2 juni 2014

Eksamen

20 deloppgaver som alle teller like mye

$\frac{5 \text{ timer}}{20 \text{ deloppgave}} = 15 \text{ min} / \text{deloppgave}$

Tilsvarende emnefordeling som prøveeksamen.

Det blir ikkje spørsmål om Fourier rekker.

Det er en TYPO på eksamensoppgave (nr 4b)

Det står $\cancel{[0.6, 0,8]}$ (3-vektor)

Det skal stå $[0.6, 0.8]$ (2-vektor)

Rekker

Geometriske rekker

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad |x|<1$$

$$(1+x+x^2+\cdots+x^n)(x-1) =$$

$$x(1+x+\cdots+x^n) \qquad x+x^2+\cdots+x^{n+1}$$

$$-1(1+x+\cdots+x^n) \qquad = -1-x-x^2-\cdots-x^n$$

$$= x^{n+1}-1$$

Deler med $x-1$:

$$1+x+x^2+\cdots+x^n = \begin{cases} \dfrac{x^{n+1}-1}{x-1} & x \neq 1 \\ n+1 \end{cases}$$

$$\lim_{n\to\infty} x^{n+1} = 0 \quad \text{når} \quad |x|<1$$

Så $\displaystyle\sum_{n=0}^{\infty} x^n = \lim_{n\to\infty} \frac{x^{n+1}-1}{x-1} = \frac{-1}{x-1} = \underline{\frac{1}{1-x}}$

$$\sum_{n=8}^{\infty} x^n = x^8 + x^9 + x^{10} + \cdots$$

$$= x^8(1+x+x^2+\cdots) = \underline{\frac{x^8}{1-x}} \qquad |x|<1$$

Avgjør om rekken konvergerer og finn summen hvis den konvergerer:

$$\sum_{n=1}^{\infty} \frac{2\cdot 4^n}{3\cdot 2^{n-1}} \cdot e^{-n}$$

n-te ledd: $\dfrac{2\cdot 4^n}{3\cdot 2^{n-1}}\cdot e^{-n} = \dfrac{2\cdot 4^n}{3\cdot 2^n\cdot 2^{-1}}\,(e^{-1})^n$

$$= \left(\frac{2}{3\cdot 2^{-1}}\right)\cdot \frac{4^n}{2^n}\cdot\left(\frac{1}{e}\right)^n = \left(\frac{2\cdot 2}{3}\right)\cdot\left(\frac{4}{2\cdot e}\right)^n$$

$$= \frac{4}{3}\cdot\left(\frac{2}{e}\right)^n$$

$$\sum_{n=1}^{\infty}\frac{2\cdot 4^n}{3\cdot 2^{n-1}}\,e^{-n} = \sum_{n=1}^{\infty}\frac{4}{3}\cdot\left(\frac{2}{e}\right)^n$$

$$= \frac{4}{3}\sum_{n=1}^{\infty}\left(\frac{2}{e}\right)^n = \frac{4}{3}\left(\frac{2}{e}\right)\left(\sum_{m=0}^{\infty}\left(\frac{2}{e}\right)^m\right)$$

Denne rekken konvergerer siden den er en geometrisk rekke med faktor $0 < \frac{2}{e} < 1$.

Summen er: $\dfrac{4}{3}\left(\dfrac{2}{e}\right)\left(\dfrac{1}{1-2/e}\right)$

$$= \underline{\underline{\frac{8}{3(e-2)}}}$$

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spørsmål...

$$\sum_{n=k}^{\infty} x^n = x^k\,(1+x+\cdots+x^n+\cdots) = \frac{x^k}{1-x}$$

$$\sum_{n=0}^{\infty} a x^n = a\sum_{n=0}^{\infty} x^n = \frac{a}{1-x}$$

Vi kan også resonere som følger:

$$\sum_{n=k}^{\infty} x^n = \sum_{n=k}^{\infty}(x^k)\cdot x^{n-k} = x^k\underbrace{\sum_{n=k}^{\infty} x^{n-k}}_{\frac{1}{1-x}}$$

p-rekker $\sum_{n=1}^{\infty} \frac{1}{n^p}$ konvergerer for $p > 1$

divergerer for $p \leq 1$.

$p = 1$ $\sum_{n=1}^{\infty} \frac{1}{n}$ divergerer (harmonisk rekke)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^8 + n^3}}{n^5 \cdot \sqrt[3]{n}}$$

n stor

$$\frac{\sqrt{n^8+n^3}}{n^5 \cdot \sqrt[3]{n}} \sim \frac{\sqrt{n^8}}{n^5 \cdot n^{1/3}} \sim \frac{n^4}{n^5 \cdot n^{1/3}} = \frac{1}{n^{1+1/3}} = \frac{1}{n^{4/3}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ konvergerer (p-rekke med $p = \frac{4}{3} > 1$)

Derfor konvergerer $\sum_{n=1}^{\infty} \frac{\sqrt{n^8+n^3}}{n^5 \sqrt[3]{n}}$.

