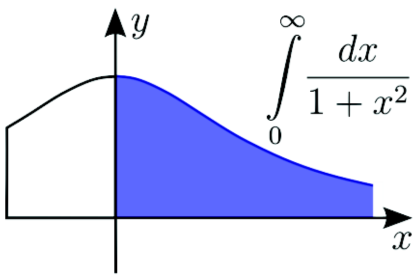
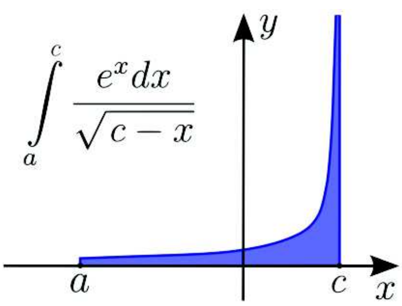
Repetisjon fra forelesning 14. mars

Uegentlige integraler (5.5)

Type 1:
Integrasjonsintervallet er ubegrenset



Type 2:
Integranden er ubegrenset



Antiderivasjon

Integralet av spesielle funksjoner

5	$\int a\,dx$	$=$	$ax + C$	Integrasjon av konstant
6	$\int x^r\,dx$	$=$	$\frac{1}{r+1}x^{r+1} + C$	$r \neq -1$ (Se 11 for $r = -1$)
7	$\int \cos(x)\,dx$	$=$	$\sin(x) + C$	
8	$\int \sin(x)\,dx$	$=$	$-\cos(x) + C$	
10	$\int e^x\,dx$	$=$	$e^x + C$	
11	$\int x^{-1}\,dx$	$=$	$\ln(\lvert x\rvert) + C$	
12	$\int \frac{1}{\sqrt{1-x^2}}\,dx$	$=$	$\arcsin(x) + C$	
13	$\int \frac{1}{1+x^2}\,dx$	$=$	$\arctan(x) + C$	

Integrasjon ved substitusjon

Integral på formen

$$\int f(u(x))u'(x)dx$$

Ved å identifisere $u(x)$ kan dette skrives

$$\int f(u)du$$

Oppskrift:

1) Velg $u(x)$.
2) Siden $u'(x) = du/dx$, erstattes dx med $du/u'(x)$
3) x-ene som er igjen i integranden skrives om til et uttrykk i u og det resulterende integralet løses (hvis mulig, ellers gå tilbake til (1) og prøv på nytt)

Delvis integrasjon

Kan brukes dersom vi har integral på formen

$\int u(x)v'(x)dx$

Produktregelen for derivasjon gir at

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

Hensikt: Velg $u(x)$ og $v'(x)$ slik at

$$\int u'(x)v(x)dx$$

er lettere å løse enn

$$\int u(x)v'(x)dx$$

Oppgaver

1. Regn ut $\int_0^\infty \frac{1}{1+4x^2}dx$. Hint: $\int \frac{1}{1+x^2}dx = \arctan x$

2. Regn ut $\int \frac{2x}{1+x^2}\ dx$

3. Regn ut $\int x \sin x\ dx$

Oppgave 1

$$\int_0^\infty \frac{1}{1+4x^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{1+4x^2}\,dx$$

$$\underset{\substack{\uparrow \\ 4x^2=(2x)^2}}{=} \lim_{b\to\infty} \int_0^b \frac{1}{1+(2x)^2}\,dx$$

$u = 2x$
$u' = 2$
$dx = \frac{du}{u'(x)} = \frac{du}{2}$

$$\begin{cases} x=0 \rightarrow u = 2\cdot 0 = 0 \\ x = b \rightarrow u = 2b \end{cases}$$

