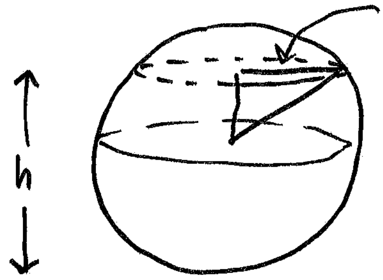
12 okt 2015

Anvendelser av integrasjon

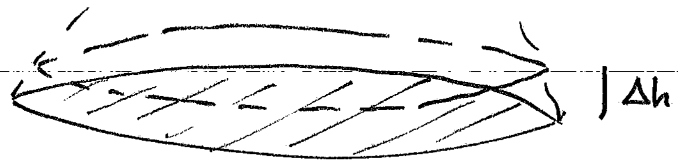
(1)

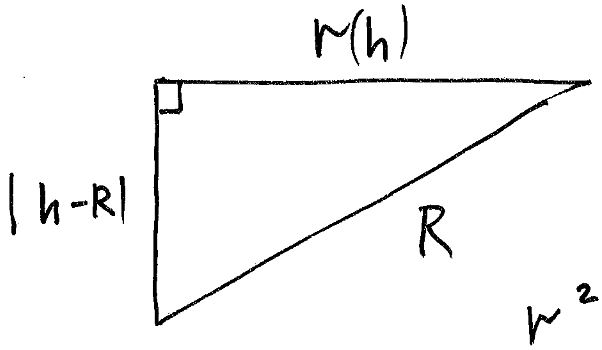


radius til tverrsnitt ved høyde h er $r(h)$

Radius R

$$\frac{dV}{dh} = \text{tverrsnittarealet ved høyde } h$$
$$= \pi \cdot r(h)^2$$





Pytagoras

$$R^2 = (r(h))^2 + (h-R)^2$$

$$r^2 = R^2 - \underbrace{(h-R)^2}_{h^2+R^2-2Rh}$$

$$\underline{r^2 = 2Rh - h^2}$$

Hva er $V(h)$?

$$\frac{dV}{dh} = \pi r(h)^2 = \pi(2Rh - h^2)$$

$$V(H) = \int_0^H \frac{dV}{dh}\, dh \quad \left(= \int_{V(0)}^{V(H)} dV \text{ substitusjon} \right)$$

$$= \int_0^H \pi(2Rh - h^2)\, dh = \pi\left[Rh^2 - \frac{h^3}{3} \right]_0^H$$

$$\underline{V(H) = \pi\left(RH^2 - H^3/3\right)}$$

sjekker svaret: $V(h) = \pi(Rh^2 - h^3/3)$

(2) $V(0) = 0$ ✓

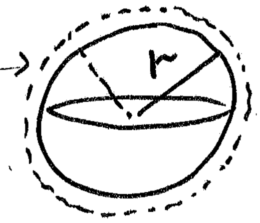
$V(R) = \pi(R^3 - R^3/3) = \frac{2\pi}{3}R^3 = \frac{1}{2}\left(\frac{4\pi R^3}{3}\right)$ ✓

$$V(2R) = \pi\left(R(2R)^2 - (2R)^3/3\right)$$
$$= \pi\left(4R^3 - \frac{8}{3}R^3\right)$$
$$= \pi R^3\left(\frac{4\cdot 3}{3} - \frac{8}{3}\right) = \underline{\underline{\frac{4\pi}{3}R^3}}$$

---

overflate arealet til en kule:

