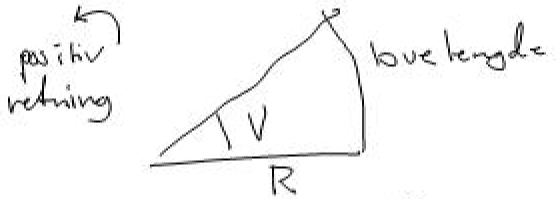
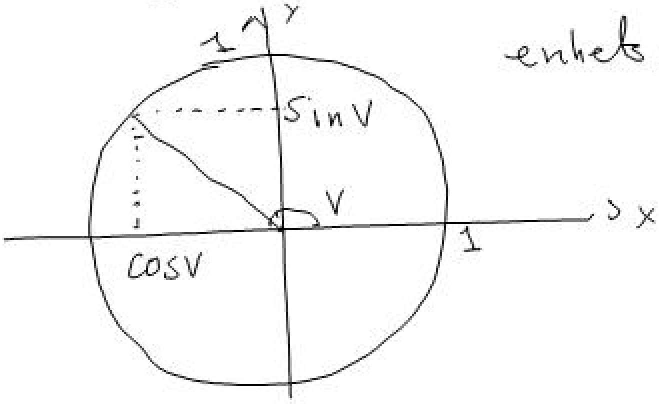
9 februar 09

10.1 og 10.2 Trigonometriske likninger

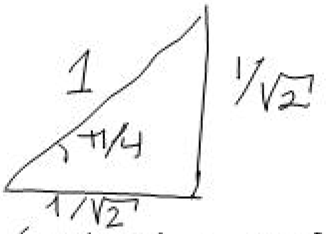


$$\text{vinkel} = \frac{\text{buelengde}}{\text{radius}}$$



enhetssirkel

Noen eksakte verdier for sinus og cosinus.

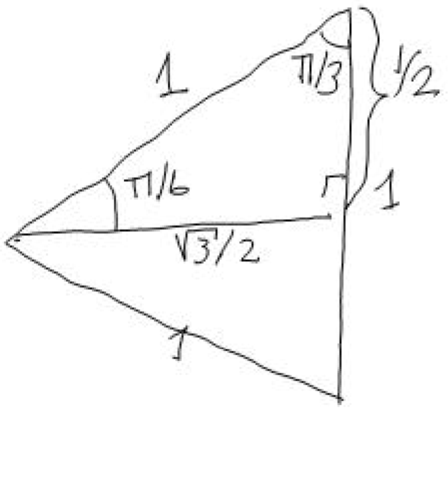


$(\pi/4 \text{ rad} = 45^\circ)$

(Pytagoras)

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$



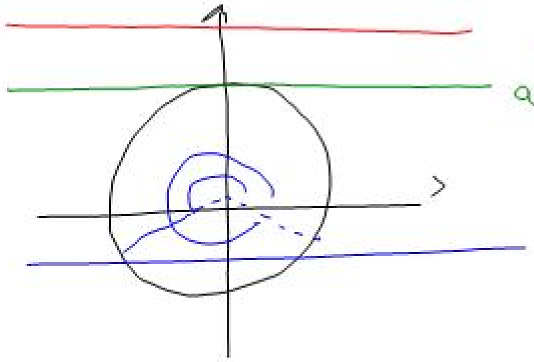
$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{3} = \frac{1}{2}$$

$$\sin\frac{\pi}{6} = \frac{1}{2}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

$\sin v = a$ likning



$a = 1.5$ ingen løsning $|a| > 1$

$a = 1$ akkurat en løsning i hver periode. $|a| = 1$

$a = -1/2$ akkurat to løsninger i hver periode $|a| < 1$

$v = \arcsin x$ for $-1 \le x \le 1$

er den unike vinkelen $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$

slik at $\sin v = x$.

Alle løsninger til $\sin v = x$:

$|x| > 1$ ingen løsning (heltall)

$|x| = 1$ $v = \arcsin x + 2\pi \cdot n$ $n \in \mathbb{Z}$

$|x| < 1$ $v = \arcsin x + 2\pi \cdot n$

$v = (\pi - \arcsin x) + 2\pi \cdot n$

Løs likningene 1) $2\sin x + 1 = 3$

$\sin x = \frac{3-1}{2} = \frac{2}{2} = 1$

$\underline{x = \frac{\pi}{2} + 2\pi \cdot n}$ n heltall

2) $2\cos x = \sqrt{3}$

$\cos x = \sqrt{3}/2$

Løsningene er $x = \frac{\pi}{6} + 2\pi \cdot n$

og $x = \frac{-\pi}{6} + 2\pi \cdot n$ n heltall.

Løsninger mellom 0 og 4π er:

$$\left\{ \frac{\pi}{6}, \frac{13\pi}{6}, \frac{11\pi}{6}, \frac{23\pi}{6} \right\}$$

3) $2\sin^2 x - 1 = 0$

$\sin^2 x = \frac{1}{2}$

så $\sin x = \frac{1}{\sqrt{2}}$ eller $\sin x = \frac{-1}{\sqrt{2}}$.

$x = \frac{\pi}{4} + 2\pi \cdot n$ eller $x = \frac{3\pi}{4} + 2\pi \cdot n$

$x = -\frac{\pi}{4} + 2\pi \cdot n$ $x = \frac{5\pi}{4} + 2\pi \cdot n$

n heltall.

