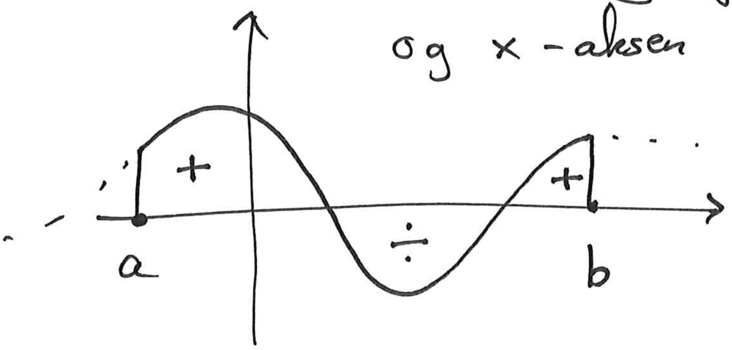
# Bestemte integral

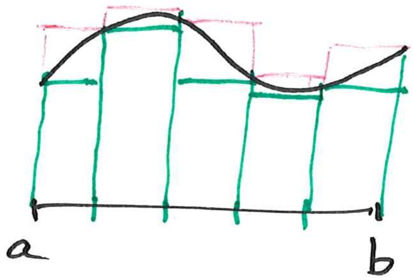
(1)

$\int_a^b f(x)\,dx$ = "areal med fortegn" mellom grafen til $f(x)$ og $x$-aksen fra $x=a$ til $x=b$.




Hva er arealet?

Estimerer med rektangler

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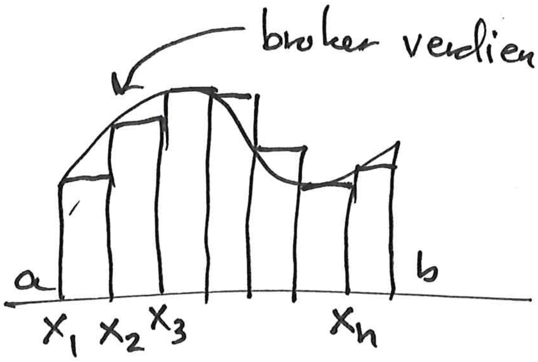
Deler intervallet $[a, b]$ i $n$-biter.




$$x_1 = a$$

$$x_2 = a + \frac{b-a}{n}$$

$$x_k = a + \left(\frac{b-a}{n}\right)(k-1)$$

$$x_n = b - \left(\frac{b-a}{n}\right).$$

$$\Delta x = \frac{b-a}{n}$$

Boken 15.1

$$S_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots f(x_n)\cdot\Delta x$$

$$= \sum_{k=1}^{n} f(x_k)\cdot\Delta x.$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n$$

$f$ kontinuerlig

$\int_a^b f(x)\,dx$ eksisterer for kontinuerlige funksjoner $f(x)$

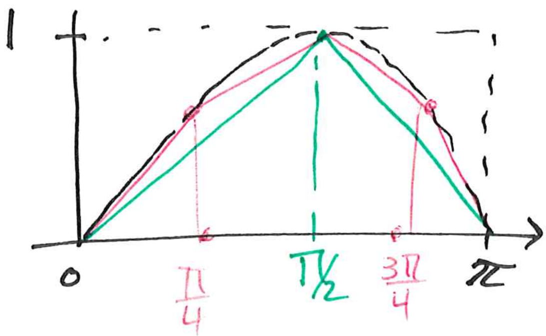
(2)

Brukte geogebra til å illustrere nedre og øvre estimat for $\int_a^b f(x)\,dx$. $f = x^2$

- LowerSum, UpperSum.

Eksempel

$\sin x$ $x \in [0, \pi]$

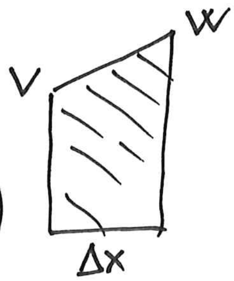



$\frac{\pi}{2} < \int_0^\pi \sin x\,dx < \pi$

trekant. rektangel

Estimat (nedre) når vi deler $[0,\pi]$ i fire biter

$\frac{\pi}{4}$ bredden




$$\frac{\pi}{4}\left(\frac{\sin(0) + 2\sin\left(\frac{\pi}{4}\right) + 2\sin\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{3\pi}{4}\right) + \sin\pi}{2}\right)$$

arealet er $\Delta x \cdot \left(\frac{V+W}{2}\right)$

Trapes

$$= \frac{\pi}{8}\left(\frac{2}{\sqrt{2}} + 2 + \frac{2}{\sqrt{2}}\right)$$

$$= \underline{\frac{\pi}{8} \cdot 2(\sqrt{2}+1)} \sim 1.896$$

Trapesmetoden 15.9

Fundamentalteoremet i kalkulus

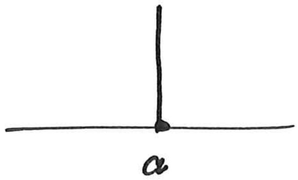
3

$$\int_a^x f(t)\,dt = F(x)$$ er

en antiderivert til $f(x)$.

