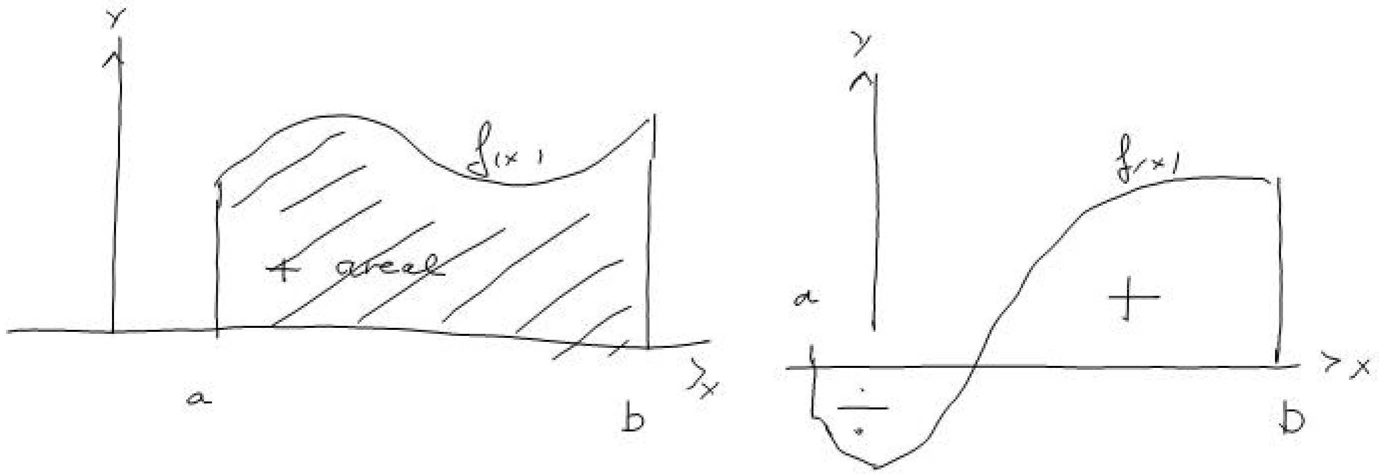
16 Bestemte integral

La $a < b$ og la $f(x)$ vere en avgrensa funksjon ($|f(x)| < M$ for $x \in [a, b]$ for en passende M)



Arealet (med fortegn!) avgrenset av grafen til $f(x)$, x-aksen og de vertikale linjene $x = a$ og $x = b$

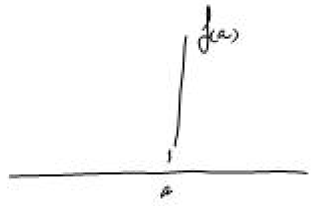
Skrives $\int_a^b f(x)\,dx$

og kalles det bestemte integralet av $f(x)$ fra a til b.

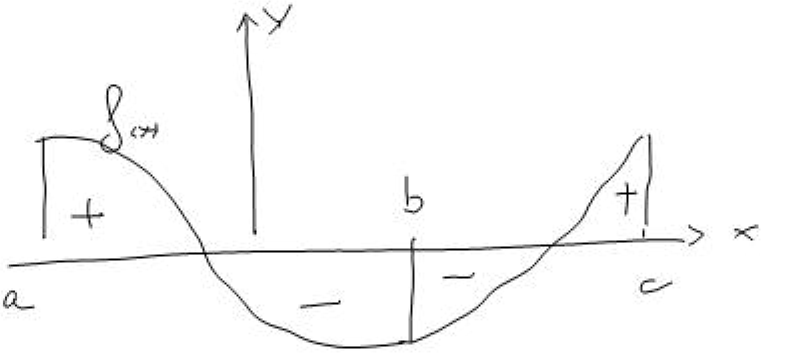
Det bestemte integralet eksisterer for en rekke funksjoner. For eksempel for alle kontinuerlige funksjoner.

Egenskaper.

