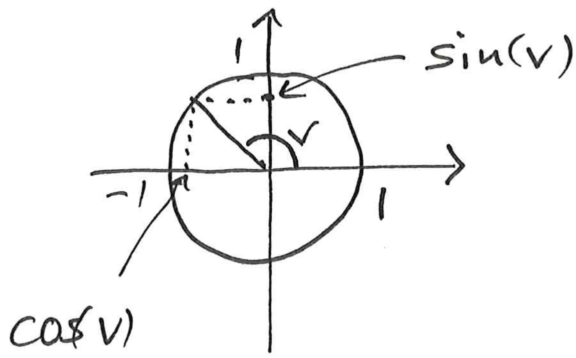
10.02.2020 Trigonometriske funksjoner

1) 11.3-6, 11.8, 11.1-2



$-1 \leq \sin(v), \cos(v) \leq 1$

$f(x)$ er en <u>periodisk</u> funksjon med periode p hvis $f(x) = f(x+p)$ alle x

$0<p$ er minst mulig med denne egenskapen.

Noen identiteter:

* Reflekterer om x-aksen. Vinkel v sendes til $-v$

$\sin(-v) = -\sin v$ $\qquad$ $\cos(-v) = \cos(v)$

* Reflekterer om y-aksen. v sendes til $\pi - v$

$\sin(\pi - v) = \sin v$ $\qquad$ $\cos(\pi - v) = -\cos v$

* Reflekterer om linjen $x=y$

Vinkel v sendes til $\frac{\pi}{2} - v$

$\cos\left(\frac{\pi}{2} - v\right) = \sin v$, $\quad \sin\left(\frac{\pi}{2} - v\right) = \cos v$

Grafen til $\cos(x)$ er grafen til $\sin x$ forskyvd med $\frac{\pi}{2}$ til venstre.

$\underline{\sin\left(\frac{\pi}{2} + x\right)} = \cos(-x) = \underline{\cos(x)}$

2)

$A\sin(kx + c) + d$

$|A|$ amplituden

$y = d$ jamvektslinje (likevekt)

$k > 0$ perioden er $\frac{2\pi}{k}$

$A\sin\left(k\left(x + \frac{c}{k}\right)\right) + d$

Faseforskyving er $-\frac{c}{k}$ ($-c$ i boka ?)

Eks $-2\sin(x) + 3$ $x \in [0, 2\pi]$

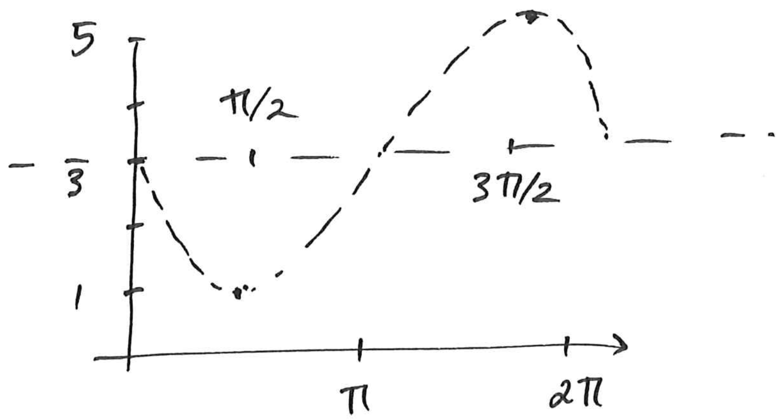
$y = 3$ likevekslinjen.

