15.1 Antideriverte – Ubestemte integral

En antiderivert $F(x)$ til $f(x)$ er en funksjon slik at $F'(x) = f(x)$

Eks $f(x) = 2x$

$F(x) = x^2$ er en antiderivert

$G(x) = x^2 - 3$ er en antiderivert

$H(x) = x^2 + c$, c konstant, er en antiderivert til $2x$.

Eks $f(x) = x^2$

$$(x^3)' = 3x^2$$

$$\frac{1}{3}(x^3)' = \frac{1}{3} \cdot 3x^2 = x^2$$

$$\left(\frac{x^3}{3}\right)' = x^2$$

Så $\frac{x^3}{3}$ er en antiderivert til x^2.

Eks $f(x) = \frac{1}{x^2} = x^{-2}$

$$(x^{-1})' = (-1) \cdot x^{-1-1} = -x^{-2}$$

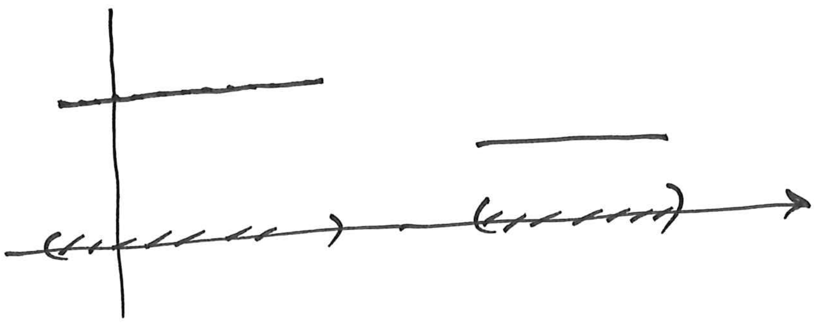
Så $\left(\frac{-1}{x}\right) = (-x^{-1})' = x^{-2} = \frac{1}{x^2}$

$F(x) = \frac{-1}{x} + c$ er antideriverte til $\frac{1}{x^2}$ for alle konstanter c.

Anta at F og G er antideriverte til $f(x)$.

$$(F(x) - G(x))' = F'(x) - G'(x)$$
$$= f(x) - f(x) = 0$$
for alle x.

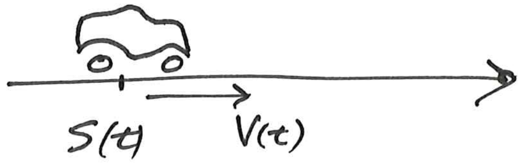
Hvis $H'(x) = 0$ for alle x, da er $H(x)$ en konstant funksjon (på hver komponent av def. mengden)



Hvis F og G er to antideriverte til $f(x)$, da må $F(x) - G(x)$ = konstant funksjon

$$F(x) = G(x) + C$$

Eks. $S(t)$



$$V(t) = S'(t)$$

$S(t)$ er ikkje bestemt av bare $V(t)$

vi må også ha en startposisjon

$$S(0) = S_0$$

La $F(t)$ være en antiderivert til $V(t)$

$$F'(t) = V(t) = S'(t)$$

så $S(t) = F(t) + C$ ← konstant.

$$S_0 = S(0) = F(0) + C$$

Dette gir $C = S_0 - F(0)$

$$\underline{S(t) = (F(t) - F(0)) + S_0}$$

Bevegelseslikningen

$S(t)$, $V(t) = S'(t)$, $a(t) = V'(t) = S''(t)$

Anta akselerasjonen er konstant

$\underline{a(t) = a}$ (som $-g \sim -9{,}8\frac{m}{s^2}$)

$V'(t) = a$ konstant

$$V(t) = a \cdot t + V_0$$

$V_0 = V(0)$

$S_0 = S(0)$

$$S'(t) = V(t) = a \cdot t + V_0$$

$$\underline{S(t) = \frac{a}{2}t^2 + V_0 \cdot t + S_0}$$

Det ubestemte integralet til $f(x)$ er samlingen av alle antideriverte til $f(x)$

hva vi integrerer med hensyn til.

