7.6 Sammensatte funksjon – kjerneregelen

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(1) $f(u(x))$ sammensatt funksjon

ytre funksjon

indre funksjon, kjerne

"først u så f"

Alternativ skrivemåte $f \circ u(x) = f(u(x))$

Eks $f(x) = x^3$ $\quad u(x) = 1 + 2x$

$$f(u(x)) = (u(x))^3 = (1+2x)^3$$

$$u(f(x)) = 1 + 2(f(x)) = 1 + 2x^3$$

Definisjonsmengden til $f(u(x))$

Alle x i D_u slik at $u(x)$ er i D_f

$\sqrt{1+x}$ $\quad f(x) = \sqrt{x}$ $\quad u(x) = 1 + x$

$D_f = [0, \infty\rangle$ $\quad D_u = \mathbb{R}$

$D_{f \circ u} = [-1, \infty\rangle$ $\quad$ (da er $u(x) = 1 + x \geq 0$)

Eks $\cos(x^2)$ $\quad u(x) = x^2$

$f(x) = \cos x$

sammensatt

$$\cos(x^2) = f(u(x))$$

$$u \circ f(x) = u(f(x)) = (f(x))^2 = (\cos x)^2$$

$$= \cos^2 x$$

Eks $\frac{1}{x^3+1} = \frac{1}{(x^3+1)} = f(u(x))$

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hvor $f(x) = \frac{1}{x}$, $u(x) = x^3+1$

Eks $f(x) = a_1 x + b_1$ $u(x) = a_2 x + b_2$

$f'(x) = a_1$ $u'(x) = a_2$

$f(u(x)) = a_1 (u(x)) + b_1 = a_1(a_2 x + b_2) + b_1$

$= \underline{a_1 a_2} x + a_1 b_2 + b_1$

Kjerneregelen $(f(u(x)))' = f'(u(x)) \cdot u'(x)$

$f'(x) = \frac{df}{dx}$ $\frac{d\, f\circ u}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}$

Leibniz notasjonen

Eks $\left(\frac{1}{x^3+1}\right)' = \frac{-1}{(x^3+1)^2} \cdot (x^3+1)'$

$= \frac{-3x^2}{(x^3+1)^2}$

Mer detaljert: $f(x) = \frac{1}{x}$ $u(x) = x^3+1$

$\left(\frac{1}{x^3+1}\right)' = (f\, u(x))' = f'(u(x)) \cdot u'(x)$

$f(x) = \frac{1}{x} = x^{-1}$ har derivert $f'(x) = -1 \cdot x^{-1-1} = \frac{-1}{x^2}$

$u(x)$ har derivert $u'(x) = (x^3+1)' = 3x^2$

setter inn: $\left(\frac{1}{x^3+1}\right)' = f'(u(x)) \cdot u'(x) = \frac{-1}{(x^3+1)^2} \cdot (3x^2)$

$= \frac{-3x^2}{(x^3+1)^2}$

Eks $(1+x^2)^3$

3 $\left(\left(1+x^2\right)^3\right)' = 3\left(1+x^2\right)^2 \cdot \underbrace{\left(1+x^2\right)'}_{2x}$

$= \underline{3 \cdot 2\left(1+x^2\right)^2 \cdot x}$

(Lettere enn å gange ut $(1+x^2)^3$
$= 1+3x^2+3x^4+x^6$
og så derivere denne.)

Oppg. Deriver 1) $(1+x)^5$
2) $(4-x)^5$

1) $\left((1+x)^5\right)' = 5(1+x)^4 \cdot \overbrace{(1+x)'}^{1} = 5(1+x)^4$

2) $\left((4-x)^5\right)' = 5(4-x)^4 \cdot \underbrace{(4-x)'}_{-1} = -5(4-x)^4$.