Amplituden $|-2| = 2$

Perioden 2π

bunnpunkt i $\left(\frac{\pi}{2}, 1\right)$

toppunkt $\left(\frac{3\pi}{2}, 5\right)$



Eks $3\sin(4x-2)$

likevekslinjen $y=0$ (x-aksen)

amplituden er $|3| = 3$.

perioden $p = \frac{2\pi}{4} = \frac{\pi}{2} \approx 1.57$

Faseforskyvning $-\frac{c}{k} = -\frac{-2}{4} = \underline{\underline{\frac{1}{2}}}$

11.8 Derivert av sin, cos og tan

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$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

vinkel i radianer

Sin med vinkel målt i grader

$$\sin\left(\frac{\pi}{180°}x\right)$$

$$\frac{d}{dx}\sin\Big(\underbrace{\frac{\pi}{180°}x}_{U(x)}\Big) = \frac{d\sin(U(x))}{dU}\cdot\frac{dU(x)}{dx}$$

$$= \cos\left(\frac{\pi}{180°}x\right)\cdot\underbrace{\frac{\pi}{180°}}_{\text{ekstra faktor}}.$$

Derfor bruker vi vinkelenheten radianer.

Eks

$$f(x) = A\sin(\overbrace{kx+c}^{u(x)}) + d$$

$$f'(x) = A\cos(kx+c)\cdot\underbrace{(kx+c)'}_{k} + \underbrace{d'}_{0}$$

$$= \underline{A\cdot k\cos(kx+c)}$$

Oppg

Deriver 1) $\sin(3x)$ 2) $\cos(x^2+1)$

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3) $\frac{1}{\sin x}$

1) $(\sin(3x))' = \cos(3x)\cdot(3x)'$

$= 3\cos(3x)$.

2) $(\cos(x^2+1))' = -\sin(x^2+1)\cdot\underbrace{(x^2+1)'}_{2x}$

$= -2x\sin(x^2+1)$

3) $((\sin x)^{-1})' = \frac{-1}{(\sin x)^2}\cdot(\sin x)'$

$= \frac{-\cos x}{(\sin x)^2}$

Eks $(\overset{u}{x}\,\overset{v}{\overbrace{\cos x}})' = (U\cdot V)' = U'\cdot V + U\cdot V'$

$= (x)'\cos x + x(\cos x)'$

$= 1\cos x + x(-\sin x)$

$= -x\sin x + \cos x$

Eks $(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = [(\sin x)((\cos x)^{-1})]'$

$\underbrace{(\sin x)'}_{\cos x}\cdot\frac{1}{\cos x} + (\sin x)(-(\cos x)^{-2}\cdot\underbrace{(\cos x)'}_{-\sin x}$

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$$(\tan x)' = 1 + \frac{\sin^2 x}{\cos^2 x}$$

$$= 1 + \left(\frac{\sin x}{\cos x}\right)^2 = \underline{1 + \tan^2 x}$$

$$1 + \tan^2 x = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \underline{\frac{1}{\cos^2 x}}$$

$$\boxed{(\tan x)' = 1 + \tan^2 x = \frac{1}{\cos^2 x}}$$

11.1-2 Addisjonsformel for sin og cos

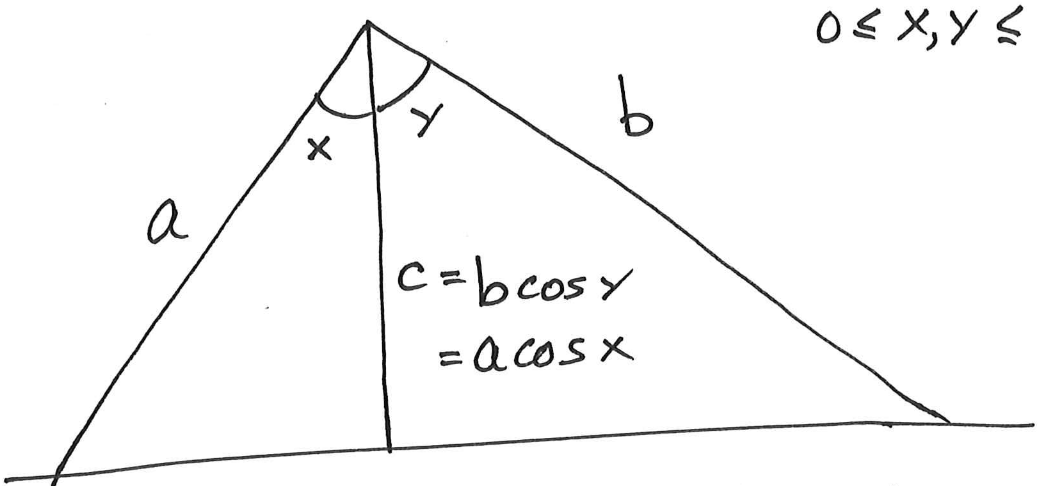
$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cdot\cos(x)$$