1) $$\int_a^a f(x)\,dx = 0$$



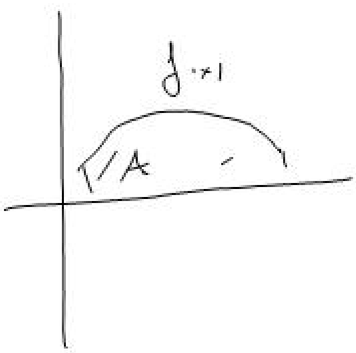
2) $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

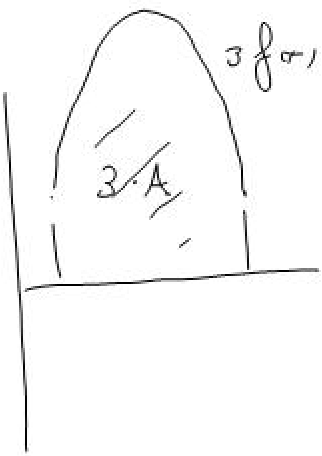


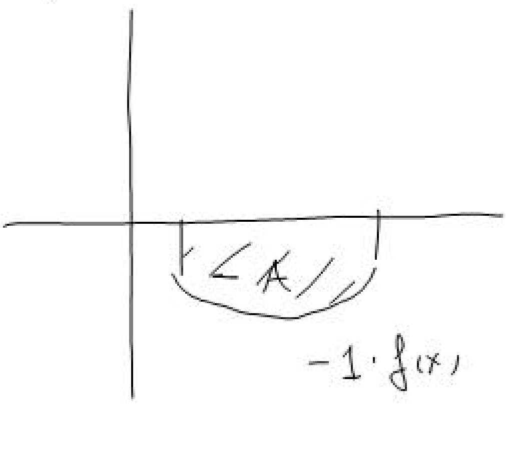
(1) følger fra 2) La $a = b < c$ ...)

3) $$\int_a^b k \cdot f(x)\,dx = k \int_a^b f(x)\,dx$$

k reell konstant

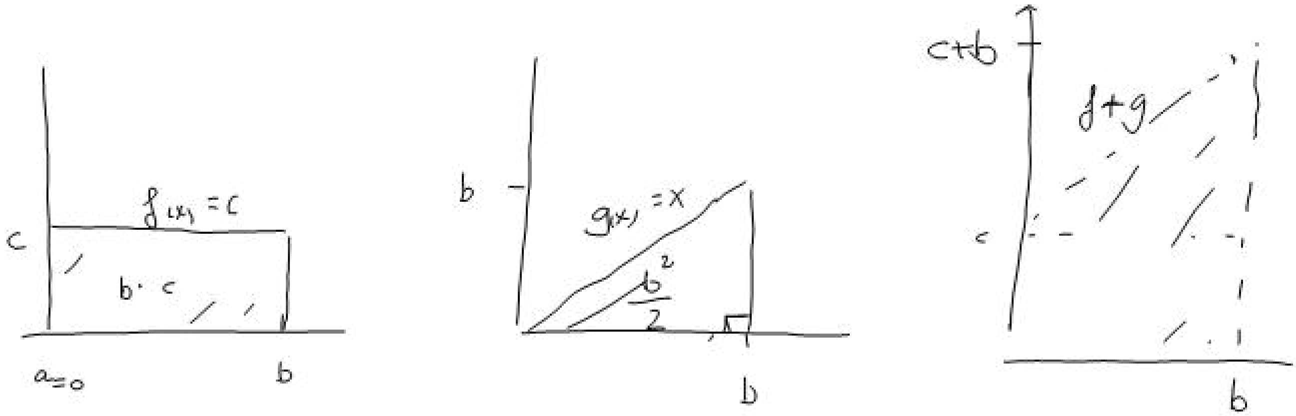






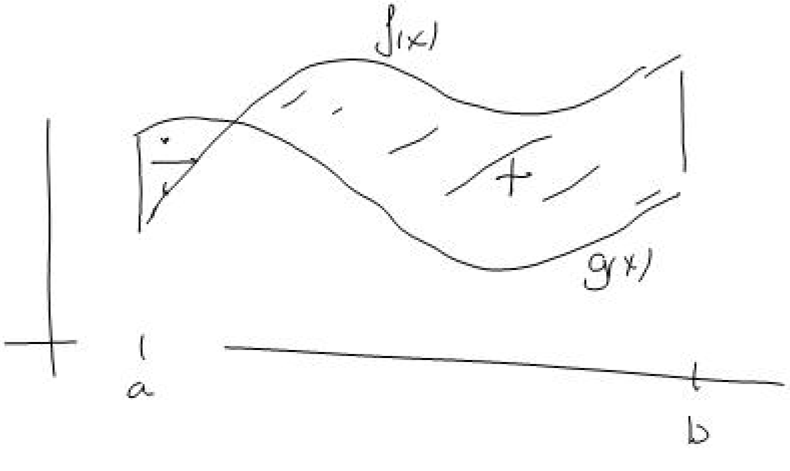
4) La $f(x)$ og $g(x)$ være funksjoner, $a<b$.