Eks $\dfrac{1}{\sqrt{2x-1}}$

$f(x) = \dfrac{1}{\sqrt{x}} = \dfrac{1}{x^{1/2}} = \left(x^{1/2}\right)^{-1}$
$= x^{-1/2}$

$u(x) = 2x-1$

$\left(\dfrac{1}{\sqrt{2x-1}}\right)' = f'(u(x)) \cdot u'(x)$

$f'(u) = \left(u^{-1/2}\right)' = \dfrac{-1}{2} \cdot u^{-3/2} = \dfrac{-1}{2u\sqrt{u}}$

$u'(x) = 2$

$\left(\dfrac{1}{\sqrt{2x-1}}\right)' = \dfrac{-1}{2(2x-1)\sqrt{2x-1}} \cdot \underbrace{(2x-1)'}_{2}$

$= \underline{\dfrac{-1}{(2x-1)\sqrt{2x-1}}}$ $\left(= -(2x-1)^{-3/2} \ldots\right)$

Eks $(x^r)^s = x^{r \cdot s}$ ytre $f(x) = x^s$

indre $u(x) = x^r$

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Den deriverte:

Direkte $(x^{r \cdot s})' = r \cdot s\, x^{r \cdot s - 1}$

Ved bruk av kjerneregelen

$$s\,(x^r)^{s-1} \cdot (x^r)' = s \cdot x^{r(s-1)} \cdot r x^{r-1}$$

$$= r \cdot s \cdot x^{r \cdot s - r + (r-1)} = r \cdot s\, x^{r \cdot s - 1}$$ ✓

Dette blir selvsagt det samme som vi får ved å derivere $x^{r \cdot s}$ direkte.

Lineær kjerne $f(u(x))$ hvor $u(x) = ax + b$ lineær funksjon

$u'(x) = a$

Kjerneregelen gir

$$\underline{\left(f(ax+b)\right)' = a \cdot f'(ax+b)}$$

Eks $\frac{1}{(3x+7)^8} = (3x+7)^{-8} = f(3x+7)$

hvor $f(u) = u^{-8}$

$$\left(\frac{1}{(3x+7)^8}\right)' = -8\, u(x)^{-9} \cdot \underline{3} = \underline{\frac{-24}{(3x+7)^9}}$$

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7.8 Produktregel

Produkt funksjon $(f\cdot g)(x) = f(x)\cdot g(x)$

Definisjonsmengden til $f\cdot g$ er de felles punktene i D_f og D_g : $D_f \cap D_g$

Produktregelen $(f\cdot g)'(x) = f'\cdot g(x) + f\cdot g'(x)$

$$(x\cdot x)' = (x^2)' = 2x = 1\cdot x + x\cdot 1 = 2x$$
$$\neq (x)'(x)' = 1.$$

$f\cdot g(x) = x^r\cdot x^s = x^{r+s}$ $f(x) = x^r,\ g(x) = x^s$

Kjerneregelen gir
$$(x^r)'\cdot x^s + x^r\cdot (x^s)'$$
$$= r x^{r-1}\cdot x^s + x^r (s x^{s-1})$$
$$= r\cdot x^{r+s-1} + s\, x^{r+s-1}$$
$$= (r+s)\, x^{r+s-1}$$

✓ Dette er $(x^{r+s})'$

Eks $\underbrace{(1-2x)^3}_{f(x)}\ \underbrace{(2x+13)^5}_{g(x)}$

$$\left((1-2x)^3(2x+13)^5\right)' = \left((1-2x)^3\right)'(2x+13)^5 + (1-2x)^3\left((2x+13)^5\right)'$$

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$$= (-2)\cdot 3(1-2x)^2(2x+13)^5$$
$$+ (1-2x)^3 \cdot 5(2x+13)^4 \cdot 2$$
$$= (1-2x)^2(2x+13)^4\Big[\underbrace{\underbrace{\underbrace{-6(2x+13)+10(1-2x)}_{-12x-78+10-20x}}_{-32x-68}}_{-4(8x+17)}\Big]$$
$$= \underline{-4(8x+17)(1-2x)^2(2x+13)^4}$$