Så løsningene er $\underline{x = \frac{\pi}{4} + \frac{\pi}{2} \cdot m}$ m heltall

4) $\cos^2 x - \sin x = 0$

$$\cos^2 x = \sin x$$

(Pytagoras : $\cos^2 x + \sin^2 x = 1$)

$$\cos^2 x = 1 - \sin^2 x = \sin x$$

$$\sin^2 x + \sin x - 1 = 0$$

$$(\sin x)^2 + \sin x - 1 = 0$$

Løser andregrads likningen :

$$\sin x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$

$$\underbrace{\sin x = \frac{-1-\sqrt{5}}{2}}_{\text{ingen løsninger}} \qquad \text{eller} \qquad \sin x = \frac{\sqrt{5}-1}{2}$$

(~ 0.618)

Gylne snitt

$$\arcsin\left(\frac{\sqrt{5}-1}{2}\right) \sim 0.666\ldots \quad (\sim 38.2^\circ)$$

Løsningene blir $x = \arcsin\left(\frac{\sqrt{5}-1}{2}\right) + 2\pi \cdot n$

eller $x = \pi - \arcsin\left(\frac{\sqrt{5}-1}{2}\right) + 2\pi \cdot n$

$\sin x + \cos x$ på formen $A\sin(x-\varphi)$

Addisjonsformelen:

$$A\sin(x-\varphi) = A\big(\sin x \cdot \cos(-\varphi) + \cos x \cdot \sin(-\varphi)\big)$$

$$= (A\cos\varphi)\cdot \sin x + (-A\sin\varphi)\cos x$$

$$= \quad \sin x \quad + \quad \cos x$$

Så $A\cdot\cos\varphi = 1$ og $-A\cdot\sin\varphi = 1$

derfor må $\tan\varphi \left(= \frac{A\sin\varphi}{A\cos\varphi}\right) = -1$.

så $\underline{\varphi = -\frac{\pi}{4}}$

$A\cdot\cos\left(-\frac{\pi}{4}\right) = 1$ så $\underline{A} = \frac{1}{1/\sqrt{2}} = \underline{\sqrt{2}}$.

$$\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$

$$b\sin x + d\cos x = A\sin(x-\varphi)$$

Addisjonsformelen og sammenlikner koeffisientene for sin x og cos x.

$$(A\cos\varphi)\sin x + (-A\sin\varphi)\cos x$$

$$= b\cdot\sin x + d\cdot\cos x$$

Anta $b \neq 0$ $\qquad \frac{d}{b} = \frac{-\cancel{A}\sin\varphi}{\cancel{A}\cos\varphi} = -\tan\varphi$

$\tan\varphi = \frac{-d}{b}$. $\qquad$ løser for φ

$$\underline{\varphi = \arctan\left(\frac{-d}{b}\right)}$$

$b = A\cos\varphi \quad (\neq 0) \qquad$ så $\quad \underline{A = \frac{b}{\cos\varphi}}$

Eksempel: Løs likningen

$$2\cos x + 2\sqrt{3}\sin x = 2\cdot\sqrt{2}$$

($b = 2\sqrt{3}$ og $d = 2$ ovenfor)

$$\varphi = \arctan\left(\frac{-2}{2\sqrt{3}}\right) = \arctan\left(\frac{-1}{\sqrt{3}}\right) = \frac{-\pi}{6}$$

$$A = \frac{2\sqrt{3}}{\cos(-\pi/6)} = \frac{2\sqrt{3}}{\sqrt{3}/2} = 4$$

$$2\cos x + 2\sqrt{3}\sin x = 4\sin(x+\pi/6) = 2\sqrt{2}$$

$\sin\left(x+\frac{\pi}{6}\right) = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$

Så $\quad x + \frac{\pi}{6} = \frac{\pi}{4} + 2\pi\cdot n$

eller $\frac{3\pi}{4} + 2\pi\cdot n$

$x = \frac{\pi}{4} - \frac{\pi}{6} + 2\pi\cdot n$

eller

$x = \frac{3\pi}{4} - \frac{\pi}{6} + 2\pi\cdot n$

Oppgave : Finn amplituden A og faseforskyvningen θ slik at

$$\sin x + \sin(x-\varphi) = A\sin(x-\theta)$$

Forslag til løsning:

$$\sin x + \sin(x-\varphi) = \sin x + (\cos\varphi)\sin x - (\sin\varphi)\cdot\cos x$$

$$= (1+\cos\varphi)\sin x + (-\sin\varphi)\cdot\cos x$$

Dette skal være lik

$$A\sin(x-\theta) = (A\cos\theta)\sin x + A(-\sin\theta)\cos x$$

så $A\cos\theta = 1+\cos\varphi$ og $A\sin\theta = \sin\varphi$

Anta $\cos\varphi \neq -1$: $\underline{\tan\theta} = \frac{A\sin\theta}{A\cos\theta} = \underline{\frac{\sin\varphi}{1+\cos\varphi}}$

$$(A\cos\theta)^2 + (A\sin\theta)^2 = A^2(\cos^2\theta + \sin^2\theta) = A^2$$

$$= (1+\cos\varphi)^2 + (\sin\varphi)^2 = 1 + 2\cos\varphi + \underbrace{\cos^2\varphi + \sin^2\varphi}_{1}$$

Så $A^2 = 2(1+\cos\varphi)$

$$A = \sqrt{2(1+\cos\varphi)}$$

Når $\varphi = 0$: $A = \sqrt{2(1+1)} = 2$

$\varphi = +\pi$ $A = \sqrt{2(1+(-1))} = 0$