radius $r+\Delta r$



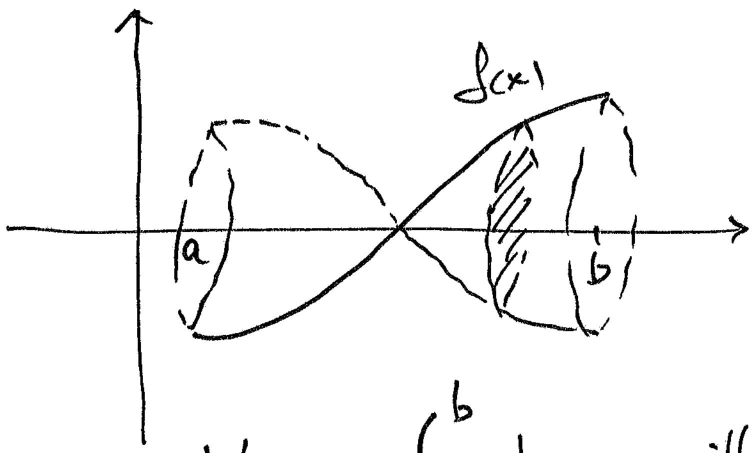
$$\lim_{\Delta r \to 0} \frac{V(r+\Delta r) - V(r)}{\Delta r} = \text{overflate-arealet til en kule med radius } r.$$

$$= \frac{dV}{dr}$$

overflate arealet er $\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4\pi r^3}{3}\right) = \underline{\underline{4\pi r^2}}$

Rotasjonslegemer

(3)



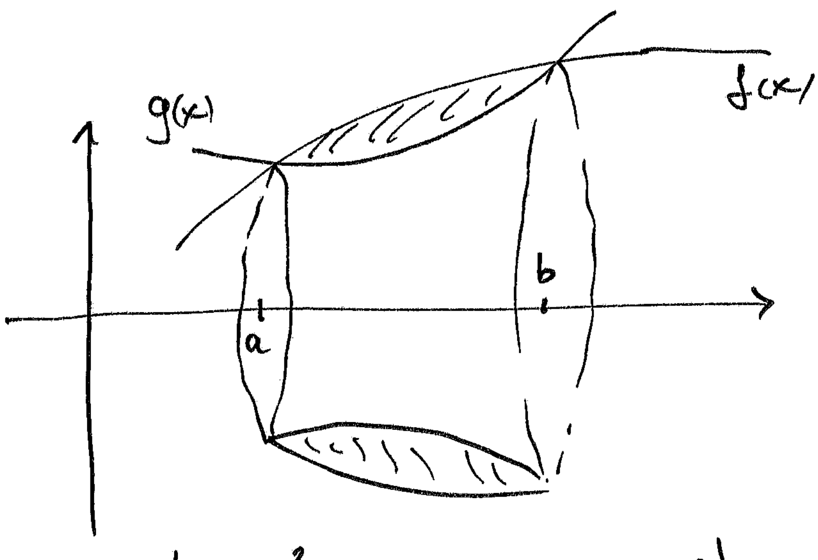
Rotasjon om x-akse.

$$V = \int_a^b \underset{\text{ved } x}{\text{tverrsnittarealet}} \, dx$$

$$= \int_a^b \pi \left(|f(x)| \right)^2 dx$$

$$\underline{V = \pi \int_a^b f^2(x) \, dx}$$

"skivemetoden"

