27 feb 2014.

10.4 Gradientvektorer

1

$f(x,y)$

gradientvektoren til $f(x,y)$ er

$$\vec{\nabla} f = [f_x, f_y] = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

∇ kalles nabla eller del

∂ kalles del, partial. (det er en $\underline{d}$)

Eksempler

1) $f(x,y) = 2x + 3y^2$

$$\frac{\partial f}{\partial x} = 2 \quad , \quad \frac{\partial f}{\partial y} = 6y$$

$$\underline{\vec{\nabla} f = [2, 6y]}$$

2) $f(x,y) = \ln(2x - y)$ def. for $2x > y$

$$\frac{\partial f}{\partial x} = \frac{2}{2x-y} \qquad \frac{\partial f}{\partial y} = \frac{-1}{2x-y}$$

$$\underline{\vec{\nabla} f = \frac{1}{2x-y}[2, -1]}$$

3) $f(x,y) = \frac{1}{|\vec{r}|} = \frac{1}{\sqrt{x^2+y^2}}$ $\qquad \vec{r} = [x,y]$

$= (x^2+y^2)^{-1/2}$ $\qquad |\vec{r}| = \sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x} = 2x \cdot \left(\frac{-1}{2}\right)(x^2+y^2)^{-3/2} = -\frac{x}{(x^2+y^2)^{3/2}} = \frac{-x}{(\sqrt{x^2+y^2})^3}$$

$$= \frac{-x}{|\vec{r}|^3}$$

$\frac{\partial f}{\partial y} = \frac{-y}{|\vec{r}|^3}$ siden $f(x,y)$ er symmetrisk i x og y.

$$\vec{\nabla} f = \frac{-1}{|\vec{r}|^3}[x,y] = \frac{-\vec{r}}{|\vec{r}|^3}.$$

(se også opg. 9 i boka)

(2) $f(x,y,z) = x^2 \cdot y \cdot z^3$

$\vec{\nabla} f = [2xyz^3,\ x^2 z^3,\ 3x^2yz^2]$.

$\vec{\nabla} f(x_1, \dots, x_n) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right]$.

én variabel $\vec{\nabla} f = \frac{d}{dx} f$

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f er kontinuerlig deriverbar i et punkt $\vec{a}$ hvis $\vec{\nabla} f$ eksisterer og er kontinuerlig i en omegn om $\vec{a}$.

$\{\vec{x} \mid |\vec{x} - \vec{a}| < \delta\}$ for en positiv δ.

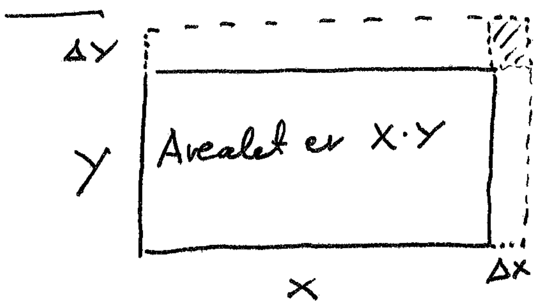
Resultat (Lineær tilnærming)

Hvis f er kont. deriverbar i $\vec{a}$:

$$f(\vec{a} + \vec{h}) = f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot \vec{h} + \epsilon(\vec{h}) \cdot |\vec{h}|$$

og $\epsilon(\vec{h}) \to 0$ når $\vec{h} \to \vec{0}$.

én variabel $f(a+h) = f(a) + \frac{df}{dx}(a) \cdot h + \epsilon(h) \cdot |h|$



$A(x,y) = x \cdot y$

$\vec{\nabla} A = [y, x]$

$\vec{h} = [\Delta x, \Delta y]$

$A(x+\Delta x, y+\Delta y) = A(x,y) + [y, x] \cdot [\Delta x, \Delta y] + \underset{\epsilon(\vec{h}) \cdot |\vec{h}|}{\cancel{\text{høgre}}}$

$= A(x,y) + y \cdot \Delta x + x \cdot \Delta y + \epsilon(\vec{h}) |\vec{h}|$.

(Eksakt verdi: $A(x+\Delta x, y+\Delta y) = \underset{A(x,y)}{x \cdot y} + y \cdot \Delta x + x \cdot \Delta y + \underbrace{\Delta x \cdot \Delta y}_{|\vec{h}| \epsilon(\vec{h})}$)

③ Relativ feil

$$f(x,y,z) = x^n y^m z^p$$

$$\vec{\nabla} f = [n x^{n-1} \cdot y^m z^p, \; m x^n y^{m-1} z^p, \; p x^n y^m z^{p-1}]$$

$$= f(x,y,z) \left[\frac{n}{x}, \frac{m}{y}, \frac{p}{z} \right].$$

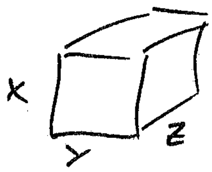
Den relative feilen til $f(x,y,z)$, når feilen til $[x,y,z]$ er $[\Delta x, \Delta y, \Delta z]$, er

$$\frac{f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)}{f(x,y,z)} = \frac{\vec{\nabla} f \cdot [\Delta x, \Delta y, \Delta z]}{f(x,y,z)}$$

$$= \frac{f}{f} \left[\frac{n}{x}, \frac{m}{y}, \frac{p}{z} \right] \cdot [\Delta x, \Delta y, \Delta z]$$

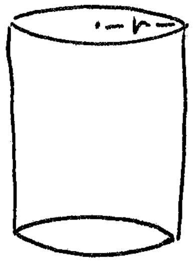
$$= \underline{n \frac{\Delta x}{x} + m \frac{\Delta y}{y} + p \frac{\Delta z}{z}}.$$

Konsekvenser: Hvis rel. unøyaktighet i x og y er 1% så er rel. unøyaktighet til arealet $x \cdot y$ lik ~2%



relativ unøyaktighet til volumet er summen av den rel. unøyaktigheten til x, y og z.