$y=0$ ✓

$y=\frac{\pi}{2}$ $\quad \sin\left(x+\frac{\pi}{2}\right) = \sin x \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} + \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} \cos x$

$= \cos(x)$ ✓

$0 \leq x, y \leq \frac{\pi}{2}$



Areal til den store trekanten er lik

$$\frac{1}{2}\cdot a\cdot b\cdot \sin(x+y)$$

= summen av de to mindre trekantene

$$= \frac{1}{2} a \cdot \underbrace{c}_{b\cos(y)} \cdot \sin(x) + \frac{1}{2} b \cdot \underbrace{c}_{a\cos(x)} \sin(y)$$

$a, b > 0$

$$\cancel{a}\cdot\cancel{b}\sin(x+y) = \cancel{a}\cdot\cancel{b}(\sin x\cdot\cos y + \sin y\cos x)$$

Dette viser addisjonsformelen for sin.

Eks $\quad \sin\left(x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(\sin x + \cos x)$

$$\sin(2x) = \sin(x)\cos(x) + \sin(x)\cos(x)$$

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$$= 2\sin(x)\cos(x)$$

$$\sin x \cdot \cos x = \frac{1}{2}\sin(2x)$$

Addisjonsformel for cos.

Deriverer addisjonsformelen for sin. m.h.t x

$$\frac{d\sin(x+y)}{dx} = \cos(x+y)$$

$$= \cos x \cdot \cos y + \sin y(-\sin x)$$

$$\underline{\cos(x+y) = \cos x \cos y - \sin x \sin y}$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

Pytagoras $\cos^2 x + \sin^2 x = 1$

så

$$\cos(2x) = 2\cos^2 x - 1$$

$$\cos(2x) = 1 - 2\sin^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Eks. $(\sin^2 x)' = ((\sin x)^2)'$

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$= 2\sin x \cdot (\sin x)' = 2\sin x \cos x$

$= \underline{\sin(2x)}$

$\frac{1}{2}(1-\cos(2x))' = -\frac{1}{2}(\cos(2x))'$

$= -\frac{1}{2}(-\sin(2x)(2x)') = \underline{\sin(2x)}$ ✓

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)}$$

$$= \frac{\sin x \cos y + \sin y \cdot \cos x}{\cos x \cdot \cos y - \sin x \cdot \sin y}$$

$$= \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$$

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Den deriverte av sin x

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \left(\cos(x) \frac{\sin(h)}{h} + \frac{\sin x(\cos(h) - 1)}{h} \right)$$

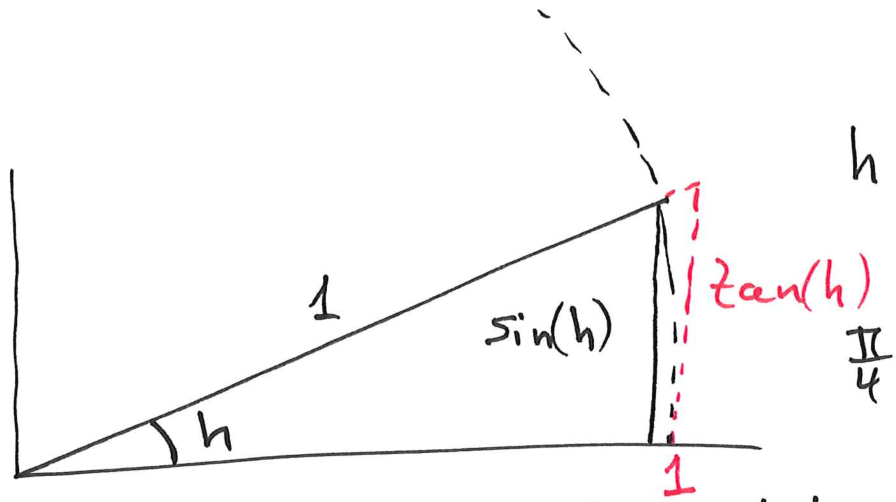
$$= \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} + \sin(x) \lim_{h\to 0} \frac{\cos(h)-1}{h}$$

