4 mars 2011

Gradienten til $f(x,y)$ er vektorfunksjon

(1) $$\vec{\nabla} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$$

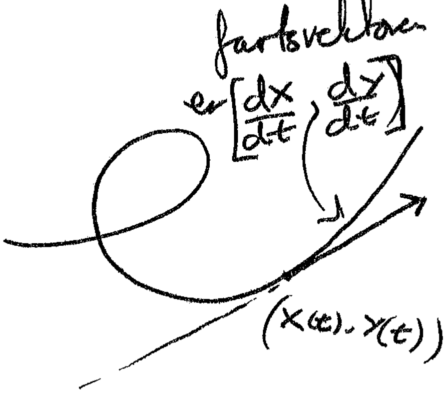
$f(x_1, \dots, x_n)$

n-vektor $$\vec{\nabla} f = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right].$$

10.5

Gitt en parametrisert kurve

$\vec{r}(t) = (x(t), y(t))$

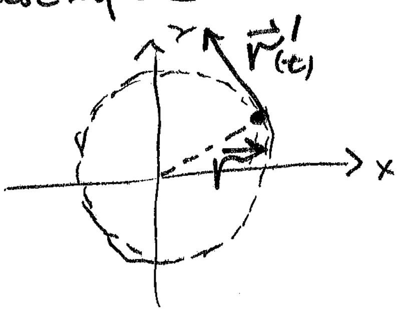


Resultat (kjerneregel)

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

når f er kontinuerlig deriverbar i $(x(t), y(t))$.

Eksempel $\vec{r}(t) = [x(t), y(t)] = [\cos t, \sin t]$



$\vec{r}'(t) = \frac{d}{dt}\vec{r} = [-\sin t, \cos t]$

$f(x,y) = 2x^2 + y^2$ 1) setter inn for den parametriserte kurven

$$f(x(t), y(t)) = 2\cos^2 t + \sin^2 t$$
$$= 1 + \cos^2 t.$$

$$\frac{d}{dt} f(x(t), y(t)) = \frac{d}{dt}(1 + \cos^2 t) = 2\cos t(\cos t)'$$
$$= \underline{-2\cos t \sin t}$$

2) Vi regner nå ut $\frac{d}{dt} f(\vec{r}(t))$ ved å benytte kjerneregelen:

2 $\quad \vec{\nabla} f = [4x, 2y] \qquad \vec{r}'(t) = [-\sin t, \cos t]$.

$$\frac{d}{dt} f(\vec{r}(t)) = \vec{\nabla} f \cdot \vec{r}'(t) = [4x, 2y] \cdot [-\sin t, \cos t]$$

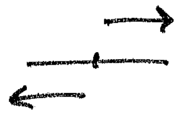
$$= 4(\cos t)(-\sin t) + 2(\sin t) \cdot \cos t$$

$$= \underline{-2 \sin t \cdot \cos t}$$

Retningsderivert

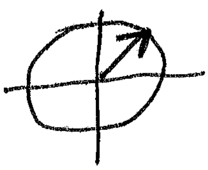
En retning er gitt ved en vektor $\vec{v}$ av lengde 1.
(enhetsvektor)

dim $n=1$



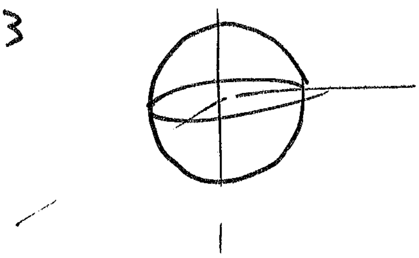
to retninger

$n=2$



retningene kan spesifiseres av et punkt på enhetssirkelen

$n=3$

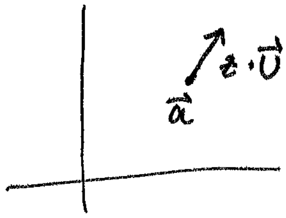


$f(x,y)$ kont. deriverbar i $\vec{a} = (a,b)$

Den retningsderiverte til $f(x,y)$ i retning $\vec{u} = (u_1, u_2)$ er endringsraten til f i retning $\vec{u}$:

$$D_u f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + \vec{u} \cdot t) - f(\vec{a})}{t}$$

