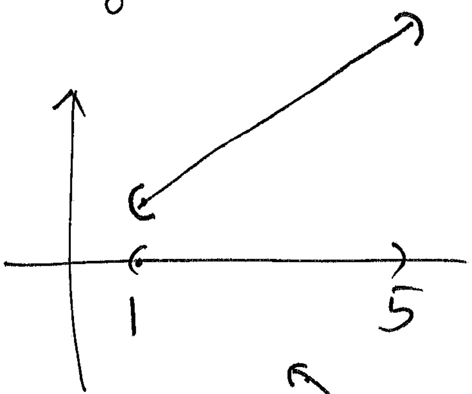
15.01.2015

1 Ekstremalverdisetningen.

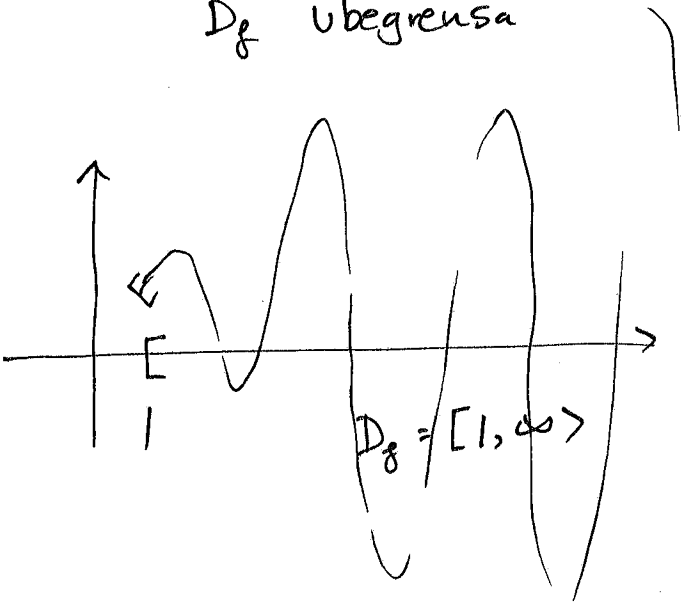
Hvis $f(x)$ er kontinuerlig med et lukka begrensa definisjonsområde, da har $f(x)$ både maksimums- og minimums punkt.

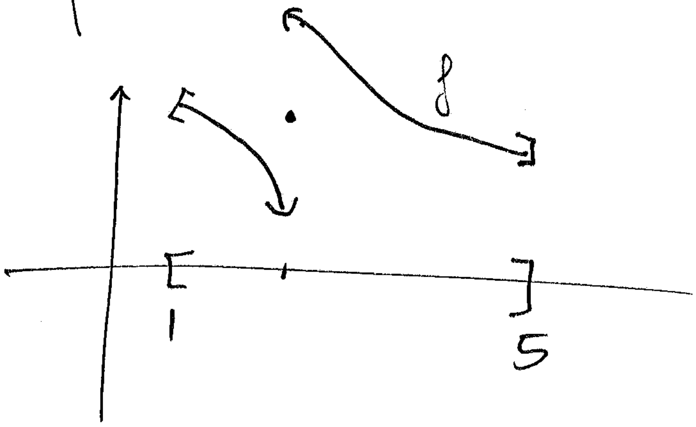
Eksempler hvor det ikke finnes ekstremalpunkt:

D_f ikke lukka



D_f ubegrensa

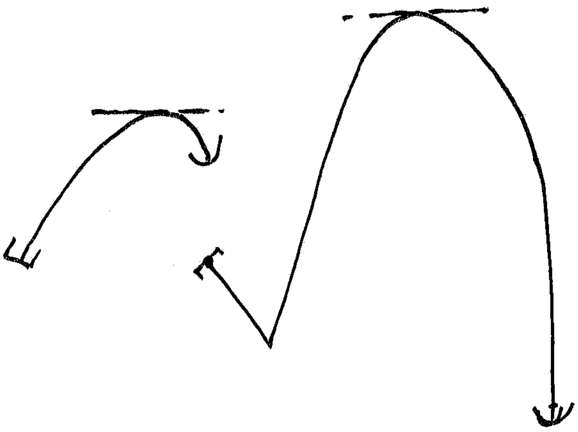




f ikke kont.