Resultat $$\lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

$$\frac{\sin(-h)}{-h} = \frac{-\sin(h)}{-h} = \frac{\sin(h)}{h}$$

Tilstrekkelig å vise $\lim_{h\to 0^+} \frac{\sin(h)}{h} = 1.$

(11)



h i radianer

$\frac{\pi}{4} > h > 0$

Arealet til sirkelsegmentet $\frac{1}{2}h$

Areal inneholt trekant $\frac{1}{2}\cos(h)\cdot\sin(h)$

Areal stor — $\frac{1}{2}\cdot 1\cdot\tan(h)$

Så

$$\frac{1}{2}\sin(h)\cos(h) < \frac{1}{2}h < \frac{1}{2}\frac{\sin(h)}{\cos(h)}$$

$$\frac{\sin(h)}{h} < \frac{1}{\cos(h)}, \quad \cos(h) < \frac{\sin(h)}{h}$$

$$\cos(h) < \frac{\sin(h)}{h} < \frac{1}{\cos(h)}$$

Siden $\lim\limits_{h\to 0^+} \cos(h) = 1 = \lim\limits_{h\to 0^+} \frac{1}{\cos(h)}$

så må $\lim\limits_{h\to 0} \frac{\sin(h)}{h} = 1$

Vi viser nå at $\lim\limits_{h\to 0} \frac{\cos(h)-1}{h} = 0$

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Grensen er lik $\lim\limits_{x\to 0} \frac{(\cos(h)-1)(\cos(h)+1)}{h\cdot(\cos(h)+1)}$

$$= \lim_{x\to 0} \frac{\cos^2(h) - 1}{h\,(1+\cos(h))} = \lim_{x\to 0} \frac{-\sin^2 h}{h\,(1+\cos(h))}$$

$$= \lim_{x\to 0} \underbrace{-\sin(h)}_{\to 0} \left(\underbrace{\frac{\sin(h)}{h}}_{\to 1}\right) \cdot \underbrace{\frac{1}{1+\cos(h)}}_{\frac{1}{2}}$$

$$= 0$$

Derfor er $(\sin x)'$

$$= \left(\lim_{h\to 0} \frac{\sin h}{h}\right) \cdot \cos x + \lim_{h\to 0} \frac{\cos(h)-1}{h} \cdot \sin x$$

$$= \underline{\underline{\cos(x)}}$$

$$\cos x = \sin\left(x+\frac{\pi}{2}\right)$$

$$(\cos x)' = \cos\left(x+\frac{\pi}{2}\right) \cdot \overbrace{\left(x+\frac{\pi}{2}\right)'}^{1}$$

$$= \cos(x) \cdot \cos\left(\frac{\pi}{2}\right) - \sin(x)\sin\left(\frac{\pi}{2}\right)$$

$$= \underline{\underline{-\sin x}}$$

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$$\lim_{h\to 0} \frac{1-\cos h}{h^2} \overset{\text{som tidlige}}{=}$$

$$= \lim_{h\to 0} \frac{1-\cos^2 h}{h^2} \cdot \frac{1}{1+\cos h}$$

$$= \lim_{h\to 0} \underbrace{\left(\frac{\sin(h)}{h}\right)^2}_{1} \cdot \underbrace{\frac{1}{1+\cos(h)}}_{\frac{1}{2}} = \frac{1}{2}.$$

h liten $\qquad \cos(h) \sim 1 - \frac{h^2}{2}$