$$\int_a^b \left(f(x) + g(x)\right) dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



Egenskapene 3 & 4 sier at det bestemte integralet fra a til b er lineært.

ex.
$$\int_a^b 2f(x) - 4g(x)\,dx = \int_a^b 2f(x)\,dx + \int_a^b (-4)\,g(x)\,dx$$
$$= 2\int_a^b f(x)\,dx - 4\int_a^b g(x)\,dx$$



$$\int_a^b \left(f(x) - g(x)\right) dx$$

= arealet med fortegn av regionen mellom $f(x)$ og $g(x)$.

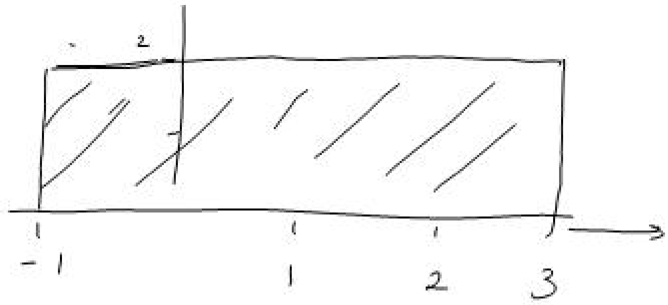
La $a < b$ Vi definerer:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Denne definisjonen gjør at egenskap 2) holder for a, b og c reelle tall

Eksempler

$$\int_{-1}^{3} 2\,dx$$



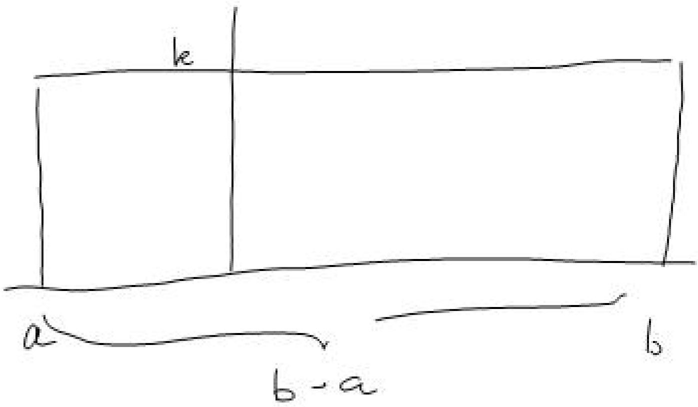
$$= 2(3-(-1))$$

$$= 2 \cdot 4 = \underline{\underline{8}}$$

$a < b$ k konstant

$$\int_a^b k\,dx$$

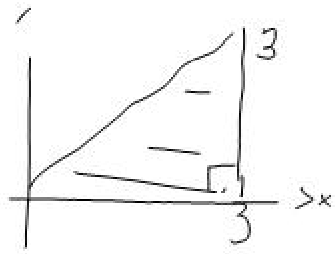
$$= \underline{k(b-a)}$$



$$\int_2^1 (-2)\,dx = -\int_1^2 -2\,dx = -(-2)\int_1^2 1\,dx$$

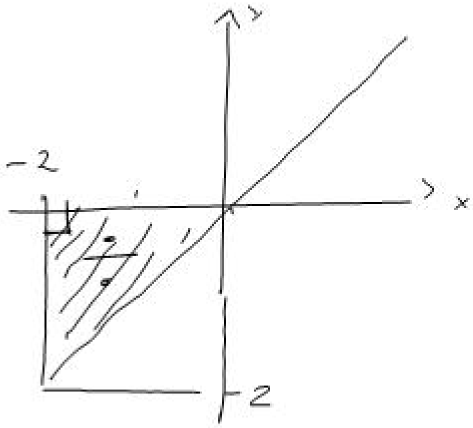
$$= 2\int_1^2 1\,dx = 2 \cdot 1 \cdot (2-1) = \underline{\underline{2}}$$