Her er $f(x)$ kontinuerlig.

$$\int_a^a f(x)\,dt = 0$$



så $F(a) = 0$ for $F$ ovenfor.

Hvis $G(x)$ er en antiderivert til $f(x)$,

da er $F(x) = G(x) + C$.

siden $F(a) = 0 = G(a) + C$

så må $C = -G(a)$

$$F(x) = \int_a^x f(t)\,dt = G(x) - G(a)$$

La $x = b$

$$\int_a^b f(t)\,dt = G(b) - G(a)$$

hvor $G$ er en antiderivert til $f(x)$.

4 $\int_0^{\pi} \sin x \, dx = -\cos(\pi) - (-\cos(0))$

( benytter $(-\cos(x))' = \sin x$ )

$$\int_0^{\pi} \sin x \, dx = -(-1) + (1) = \underline{\underline{2}}$$

$$\int_0^1 x^2 dx = \frac{x^3}{3}(x=1) - \frac{x^3}{3}(x=0)$$

$$= \frac{1}{3} - 0 = \underline{\underline{\frac{1}{3}}}$$

skrivemåte $\qquad F(b) - F(a) = F(x) \Big|_a^b$

$$\int_a^b f(x) dx = F(x) \Big|_a^b$$

Presentasjon fra MEK

⑤ $\int_0^1 e^{3x}\,dx = \frac{1}{3}e^{3x}\Big|_0^1$

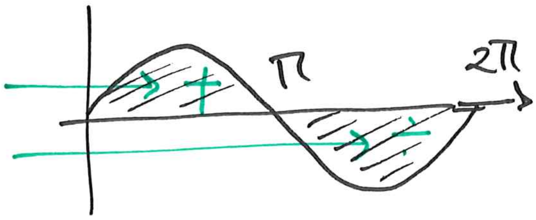
$= \frac{1}{3}\left(e^{3\cdot 1} - e^{3\cdot 0}\right) = \underline{\frac{1}{3}\left(e^3 - 1\right)}$

oppg $\int_0^1 x^5\,dx = \frac{x^6}{6}\Big|_0^1$

$= \frac{1^6}{6} - \frac{0^6}{6} = \underline{\underline{\frac{1}{6}}}$

oppg. Finn arealet mellom x-aksen og grafen til $\sin x$ $x \in [0, 2\pi]$

like store




$\int_0^{2\pi} \sin x\,dx = 0$

Arealet er : $\int_0^{2\pi} |\sin x|\,dx$

$= \underbrace{\int_0^{\pi} \sin x\,dx}_{2} - \underbrace{\int_{\pi}^{2\pi} \sin x\,dx}_{-2}$

$= \underline{\underline{4}}$

# Egenskaper til integraler

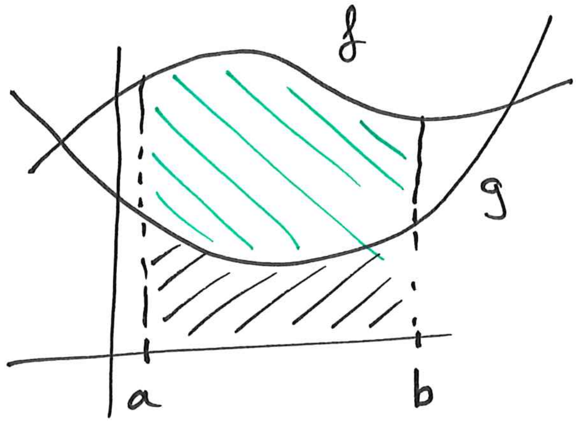
(6)

$$\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

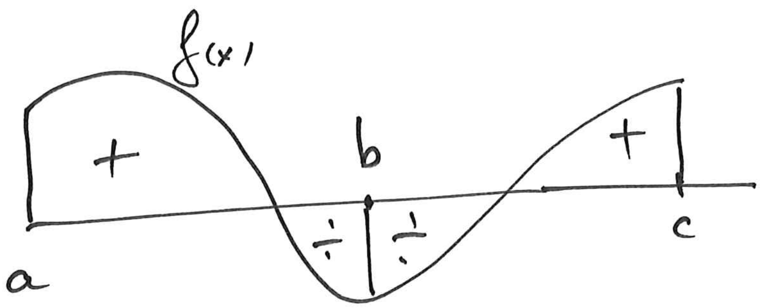
Bestemte integral er lineære.