Integrasjons-grenser

$$= \lim_{b\to\infty} \int_0^{2b} \frac{1}{1+u^2}\,\frac{du}{2}$$

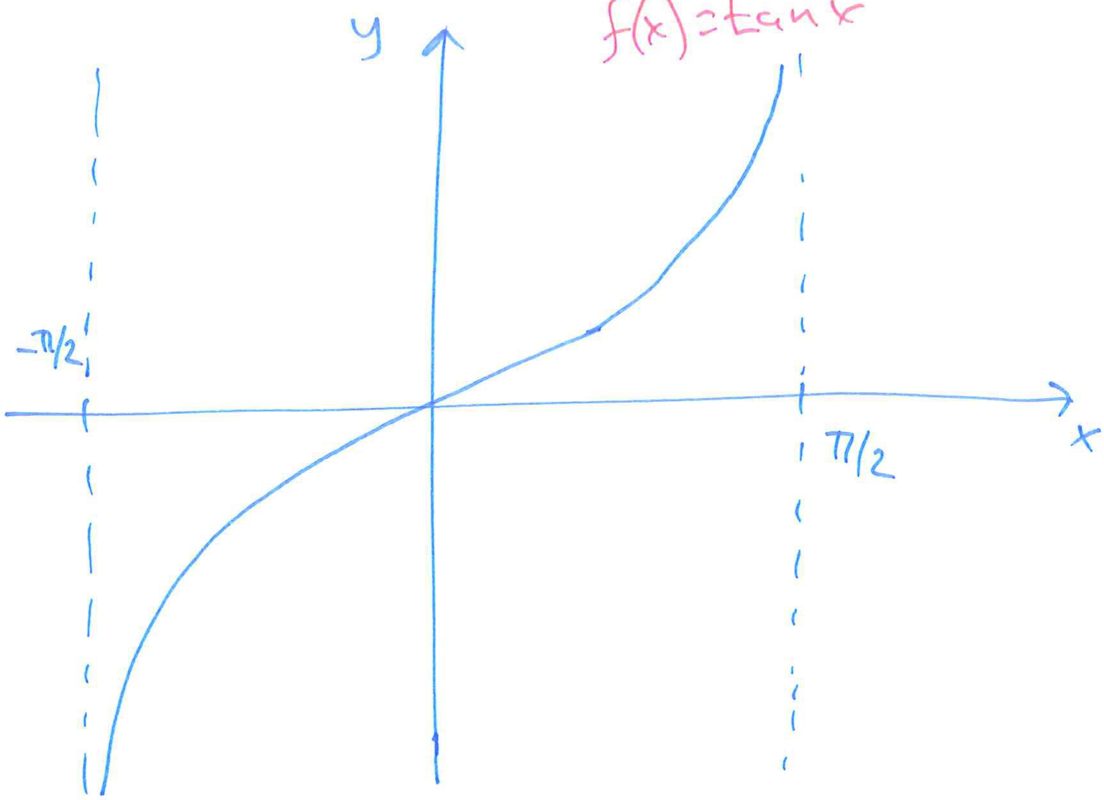
$$= \frac{1}{2}\lim_{b\to\infty} \int_0^{2b} \frac{1}{1+u^2}\,du$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

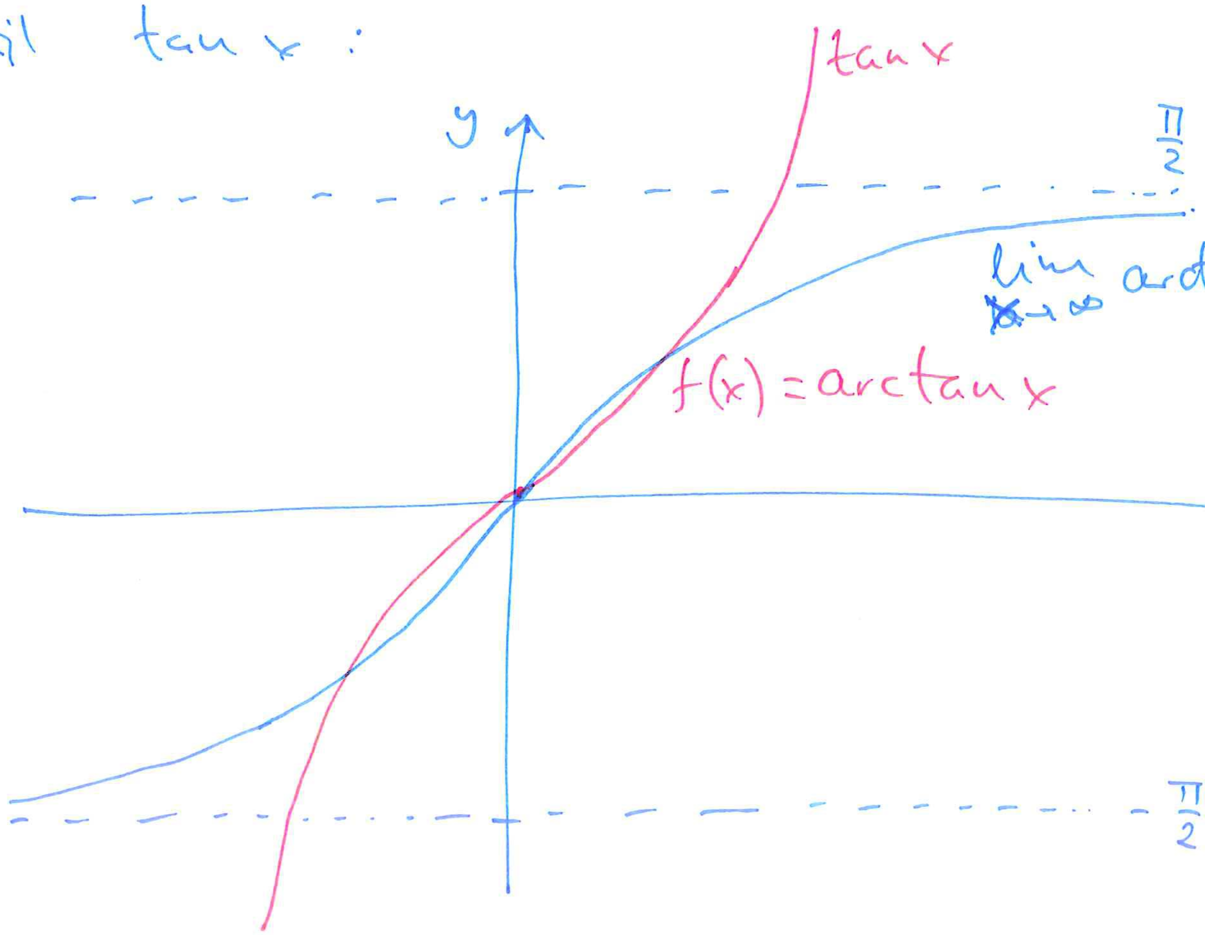
$$= \frac{1}{2}\lim_{b\to\infty} \Big[\arctan u\Big]_0^{2b}$$

