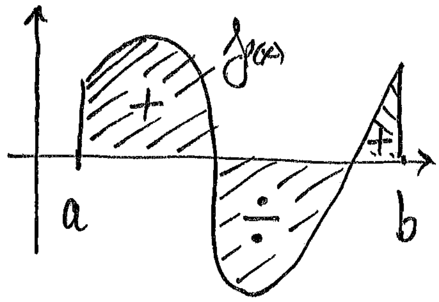
21. mars 2013

Bestemte integral

1

$\int_a^b f(x)\,dx$ "det bestemte integralet av $f(x)$ fra a til b".



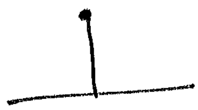
areal (med fortegn) til regionen mellom $y=f(x)$, $y=0$ (x-aksen) avgrenset av $x=a$, $X=b$.

Vi bruker et intuitivt areal-begrep inntil videre.

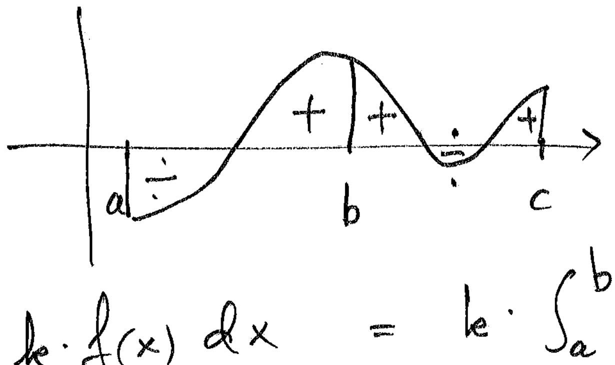
$\int_a^b f(x)\,dx$ eksisterer for alle kontinuerlige funksjoner på $[a,b]$. (og mange andre)

Egenskaper:

1) $\int_a^a f(x)\,dx = 0$

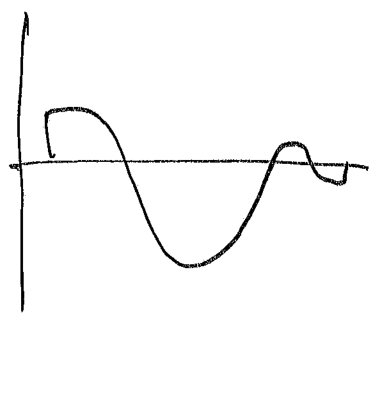


2) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

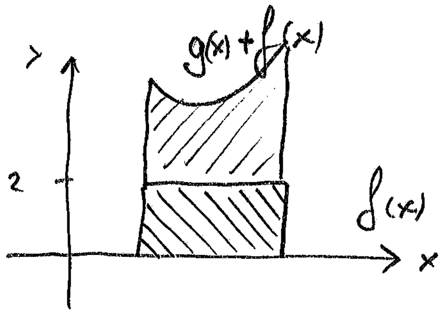


$k=1$

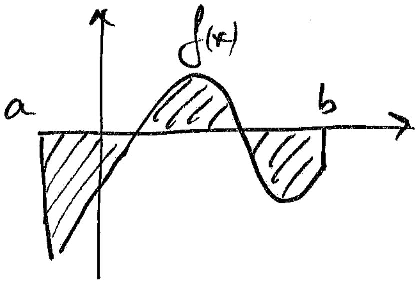
3) $\int_a^b k\cdot f(x)\,dx = k\cdot \int_a^b f(x)\,dx$



4) $$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b f(x)+g(x)\,dx$$



3) og 4) sier at bestemte integraler er lineære.

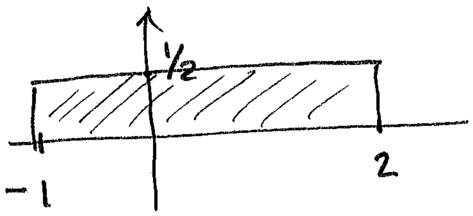


Arealet mellom $y=f(x)$ og x-aksen begrenset av $x=a$ og $x=b$ er

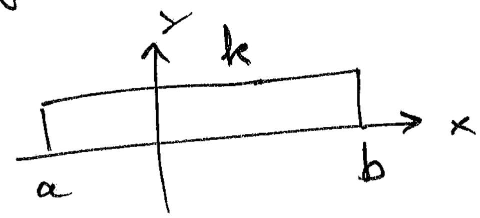
$$\int_a^b |f|\,dx$$

Eksempler

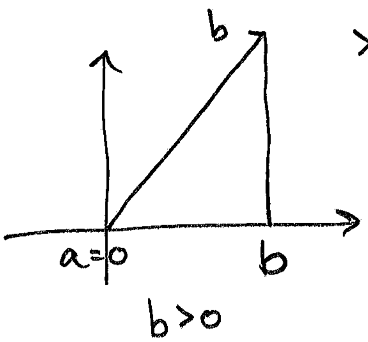
$f(x) = \frac{1}{2}$ konstant.



