LØSNINGSFORSLAG TIL EKSAMEN I MATEMATIKK 1000, 01.06.2011

Oppgave 1 :

a) i) $\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{2x}\right)} = \sqrt{e}$ , siden

$$\lim_{x\to\infty} x\ln\left(1+\tfrac{1}{2x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{2x}\right)}{\frac{1}{x}} = \lim_{t\to 0^+}\frac{\ln\left(1+\frac{t}{2}\right)}{t} = \lim_{t\to 0^+}\frac{\frac{1/2}{1+t/2}}{1} = \frac{1}{2}$$

ii) $\lim_{x\to 1}\frac{\int_1^x t^2\,dt}{3\,(x-1)} = \lim_{x\to 1}\frac{x^2}{3} = \frac{1}{3}$

b) i) $y' = 1\cdot\tan(x^2) + x\cdot\frac{2x}{\cos^2(x^2)} = \tan(x^2) + \frac{2x^2}{\cos^2(x^2)}$

ii) $2xy^3 + y^2 + x^2 = 0$

Implisitt derivasjon gir

$2\cdot y^3 + 2x\cdot 3y^2\,y' + 2y\,y' + 2x = 0$

$(3xy^2 + y)\,y' = -(y^3 + x)$

$y' = -\frac{y^3+x}{3xy^2+y}$

Oppgave 2 :

a) $S(\begin{bmatrix}1\\0\\0\end{bmatrix}) = \begin{bmatrix}0\\0\\0\end{bmatrix}$ , $S(\begin{bmatrix}0\\1\\0\end{bmatrix}) = \begin{bmatrix}0\\1\\0\end{bmatrix}$ og $S(\begin{bmatrix}0\\0\\1\end{bmatrix}) = \begin{bmatrix}0\\0\\0\end{bmatrix}$

gir at $A = \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}$ .

$T(\begin{bmatrix}1\\0\\0\end{bmatrix}) = \begin{bmatrix}3\\0\\1\end{bmatrix}$ , $T(\begin{bmatrix}0\\1\\0\end{bmatrix}) = \begin{bmatrix}-2\\1\\0\end{bmatrix}$ og $T(\begin{bmatrix}0\\0\\1\end{bmatrix}) = \begin{bmatrix}1\\2\\-5\end{bmatrix}$

gir at $B = \begin{bmatrix}3&-2&1\\0&1&2\\1&0&-5\end{bmatrix}$ .

b) Matrisen til transformasjonen $ST$ er

$$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} .$$

Matrisen til transformasjonen $TS$ er

$$BA = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

c) Alle vektorer $v \neq 0$ langs y-aksen ligger i ro når vi bruker $S$. De er derfor egenvektorer med egenverdi $1$.

$$\lambda = 1: \; v = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{hvor } t \neq 0.$$

Alle vektorer $v \neq 0$ i xz-planet blir avbildet på 0-vektoren når vi bruker $S$. De er derfor egenvektorer med egenverdi $0$.

$$\lambda = 0: \; v = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{hvor } s \text{ og } t \text{ ikke begge er } 0 .$$

Kontroll :

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} s \\ 0 \\ t \end{bmatrix}$$

Oppgave 3 :

a) i) $\displaystyle\int_0^b x\,e^{-x}\,dx = -\left[x\,e^{-x}\right]_0^b + \int_0^b e^{-x}\,dx = -b\,e^{-b} - \left[e^{-x}\right]_0^b = -(b+1)\,e^{-b} + 1$

$u = x \;,\; v' = e^{-x}$

$u' = 1 \;,\; v = -e^{-x}$

$$\int_0^\infty x\,e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x\,e^{-x}\,dx = 1 - \lim_{b\to\infty}\frac{b+1}{e^b} = 1 - \lim_{b\to\infty}\frac{1}{e^b} = 1$$

ii) $\displaystyle\int_1^e x^{-1}\ln x\,dx = \int_0^1 z\,dz = \left[\tfrac{1}{2}z^2\right]_0^1 = \tfrac{1}{2}$