$$= \frac{1}{2}\lim_{b\to\infty} \Big(\arctan(2b) - \underbrace{\arctan 0}_{=0}\Big)$$

$$= \frac{1}{2}\,\frac{\pi}{2} = \underline{\underline{\frac{\pi}{4}}}$$



arctan x er den omvendte funksjonen til tan x:



$$\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$$

Oppgave 2

$$\int \frac{2x}{1+x^2}\,dx$$

Naturlig å velge $u(x) = \underline{1+x^2}$.

$$u'(x) = 2x$$

Erstatter dx med $\frac{du}{u'(x)} = \frac{du}{2x}$:

$$\int \frac{2x}{1+x^2}\,dx = \int \frac{\cancel{2x}}{u}\,\frac{du}{\cancel{2x}} = \int \frac{1}{u}\,du$$

$$= \ln(u) + C$$

$$= \underline{\underline{\ln(1+x^2) + C}}$$

Kontroll: $\left(\ln(1+x^2)\right)' = \frac{1}{1+x^2} \cdot 2x$

$$= \frac{2x}{1+x^2}. \qquad \text{OK.}$$

Oppgave 3

$\int u'v\,dx = uv - \int uv'\,dx$

$$\int \underbrace{x}_{v}\,\underbrace{\sin x}_{u'}\,dx$$

$v(x) = x \Rightarrow v'(x) = 1$

$u'(x) = \sin x \Rightarrow u(x) = -\cos x$

$$\int x \sin x\,dx = -\cos x \cdot x - \int (-\cos x) \cdot 1\,dx$$

$$= -x \cdot \cos x + \underbrace{\int \cos x\,dx}_{=\sin x}$$

$$= \underline{\underline{-x\cos x + \sin x + C}}$$

Kontroll:

$(-x\cos x + \sin x)'$

$= -\left(1 \cdot \cos x + x \cdot (-\sin x)\right) + \cos x$

$= -\cancel{\cos x} - x(-\sin x) + \cancel{\cos x}$

$= \underline{x \sin x}$ ok

I dag

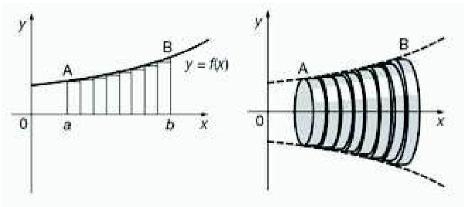
- Anvendelser av integrasjon
  - Volum av omdreiningslegemer (6.1)
  - Buelengde (6.2)