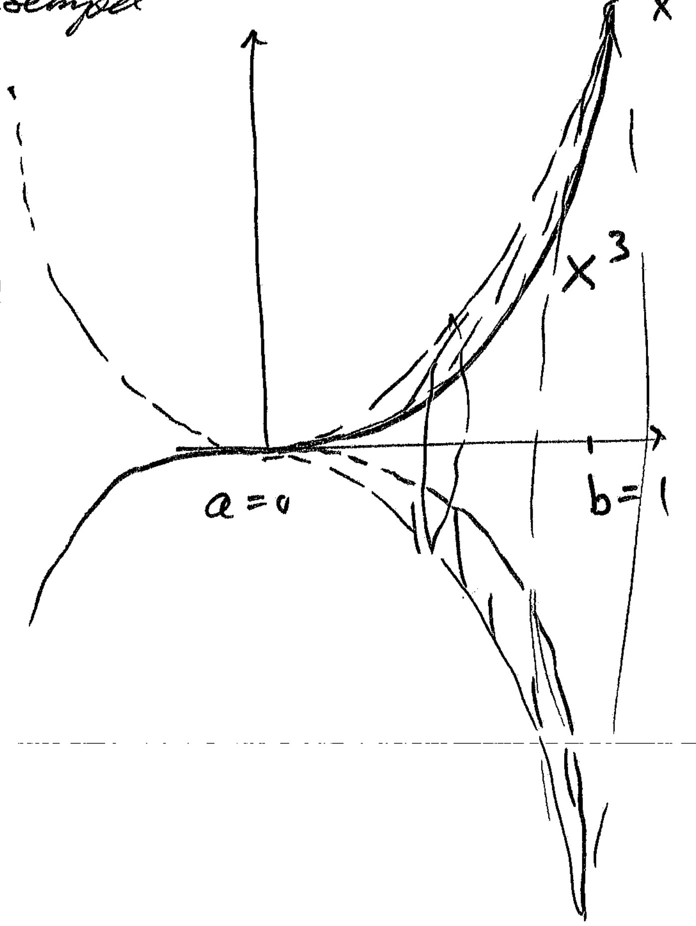


$$V = \pi \int_a^b f^2(x) \, dx - \pi \int_a^b g^2(x) \, dx$$

$$\underline{= \pi \int_a^b f^2(x) - g^2(x) \, dx}$$

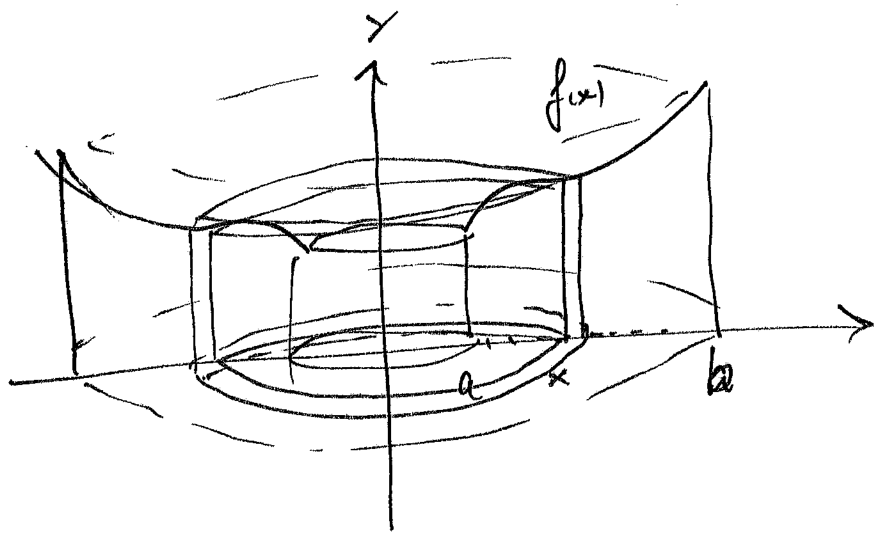
Eksempel

4



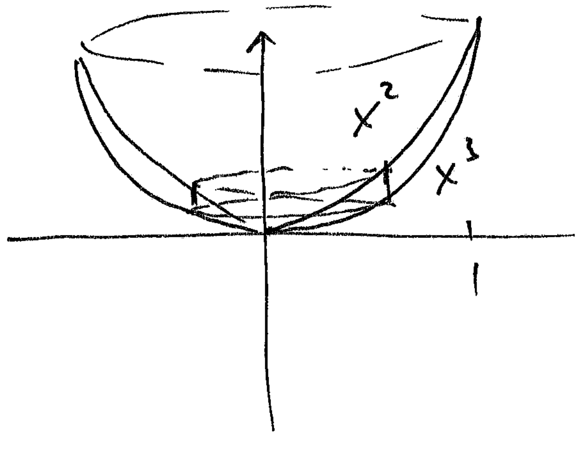
$$V = \pi \int_0^1 (x^2)^2 - (x^3)^2 \, dx$$
$$= \pi \int_0^1 x^4 - x^6 \, dx$$
$$= \pi \left(\frac{x^5}{5} - \frac{x^7}{7} \right) \Big|_0^1$$
$$= \pi \left(\frac{1}{5} - \frac{1}{7} \right)$$
$$= \underline{\frac{2\pi}{35}}$$

Roterer om y-aksen



"skallmetoden"

