24.04.2012

Delvis integrasjon

(1) $\int \frac{1}{4x^2-9}\,dx$

$$4x^2-9 = (2x-3)(2x+3)$$

$$\frac{1}{4x^2-9} = \frac{A}{2x-3} + \frac{B}{2x+3}$$ delbrøks oppspalting

Felles nevner $$\frac{1}{4x^2-9} = \frac{A(2x+3)}{4x^2-9} + \frac{B(2x-3)}{4x^2-9}$$

Tellerne må være like $$1 = A(2x+3) + B(2x-3)$$

$$= 2(A+B)x + 3(A-B)$$

Så $A+B=0$ $\quad$ $B=-A$

$3(A-B)=1$ $\quad$ $3(A-(-A))=1$

$3\cdot 2\cdot A=1$

$A=\frac{1}{6}$ og $B=\frac{-1}{6}$.

$$\int \frac{1}{4x^2-9}dx = \int \frac{1}{6}\left(\frac{1}{2x-3} - \frac{1}{2x+3}\right)dx$$

$$= \frac{1}{6}\left(\int \frac{1}{2x-3} - \frac{1}{2x+3}\,dx\right)$$

$$= \frac{1}{6}\cdot\left(\frac{1}{2}\ln|2x-3| - \frac{1}{2}\ln|2x+3|\right) + c$$

$$= \underline{\frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + c}$$

(2)

$$\int \frac{1}{x(x+1)^2}\,dx$$

Delbrøksoppspalting

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$= \frac{1}{x(x+1)^2}\left[A(x+1)^2 + Bx(x+1) + Cx\right]$$

Sammenligner tellerne:

$$0x^2+0x+1 = A(x+1)^2 + Bx(x+1) + Cx$$

$$= (A+B)x^2 + (2A+B+C)x + A.$$

(alternativt kan viselte inn verdier for x)

$A = 1$

$A+B=0 \qquad B=-A=-1$

$2A+B+C=0 \qquad C=-B-2A = -(-1)-2 = -1.$

$$\int \frac{1}{x(x+1)^2}\,dx = \int \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}\,dx$$