Hvordan finne ekstremalpunkter?

Et kritisk punkt er et punkt hvor $f'(x) = 0$, eller $f'(x)$ ikke eksisterer eller et endepunkt.



Alle ekstremalpunkt er kritiske punkt. (så for å finne ekstremalpunkt trenger vi bare lete blant de kritiske punktene.)

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Eksempler på grenser

$p(x)$ polynom

$$\lim_{x \to a} p(x) = p(a) \qquad \text{(grensesetningene)}$$

$$\lim_{x \to 0} \frac{kx}{x} = \lim_{x \to 0} k = k$$

$$\lim_{x \to -2} \frac{x^2 - 4}{x+2} \qquad \text{"type } \tfrac{0}{0}\text{"}$$

$$\lim_{x \to -2} \frac{(x-2)(x+2)}{(x+2)} = \lim_{x \to -2} (x-2) = -2-2 = \underline{\underline{-4}}$$

(generelt: p, q polynomer.

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \lim_{x \to a} \frac{(x-a)^n \bar{p}(x)}{(x-a)^m \bar{q}(x)} \qquad \bar{p}(a) \neq 0 \quad \bar{q}(a) \neq 0$$

faktorisert ut alle faktorer $(x-a)$.

Utfører polynomdivisjon med $(x-a)$ til div. ikke går opp.)

$$\lim_{x \to 1} \frac{x^3 - 1}{(x-1)^2} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)^2}$$

$$= \lim_{x \to 1} \frac{x^2+x+1}{x-1} \quad \text{eksisterer ikke}$$

$$\lim_{x \to 1^+} \frac{x^2+x+1}{x-1} = +\infty \qquad \lim_{x \to 1^-} \frac{x^2+x+1}{x-1} = -\infty$$

(vertikal asymptote $x = 1$ til grafen av $\frac{x^3-1}{(x-1)^2}$)

$$\lim_{x\to 0} \frac{1}{x^2+x} - \frac{1}{x}$$

(3)

$$= \lim_{x\to 0} \frac{1}{x(x+1)} - \frac{1\cdot(x+1)}{x(x+1)} = \lim_{x\to 0} \frac{1-(x+1)}{x(x+1)}$$

$$= \lim_{x\to 0} \frac{-x}{x(x+1)} = \lim_{x\to 0} \frac{-1}{x+1}\cdot\left(\frac{x}{x}\right) = \frac{-1}{1} = \underline{\underline{-1}}$$

$$\lim_{x\to\infty} \frac{x^3-2x^2+4}{4x^3-3x-2} = \lim_{x\to\infty} \frac{(x^3-2x^2+4)/x^3}{(4x^3-3x-2)/x^3}$$

$$= \lim_{x\to\infty} \frac{1-\frac{2}{x}+\frac{4}{x^3}}{4-\frac{3}{x^2}-\frac{2}{x^3}} = \underline{\underline{\frac{1}{4}}}$$

$$\lim_{h\to 0} \frac{\left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right)}{h} \overset{\text{felles nevner}}{=} \lim_{h\to 0} \frac{1}{h}\left[\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right]$$

Kvadratsetningen $(b-a)(b+a) = b^2 - a^2$.

$$\frac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{(\sqrt{x}+\sqrt{x+h})} = \frac{x-(x+h)}{(\sqrt{x}+\sqrt{x+h})} = \frac{-h}{\sqrt{x}+\sqrt{x+h}}$$

Setter dette inn i grensen:

$$\lim_{h\to 0} \frac{1}{h}\,\frac{-h}{\sqrt{x}+\sqrt{x+h}}\cdot\frac{1}{\sqrt{x}\sqrt{x+h}} = \lim_{h\to 0} \frac{-1}{(\sqrt{x}+\sqrt{x+h})(\sqrt{x}\cdot\sqrt{x+h})}$$

$$= \frac{-1}{(2\sqrt{x})(\sqrt{x})^2} = \underline{\frac{-1}{2x^{3/2}}} = \underline{\frac{-1}{2}x^{-3/2}}$$

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}$$ ("type $\frac{0}{0}$")