$$\int_a^b f(x)-g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$




$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$




$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

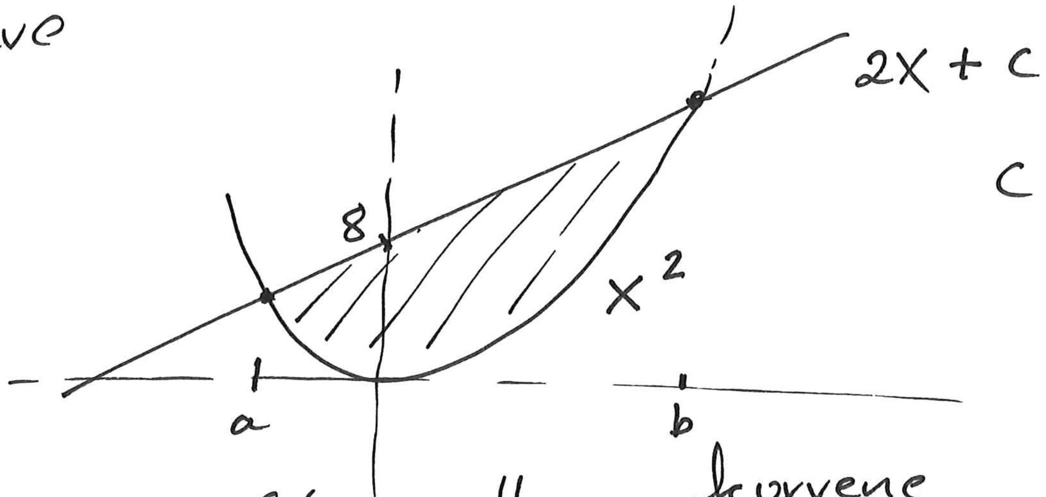
(7) $\int_0^1 3x - 5 + x^2 \, dx$

$$= \frac{3x^2}{2} - 5x + x^3/3 \Big|_0^1$$

$$= \frac{3}{2} - 5 + \frac{1}{3} = \frac{9}{6} - \frac{30}{6} + \frac{2}{6}$$

$$= \underline{\frac{-19}{6}}$$

oppgave




$c = 8$

Finn arealet mellom kurvene

Kurvene møtes der hvar $x^2 = 2x + c$

$$x^2 - 2x - c = 0$$

$$(x-1)^2 - 1 - c = 0$$

$$(x-1)^2 = 1 + 8 = 9 = 3^2$$

$x - 1 = \pm 3$ $\qquad$ $x = 1 \pm 3$

$a = 1 - 3 = \underline{-2}$ $\qquad$ $b = 1 + 3 = \underline{4}$

$$A = \int_{-2}^{4} (2x+8) - x^2 \, dx$$

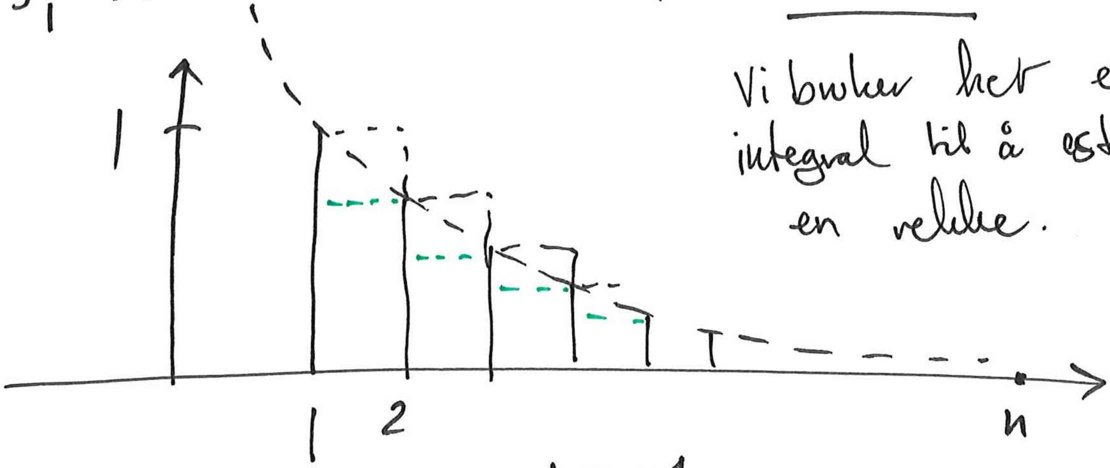
$$= x^2 + 8x - \frac{x^3}{3} \Big|_{-2}^{4} = 16 - 4 + 8(4-(-2)) - \frac{1}{3}(4^3 - (-2)^3)$$