$$\int \underbrace{f(x)}_{\text{integranden}} \, dx = F(x) + C$$

integral tegn

én antiderivert

Eks

$$\int 2x \, dx = x^2 + C$$

$$\int \frac{1}{x^2} \, dx = \frac{-1}{x} + C \qquad x \neq 0$$

(kan velge forskjellige konstanter for $x<0$ og for $x>0$)

$$\int x^6 \, dx = \frac{x^7}{7} + C$$

(siden $(x^7)' = 7 \cdot x^6$)

$$\left(\tfrac{1}{7}x^7\right)' = \tfrac{1}{7}(x^7)' = \tfrac{1}{7} \cdot 7x^6 = x^6$$

oppgave

$$\int 3x^2\,dx = x^3 + C$$

$$\int x^2\,dx = \frac{x^3}{3} + C$$

Resultater

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,f(x)\,dx = k\int f(x)\,dx \quad (+c)$$

Ubestemte integraler er lineære.

$$F'(x) = f(x) \qquad G'(x) = g(x)$$

$$(F+G)' = F' + G' = f(x) + g(x)$$

$$\left.\begin{aligned}\int f(x)\,dx + \int g(x)\,dx &= F(x) + G(x) + C\\ \int f(x) + g(x)\,dx &= (F(x) + G(x)) + C\end{aligned}\right]\text{like}$$

$$(k \cdot F(x))' = k(F'(x)) = k\,f(x)$$

så
$$\int k \cdot f(x)\,dx = k \cdot F(x) + C$$
$$= k(F(x) + C) \quad (+C')$$

Eks $\int 7x\,dx = 7\int x\,dx$

$$= 7\left(\frac{x^2}{2}\right) + c$$

$$= \underline{\frac{7}{2}\cdot x^2 + c}$$

$$\int \underbrace{0\cdot x}_{0}\,dx = 0\cdot\int x\,dx = c$$

konstante funksjoner

$$\int x^6 + x^2\,dx = \int x^6 dx + \int x^2 dx$$

$$= \frac{x^7}{7} + \frac{x^3}{3} + c$$

Resultat $$\int x^r dx = \begin{cases} \frac{x^{r+1}}{r+1} + c, & r \neq -1 \\ \ln|x| + c, & r = -1 \end{cases}$$

$(x^{s})' = s\cdot x^{s-1} \Rightarrow s = r+1$

$(x^{r+1})' = (r+1)x^r$ $r \neq -1$ deler med $r+1$

$$\left(\frac{x^{r+1}}{r+1}\right)' = x^r$$

$(\ln x)' = \frac{1}{x}$ $x > 0$

$r = -1$

$(\ln(-x))' = \frac{1}{-x}(-x)' = \frac{-1}{-x} = \frac{1}{x}$

for $x < 0$

så $(\ln|x|)' = \frac{1}{x} \quad x \neq 0.$

Eks.

$$\int x^{17}\,dx = \frac{x^{18}}{18} + c$$

$$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx = \frac{x^{-4}}{-4} + c = \underline{\frac{-1}{4x^4} + c}$$

$$\int x\cdot\sqrt{x}\,dx = \int x^1\cdot x^{1/2}\,dx$$

$$= \int x^{1+\frac{1}{2}}\,dx = \int x^{3/2}\,dx$$

$$= \frac{x^{5/2}}{5/2} + c = \underline{\frac{2x^{5/2}}{5} + c}$$

$$= \frac{2}{5}x^{2+\frac{1}{2}} + c = \underline{\frac{2}{5}x^2\sqrt{x} + c}$$

$$\int \frac{3}{x^{7/5}}\,dx = 3\int x^{-7/5}\,dx$$

$$= 3 \cdot \frac{x^{-2/5}}{-2/5} + c$$

$$= \frac{-3\cdot 5}{2} \cdot x^{-2/5} + c = \underline{\frac{-3\cdot 5}{2x^{2/5}} + c}$$

oppg

$$\int 3x^4\,dx = 3\int x^4\,dx$$

$$= 3\,\frac{x^5}{5} + c = \underline{\frac{3}{5}\cdot x^5 + c}$$

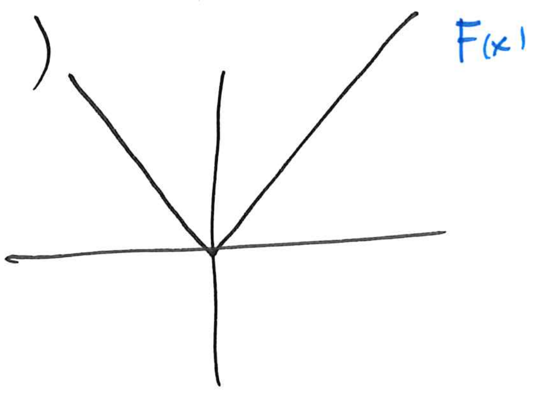
$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx$$

$$= \frac{x^{1/2}}{1/2} + c = 2\cdot x^{1/2} + c$$

$$= \underline{2\sqrt{x} + c}$$

Resultat (begrensa) kontinuerlige funksjoner har antideriverte.