Benytter den "utvida konjugatsetningen" $b^3-a^3=(b-a)(b^2+ab+a^2)$

Så $\sqrt[3]{x}-1 = \dfrac{(\sqrt[3]{x})^3-1^3}{(\sqrt[3]{x})^2+\sqrt[3]{x}+1} = \dfrac{x-1}{x^{2/3}+x^{1/3}+1}$

$(b-a) = \dfrac{b^3-a^3}{b^2+ab+a^2}$

grensen er lik $\lim_{x\to 1}\dfrac{1}{x-1}\cdot\dfrac{x-1}{x^{2/3}+x^{1/3}+1}$

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$= \lim_{x\to 1}\dfrac{1}{x^{2/3}+x^{1/3}+1} = \dfrac{1}{1+1+1} = \underline{\underline{\dfrac{1}{3}}}$

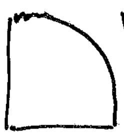
$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$ (x radianer)

(hvis vi hadde brukt grader hadde grensen blitt: $\frac{\pi}{180}$)

Buelengde og areal av sirkelsektorer

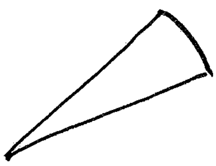
omkretsen : $2\pi\cdot r$

Areal : πr^2

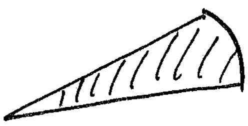


buelengden $\dfrac{2\pi r}{4} = \dfrac{\pi}{2}\cdot r$

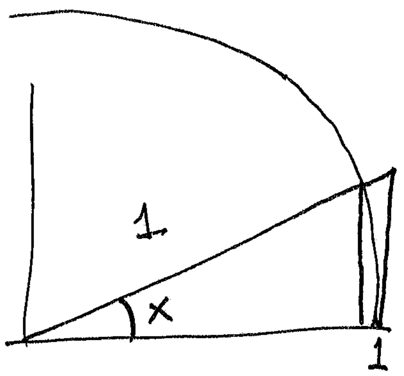
Arealet $\dfrac{\pi r^2}{4}$



$\dfrac{\text{buelengde}}{r}$ = vinkel i radianer



arealet er $\frac{1}{2}r^2\cdot$ vinkelen.



$x > 0$

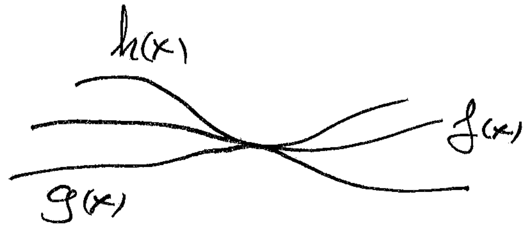
minste trekanten $\le$ areal sirkelsegment $\le$ største trekanten

$$\frac{1}{2}\sin x \cdot \cos x \le \frac{1}{2}x \le \frac{1}{2}\cdot\frac{1}{\cos x}\cdot \sin x$$

$$\frac{\sin x}{x}\cdot \cos x \le 1 \le \frac{1}{\cos x}\cdot\frac{\sin x}{x}$$

Så $\quad \cos x \le \dfrac{\sin x}{x} \le \dfrac{1}{\cos x} \qquad x > 0$

Skvisteoremet



Anta $\quad h(x) \ge f(x) \ge g(x)$

nær $\quad x = c$.

Hvis $\displaystyle\lim_{x\to c} h(x) = \lim_{x\to c} g(x) = L$, da er

$$\lim_{x\to c} f(x) = L.$$

$\displaystyle\lim_{x\to 0} \cos x = 1$ og da $\displaystyle\lim_{x\to 0}\frac{1}{\cos x} = 1$

Ved skvisteoremet er

$$\lim_{x\to 0}\frac{\sin x}{x} = \underline{\underline{1}}$$

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$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

bevis: $\lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x^2 \cdot (1+\cos x)} = \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)}$

(Pytagoras $\sin^2 x + \cos^2 x = 1$)

$$= \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \frac{1}{1+\cos x}$$

$$= 1^2 \cdot \frac{1}{2} = \underline{\underline{\frac{1}{2}}}$$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \left(\frac{1-\cos x}{x^2} \cdot x\right) = 0$$