$$= 12 + 8 \cdot 6 - \frac{1}{3}(64 + 8) = 12 + 48 - \frac{1}{3} 72 = \underline{60 - \frac{72}{3}}$$

8

$$\int_1^n \frac{1}{x}\,dx = \ln x \Big|_1^n = \underline{\ln(n)}$$

Vi bruker her et kjent integral til å estimere en rekke.




$\Delta x = 1$

$$\sum_{k=1}^{n-1} \frac{1}{k} > \underbrace{\int_1^n \frac{1}{x}\,dx}_{\ln(n)} > \sum_{k=2}^{n} \frac{1}{k}$$

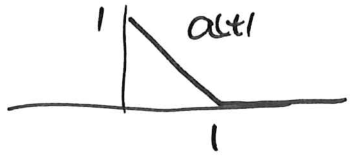
"venstre sum" "høyre sum"

$$\ln(n) < \ln(n+1) \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq \ln(n) + 1$$

9

$$a(t) = \begin{cases} 1 - t & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$

aksellerasjon




$S_0 = 0$

Finn $V(t)$ og $S(t)$. $\boxed{3}$ $V_0 = 0$

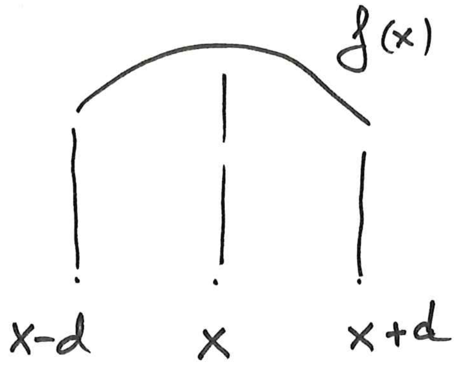
$$V(t) = \begin{cases} t - t^2/2 + V_0 & 0 \le t \le 1 \\ 1/2 + V_0 & t \ge 1 \end{cases}$$

$$= \begin{cases} t - t^2/2 & 0 \le t \le 1 \\ 1/2 & t > 1 \end{cases}$$

$$S(t) = \begin{cases} t^2/2 - t^3/6 + S_0 & 0 \le t \le 1 \\ 1/2\, t - 1/6 + S_0 & t \ge 1 \end{cases}$$

15.9 Simpsons metode

(10)



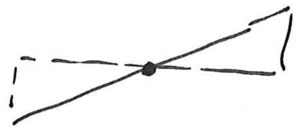
$$\int_{x-d}^{x+d} f(x)\,dx$$

$$= \underset{\substack{\uparrow \\ \text{bredde}}}{2d} \cdot \frac{1}{6}\left(f(x-d) + 4f(x) + f(x+d)\right)$$

når $f(x)$ er et 2. grads uttrykk.

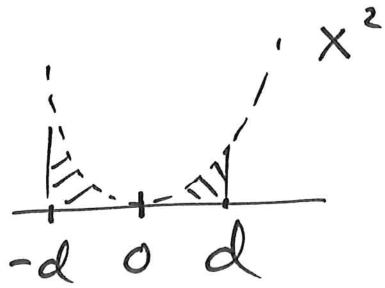
ok for $f(x) = c$ konstant.

ok for $f(x) = x$



Tilstrekkelig å sjekke for:

(forskyver horisontalt ved å legge til lineære uttrykk)



$$\int_{-d}^{d} x^2\,dx = \frac{x^3}{3}\Big|_{-d}^{d} = \frac{2}{3}d^3.$$

Dette er lik $\underset{\text{bredde}}{2d} \cdot \dfrac{\left(1\cdot(-d)^2 + 4\cdot 0^2 + 1\cdot d^2\right)}{6}$

$$= \frac{2d^3}{3} \checkmark$$

n jevn

1 4 1

1 4 1

1 4 1 ...

Vektingen: 1 4 2 4 2 4 2 - ... 2 4 1

Vi benytta geogebra til å estimere $\int_0^1 x^n\,dx$ og $\int_0^{\pi} \sin x\,dx$ med simpsonsmetode