(Grensesammenlikningstesten ...)

Forslag fra dere:

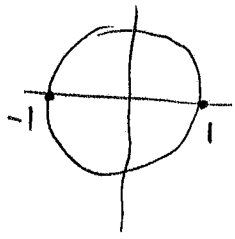
$$\sum_{n=1}^{\infty} \frac{\left(\sqrt[100]{n} + \sqrt[99]{n}\right) \cancel{/\sqrt[99]{n}}}{n^{1+1/99} \cancel{/\sqrt[99]{n}}} = \sum_{n=1}^{\infty} \frac{1 + \frac{1}{\sqrt[9900]{n}}}{n}$$

$$\left(\begin{array}{l} \sqrt[99]{n} = n^{1/99} \qquad \sqrt[100]{n} = n^{1/100} \\ \sqrt[100]{n} + \sqrt[99]{n} = n^{1/99}\Bigg(\underbrace{\underbrace{n^{\frac{1}{100} - \frac{1}{99}}}_{\frac{1}{n^{1/9900}}}}_{\frac{1}{\sqrt[9900]{n}}} + 1 \Bigg) \end{array} \right)$$

Rekken divergerer.

Rekker kan av og til relateres til kjente rekker, som Taylor rekker.

$$\sum_{n=0}^{\infty} \frac{2^n}{(2n)!} \cos(\pi \cdot n)$$



$$\cos(\pi \cdot n) = (-1)^n$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^n}{(2n)!}$$

$$\left(\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - + \cdots \right)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(2^{1/2}\right)^{2n} = \cos\left(2^{1/2}\right) = \cos\left(\sqrt{2}\right)$$

$$\left(= \cos(-\sqrt{2}) \right)$$

Alternerende rekker

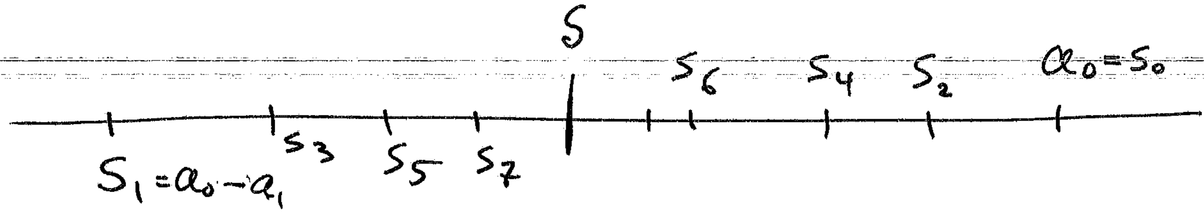
$$\sum_{n=0}^{\infty} (-1)^n a_n \qquad a_n > 0$$

$$= a_0 - a_1 + a_2 - a_3 + \cdots$$

En alternerende rekke $\sum_{n=0}^{\infty} (-1)^n a_n$, $a_n > 0$ konvergerer hvis a_n er avtagende $(0 < a_{n+1} < a_n)$ og $\lim_{n \to \infty} a_n = 0$.

Eksempel $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = \frac{-1}{1} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + - + \cdots$

Alternerende rekke $a_n = \frac{1}{\sqrt{n}}$ er avtagende og $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$, så rekken konvergerer (betinget)



n-te delsum $\quad S_n = \sum_{i=0}^{n} (-1)^i a_i \qquad S = \sum_{i=0}^{\infty} (-1)^i a_i$

$|S_n - S| < a_{n+1}$ Gyldig når a_n er avtagende.

Vis at $\quad \left| \cos(x) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \right| < \frac{1}{720}$

når $|x| \leq 1$.

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + - \cdots = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!}$$

Dette er en alternerende rekke med

$a_n = \frac{x^{2n}}{(2n)!}$. Dette er avtagende

ledd og $\lim_{n\to\infty} a_n = 0$ når $|x| \leq 1$.

Derfor er $\quad \left| \cos(x) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \right| < \frac{x^6}{720} \leq \frac{1}{720}$

for $|x| \leq 1$.