Fra kjerneregelen er dette:



(3)

$$D_u f(\vec{a}) = \frac{\partial f}{\partial x}(\vec{a}) \cdot \frac{d}{dt}(a + u_1 t) + \frac{\partial f}{\partial y}(\vec{a}) \cdot \frac{d}{dt}(b + u_2 \cdot t)$$

$$= \frac{\partial f}{\partial x}(\vec{a}) \cdot u_1 + \frac{\partial f}{\partial y}(\vec{a}) \cdot u_2$$

$$D_u f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u}$$

spesielt:

$\vec{u} = [1, 0]$ enhetsvektor i x-retning

$$D_{\vec{e}_1} f(\vec{a}) = \frac{\partial f}{\partial x}(\vec{a})$$

La θ er vinkelen mellom gradienten i $\vec{a}$

$\vec{\nabla} f(\vec{a})$ og retningsvektoren $\vec{u}$

så $D_{\vec{u}} f(\vec{a}) = |\vec{\nabla} f(\vec{a})| \cdot \cos\theta$ (siden $|\vec{u}| = 1$)

Resultat

Den retningsderiverte er størst i retning til gradienten $\vec{\nabla} f(\vec{a})$. Den er der lik $|\vec{\nabla} f(\vec{a})|$

4 La $f(x,y) = 2x\cos(x+y)$ og $\vec{u} = \frac{[1,1]}{\sqrt{2}}$

Finn $D_u f$ i punktet $(\frac{\pi}{8}, \frac{\pi}{8})$

$|\vec{u}| = 1$ retningsvektor

$$\vec{\nabla} f = \left[2\cdot\cos(x+y) + 2x(-\sin(x+y))\cdot 1,\ 2x(-\sin(x+y))\cdot 1\right]$$

$$\vec{\nabla} f\left(\frac{\pi}{8}, \frac{\pi}{8}\right) = \left[2\frac{1}{\sqrt{2}} + \frac{2\pi}{8}\left(-\frac{1}{\sqrt{2}}\right),\ \frac{2\pi}{8}\left(\frac{-1}{\sqrt{2}}\right)\right]$$

minus

$$= \left[\underline{\sqrt{2} + \frac{\sqrt{2}\pi}{8}},\ \underline{-\frac{\sqrt{2}\pi}{8}}\right]$$

$$D_u f\left(\frac{\pi}{8}, \frac{\pi}{8}\right) = \vec{\nabla} f\left(\frac{\pi}{8}, \frac{\pi}{8}\right) \cdot \frac{[1,1]}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \cdot \sqrt{2}\left(\left(1 - \frac{\pi}{8}\right) + \left(-\frac{\pi}{8}\right)\right)$$

$$= 1 - \frac{2\pi}{8} = \underline{\underline{1 - \frac{\pi}{4}}}$$

I hvilke retning vokser $f(x,y) = x^2 e^{xy}$ raskest i punktet $(1, \ln 2)$?

$$\frac{\partial}{\partial x} f = 2x e^{xy} + x^2(y e^{xy})$$

$$\frac{\partial}{\partial y} f = x^2(x e^{xy})$$

$$\vec{\nabla} f = [2x + x^2\cdot y,\ x^3]\, e^{xy}$$

$$\vec{\nabla} f(1, \ln 2) = [2 + \ln 2,\ 1] \cdot 2.$$

Retningen den vokser raskest er $\frac{[2+\ln 2, 1]}{\sqrt{1 + 4 + 4\ln 2 + (\ln 2)^2}}$.

Eksemplene er hentet fra: Matematikk for ingeniørfag (Kro, Kleppe, Vatne, Gulbrandsen)