$

$$= 2 - 2 = \underline{0}.$$

Dvs: B ikke inverterbar!

Kolonnevektorer:

$$x_1\begin{pmatrix}1\\2\\0\end{pmatrix} + x_2\begin{pmatrix}1\\1\\1\end{pmatrix} + x_3\begin{pmatrix}0\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + x_2 - x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

$$B\vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \overset{-2}{} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \overset{1}{}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$B\vec{x} = \vec{0}$:

$0 \cdot x_3 = 0$: x_3 fri variabel

$-x_2 - x_3 = 0$

$$x_2 = -x_3$$

$x_1 + x_2 = 0$

$$x_1 = -x_2 = x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$