oppg. Deriver $\underbrace{x}_{f(x)}\underbrace{(1-x^2)^7}_{g(x)}$

$$f'(x) = (x)' = 1$$
$$g'(x) = \left((1-x^2)^7\right)' = 7(1-x^2)^6 \cdot \underbrace{(1-x^2)'}_{(-2x)}$$
$$= -14x(1-x^2)^6$$

produktregelen gir

$$\left(x(1-x^2)^7\right)' = (f\cdot g)' = f'\cdot g + f\cdot g'$$
$$1(1-x^2)^7 + x\left(-14x(1-x^2)^6\right)$$
$$= (1-x^2)^6\left[(1-x^2) - 14x^2\right]$$
$$= \underline{(1-15x^2)(1-x^2)^6}$$

⑦

Eks $\frac{x^3}{(3x^2+1)^4} = x^3 \cdot \frac{1}{(3x^2+1)^4}$

$(x^3)' = 3x^2$

$\left(\frac{1}{(3x^2+1)^4}\right)' = ((3x^2+1)^{-4})' = -4(3x^2+1)^{-5} \cdot 6 \cdot x$

kjerneregelen.

Produktregelen gir

$\left(\frac{x^3}{(3x^2+1)^4}\right)' =$

$3x^2 \cdot (3x^2+1)^{-4} + x^3\left(-24x(3x^2+1)^{-5}\right)$

$= (3x^2+1)^{-5}\left[\underbrace{3x^2(3x^2+1) - 24x^4}_{\underbrace{9x^4+3x^2-24x^4}_{\underbrace{-15x^4+3x^2}_{3(-5x^4+x^2)}}}\right]$

$3(\cancel{-x^4+}$

$= \frac{3x^2(1-5x^2)}{(3x^2+1)^5}$

Eks $\quad x\sqrt{x-1} = x \cdot (x-1)^{1/2}$

$$\left(x\sqrt{x-1}\right)' = 1\sqrt{x-1} + x\underbrace{\left(\left((x-1)^{1/2}\right)'\right)}_{\frac{1}{2}(x-1)^{-1/2}(x-1)'}$$

$$= \sqrt{x-1} + x\left(\frac{1}{2}(x-1)^{-1/2} \cdot 1\right)$$

$$= \sqrt{x-1} + \frac{x}{2\sqrt{x-1}}$$

$$= \frac{2(x-1)}{2\sqrt{x-1}} + \frac{x}{2\sqrt{x-1}}$$

$$= \underline{\frac{3x-2}{2\sqrt{x-1}}}$$

7.9 Kvotientregelen

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}.$$

Utleding:

$$\left(f \cdot \frac{1}{g}\right)' \overset{\text{prod.}}{=} f' \cdot \frac{1}{g} + f \cdot \underbrace{\left(\frac{1}{g}\right)'}_{\text{kjerneregelen}}$$

gir $\left((g(x))^{-1}\right)' = -1 \cdot (g(x))^{-2} \cdot g'(x)$

$$= f' \cdot \frac{1}{g} + f \cdot \frac{-g'}{g^2} = \frac{f' \cdot g - f g'}{g^2}$$

Eks $\left(\frac{x^2+10x+21}{x+3}\right)'$

$$= \frac{(x^2+10x+21)'\cdot(x+3) - (x^2+10x+21)(x+3)'}{(x+3)^2}$$

$$= \frac{(2x+10)(x+3) - (x^2+10x+21)\cdot 1}{(x+3)^2}$$

$$= \frac{2x\cdot x + 2x\cdot 3 + 10\cdot x + 30 - x^2 - 10x - 21}{(x+3)^2}$$

$$= \frac{x^2+6x+9}{(x+3)^2} = 1$$ siden $(x+3)^2 = x^2+6x+9$.

Dette var mye arbeid. I dette tilfellet kan vi regne ut den deriverte på en mye enklere måte ved først å forenkle uttrykket vårt

polynom divisjon gir $\frac{x^2+10x+21}{x+3} = \underline{x+7}$

Det er da umiddelbart klart at den deriverte må være $\underline{1}$