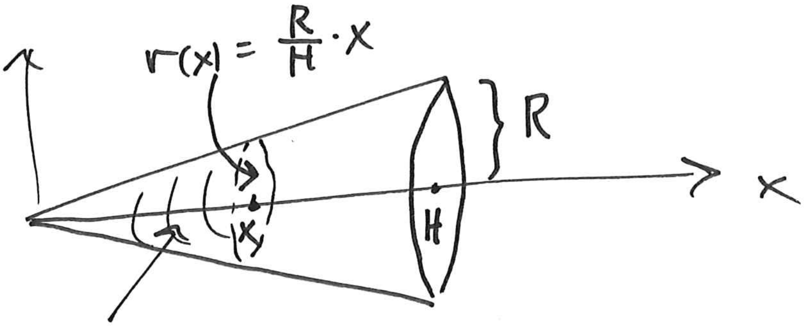
$$f(x) = \begin{cases} -1 & x<0 \\ 1 & x \geq 0 \end{cases}$$ har ikke en antiderivert (i $x=0$)



$F'(x) = f(x)$ for $x \neq 0$

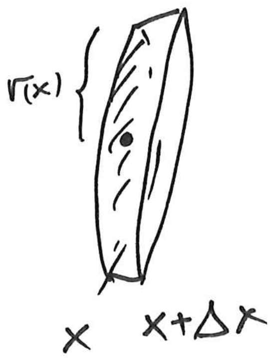
$$\int |x|\,dx = \begin{cases} -x^2/2 + c & x<0 \\ x^2/2 + c & x \geq 0 \end{cases}$$

Eksempel. Volum til kjegle



$V(x)$

Volumet til kjeglen frem til x.



$$V'(x) = \lim_{\Delta x \to 0} \frac{V(x+\Delta x) - V(x)}{x}$$

$\frac{\Delta V(x)}{\Delta x}$ = tverrsnitt-arealet i x

$$= \pi \cdot r(x)^2$$

$$V'(x) = \pi\left(\frac{R}{H}\cdot x\right)^2 = \left(\frac{\pi R^2}{H^2}\right)\cdot x^2, \quad V(0) = 0$$

$$V(x) = \frac{\pi R^2}{H^2}\cdot \underbrace{\frac{x^3}{3}}_{\text{antiderivert til } x^2,} + C$$

siden $V(0) = 0$ må $C = 0$

$$V(x) = \frac{\pi R^2}{3H^2}\cdot x^3$$

setter inn $x = H$

$$V(H) = \frac{\pi R^2}{3H^2}\cdot H^3 = \underline{\underline{\frac{\pi R^2 H}{3}}}$$

volumet til en kjegle med høyde H og grunnflate en disk med radius R.

15.3

$$(e^x)' = e^x$$

$$\int e^x dx = e^x + c$$

$$(e^{a\cdot x+b})' = e^{a\cdot x+b}\cdot(a\cdot x+b)'$$

$$= a\cdot e^{ax+b} \qquad a \neq 0$$

$$\left(\frac{1}{a}e^{ax+b}\right)' = e^{ax+b}$$

$$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$$

$$\int e^{2x}dx = \frac{1}{2}e^{2x} + c$$

$$\int e^{x+3}dx = e^{x+3} + c$$

(alternativt $e^{x+3} = e^x\cdot e^3 = e^3\cdot e^x$

$$\int e^{x+3}dx = \int e^3\cdot e^x dx = e^3\int e^x dx =$$

$$e^3 e^x + c = e^{x+3} + c$$)

$$\int e^{-3x+2} dx = \underline{\frac{1}{-3}e^{-3x+2} + c}$$

$$(e^{u(x)})' = u'(x)\cdot e^{u(x)}$$

$$\int u'(x)e^{u(x)}dx = e^{u(x)} + c$$

Så

$$\int \underbrace{2x}_{u'} \cdot \underbrace{e^{x^2}}_{e^u}\, dx = e^{x^2} + c$$

