23.01.2013 Matte 2000

H Fausk 7.5

(1) Taylor rekker lar oss estimere bestemte integral

$$\int_0^x e^{-t^2}\,dt$$

$$e^y = \sum_{n=0}^{\infty} \frac{\cancel{x}^n}{n!} \qquad \text{setter } y = -t^2$$

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}$$

$$\int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} (-1)^n \underbrace{\int_0^x \frac{t^{2n}}{n!}\,dt}_{\left.\frac{t^{2n+1}}{(2n+1)\cdot n!}\right|_0^x}$$

$$\int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\cdot n!}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + - \cdots$$

—

Hva er n-te deriverte til $f = e^{-x^2}$ når $x=0$?

$f(0) = 1$ $\qquad f' = -2x e^{-x^2}$ $\qquad f'(0) = 0$

$f'' = -2e^{-x^2} + 4x^2 e^{-x^2}$ $\qquad f''(0) = -2$

$\vdots$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \text{det er og lik } \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

Sammenligner koeffisientene : $f^{(n)}(0) = 0$ n oddetall

$$f^{(2n)}(0)/(2n)! = (-1)^n/n! \quad \text{så} \quad f^{(2n)}(0) = (-1)^n \frac{(2n)!}{n!}$$

(2) $f^{(10)}(0) = (-1)^5 \dfrac{10!}{5!} = \underline{-1 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}$

—

La $f(x) = x^7 e^{-x} = x^7 \sum_{n=0}^{\infty} \dfrac{(-x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{n+7}}{n!}$

Hva er $f^{(5)}(0) = 0$

$f^{(10)}(0) = -10!/3! = -4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10$

$\cdots + 0 \cdot x^5 + 0 \cdot x^6 + \cdots$

$$f(x) = \frac{x^7}{0!} - \frac{x^8}{1!} + \frac{x^9}{2!} - \frac{x^{10}}{3!} + - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$\dfrac{f^{(5)}(0)}{5!} = 0$ så $f^{(5)}(0) = 0$

$\dfrac{f^{(10)}(0)}{10!} = \dfrac{-1}{3!} = \dfrac{-1}{6}$ så $f^{(10)}(0) = -\dfrac{10!}{3!}$

—

Mer eksplisitt:

$$x^7 e^{-x} = x^7\left(\frac{1}{0!} - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - + \cdots\right)$$

$$= \frac{x^7}{0!} - \frac{x^8}{1!} + \frac{x^9}{2!} - \frac{x^{10}}{3!} + \frac{x^{11}}{4!} - + \cdots$$

Dette er også lik

$$\frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

3 Taylor rekkeutvikling til å bestemme grenser.

Grensen $\lim_{x \to 0} \frac{\sin x}{x}$ er lik 1.

Hvor raskt går $\frac{\sin x}{x}$ mot 1?

$$\lim_{x \to 0} \frac{x - x^3/6 + x^5/120 - + \cdots}{x}$$

$$= \lim_{x \to 0} \frac{1 - x^2/6 + x^4/120 - + \cdots}{1}$$

$$= 1$$

$$\left(\frac{\sin x}{x} - 1\right) \sim -\frac{x^2}{6} \qquad x \text{ liten}$$

$\frac{\sin x}{x}$ går kvadratisk mot 1.

$$\lim_{x \to 0} \frac{e^{3x^2} - 1 - 3x^2}{x^4}$$

$$e^{3x^2} = 1 + 3x^2 + \frac{(3x^2)^2}{2!} + \frac{(3x^2)^3}{3!} + \cdots$$

Så $e^{3x^2} - 1 - 3x^2 = \frac{9x^4}{2} + \frac{27x^6}{3!} + \cdots$

$$\lim_{x \to 0} \frac{e^{3x^2} - 1 - 3x^2}{x^4} = \lim_{x \to 0} \frac{9}{2} + \frac{9}{2}x^2 + \cdots$$

$$= \underline{\underline{\frac{9}{2}}}$$

(4) $$\lim_{x\to 0^+} \frac{e^x - x + \cos x - 1}{\sin x - x}$$

teller $\left(1+x+x^2/2+x^3/3!+\cdots\right) - x + \left(1-\frac{x^2}{2}+\frac{x^4}{24}-+\cdots\right)-1$

$1 + 0\cdot x + 0\cdot x^2 + x^3/6 + \cdots$

nevner $\left(x - \frac{x^3}{6} + \frac{x^5}{120} -+\cdots\right) - x = \frac{-x^3}{6} + \cdots$

$$\lim_{x\to 0^+} \frac{1 + x^3/6 + \cdots}{-x^3/6 + \cdots} = -\infty$$

divergerer mot $-\infty$

(som $\frac{-6}{x^3}$)

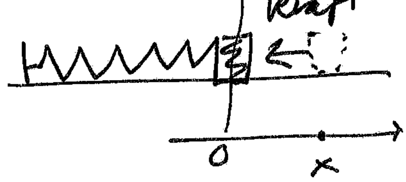
Eksempel på bruk av Potensrekker til å løse diff. likninger

$y'' + y = 0$

Harmonisk svingning

likevekt

kraft $-k\cdot x$



$F = m\cdot a = -k\cdot x$

$m\cdot x'' = -k\cdot x$

$x'' = -\frac{k}{m}\cdot x$

$$x'' + \left(\frac{k}{m}\right)\cdot x = 0$$

En generell løsning til $y''+y=0$ er

$y = A\cdot \sin t + B\cos t$, A, B konstanter

⑤ Anta en løsning kan skrives som en potensrekke $y(t) = \sum_{n=0}^{\infty} a_n t^n$.

Den dobbeldeiverte har potensrekke: $y''(t) = \left(\sum_{n=0}^{\infty} a_n \cdot n t^{n-1} \right)'$

$$= \sum_{n=0}^{\infty} a_n \cdot n(n-1) t^{n-2}.$$

Vi setter dette inn i diff. likeningen

$$\sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^n = 0$$

(erstatter n med n+2)

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} t^n + \sum a_n t^n = 0$$

$$\sum_{n=0}^{\infty} \left((n+1)(n+2) a_{n+2} + a_n \right) t^n = 0$$

Så $(n+1)(n+2) a_{n+2} = -a_n \quad n \geq 0$

$$a_{n+2} = \frac{-a_n}{(n+1)(n+2)} \quad n \geq 0$$

Vi får at $a_{2n} = \frac{(-1)^n}{(2n)!} a_0$

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1.$$

Løsningen er $a_0 \cos t + a_1 \sin t$.