Volum av omdreiningslegemer

Sirkulær symmetri: vi kan bruke integraler til å beregne volum

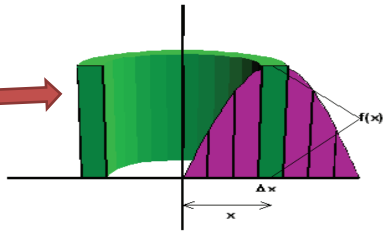
Volum av omdreiningslegemer

Sirkulær symmetri: vi kan bruke integraler til å beregne volum



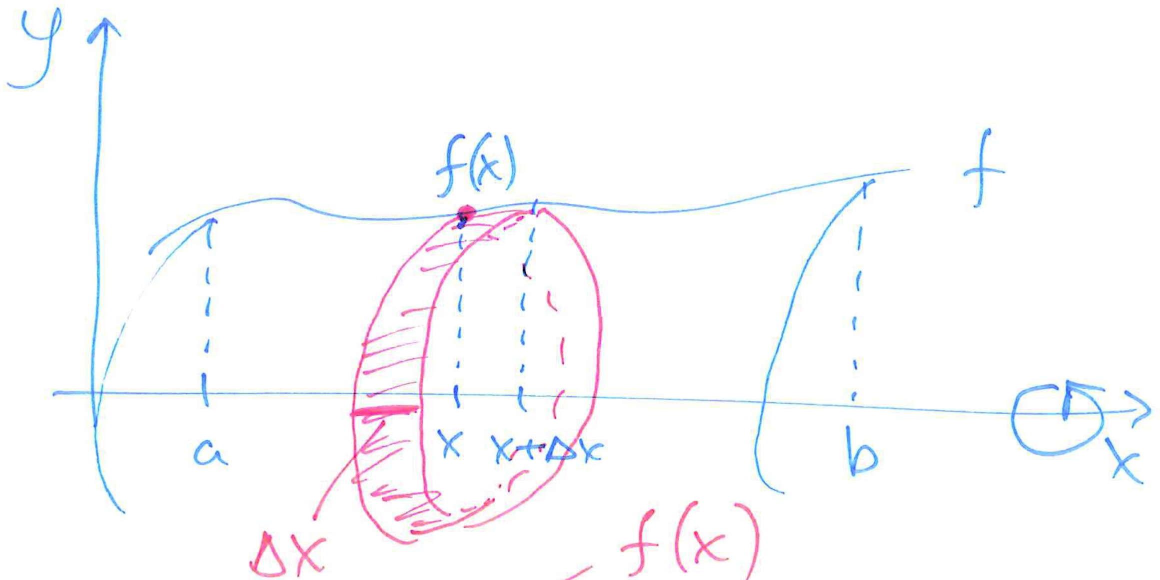
Rotasjon om *x*-aksen: skivemetoden

Rotasjon om *y*-aksen: sylinderskallmetoden



Anvendelser av integralet (6)

Volum av omdreiningslegemer (6.1)



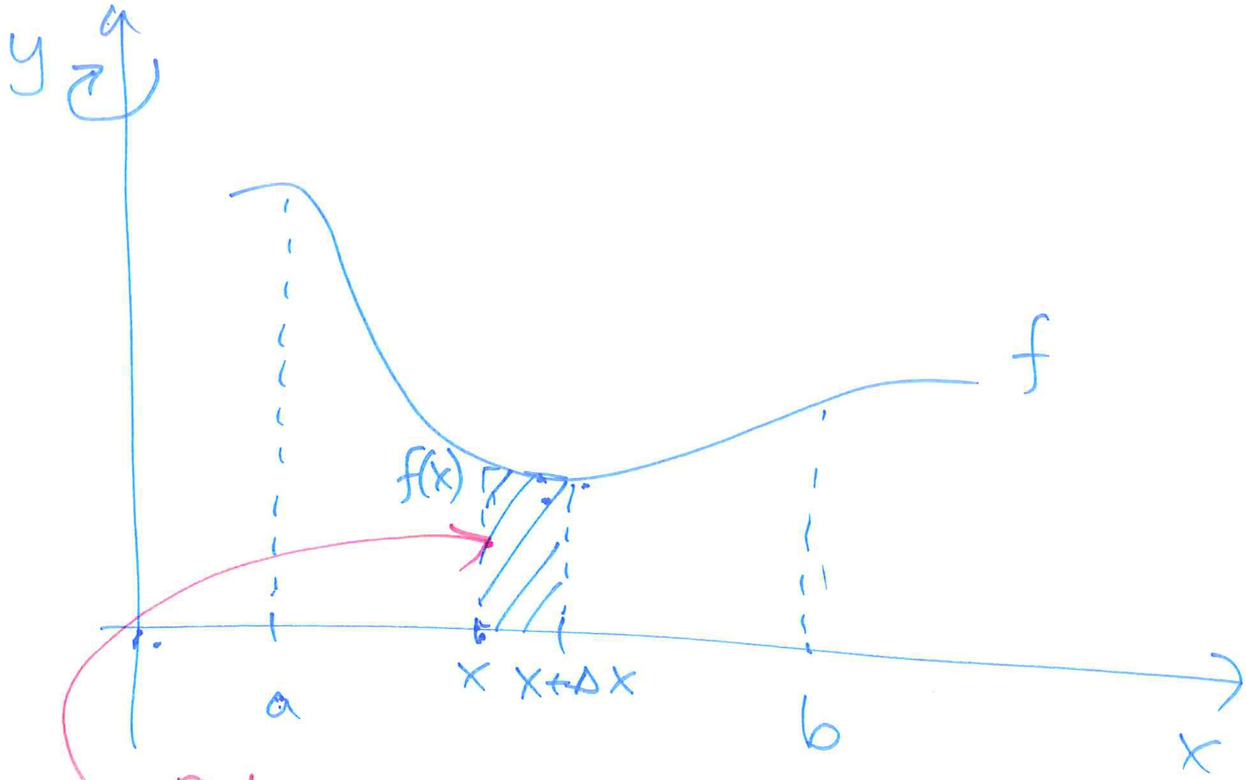
$$\Delta V_k = \pi \overset{f(x)}{r}^2 \overset{\Delta x}{h}$$

$$= \pi \left[f(x_k) \right]^2 \Delta x$$

$$V = \sum_k \Delta V_k = \pi \sum_k \left[f(x_k) \right]^2 \Delta x$$

$$\xrightarrow[\substack{\text{Ant. skiver} \to \infty \\ \Delta x \to 0}]{} \boxed{\pi \int_a^b \left(f(x) \right)^2 dx}$$

Volum til omdreiningslegeme, dreining om x-aksen.