$$= \ln|x| - \ln|x+1| + \frac{1}{x+1} + c$$

$$= \underline{\ln\left|\frac{x}{x+1}\right| + \frac{1}{x+1} + c}$$

③ $\int \frac{1}{1+x^2}\,dx = \arctan x + c$

$$x = \tan U$$

$$\frac{dx}{dU} = \frac{1}{\cos^2 U} = \frac{\sin^2 U + \cos^2 U}{\cos^2 U}$$

$$= 1 + \tan^2 U$$

$$= 1 + x^2$$

$$dx = 1 + x^2\, dU \qquad \text{(deler med } 1+x^2\text{)}$$

$$\frac{dx}{1+x^2} = dU$$

$$\int \frac{1}{1+x^2}\,dx = \int 1 \cdot dU = U + c$$

$$= \underline{\arctan x + c}$$

$$\int \frac{1}{x^2+4x+5}\,dx$$

Fullføring av kvadrat $\quad x^2+4x+5 = (x+2)^2 + 1.$

$$\int \frac{1}{(x+2)^2+1}\,dx$$

Lineær substitusjon

$$U = x+2$$

$$dU = dx$$

$$= \int \frac{1}{U^2+1}\,dU$$

$$= \arctan(U) + c$$

$$= \underline{\arctan(x+2) + c}$$

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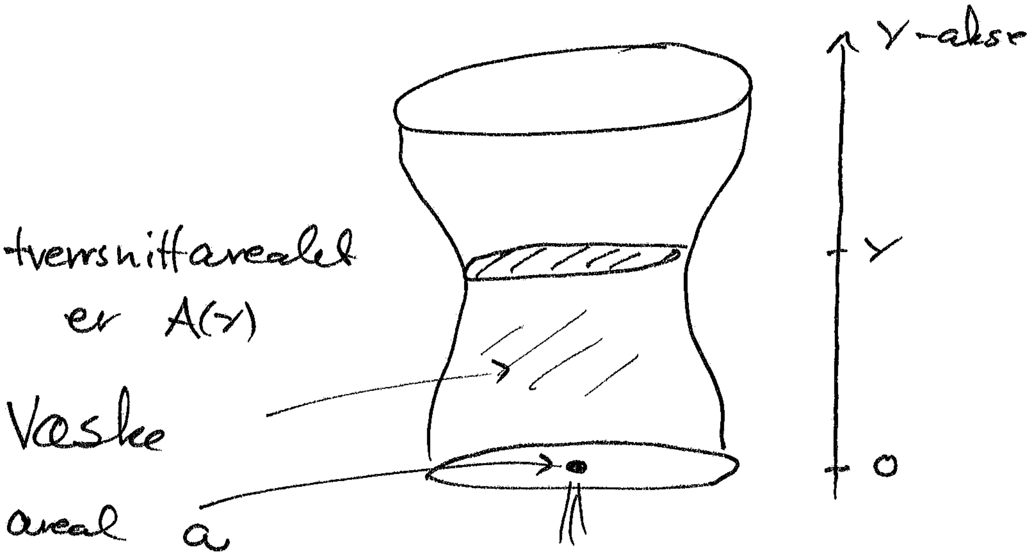
Stort eksempel

Torricellis Lov



Vi går utifra at $a \ll A(y)$ "mye mindre enn".

Ser bort fra friksjon etc.

Ved tiden $t=0$ er væsken fylt opp til høyde H.

Hva er høyden som en funksjon av tiden t?

I et kort tidspunkt Δt renner det ut Δm (masse) av væsken.

Tap i potensiell energi : $\Delta m \cdot g\, h(t)$

= kinetisk energi til væska som renner ut

$\frac{1}{2} \Delta m \cdot V^2$ V farten til væska som renner ut.

Bevaring av energi

$$\Delta m g h(t) = \frac{1}{2} \Delta m V^2$$

$$\underline{2g \cdot h(t) = V^2} \qquad \underline{V = \sqrt{2gh}}$$

Relatere $h'(t)$ til $V(t)$.

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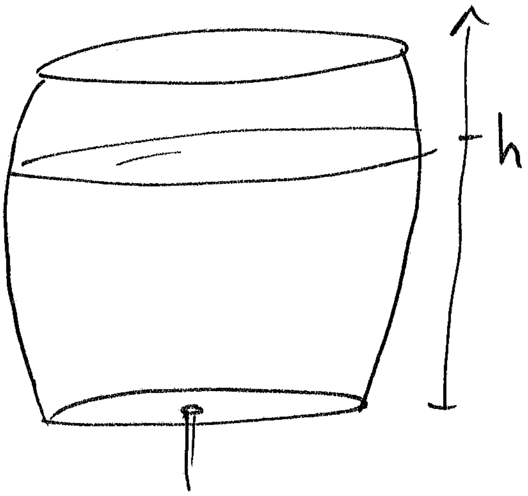


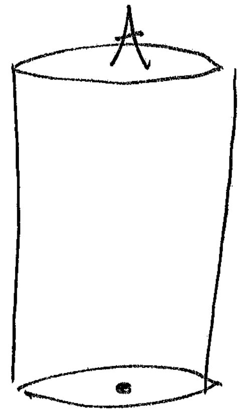
$$\frac{dh}{dt} < 0$$

$-A(h)\cdot\left(\frac{dh}{dt}\right)$ volumet som renner ut/per tid.

$= a \cdot V(t)$

$$\boxed{-A(h)\,\frac{dh}{dt} = a\sqrt{2g\cdot h}}$$

Eksempel Rett sylinder



$$-A\cdot h' = a\sqrt{2gh}$$

$$h' = \underbrace{\left(\frac{-a}{A}\sqrt{2g}\right)}_{\text{konstant}}\cdot\sqrt{h}.$$

$$\frac{1}{\sqrt{h}}\cdot h' = \left(\frac{-a}{A}\sqrt{2g}\right)$$

(6) $\int \frac{1}{\sqrt{h}}\,dh = \int \left(\frac{-a}{A}\sqrt{2g}\right) dt$ (substitusjon)

$$\int h^{-1/2}\,dh = \frac{h^{1/2}}{1/2} + c$$

$$= \left(\frac{-a}{A}\sqrt{2g}\right) t$$

(deler med 2)

$$h^{1/2} = \left(\frac{-a}{2A}\sqrt{2g}\,t + c\right)$$

$\underline{h(0) = H}$

initialverdien

$$h(0)^{1/2} = 0 + c$$

så $c = \sqrt{H}$.

$$h^{1/2} = \left(\sqrt{H} - \frac{a}{A}\sqrt{\frac{g}{2}}\,t\right)$$

$$h = \left(\sqrt{H} - \frac{a}{A}\sqrt{\frac{g}{2}}\,t\right)^2$$

gyldig inntill h blir 0.

