Delbrøksoppspalting

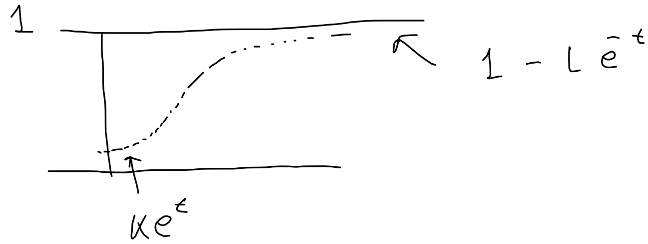
Eksempel Logistisk diff. likning

$$y' = ky\left(1 - \frac{y}{N}\right)$$

$k = 1, \quad N = 1$

$$y' = y(1 - y)$$

Forventer



Vi løser diff likningen:

$$\frac{dy}{dx} = y(1-y)$$

$$\int \frac{1}{y(1-y)}\,dy = \int 1 \cdot dx$$

Vi benytter delbrøksoppspaltingen:

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$$

$$\left(\frac{1-y}{y(1-y)} + \frac{y}{y(1-y)}\right)$$

$$\int \frac{1}{y(1-y)}\,dy = \int \frac{1}{y} + \frac{1}{1-y}\,dy$$

$$= \ln|y| - \ln|1-y| + c$$

$$= \ln\left|\frac{y}{1-y}\right| + c$$

Dette skal være lik $\int 1\,dx = x + c$,

$$\ln\left|\frac{y}{1-y}\right| = x + c$$

$$\left|\frac{y}{1-y}\right| = e^{\ln\left|\frac{y}{1-y}\right|} = e^{x+c} = e^{c} \cdot e^{x}$$

$$\frac{y}{1-y} = Ke^{x} \qquad K \in \mathbb{R}$$

Løser ut y

$$\frac{y}{1-y} = ke^x$$

$$y = (1-y)ke^x$$

$$y(1+ke^x) = ke^x$$

$$y(x) = \frac{ke^x}{1+ke^x} = 1 - \frac{1}{1+ke^x}$$

Oppgave: Finn løsningene til

$$y' = ky\left(1 - \frac{y}{N}\right) \quad \text{for } k, N > 0.$$

$\frac{1}{x(x-2)}$ kan skrives som

$$\frac{A}{x} + \frac{B}{x-2}$$

Løser for A, B.

$$\frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} = \frac{A(x-2)}{x(x-2)} + \frac{Bx}{x(x-2)}$$

$$= \frac{A(x-2) + Bx}{x(x-2)}$$

Så $1 = A(x-2) + Bx$ for <u>alle</u> x

(for $x \neq 2$ og 0 like for vendelig mange verdier ⇒ like. Alle ikke-null polynomer har et endelig antall røtter)

Metode 1: setter inn verdier for x som gjør det lett å finne koeffisientene.

$x = 0$: $1 = A(0-2) = -2A$, $A = -\frac{1}{2}$

$x = 2$: $1 = 2 \cdot B$ $B = \frac{1}{2}$

Metode 2: Sammenligner koeffisientene til monomene x^i

$$\underbrace{0 \cdot x + \overbrace{1 = (A+B) \cdot x - 2 \cdot A}}$$

$A + B = 0$

$-2A = 1$ gir $A = -\frac{1}{2}$, $B = \frac{1}{2}$

$$\frac{1}{x(x-2)} = \frac{-1/2}{x} + \frac{1/2}{x-2} = \frac{1}{2}\left[\frac{-1}{x} + \frac{1}{x-2}\right]$$

$$\int \frac{1}{x(x-2)}\,dx = \frac{1}{2}\int \frac{-1}{x} + \frac{1}{x-2}\,dx$$

$$= \frac{1}{2}\left[-\ln|x| + \ln|x-2|\right] + C$$

$$= \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C$$

$$\left(= \ln\sqrt{\left|1 - \frac{2}{x}\right|} + C \right)$$

Finn $\int \frac{1}{x^2-1}\,dx$

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$= \frac{A(x-1)}{(x+1)(x-1)} + \frac{B(x+1)}{(x+1)(x-1)}$$

$\Leftrightarrow$ $\quad 1 = A(x-1) + B(x+1)$ for alle x.

$x = 1$ $\quad 1 = 2 \cdot B$ $\quad B = 1/2$

$x = -1$ $\quad 1 = -2 \cdot A$ $\quad A = -1/2$

$$\int \frac{1}{x^2-1}\,dx = \int \frac{1}{2}\left[\frac{-1}{x+1} + \frac{1}{x-1}\right]dx$$

$$= \frac{1}{2}\left[-\ln|x+1| + \ln|x-1|\right] + C$$

$$= \underline{\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C}$$

oppg. $\int \frac{3x-2}{x^2-4}\,dx$

$$\frac{3x-2}{x^2-4} = \frac{3x-2}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$= \frac{A(x-2)}{(x+2)(x-2)} + \frac{B(x+2)}{(x+2)(x-2)}$$

$$3x-2 = A(x-2) + B(x+2) \quad \text{for alle } x$$

$x = 2 \qquad 3\cdot 2 - 2 = 4 = 4B \qquad B = 1$

$x = -2 \qquad 3(-2) - 2 = -8 = -4\cdot A \qquad A = +2$

$$\int \frac{3x-2}{x^2-4}\,dx = \int \frac{2}{x+2} + \frac{1}{x-2}\,dx$$

$$= 2\int \frac{1}{x+2}\,dx + \int \frac{1}{x-2}\,dx = \underline{2\ln|x+2| + \ln|x-2| + c}$$

$$\left(= \ln\left(|x+2|^2\cdot|x-2|\right) + c = \ln\left|(x+2)(x^2-4)\right| + c\right)$$

$$\int \frac{1}{x(x-1)^2}\,dx$$

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$= \frac{A(x-1)^2}{x(x-1)^2} + \frac{Bx(x-1)}{x(x-1)^2} + \frac{C\cdot x}{(x-1)^2\cdot x}$$

$$1 = A(x-1)^2 + Bx(x-1) + C\cdot x \quad \text{for alle } x$$

$x=0$: $\quad 1 = A(-1)^2 = A \qquad \underline{A=1}$

$x=1 \quad 1 = C\cdot 1 \qquad \underline{C=1}$

$x=2$ (eller et annet tall) $\quad 1 = A(2-1)^2 + 2\cdot B + 2\cdot C$

$$2B = 1 - A - 2C = 1-1-2 = -2$$

$$\underline{B=-1}$$

$$\int \frac{1}{x(x-1)^2}dx = \int \frac{1}{x} + \frac{-1}{x-1} + \frac{1}{(x-1)^2}\,dx$$

$$= \underline{\ln|x| - \ln|x-1| + \frac{-1}{x-1} + C}$$