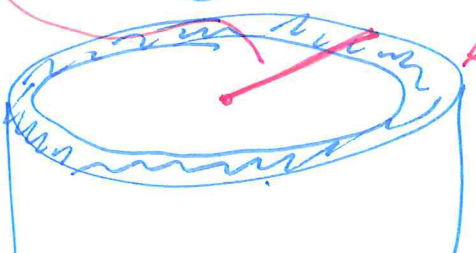
Roteres rundt y-aksen

$\Rightarrow$ Sylinderskall:

høyde $= f(x)$

tykkelse $= \Delta x$

radius $= x$



Omkretsen $= 2\pi x$

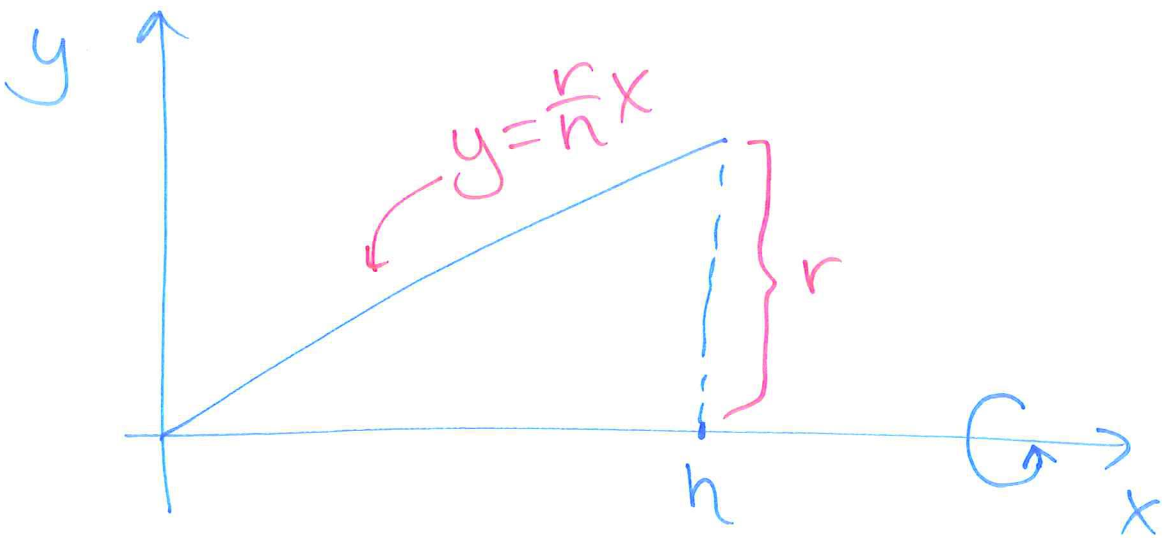
$$\Rightarrow \Delta V = 2\pi x \cdot f(x)\,\Delta x$$

$$V = \sum_k \Delta V_k = 2\pi \sum_k x_k f(x_k)\,\Delta x$$

Volum av omdreiningsleg. dreining om y-aksen:

$$\boxed{2\pi \int_a^b x\, f(x)\, dx}$$

Eksempel (6.1.4)



$$f(x) = \frac{r}{h}x, \quad 0 \le x \le h.$$

$$V = \pi \int_0^h \left(f(x)\right)^2 dx$$

$$= \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx$$

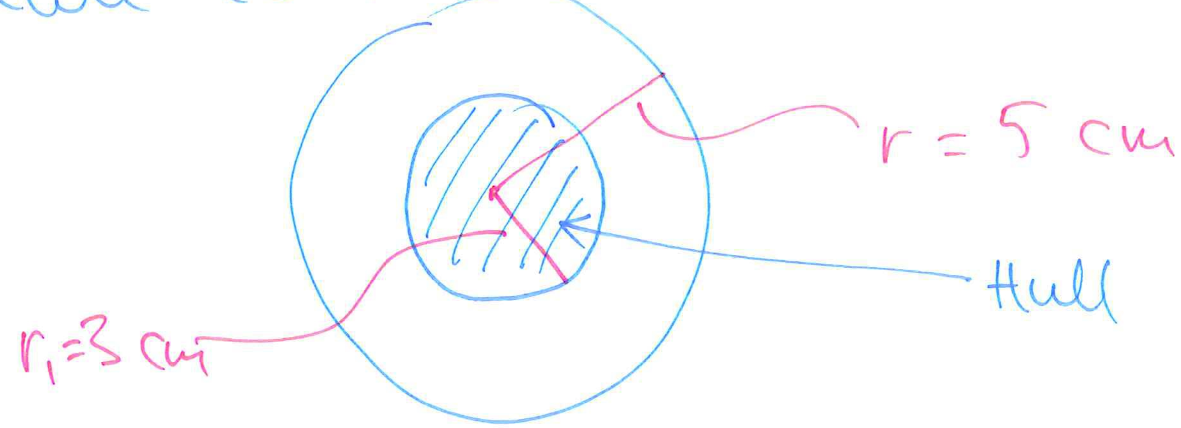
$$= \pi \left(\frac{r}{h}\right)^2 \int_0^h x^2 dx$$

$$= \pi \frac{r^2}{h^2} \left[\frac{1}{3}x^3\right]_0^h = \pi \frac{r^2}{\cancel{h^2}} \cdot \frac{1}{3} h^{\cancel{3}}$$