(legg merke til : $\frac{1}{2}\sqrt{2g} = \sqrt{\frac{2g}{4}} = \sqrt{\frac{g}{2}}$)

Hvor lang tid tar det før tanken tømmes?

$h = 0$: $\sqrt{H} - \frac{a}{A}\sqrt{\frac{g}{2}}\,T = 0$

$$\underline{T = \sqrt{H}\cdot\frac{A}{a}\sqrt{\frac{2}{g}}}.$$

(7) Hva må $A(y)$ være for at $h(t)$ skal være lineær?

$$h(t) = H - k \cdot t$$

$$h'(t) = -k$$

$$\frac{1}{\sqrt{h(t)}} \cdot (-k) = \frac{-a}{A(h)} \sqrt{2g}$$

$$A(h) = \frac{-a\sqrt{2g}}{-k} \sqrt{h}$$

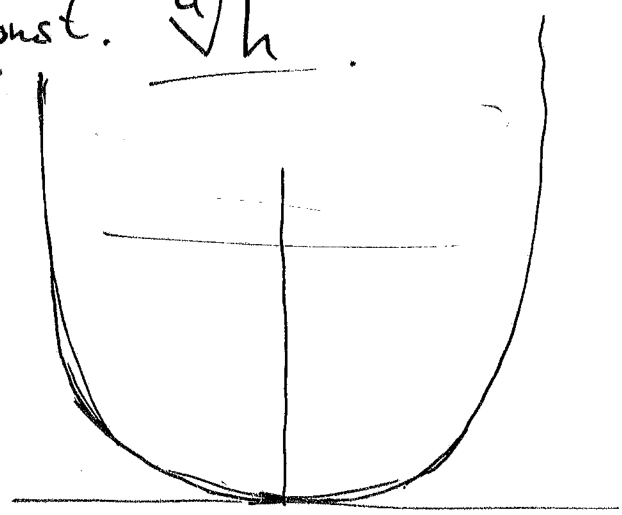
$$A(h) = \frac{a}{k}\sqrt{2g}\,\sqrt{h}$$

Gå utfra at tanken er sylindsymmetrisk

$$A(h) = \pi\, r(h)^2$$

$$r(h) = \sqrt{\frac{a}{\pi k}\sqrt{2g}\,\sqrt{h}}$$

$$= \text{konst.}\ \underline{\sqrt[4]{h}}\,.$$



KLADD

$$\frac{Dx+E}{(x+1)^2} = \frac{Dx+D \quad +E-D}{(x+1)^2}$$

$$\frac{D(x+1) \quad +(E-D)}{(x+1)^2} = \frac{D}{x+1} + \frac{E-D}{(x+1)^2}.$$

$$A(\underline{x}^2+2x+1) + B(\underline{x}^2+x) \quad + Cx$$

$$(A+B)x^2 + (2A+B+C)x \quad + A$$

$$\int \frac{1}{(x+1)^2}\,dx$$

La $U = x+1$
$dU = dx$

$$= \int \frac{1}{U^2}\,dU$$

$$= \int U^{-2}\,dU = \frac{U^{-2+1}}{-1} + c$$

$$= -U^{-1} + c$$

$$= \frac{-1}{x+1} + c$$

Fullføring av kvadrat

$$\boxed{\begin{aligned} &x^2+bx+c \\ &\left(x+\frac{b}{2}\right)^2 + c-\left(\frac{b}{2}\right)^2 \end{aligned}} = (x+d)^2 + e$$

$$x^2+bx+c = 0$$

$$\left(x+\frac{b}{2}\right)^2 = \frac{b^2}{4} - c = \frac{b^2-4c}{4}$$

$$x + \quad = \frac{-b}{2} \pm \sqrt{\frac{b^2-4c}{4}}$$