$z = \ln x \;,\; dz = x^{-1}dx$

b) $\displaystyle A = \int_0^1 [f(x) - g(x)]\,dx = \int_0^1 (x - x^2)\,dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}$

$\displaystyle V = \int_0^1 2\pi x[f(x) - g(x)]\,dx = 2\pi\int_0^1 (x^2 - x^3)\,dx = 2\pi\left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^1$

$= 2\pi\left(\tfrac{1}{3} - \tfrac{1}{4}\right) = 2\pi\cdot\tfrac{1}{12} = \tfrac{\pi}{6}$

Oppgave 4 :

a) $3A = 3\begin{bmatrix} 4 & 1 & -2 \\ -3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 12 & 3 & -6 \\ -9 & 6 & 15 \end{bmatrix}$

$7A - B^4$ er ikke definert.

$$AB = \begin{bmatrix} 4 & 1 & -2 \\ -3 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 0 & 4 \\ 1 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 0 \\ 0 & -1 & 4 \end{bmatrix}$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & -2 & 1 & 0 & 0 \\ -1 & 0 & 4 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \quad 1 \to R_2,\; -1 \to R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \quad \tfrac{1}{2}$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & \tfrac{1}{2} & \tfrac{1}{2} & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \quad 1 \to R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \begin{array}{l} \leftarrow \\ \leftarrow \\ -1 \quad 2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \begin{array}{l} \leftarrow \\ -2 \\ \phantom{0} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 4 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right]$$

$$B^{-1} = \frac{1}{2}\begin{bmatrix} -4 & 2 & 8 \\ 2 & 0 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

b) $$\begin{vmatrix} a+2 & 2 & -2 \\ -1 & a+1 & -(a-3) \\ 1 & 1 & a-1 \end{vmatrix} = \begin{vmatrix} a & 0 & -2a \\ -1 & a+1 & -(a-3) \\ 1 & 1 & a-1 \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ -1 & a+1 & -(a-1) \\ 1 & 1 & a+1 \end{vmatrix}$$

$$= a\begin{vmatrix} a+1 & -(a-1) \\ 1 & a+1 \end{vmatrix} = a[(a^2+2a+1)+(a-1)] = a(a^2+3a) = a^2(a+3)$$

Eksakt en løsning når $a \neq 0$ og $a \neq -3$.

<u>$a = 0$</u> :

$$\begin{bmatrix} 2 & 2 & -2 & -2 \\ -1 & 1 & 3 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{array}{l} \leftarrow \\ \\ \leftarrow \end{array}$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 3 & 1 \\ 2 & 2 & -2 & -2 \end{bmatrix} \begin{array}{l} 1 \quad -2 \\ \leftarrow \\ \leftarrow \end{array}$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \tfrac{1}{2}$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -1$$

$$\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Uendelig mange løsninger når $a = 0$.

$$\begin{array}{rcrcr} x & - & 2z & = & -1 \\ y & + & z & = & 0 \end{array}$$

$x = -1 + 2z$

$y = -z$

$x = -1 + 2t$
$y = -t$
$z = t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\underline{a = -3}$ :

$$\begin{bmatrix} -1 & 2 & -2 & -2 \\ -1 & -2 & 6 & 1 \\ 1 & 1 & -4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -4 & -1 \\ -1 & -2 & 6 & 1 \\ -1 & 2 & -2 & -2 \end{bmatrix} \quad 1 \quad 1$$

$$\begin{bmatrix} 1 & 1 & -4 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & 3 & -6 & -3 \end{bmatrix} \begin{matrix} \\ -1 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & -4 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -6 & -3 \end{bmatrix} \begin{matrix} \\ -3 \urcorner \\ \leftarrow\lrcorner \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & -4 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Ingen løsning når $a = -3$.

c) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1} \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -4 & 2 & 8 \\ 2 & 0 & -2 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$x = 1$ , $y = -1$ , $z = \frac{1}{2}$ .

Oppgave 5 :

Arealet av trekanten er :

$A(x) = \frac{1}{2} \cdot 2x \cdot x = x^2$

Vi deriverer med hensyn på tiden t.