$$\int_a^b f(x)\,dx = \int_{-1}^{2} \frac{1}{2}\,dx = 3\cdot\frac{1}{2}$$ (bredde × høyde)



$$\int_a^b k\,dx = k(b-a)$$

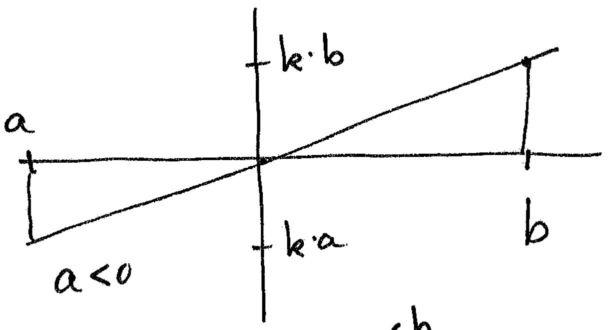


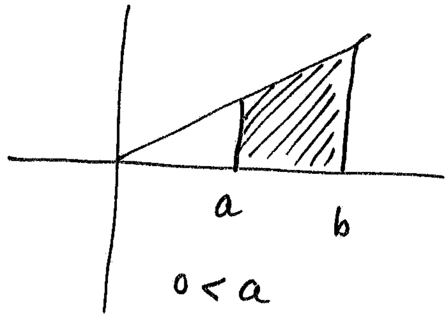
$y = x$

$$\int_0^b x\,dx = \frac{b^2}{2}.$$

③

$y = k \cdot x$





$$\int_0^b y(x) = \frac{k \cdot b^2}{2}$$

$a < 0$: $\quad \int_a^0 y\,dx = -\frac{k \cdot a^2}{2}$

$a > 0$ $\quad \int_a^b y\,dx = \int_0^b y\,dx - \int_0^a y\,dx$

$$\int_a^b (k \cdot x)\,dx = \underline{\frac{k}{2}(b^2 - a^2)}.$$

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$a < b$ $\qquad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

gir at ~~regel~~ egenska 2) er gyldig.

$$\int_3^{-2} f(x)\,dx = -\int_{-2}^3 f(x).$$

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$$\int_{-2}^{1} 5x - 3\ dx = 5\int_{-2}^{1} x\,dx - 3\int_{-2}^{1} 1\,dx$$

$$= 5\left(\frac{1^2 - (-2)^2}{2}\right) - 3(1 - (-2))$$

$$= 5\left(\frac{-3}{2}\right) - 9 = \frac{-15}{2} - \frac{18}{2} = \underline{\underline{\frac{-33}{2}}}$$

4

Fundamentalteoremet i kalkulus

Anta $f(x)$ er en kontinuerlig funksjon i $[a,b]$

Da er $\int_a^z f(x)\,dx$ en antiderivert til $f(x)$ i (a,b).

$$\frac{d}{dz}\int_a^z f(x)\,dx = f(z) \qquad z \text{ i } (a,b).$$

* Alle kontinuerlige funksjoner har en antiderivert.
* Hvis $F(x)$ er en antiderivert til $f(x)$

da er
$$\int_a^b f(x)\,dx = F(b) - F(a)$$
$$= F(x)\Big|_a^b \quad \text{notasjon.}$$

$$\left(\int_a^z f(x)\,dx = F(z) + C \right.$$

La $z = a$: $\int_a^a f(x)\,dx = 0 = F(a) + C$

Så $C = -F(a)$

setter inn for C og lar $z = b$:

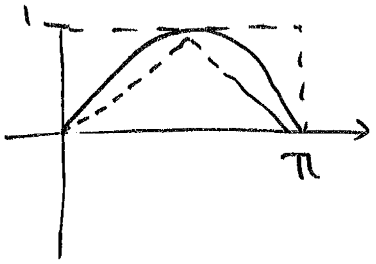
$$\left. \int_a^b f(x)\,dx = F(b) - F(a). \right)$$

