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# 7.5 Anvendelser av Taylor rekker

1 Nye rekker fra kjente rekker

- substitusjon
- multiplikasjon, addisjon
- derivasjon, integrasjon

Geometriske rekker: $1 + x + x^2 + \cdots = \frac{1}{1-x}$ $|x| < 1$

Hva er $\frac{4}{x^2} + \frac{4}{x} + 4 + 4x + \cdots$

$$= \frac{4}{x^2}\left(1 + x + x^2 + \cdots\right) = \underline{\frac{4}{x^2} \cdot \frac{1}{1-x}} \quad 0 < |x| < 1$$

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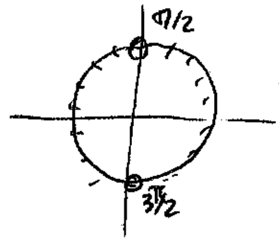
Finn et enkelt uttrykk for rekken

$$\sum_{n=3}^{\infty} (-1)^{n+1} \sin^n x = \sin^3 x - \sin^4 x + \sin^5 x - + \cdots$$

$$= \sin^3 x \left(1 - \sin x + (-\sin x)^2 + (-\sin x)^3 + \cdots\right)$$

$$= \sin^3 x \cdot \frac{1}{1-(-\sin x)} \quad \text{når } |-\sin x| < 1$$

$$= \underline{\frac{\sin^3 x}{1 + \sin x}} \quad \text{når } |\sin x| < 1$$

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$$t^5 - 3t^2 + 9/t - 27/t^4 - + \cdots$$

$$= t^5\left(1 - 3t^{-3} + (-3t^{-3})^2 + (-3t^{-3})^3 + \cdots\right)$$

$$= t^5 \cdot \frac{1}{1-(-3/t^3)} \quad \left|-\frac{3}{t^3}\right| < 1 \quad \begin{pmatrix} a_0 = t^5 \\ r = -3/t^3 \end{pmatrix}$$

$$= \underline{\frac{t^8}{t^3 + 3}} \quad \underline{|t| > \sqrt[3]{3}}$$

$$-x^5 + 3x^7 - 9x^9 + - \cdots$$

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$$= -x^5\left(1 - 3x^2 + (-3x^2)^2 + (-3x^2)^3 + \cdots\right)$$

$$= -x^5 \cdot \frac{1}{1-(-3x^2)} \qquad |-3x^2| < 1$$

$$= \underline{\frac{-x^5}{1+3x^2}} \qquad \underline{|x| < \frac{1}{\sqrt{3}}}$$

Den deriverte til den geometriske rekken

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \underline{\sum_{n=1}^{\infty} n x^{n-1}} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \underline{\frac{1}{(1-x)^2}}$$

$$|x| < 1$$

$$\frac{d^2}{dx^2}\sum_{n=0}^{\infty} x^n = \underline{\sum_{n=2}^{\infty} n(n-1)x^{n-2}} = \underline{\frac{2}{(1-x)^3}} = \left(\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right)\right)$$

Finn et enkelt uttrykk for $\sum_{n=1}^{\infty} n x^{5+2n}$

og avgjør når den konvergerer

"Prøver oss fram"

$$\sum_{n=1}^{\infty} n \cdot \underbrace{(x^2)^{n-1}}_{x^{2n-2}} = \frac{1}{(1-x^2)^2} \qquad |x^2| < 1$$

$$x^7 \cdot \sum_{n=1}^{\infty} n(x^2)^{n-1} = \sum_{n=1}^{\infty} n \cdot x^{2n-2+7} = \sum_{n=1}^{\infty} n x^{2n+5}$$

Så $\sum_{n=1}^{\infty} n x^{2n+5} = \underline{\frac{x^7}{(1-x^2)^2}} \qquad \underline{|x| < 1}$

$$\sum_{n=0}^{\infty} (n^2 + 2n - 1) x^n$$ tar ut faktoren $x^2$

(3) $$\sum_{n=0}^{\infty} n(n-1) \underbrace{x^n}_{x^{n-2+2}} = x^2 \sum_{n=0}^{\infty} n(n-1) x^{n-2} = \frac{2x^2}{(1-x)^3}$$

$$\sum_{n=0}^{\infty} n x^n = \sum_{n=0}^{\infty} n x^{n-1} \cdot x = \frac{1}{(1-x)^2} \cdot x$$

Uttrykker $n^2 + 2n - 1$ ved $n(n-1)$, $n$ og $1$

$$n^2 + 2n - 1 = n(n-1) + 3n - 1$$

$$\sum_{n=0}^{\infty} (n^2+2n-1) x^n = \underbrace{\sum_{n=0}^{\infty} n(n-1) x^n}_{\frac{2x^2}{(1-x)^3}} + \sum_{n=0}^{\infty} 3n x^n - \sum_{n=0}^{\infty} x^n$$

$$\frac{2x^2}{(1-x)^3} + 3\frac{x}{(1-x)^2} - \frac{1}{1-x}$$

$$= \frac{2x^2}{(1-x)^3} + \frac{3x}{(1-x)^2} - \frac{1}{1-x}$$

$$= \frac{-2x^2 + 5x - 1}{(1-x)^3}$$

$|x| < 1.$

Mer generelle potensrekker enn den geometriske

(4) Hva er $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{z^{3n+2}}{(2n+1)!}$ ?

Vi kjenner $\displaystyle \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - + = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$$\sin\left(z^{3/2}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(z^{3/2}\right)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{3n+(3/2)}}{(2n+1)!}$$

"mangler en faktor $z^{1/2}$" ganger med $\sqrt{z}$

$$\sqrt{z}\,\sin\left(z^{3/2}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{3n+2}}{(2n+1)!} \qquad |z|<1$$

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$$\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{(x^2)^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \underline{e^{-x^2} - 1} \qquad (\text{for alle } x)$$

(Husk at $\displaystyle e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$)

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-x^2)^n}{n!}$$