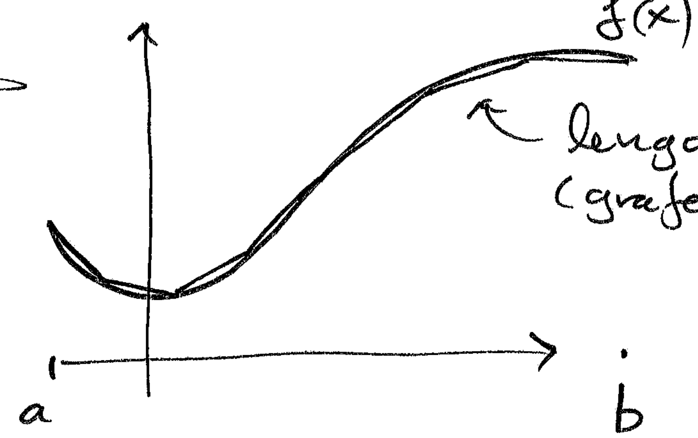
$$\underline{V = \int_a^b 2\pi x \cdot |f(x)| \, dx}$$



$$V = \int_0^1 2\pi x \left| (x^2 - x^3) \right| dx$$
$$= 2\pi \int_0^1 x^3 - x^4 \, dx$$
$$= 2\pi \left(\frac{x^4}{4} - \frac{x^5}{5} \right) \Big|_0^1$$
$$= 2\pi \left(\frac{1}{4} - \frac{1}{5} \right)$$
$$= \underline{\frac{\pi}{10}}$$

Buelengde

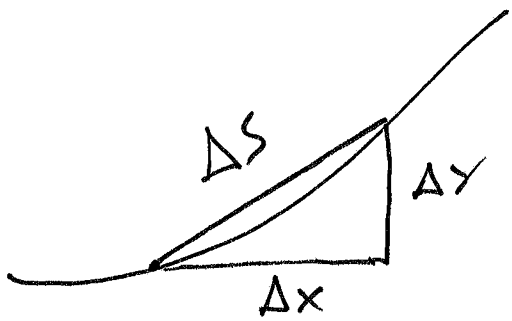
5



lengden til kurven (grafen) kalles buelengden.

$$L = \int_a^b \sqrt{1+(f'(x))^2}\, dx \quad \left(= \int_{S(a)}^{S(b)} ds \right)$$

(ofte vanskelig å finne en antiderivert)



$$(\Delta s)^2 \sim (\Delta x)^2 + (\Delta y)^2$$

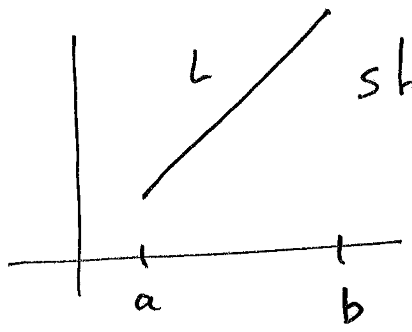
$$\sim (\Delta x)^2 \left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)$$

$\Delta x > 0$:

$$\Delta s \sim \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

$\Delta x \to 0$

$$ds = dx \sqrt{1 + (f'(x))^2}$$



stigningstall T

Lengden : L

$$L^2 = (b-a)^2 + ((b-a)\cdot T)^2$$

$$L = |b-a| \sqrt{1+T^2}.$$

$$L = \int_a^b \sqrt{1+(f'(x))^2}\, dx = \int_a^b \sqrt{1+T^2}\, dx$$

$$= (b-a)\sqrt{1+T^2}.$$

siden $f'(x) = T$.

Kinetisk energi $\frac{m}{2}V^2$.