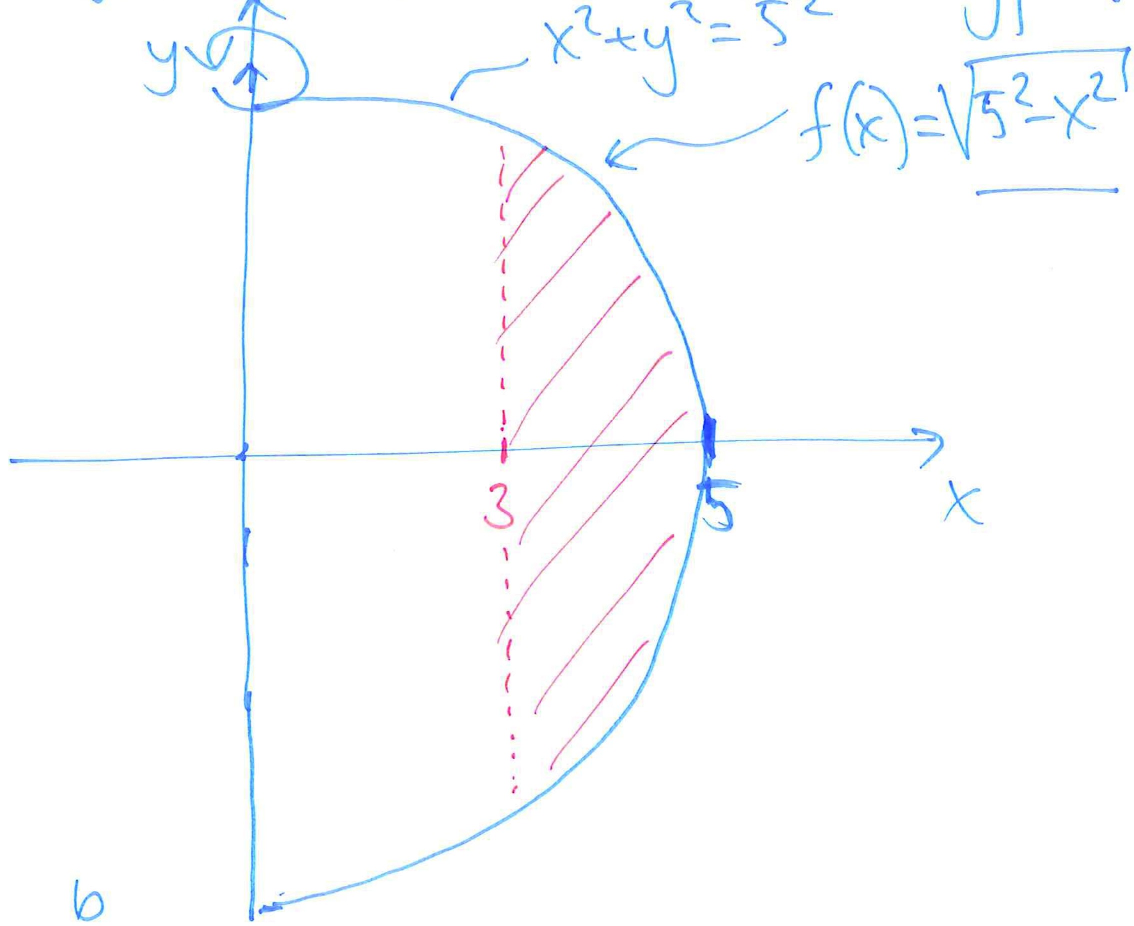
$$= \underline{\frac{\pi}{3} r^2 h} \longleftarrow \text{Formel for volumet til en kjegle.}$$

Eksempel (6.1.7)

Kule (sett ovenfra):



Hva er volumet av det som er igjen?



$$V = 2\pi \int_a^b x\, f(x)\, dx$$

$$= 2\pi \int_3^5 x\sqrt{25-x^2}\, dx$$

$$\int x\sqrt{25-x^2}\,dx$$

Substitusjon: $u = 25 - x^2$

$\Rightarrow u' = -2x$

$\Rightarrow dx = \frac{du}{u'(x)} = \frac{du}{-2x}$

$$= \int \cancel{x}\sqrt{u}\,\frac{du}{-2\cancel{x}} = -\frac{1}{2}\int \underbrace{\sqrt{u}}_{u^{1/2}}\,du$$

$$= -\frac{1}{2}\int u^{1/2}\,du = -\frac{1}{2}\,\frac{1}{1+1/2}\,u^{1/2+1} + C$$

