19 oktober 2015

Integrasjons teknikker

(1)

$$\int x\,(\underbrace{1+x^2}_{u})^5\,dx$$

$u = 1+x^2$

$u' = 2x$

så $x = \frac{1}{2}u'$

$u' = \frac{du}{dx}$

$\underline{u'dx = du}$

substitusjon

$$= \int \tfrac{1}{2}u'\; u^5\, dx$$

$$= \tfrac{1}{2}\int u^5 \underbrace{u'dx}_{du}$$

$$= \tfrac{1}{2}\int u^5\, du$$

$$= \tfrac{1}{2}\left(\frac{u^6}{6}\right) + c = \underline{\frac{(1+x^2)^6}{12} + c}$$

$$\int (1+x^2)^5\,dx$$

her må vi gange ut polynomel og så integrere!

Delvis integrasjon:

$$\int u'\cdot v\,dx = u\cdot v - \int u\cdot v'dx$$

$$\int \underbrace{x}_{v}\,\underbrace{\sin(x)}_{u'}\,dx$$

La $u = -\cos(x)$

$$\int x\sin(x)\,dx = x(-\cos x) - \int \overset{v'}{1}\cdot(-\cos x)\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= \underline{-x\cos x + \sin x + c}$$

Kombinasjon av substitusjon og delvis integrasjon

(2) $\int x^3 \sin(x^2)\,dx$ $\qquad u = x^2$

$\qquad u' = 2x$

$$\int \underbrace{x}_{\frac{1}{2}u'} \cdot \underbrace{x^2}_{u} \underbrace{\sin(x^2)}_{\sin(u)}\,dx$$

$$= \frac{1}{2}\int u \sin(u) \underbrace{u'dx}_{du} = \frac{1}{2}\int u\sin(u)\,du$$

$$= \frac{1}{2}\left[-u\cos(u) + \sin(u)\right] + c$$

$$\int x^3 \sin(x^2)\,dx = \underline{\frac{1}{2}\left[-x^2\cos(x^2) + \sin(x^2)\right] + c}$$

$$\int \underbrace{e^x}_{u'} \underbrace{\sin(x)}_{v}\,dx$$ Velger $u = e^x$

$$= e^x \sin(x) - \int \underbrace{e^x}_{w'} \underbrace{\cos x}_{z}\,dx$$ $\qquad w = e^x$, $z = \cos x$, $z' = -\sin x$

$$= e^x \sin x - \left[e^x \cos x - \int e^x(-\sin x)\,dx\right]$$

$$= e^x \sin(x) - e^x\cos(x) - \int e^x \sin x\,dx$$

flytter over til venstre side!

$$2\int e^x \sin x\,dx = e^x \sin x - e^x\cos x + c$$

deler med 2:

$$\int e^x \sin(x)\,dx = \underline{\frac{1}{2}e^x(\sin x - \cos x) + c}$$

$\int e^{ax} \sin(bx)\, dx$

Bestem integralet.

(se forelesning M1000 18 mars 2015.)

③

Delbrøksoppspalting se 4.5 i boka

$\int \frac{1}{x^2 - x}\, dx$ (mulig ved delbrøksoppspalting)

$$\frac{1}{x^2 - x} = \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$= \frac{A(x-1)}{x(x-1)} + \frac{B \cdot x}{x(x-1)}$$

Så $1 = A(x-1) + Bx$

$$= (A+B) \cdot x \quad (-A) \cdot 1$$

Derfor må $A + B = 0$, $-A = 1$ $\quad B = -A = \underline{1}$, $A = \underline{-1}$

(Alternativt: sett inn "gunstige" verdier for x:

sett $x = 1$ gir $A \cdot 0 + B \cdot 1 = 1$ så $B = 1$

$x = 0$ gir $A(-1) + B \cdot 0 = 1$ så $A = -1$)

$$\frac{1}{x^2 - x} = \frac{1}{x-1} - \frac{1}{x}$$

$$\int \frac{1}{x^2 - x}\, dx = \int \frac{1}{x-1} - \frac{1}{x}\, dx$$

$$= \ln|x-1| - \ln|x| + c$$

$$= \ln|x-1| + \ln(|x|^{-1}) + c$$

$$= \underline{\ln\left|\frac{x-1}{x}\right| + c}$$

Alle rasjonale funksjoner har en elementær antiderivert.