6 b) $D = \begin{bmatrix} 3 & 0 \\ 0 & i \end{bmatrix}$

$$\sum_{m=0}^{\infty} \frac{D^{2m}}{(2m)!}$$

$$D^2 = \begin{bmatrix} 3 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & -1 \end{bmatrix}$$

$$D^{2m} = (D^2)^m = \begin{bmatrix} 9^m & 0 \\ 0 & (-1)^m \end{bmatrix}$$

$$\sum_{m=0}^{\infty} \frac{1}{(2m)!} \begin{bmatrix} 9^m & 0 \\ 0 & (-1)^m \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{m=0}^{\infty} \frac{1}{(2m)!} \cdot 9^m & 0 \\ 0 & \underbrace{\sum_{m=0}^{\infty} \frac{1}{(2m)!} (-1)^m}_{\cos(1)} \end{bmatrix} = \underline{\underline{\begin{bmatrix} \cosh(3) & 0 \\ 0 & \cos(1) \end{bmatrix}}}$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$
$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \left(\frac{x^n}{n!} + (-1)^n \frac{x^n}{n!} \right) \right] = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}$$

$$= \frac{1}{2}\left[1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \right.$$
$$\left. + 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots \right]$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$$

Diagonaliser matrisen

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbb{1}$$

Karakteristiske likningen

$$\det(A - \lambda \cdot \mathbb{1}_2) = 0$$

$$\det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = (-\lambda)^2 - (1)(-1) = \lambda^2 + 1 = 0$$

$\lambda = +i$ og $\lambda = -i$.

Egenvektorer til $\lambda = i$:

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \vec{V} = 0$$

$$\vec{V} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{matrix} -i \cdot x - y = 0 \\ (x - iy = 0) \end{matrix}$$

La $x = 1$: $y = -i \cdot x = -i$

En egenvektor $(\neq \vec{0})$ er $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

en annen egenvektor er $\begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}$

(sjekker: $A \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}$) ✓

kompleks konj.

$\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}^*$ er en egenvektor til $\underline{-i}$.

(siden A er en reell matrise)

Vi diagonaliserer A

$$D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

$$P^{-1} = \frac{1}{2i} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}$$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \cdot \frac{1}{2i} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}.$$

System av lineære differensialdikninger

$$x' = -y \qquad \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0, & -1 \\ 1, & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$y' = x$$

$\lambda_1 = i \quad \lambda_2 = -i.$ så $P^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_1 e^{it} \\ k_2 e^{-it} \end{bmatrix}.$

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} k_1 e^{it} \\ k_2 e^{-it} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} k_1 e^{it} \\ k_2 e^{-it} \end{bmatrix} = \underline{\begin{bmatrix} k_1 e^{it} + k_2 e^{-it} \\ -i(k_1 e^{it} - k_2 e^{-it}) \end{bmatrix}}$$

Hvis $k_1 = k_2 = k$: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2k\cos t \\ 2k\sin t \end{bmatrix}$

$k_1 = -k_2 = k$ $\quad \begin{bmatrix} x \\ y \end{bmatrix} = 2k \cdot i \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + b \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$$

konstanter a, b.

Her har vi benyttet Eulers formel

$$e^{it} = \cos t + i \sin t$$
$$e^{-it} = \cos t - i \sin t$$

Alternativ løsningsmetode:

$$x' = -y \qquad y' = x$$

$$x'' = (x')' = (-y)' = -y'$$

dette er også lik $-x$.

$$x'' = -x$$

Tilsvarende $\quad y'' = -y$.

$x'' = -x$ har løsningene:

$$x = \underline{a \cos t + b \sin t}.$$

$$\underline{y} = -x' = \underline{a \sin t - b \cos t}$$

Andrederivert testen

f kont. deriverbar i et punkt P.

$\vec{\nabla} f(P) = 0$

Hessematrisen $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$

$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ $\qquad$ $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

Diskriminanten $\Delta = \det H$

$$= f_{xx} \cdot f_{yy} - (f_{xy})^2$$

da har $f(x,y)$

Hvis $\Delta > 0$: $f_{xx} > 0$, bunnpunkt i P

$f_{xx} < 0$, toppunkt i P

$\Delta < 0$ sadelpunkt

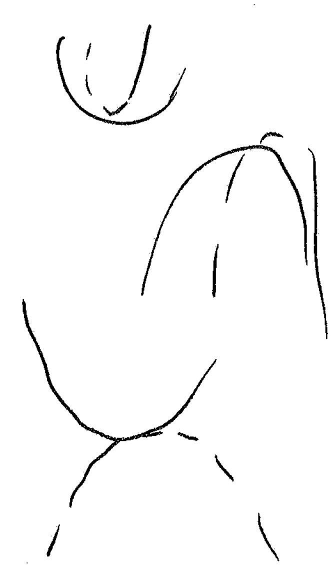
$\Delta = 0$?

Tenk på eksemplene

$f = x^2 + y^2$

$f = -(x^2 + y^2)$

$f = x^2 - y^2$



$\Delta = 4$

$f_{xx} = 2 > 0$

$f_{xx} = -2 < 0$

$\Delta = -4$