(4) Oppgave



sylinder. $r = 10\,cm$

$h = 25\,cm$

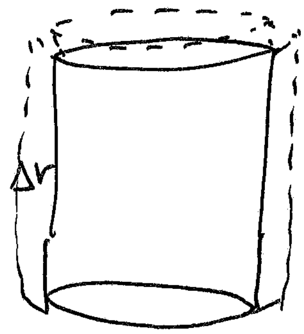
Relativ unøyaktighet til r og h er:

$$\frac{\Delta r}{r} = 2\% \qquad \frac{\Delta h}{h} = 3\%$$

Hva er relativ unøyaktighet til volumet til sylinderen? $(V = \pi r^2 \cdot h)$

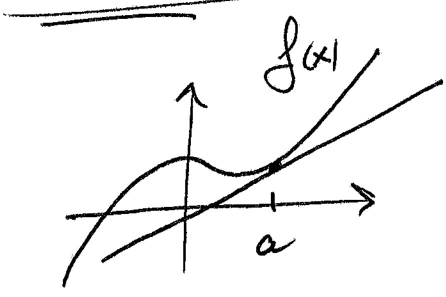
Fra foregående betraktning er rel. unøyaktighet i Volumet V : $2\frac{\Delta r}{r} + 1 \cdot \frac{\Delta h}{h} \cong 7\%$ ..

Geometrisk :



$$\Delta V = \pi r^2 \cdot \Delta h + 2\pi r \cdot \Delta r \cdot h + \cdots$$

$$\frac{\Delta V}{V} = \underline{\frac{\Delta h}{h} + 2\frac{\Delta r}{r}}$$



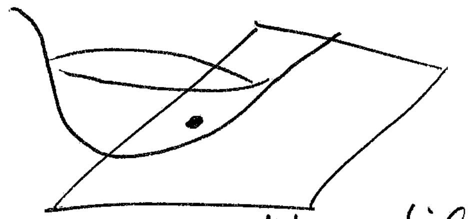
<u>Tangentlinjen</u> til f(x) er den lineære tilnærmingen til f(x) i (a, f(a)).

$$y = f'(a)(x-a) + f(a)$$

(1. ordens Taylor ~~rekke~~ polynom om a.)

$z = f(x,y)$

(5)



tangent plan til $f(x,y)$ i $(a, b, f(a,b))$.

Likningen til tangentplanet:

$$f(a,b) + \vec{\nabla} f(a,b) \cdot [x-a, y-b] = z$$

z er en Linear funksjon i x og y.

to parametre a og b

Eksempler 1) $f(x,y) = ax + by$ plan.

$$\vec{\nabla} f = [a, b]$$

Tangentplanet

$$z = \vec{\nabla} f \, [x-x_0, y-y_0] + f(x_0,y_0)$$

$$= a(x-x_0) + b(y-y_0) + ax_0 + by_0$$

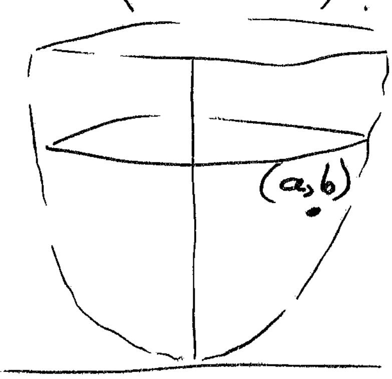
$$= \underline{ax + by}$$

Tangentplanet til et plan er like planet selv (i alle punkt (x_0, y_0) på planet).

2) $f(x,y) = x^2 + y^2$

$$\vec{\nabla} f = [2x, 2y]$$

$$\vec{\nabla} f(a,b) = [2a, 2b].$$



Tangentplanet er gitt ved

$$z = \vec{\nabla} f(a,b) \cdot [x-a, y-b] + f(a,b)$$

$$= 2a(x-a) + 2b(y-b) + a^2 + b^2$$

$$= \underline{2ax + 2by - (a^2 + b^2)}.$$

Oppgave

6 Finn tangentplanet til

$$f(x,y) = z = \sin x + \frac{x}{y}$$

i (a,b). $\qquad b \neq 0$.

Hva er tangentplanet i $\left(\frac{\pi}{2}, 1\right)$?

$$\vec{\nabla} f = \left[\cos x + \frac{1}{y}, \ \frac{-x}{y^2}\right]$$

Tangentplanet $\quad z = \vec{\nabla} f(a,b) \cdot [x-a, y-b] + f(a,b)$

$$z = \left(\cos(a) + \frac{1}{b}\right)(x-a) + \frac{-a}{b^2}(y-b) + \sin a + \frac{a}{b}$$

$$= \left(\cos(a) + \frac{1}{b}\right)x - a\cos(a) \underbrace{- \frac{a}{b} + \frac{a \cdot b}{b^2}}_{0} - \frac{a}{b^2} \cdot y + \sin a + \frac{a}{b}$$

$$= \left(\cos(a) + \frac{1}{b}\right)x - \frac{a}{b^2} y - a\cos(a) + \sin(a) + \frac{a}{b}.$$

i punktet $\left(\frac{\pi}{2}, 1\right)$:

$$z = x - \frac{\pi}{2} y + 1 + \cancel{si}\ \frac{\pi}{2}$$