④
$$\int \frac{x^2}{x^2-4}\,dx = \int \frac{x^2-4+4}{x^2-4}\,dx$$

$$= \int 1 + \frac{4}{x^2-4}\,dx$$

$$x^2-4 = x^2-2^2 = (x+2)(x-2)$$

$$\frac{4}{x^2-4} = \frac{1}{x-2} + \frac{-1}{x+2}$$

så $\displaystyle\int \frac{x^2}{x^2-4}\,dx = \int 1 + \frac{1}{x-2} - \frac{1}{x+2}\,dx$

$$= x + \ln|x-2| - \ln|x+2| + c$$

$$= x + \ln\left|\frac{x-2}{x+2}\right| + c$$

$$\int \frac{1}{x^3+9x}\,dx$$

nevneren faktorisert $\quad x(x^2+9)$

$$\frac{1}{x^3+9x} = \frac{A}{x} \quad \frac{B+Cx}{x^2+9}$$

felles nevner

$$= \frac{A(x^2+9) + (B+Cx)\cdot x}{x(x^2+9)}$$

sammenlikner tellerne

$$1 = Ax^2 + 9A + B \cdot x + Cx^2$$

$$0 \cdot x^2 + 0 \cdot x + 1 = (A+C)x^2 + B \cdot x + 9 \cdot A$$

5 $A + C = 0, \quad B = 0 \quad$ og $\quad 9A = 1$

$A = \frac{1}{9}, \qquad C = -A = \frac{-1}{9}.$

så $\quad \frac{1}{x^3+9x} = \frac{1}{9}\left[\frac{1}{x} - \frac{x}{x^2+9} \right]$

$$\int \frac{1}{x^3+9x}\,dx = \frac{1}{9}\int \frac{1}{x} - \frac{x}{x^2+9}\,dx$$

$$= \frac{1}{9}\left[\int \frac{1}{x}\,dx - \int \frac{x}{x^2+9}\,dx \right]$$

$u = x^2+9$
$u' = 2x$
$x = \frac{1}{2}u'$

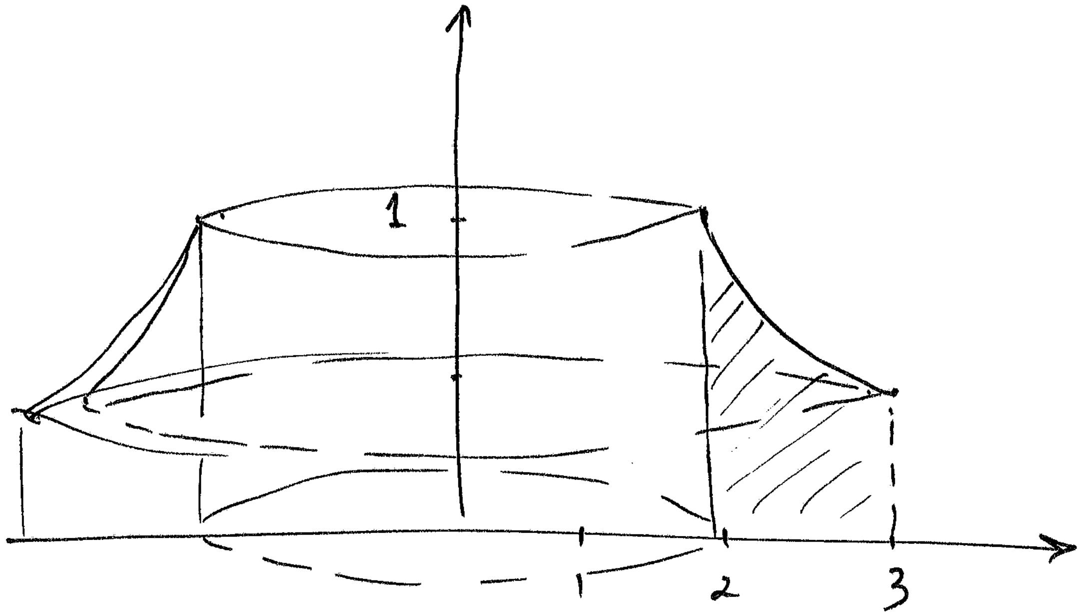
$$= \frac{1}{9}\left[\ln|x| - \int \frac{1}{u}\,\frac{1}{2}\underbrace{u'dx}_{du} \right]$$

$$= \frac{1}{9}\left[\ln|x| - \frac{1}{2}\int \frac{1}{u}\,du \right]$$

$$= \frac{1}{9}\left[\ln|x| - \frac{1}{2}\ln(x^2+9) \right] + C$$

Finn volumet til rotasjonslegemet vi får ved å rotere grafen til $\frac{1}{x-1}$ $\quad 2 \leq x \leq 3$ om y-aksen. Avgrenset av grafen, x-aksen og de vertikale linjene $x = 2$, $x = 3$.

⑥



$$V = \int_2^3 (2\pi \cdot x) \cdot \frac{1}{x-1}\, dx$$

$$= 2\pi \int_2^3 \frac{x}{x-1}\, dx = 2\pi \int_2^3 \frac{x-1+1}{x-1}\, dx$$

(polynomdivisjon)

$$= 2\pi \int_2^3 1 + \frac{1}{x-1}\, dx$$

$$= 2\pi \left[x + \ln|x-1| \right]\Big|_2^3$$

$$= 2\pi \left[3 - 2 + \ln(3-1) - \ln(2-1) \right]$$

$$= \underline{\underline{2\pi \left(1 + \ln(2) \right)}}$$

$$\int \frac{x+2}{(x-1)^2}\,dx$$

(7) $$\frac{x+2}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

(Faktoren $(x-1)$ forekommer to ganger i nevner)