$$= -\frac{1}{3}u^{3/2} + C = \underline{-\frac{1}{3}(25-x^2)^{3/2} + C}$$

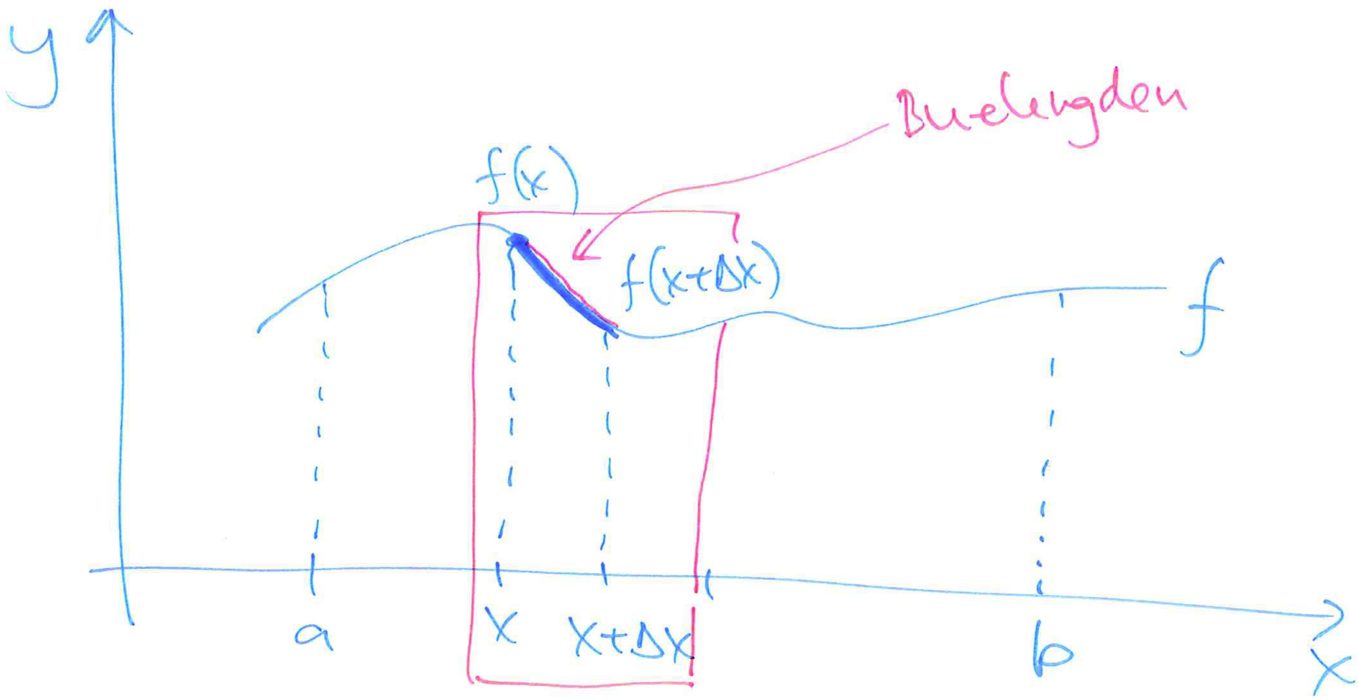
$$V = 2\pi\int_3^5 x\sqrt{25-x^2}\,dx$$

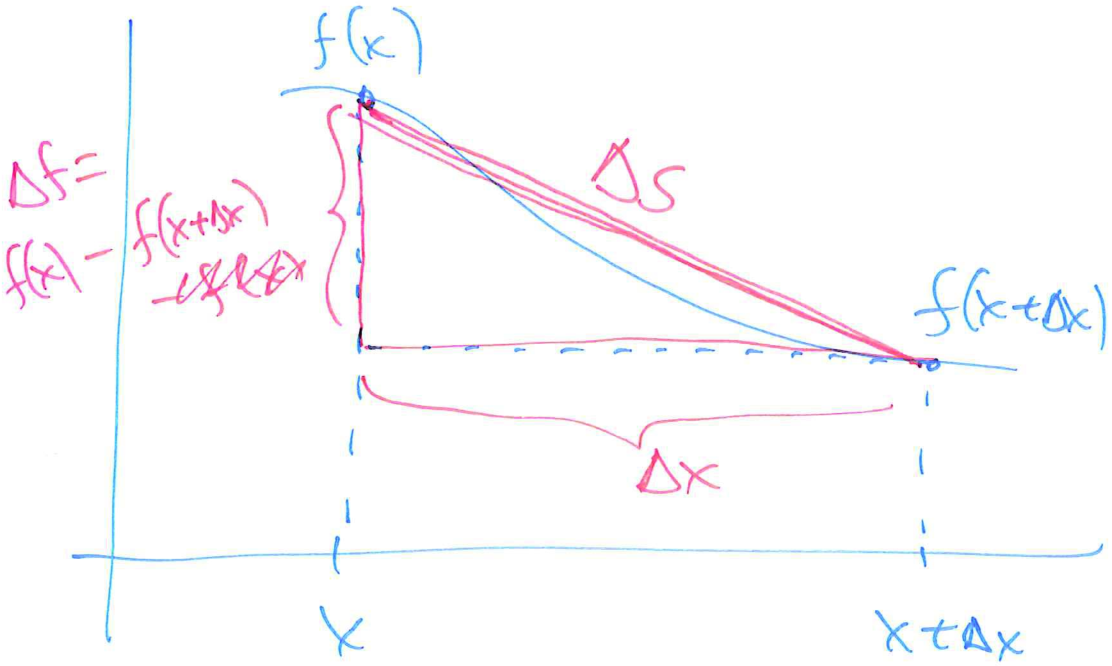
$$= 2\pi\left[-\frac{1}{3}(25-x^2)^{3/2}\right]_3^5$$