Arbeid er kraft × vei.

$$m\frac{dV}{dt} = F \quad \text{kraft.}$$

(6)

$$\xrightarrow{V}$$

$$V = \frac{dx}{dt} \qquad dx = V \cdot dt$$

$$F \cdot dx$$

$$F \cdot V\,dt \qquad \text{setter inn } F = m \cdot \frac{dV}{dt}$$

$$= m \cdot \frac{dV}{dt} \cdot V\,dt$$

substitusjon

$$\text{Arbeidet utført: } \int_0^T m \cdot V \cdot \underbrace{V'dt}_{dV}$$

$$= \int_0^V m\,V\,dV$$

$$= m \cdot \frac{V^2}{2}\Big|_0^V = \underline{\underline{\frac{mV^2}{2}}}$$

Uegentlige integraler

7

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{N\to\infty} \int_1^N \frac{1}{x^2}\,dx$$

$$= \lim_{N\to\infty} \left(\frac{x^{-1}}{-1} \Big|_1^N \right) = \lim_{N\to\infty} \left(\frac{-1}{N} + 1 \right)$$

$$= \underline{\underline{1}}$$

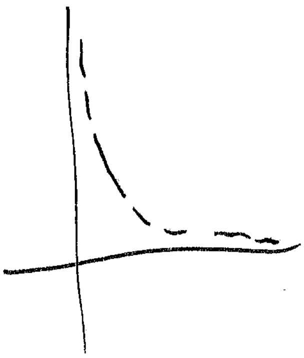
$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ ikkje Riemann integrerbar

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{d\to 0^+} \int_d^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{d\to 0^+} \left(2\cdot x^{1/2} \Big|_d^1 \right)$$

$$= \lim_{d\to 0^+} \left(2 - 2\sqrt{d} \right)$$

$$= \underline{\underline{2}}$$



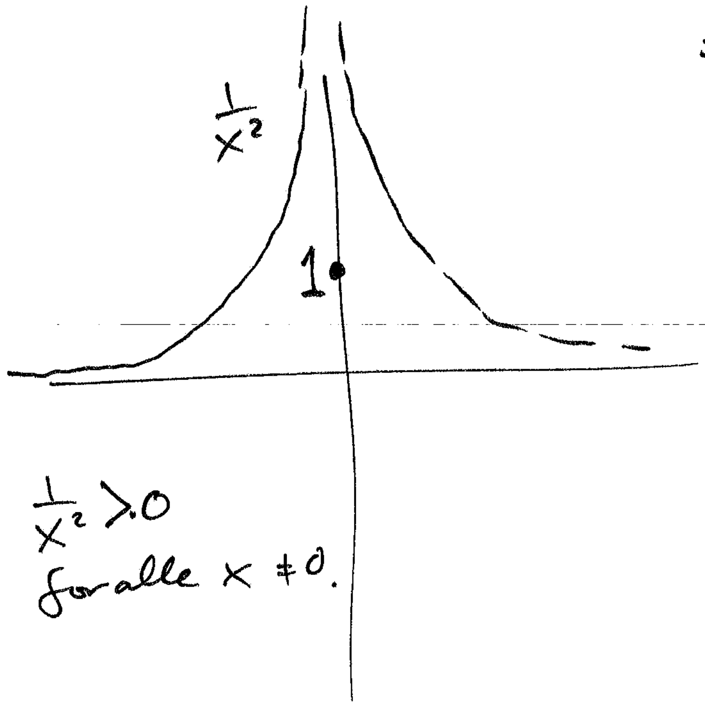
$$\int_{-3}^{2} \frac{1}{x^2}\, dx = \int_{-3}^{2} x^{-2}\, dx$$

(8)

$$\frac{x^{-1}}{-1}\Big|_{-3}^{2} = -\left(\frac{1}{2} - \frac{1}{-3}\right)$$

$$= -\left(\frac{1}{2} + \frac{1}{3}\right)$$

$$= -\frac{5}{6}$$



$\frac{1}{x^2} > 0$
for alle $x \neq 0$.

Integralet
$$\int_{-3}^{2} \frac{1}{x^2}\, dx$$
eksisterer ikkje.

↓

La oss se på $\int_{0}^{2} x^{-2}\, dx$

Ikke Riemann int.

$$\lim_{d \to 0^+} \int_{d}^{2} x^{-2}\, dx = \lim_{d \to 0^+} \frac{-1}{x}\Big|_{d}^{2}$$

$$= \lim_{d \to 0^+} \left(\frac{-1}{2} + \frac{1}{d}\right)$$

eksisterer ikkje.