$\frac{dA}{dt} = \frac{dA}{dx}\frac{dx}{dt} = 2x\frac{dx}{dt}$

Når $x = 1$ og $\frac{dA}{dt} = -0.1$ , er

$\frac{dx}{dt} = \frac{dA/dt}{2x} = \frac{-0.1}{2 \cdot 1} = -0.05$

Høyden avtar med 0.05 m/s.

Når $x = 3$ og $\frac{dx}{dt} = 1$ , er

$\frac{dA}{dt} = 2x\frac{dx}{dt} = 2 \cdot 3 \cdot 1 = 6$

Arealet øker med 6 $m^2/s$.

Oppgave 6 :

a) $p(\lambda) = \begin{vmatrix} \lambda - 5 & 4 \\ 3 & \lambda - 1 \end{vmatrix} = (\lambda - 5)(\lambda - 1) - 12 = \lambda^2 - 6\lambda - 7 = 0$

$\lambda = \frac{6 \pm \sqrt{36+28}}{2} = \frac{6 \pm 8}{2} = 3 \pm 4$

$A$ har egenverdiene $\lambda = -1$ og $\lambda = 7$.

$\underline{\lambda = -1}$ :

$$\begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3x - 2y = 0 \ , \ y = \frac{3}{2}x$

$x = 2t$
$y = 3t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 2 \\ 3 \end{bmatrix} \ , \ t \neq 0$$

$\underline{\lambda = 7}$ :

$$\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x + 2y = 0 \ , \ y = -\frac{1}{2}x$

$x = 2t$
$y = -t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \end{bmatrix} \ , \ t \neq 0$$

$$P = \begin{bmatrix} 2 & 2 \\ 3 & -1 \end{bmatrix} \ , \ D = \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix}$$

b) $\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$

La $\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} u \\ v \end{bmatrix}$ .

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = P^{-1}AP\begin{bmatrix} u \\ v \end{bmatrix} = D\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -u \\ 7v \end{bmatrix}$$

$u = c_1\, e^{-t}$
$v = c_2\, e^{7t}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = P\begin{bmatrix} u \\ v \end{bmatrix} = c_1\, e^{-t}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2\, e^{7t}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$x = 2\, c_1\, e^{-t} + 2\, c_2\, e^{7t}$
$y = 3\, c_1\, e^{-t} - c_2\, e^{7t}$

Oppgave 7 :

a) $x^2y' + e^{-y} = 0$

$x^2y' = -e^{-y}$

$\int e^y\, dy = -\int x^{-2}\, dx$

$e^y = x^{-1} + C$

Generell løsning :

$y = \ln(x^{-1} + C)$

$y(1) = \ln(1 + C) = 1\ , \quad C = e - 1$

Løsning av initialverdiproblemet :

$y = \ln(x^{-1} + e - 1)$

b) $y'' + y = \cos(2x) \qquad (1)$

Tilhørende homogene likning :

$y'' + y = 0 \qquad (2)$

Karakteristisk likning :

$r^2 + 1 = 0$

$r = \pm i$

Generell løsning av (2) :

$y_h = A\cos x + B\sin x$

$y_p = C\cos 2x + D\sin 2x$

$y_p' = -2C\sin 2x + 2D\cos 2x$

$y_p'' = -4C\cos 2x - 4D\sin 2x$

Vi setter inn i differensiallikningen (1) :

$(-4C\cos 2x - 4D\sin 2x) + (C\cos 2x + D\sin 2x) = \cos 2x$

$-3C\cos 2x - 3D\sin 2x = \cos 2x$

$-3C = 1$ , $C = -\frac{1}{3}$

$-3D = 0$ , $D = 0$

$y_p = -\frac{1}{3}\cos 2x$

Generell løsning av (1) :

$y = A\cos x + B\sin x - \frac{1}{3}\cos 2x$

$y' = -A\sin x + B\cos x + \frac{2}{3}\sin 2x$

$y(0) = A - \frac{1}{3} = 1$ , $A = \frac{4}{3}$

$y'(0) = B = 1$

Løsning av initialverdiproblemet :

$y = \frac{4}{3}\cos x + \sin x - \frac{1}{3}\cos 2x$