$$\frac{x-1+1+2}{(x-1)^2} = \frac{x-1}{(x-1)^2} + \frac{3}{(x-1)^2}$$

$$= \frac{1}{x-1} + \frac{3}{(x-1)^2}$$

$u = x-1$
$du = dx$

$$\int \frac{x+2}{(x-1)^2}\,dx = \int \frac{1}{x-1} + \frac{3}{(x-1)^2}\,dx$$

$$= \int \frac{1}{u} + \frac{3}{u^2}\,du$$

$$= \ln|u| + 3\int u^{-2}\,du$$

$$= \ln|u| + 3\cdot\left(\frac{u^{-1}}{-1}\right) + c$$

$$= \ln|x-1| + 3\frac{(-1)}{x-1} + c$$

$$= \underline{\underline{\ln|x-1| - \frac{3}{x-1} + c}}$$

$\int \frac{1}{x^4+1}\,dx$ Dette eksempelet krever noe regning.

(Vanskeligere enn eksamensoppgaver)

8

$$x^4+1 = (x^2+1)^2 - 2x^2$$
$$= (x^2+1)^2 - (\sqrt{2}\,x)^2$$
$$= (x^2+1+\sqrt{2}\cdot x)(x^2+1-\sqrt{2}\cdot x)$$

$$\frac{1}{x^4+1} \overset{\text{delbrøksoppspalting}}{=} \frac{Ax+B}{x^2+1+\sqrt{2}\,x} \qquad \frac{Cx+D}{x^2+1-\sqrt{2}\,x}$$

Felles nevner $(Ax+B)(x^2+1-\sqrt{2}\,x)+(Cx+D)(x^2+1+\sqrt{2}\,x)=1$

$$(A+C)x^3+(B+D+\sqrt{2}(C-A))x^2+(A+C+\sqrt{2}(D-B))x+B+D=1$$

så $A+C=0$, $B+D=1$

$$A+C+\sqrt{2}(D-B) = \sqrt{2}(D-B)=0$$

og $B+D+\sqrt{2}(C-A) = 1+\sqrt{2}(C-A) = 1-2\sqrt{2}A = 0$

Dette gir $B=D=\frac{1}{2}$ $\quad A=\frac{1}{2\sqrt{2}} = -C$

$$\frac{1}{x^2+1\pm\sqrt{2}\,x} = \frac{1}{\left(x\pm\frac{\sqrt{2}}{2}\right)^2-\frac{1}{2}+1}$$
$$= \frac{1}{\left(x\pm\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}$$

$$\frac{1}{x^4+1} = \frac{1}{2}\left[\frac{\frac{x}{\sqrt{2}}+1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} + \frac{\frac{-x}{\sqrt{2}}+1}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\right]$$

gange teller og nevner med $(\sqrt{2})^2=2$:

$$= \frac{1}{2}\left[\frac{\sqrt{2}\,x+1+1}{(\sqrt{2}\,x+1)^2+1} + \frac{-\sqrt{2}\,x+1+1}{(\sqrt{2}\,x-1)^2+1}\right]$$

Vi benytter substitusjonene

$$u = \sqrt{2}x + 1 \qquad du = \sqrt{2}\,dx$$

$$v = \sqrt{2}x - 1 \qquad dv = \sqrt{2}\,dx$$

(9)

$$\int \frac{1}{x^4+1}\,dx = \frac{1}{2}\int \frac{u+1}{u^2+1}\frac{1}{\sqrt{2}}\,du + \frac{1}{2}\int \frac{-v+1}{v^2+1}\frac{1}{\sqrt{2}}\,dv$$

Nå benytter vi at

$$\int \frac{1}{x^2+1}\,dx = \arctan(x) + C \qquad \text{og}$$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C$$

(ved substitusjonen x^2+1 ...)

$$\int \frac{1}{x^4+1}\,dx = \frac{1}{2\sqrt{2}}\left(\frac{1}{2}\ln(u^2+1) + \arctan(u)\right) + \frac{1}{2\sqrt{2}}\left(-\frac{1}{2}\ln(v^2+1) + \arctan(v)\right) + C$$

$$= \frac{1}{4\sqrt{2}}\ln\left(\frac{2x^2+2\sqrt{2}x+2}{2x^2-2\sqrt{2}x+2}\right) + \frac{1}{2\sqrt{2}}\left(\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x-1)\right) + C$$

$$= \frac{1}{4\sqrt{2}}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \frac{1}{2\sqrt{2}}\left(\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x-1)\right) + C$$

Bonus:

(10) Forsøk gjerne å benytte en maskin som kan regne symbolsk til å finne antiderivete. Et slikt program kalles gjerne CAS (Computer algebra system)

Geogebra har en enkel CAS:

Forsøk å skrive:

integral[1/(1+x^4)] (eg benyttet dette til å sjekke utregningen ovenfor.)

integral[(1+x^2)^30] (langt uttrykk!)

integral[x·(1+x^2)^30] (kort)

integral[sin(x^2)] ?

integral[1/sin(x)] etc.