7.8 Produktregelen

To funksjoner f og g

Produktet $f \cdot g$ er definert ved

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Summen av f og g er $f+g$

$$(f+g)(x) = f(x) + g(x)$$

Derivasjon er lineær:

$$(f+g)' = f' + g'$$

$$(c \cdot f)' = c \cdot f' \qquad c \text{ konstant funksjon}$$

Typisk er $(f \cdot g)'$ ulik $f' \cdot g'$

$f = x^2 \qquad g = x^3$

$f' = 2x \qquad g' = 3x^2$

$f' \cdot g' = 2x \cdot 3x^2 = 6x^3 \neq 5x^4 = (\underbrace{x^2 \cdot x^3}_{x^5})'$

Produktregelen

$$(f \cdot g)' = f \cdot g' + f' \cdot g$$

Prøver med $f = x^2$ og $g = x^3$

$$(f \cdot g)' = \underset{f}{x^2} \cdot \underset{g'}{3x^2} + \underset{f'}{2x} \cdot \underset{g}{x^3}$$

$$= 3x^4 + 2x^4 = \underline{5x^4}$$

Anta g er en konstant funksjon

$g'(x) = 0$ alle x

$$(f \cdot g)' = f \cdot \underbrace{g'}_{0} + f' \cdot g = f' \cdot g$$

Eks. Deriver $\underbrace{(1-x^3)^2}_{f} \cdot \underbrace{x}_{g}$

$$(f \cdot g)' = \underbrace{f'} \cdot g + f \cdot g'$$

$$= 2(1-x^3) \cdot (-3x^2) \cdot x + (1-x^3)^2 \cdot 1$$

$$= (1-x^3)[-6x^3 + 1 - x^3]$$

$$= \underline{(1-x^3)(1-7x^3)}$$

Deriver $\sqrt{2x-1} \cdot x^3$ ved bruk av produktregelen. La $f = \sqrt{2x-1} = (2x-1)^{1/2}$

$g = x^3$

$$f' = \left((2x-1)^{1/2}\right)' = \frac{1}{2}(2x-1)^{-1/2} \cdot \overbrace{(2x-1)'}^{2}$$

$$= (2x-1)^{-1/2} = \frac{1}{\sqrt{2x-1}}$$

$$g' = \frac{dg}{dx} = (x^3)' = 3x^2$$

Produktregelen gir

$$\left(\sqrt{2x-1} \cdot x^3\right)' = \frac{1}{\sqrt{2x-1}} \cdot x^3 + \sqrt{2x-1} \cdot 3x^2$$

$$= \frac{1}{\sqrt{2x-1}}\left[x^3 + \underbrace{\left(\sqrt{2x-1}\right)^2}_{2x-1} \cdot 3x^2\right]$$

$$= \frac{1}{\sqrt{2x-1}}\left[7x^3 - 3x^2\right]$$

Deriver $\underbrace{(1-x)^3}_{f} \cdot \underbrace{x^5}_{g}$

$$g' = (x^5)' = 5x^4$$

$$f' = \left(\left(1-x\right)^3\right)' = 3(1-x)^2 \cdot \underbrace{(1-x)'}_{-1}$$

$$f' = -3(1-x)^2$$

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$= -3(1-x)^2 \cdot x^5 + (1-x)^3 \cdot 5x^4$$

$$= (1-x)^2 \cdot x^4 \left[\underbrace{-3x + 5(1-x)}_{5-8x}\right]$$

$$= \underline{(1-x)^2 x^4 (5-8x)}$$

Deriver $\dfrac{x}{2x-1} = x \cdot \dfrac{1}{2x-1}$

$$= x \cdot (2x-1)^{-1}$$

$$\left(\frac{x}{2x-1}\right)' = (x)' \cdot \frac{1}{2x-1} + x \cdot \left((2x-1)^{-1}\right)'$$

$$= 1 \cdot \frac{1}{2x-1} + x \cdot \frac{-1}{(2x-1)^2} \cdot 2$$

Finner felles nevner

$$\frac{1}{(2x-1)^2}\left[(2x-1) - 2x\right] = \underline{\underline{\frac{-1}{(2x-1)^2}}}$$

Kvotientregelen

$$\left(\frac{f}{g}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2}$$

bevis:

$$\left(f\cdot\frac{1}{g}\right)' = f'\cdot\frac{1}{g} + f\cdot\left(\frac{1}{g}\right)'$$

(Kjerneregelen gir: $\left(\frac{1}{g}\right)' = \left((g)^{-1}\right)'$

$$= -1\cdot g^{-2}\cdot g' = \frac{-g'}{g^2}$$)

Så: $\left(\frac{f}{g}\right)' = f'\cdot\frac{1}{g} + f\cdot\left(\frac{-g'}{g^2}\right)$ finner felles nevner

$$= f'\cdot\frac{1}{g}\cdot\frac{g}{g} + \frac{-f\cdot g'}{g^2}$$

$$= \frac{f'\cdot g - f\cdot g'}{g^2}$$

Bevis for produktregelen

$$\Delta f = f(x+h) - f(x) \quad \text{så} \quad f(x+h) = f(x) + \Delta f$$

$$\Delta g = g(x+h) - g(x) \quad \ldots$$

$$(f \cdot g)' \overset{\text{def}}{=} \lim_{h \to 0} \frac{(f \cdot g)(x+h) - (f \cdot g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{(f(x) + \Delta f)(g(x) + \Delta g) - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x) \cdot \Delta g + g(x) \cdot \Delta f + \Delta f \cdot \Delta g + \overbrace{f(x) \cdot g(x) - f(x) g(x)}^{0}}{h}$$

$$= \lim_{h \to 0} f(x) \cdot \frac{\Delta g}{h} + \lim_{h \to 0} g(x) \cdot \frac{\Delta f}{h} + \lim_{h \to 0} \Delta f \cdot \frac{\Delta g}{h}$$

$$= f(x) \lim_{h \to 0} \frac{\Delta g}{h} + g(x) \lim_{h \to 0} \frac{\Delta f}{h} + \lim_{h \to 0} \Delta f \cdot \lim_{h \to 0} \frac{\Delta g}{h}$$

uavhengige av h.

$$= f(x) \cdot g'(x) + g(x) \cdot f'(x) + 0$$