$$\int_0^3 x\,dx = \frac{3\cdot 3}{2} = \frac{9}{2}$$



$$\int_{-2}^0 x\,dx = -\frac{2^2}{2}$$

$$= \underline{\underline{-2}}$$



$$\int_a^b x\,dx = \underline{\frac{1}{2}\left(b^2 - a^2\right)}$$

$$\int_{-2}^1 2 - 3x\,dx = \int_{-2}^1 2\,dx - 3\int_{-2}^1 x\,dx$$

$$= 2\,(1-(-2)) \quad - 3\left(\frac{1^2}{2} - \frac{(-2)^2}{2}\right)$$

$$= 2\cdot 3 \quad - 3\left(\frac{1}{2} - 2\right) = 6 + \frac{9}{2}$$

$$= \underline{\underline{\frac{21}{2}}}$$

Fundamentalteoremet i kalkulus

Hvis $F(x)$ er en antiderivert til $f(x)$,

så er funksjonen $\int_a^z f(x)\,dx = F(z) - F(a)$

Dette gir en effektiv måte å finne bestemte integraler.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

hvor $F(x)$ er en antiderivert til $f(x)$.

$F(b) - F(a)$ skrives ofte som $F(x)\Big|_a^b$.

antiderivert til $3x^2$

$$\int_1^2 3x^2\,dx = x^3\Big|_1^2 = 2^3 - 1^3 = 8 - 1 = \underline{\underline{7}}$$

$$\int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \underline{\frac{b^2 - a^2}{2}}$$

(antiderivert til x)

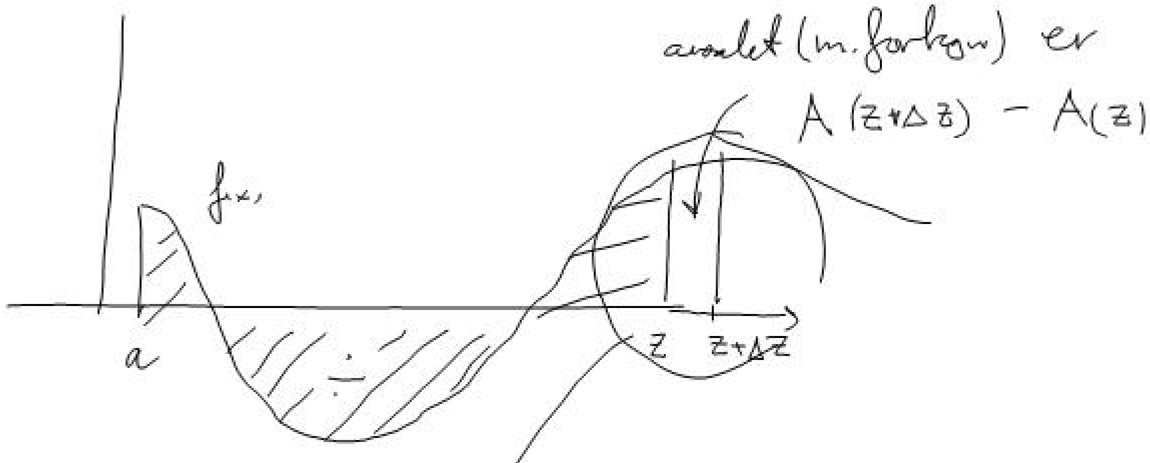
$$\int_a^b k\,dx = k \cdot x\Big|_a^b = k \cdot b - k \cdot a = \underline{k(b-a)}$$

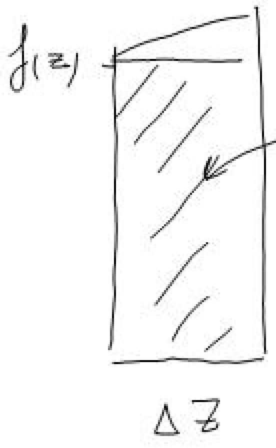
"bevis for fundamental teoremet"

$$A(z) = \int_a^z f(x)\,dx$$

Deriverer $A(z)$:

$$\frac{d}{dz}A(z) = \lim_{\Delta z \to 0} \frac{A(z+\Delta z) - A(z)}{\Delta z}$$





arealet til rektangelet er $\Delta z \cdot f(z)$.

$$A'(z) = \lim_{\Delta z \to 0} \frac{A(z+\Delta z) - A(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{f(z) \cdot \Delta z}{\Delta z}$$

$$= f(z)$$

Derfor er $A(z) = F(z) + C$

siden $A(a) = \int_a^a f(x)\,dx = 0 = F(a) + C$, så $C = -F(a)$

$$A(z) = F(z) - F(a)$$