Dette gir en effektiv måte å regne ut $\int_a^b f(x)\,dx$ når vi kan finne en antiderivert for $f(x)$.

$$\int_a^b k\,dx = kx\Big|_a^b = k(b) - ka = \underline{k(b-a)}$$

$$\int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2}{2} - \frac{a^2}{2} = \underline{\frac{1}{2}(b^2 - a^2)}$$

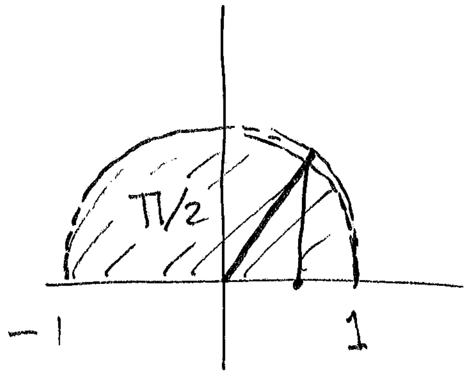
(5) $f(x) = \sin x$ $\qquad [a,b] = [0,\pi]$.



$$\frac{\pi}{2} < \int_0^{\pi} \sin x\,dx < \pi$$

Fundamentalteoremet:

$$\int_0^{\pi} \sin x\,dx = (-\cos x)\Big|_0^{\pi}$$
$$= -\cos(\pi) - (-\cos(0))$$
$$= -(-1) + 1 = \underline{\underline{2}}$$



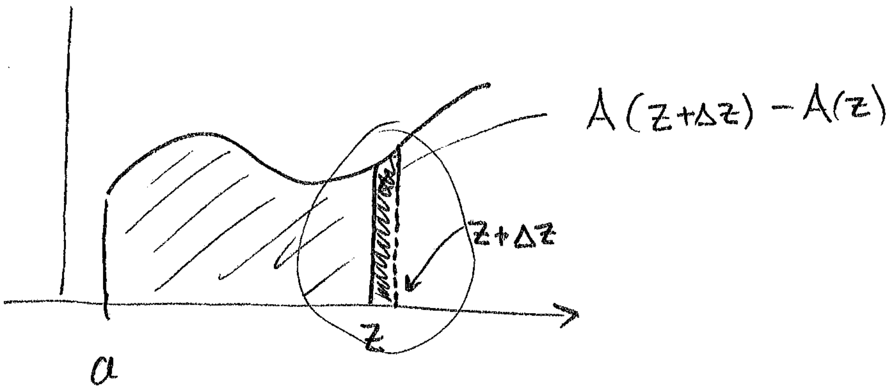
$$x^2 + y^2 = 1$$
$$y^2 = 1 - x^2$$
$$y \geq 0 : \quad y = \sqrt{1-x^2}.$$

Vi relaterte dette til numeriske estimat ved å bruke <u>SumOver</u> og <u>SumUnder</u> i geogebra.

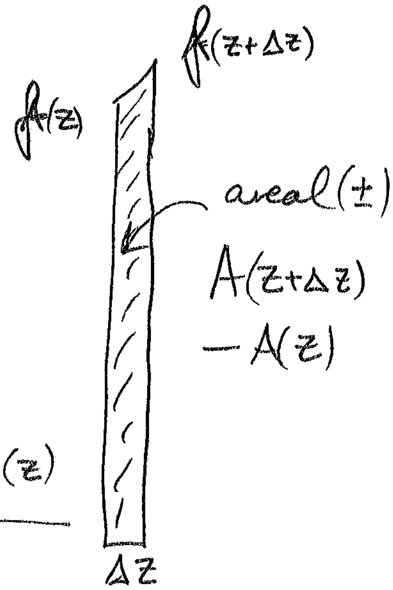
Bevisskisse for fundamentalteoremet.

(6)

$$A(z) = \int_a^z f(x)\,dx$$



$$\underline{\frac{d}{dz}A(z)} = \lim_{\Delta z \to 0} \underbrace{\frac{A(z+\Delta z) - A(z)}{\Delta z}}_{\text{gjennomsnittlig høyde i intervallet } [z,\, z+\Delta z].} = \underline{f(z)}$$



f(x) er kontinuerlig så gjennomsnittshøyde i [z, z+Δz]
går mot f(z) når Δz → 0.