$$= 2\pi\left(-\frac{1}{3}\right)\left((\underbrace{25-5^2}_{0})^{3/2} - (\underbrace{25-3^2}_{25-9=16})^{3/2}\right)$$

$$= -\frac{2\pi}{3}\left(-(16)^{3/2}\right) = \frac{2\pi}{3}\cdot 4\cdot 16 = \underline{\underline{\frac{256\pi}{3}\,\text{cm}^3}}$$

Buelengde (6.2)





$$\Delta s = \sqrt{\Delta x^2 + \Delta f^2}$$
$$= \sqrt{(\Delta x)^2 \left(1 + \left(\frac{\Delta f}{\Delta x}\right)^2\right)}$$
$$= \Delta x \sqrt{1 + \left(\frac{\Delta f}{\Delta x}\right)^2}$$

$$\sum_k \Delta s_k = \sum_k \Delta x \sqrt{1+\left(\frac{\Delta f_k}{\Delta x}\right)^2}$$

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = f'(x)$$

Ant. delint. $\to \infty$

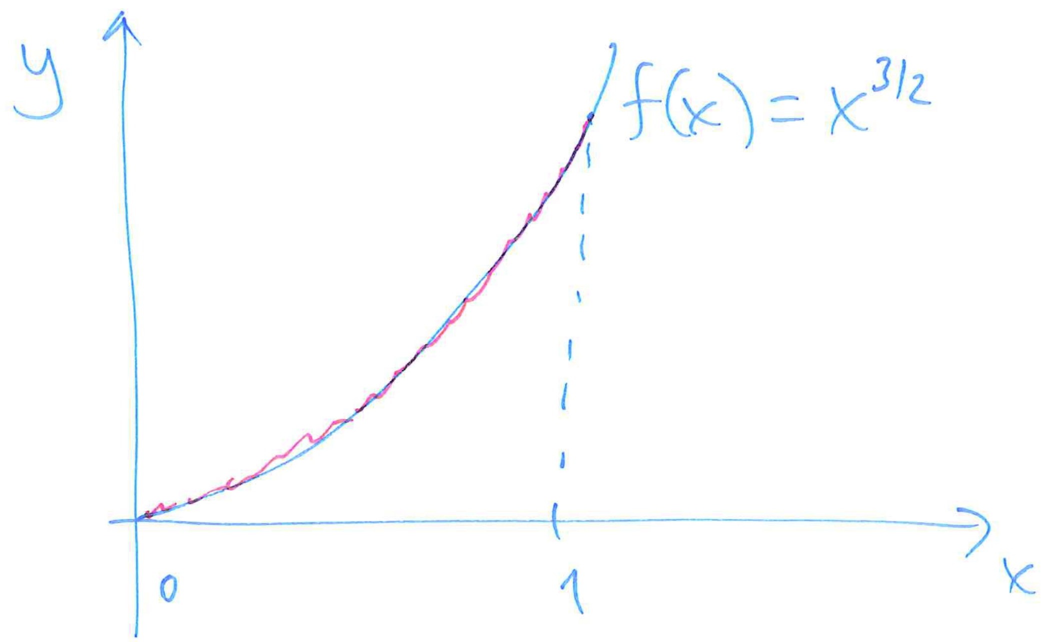
$\Delta x \to 0$

$$\int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx$$

$$\boxed{L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx}$$

Buelengde.

Eksempel (6.2.2)



$$f'(x) = \frac{3}{2} x^{\frac{3}{2}-1} = \frac{3}{2} x^{\frac{1}{2}}$$

$$\int u^r \, dr = \frac{1}{r+1} u^{r+1} + C$$

$$L = \int_0^1 \sqrt{1 + \left(\frac{3}{2} x^{\frac{1}{2}}\right)^2} \, dx$$

$$= \int_0^1 \sqrt{1 + \frac{9}{4} x} \, dx$$

$u = 1 + \frac{9}{4} x$

$\Rightarrow u' = \frac{9}{4}$

$\Rightarrow dx = \frac{du}{9/4}$

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = \frac{13}{4}$

$$= \int_1^{13/4} \sqrt{u} \, \frac{du}{9/4} = \frac{4}{9} \int_1^{13/4} u^{\frac{1}{2}} du$$

$$= \frac{4}{9} \cdot \frac{1}{\frac{1}{2}+1} \left[u^{3/2} \right]_1^{13/4} = \dots \approx \underline{1{,}81}$$