$$\int x^2 e^{x^3} dx$$

$u = x^3$

$u' = 3x^2$

$$= \int \frac{1}{3} \cdot \underbrace{(3x^2)}_{u'} \cdot \underbrace{e^{x^3}}_{e^u} dx$$

$$= \frac{1}{3} \int (3x^2) e^{x^3} dx = \underline{\frac{1}{3} e^{x^3} + c}$$

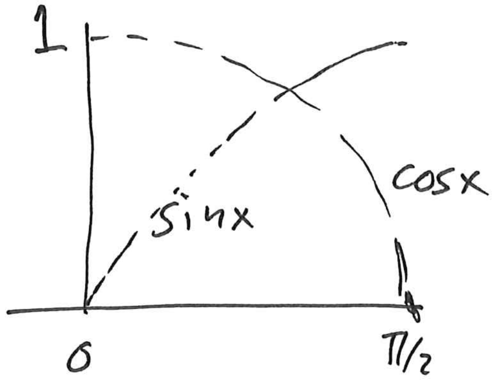
$$\int e^{x^2} dx$$

Har ikke en antiderivert som er en elementær funksjon.

15.4

$(\sin x)' = \cos x$

$(\cos x) = -\sin x$



$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$(\sin(ax+b))' = \cos(ax+b)\,(ax+b)'$$
$$= a \cdot \cos(ax+b) \qquad a \neq 0$$

$$\left(\frac{1}{a}\sin(ax+b)\right)' = \cos(ax+b)$$

$$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$$

tilsvarende $\int \sin(ax+b)dx = \underline{\frac{-1}{a}}\cos(ax+b) + C$

$$\int \sin(\pi x - 1)dx$$
$$= \frac{-1}{\pi}\cos(\pi x - 1) + C$$

$$\int \underbrace{2\sin(x)\cdot\cos(x)}_{\sin 2x}\,dx \quad \text{tig. identitet}$$

$$= \int \sin(2x)\,dx = -\frac{1}{2}\cos(2x) + C$$

oppg

$$\int \cos\left(\frac{x}{3}\right)dx = \frac{1}{1/3}\sin\left(\frac{x}{3}\right) + C$$

$$= 3\sin\left(\frac{x}{3}\right) + C$$

$$\int \cos^2(x)\,dx$$

Hint: $\cos(2x) = \cos^2 x - \sin^2 x$

$1+\cos(2x) = \cos^2 x + (1-\sin^2 x) = 2\cos^2 x$

$\cos^2 x = \frac{1}{2}(1+\cos(2x))\ \ldots$

$\int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos(2x))\,dx$

$\frac{1}{2}\int 1+\cos(2x)\,dx = \frac{1}{2}\left[\int 1\,dx + \int \cos(2x)\,dx\right]$

$= \frac{1}{2}\left[x + \frac{\sin(2x)}{2}\right] + C$

$= \frac{x}{2} + \frac{\sin(2x)}{4} + C$

15.2 $\int \frac{1}{x}\,dx = \ln|x| + c$

$$\left(\ln|u(x)|\right)' = \frac{1}{u(x)} \cdot u'(x)$$

$$\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + c$$

$$\int \frac{1}{3x+1}\,dx$$

$u = 3x+1$

$u' = 3$

$$\int \frac{\frac{1}{3} \cdot 3}{3x+1}\,dx = \frac{1}{3} \int \underbrace{\frac{3}{3x+1}}_{u'/u}\,dx$$

$$= \frac{1}{3} \ln|u(x)| + c$$

$$= \underline{\frac{1}{3} \ln|3x+1| + c}$$

$$\int \frac{2x}{x^2+1}\,dx = \int \frac{u'}{u}\,dx$$

hvor $u(x) = x^2+1$

$$= \ln|u(x)| + c$$

$$= \underline{\ln|x^2+1| + c}$$

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$$

$$\left(u = \cos(x), \quad u' = -\sin x \right)$$

$$= \int \frac{-u'}{u}\,dx = -\int \frac{u'}{u}\,dx = -\ln|u(x)| + c$$

$$\int \tan(x)\,dx = \underline{-\ln|\cos(x)| + c}$$

Eks.

$$\int 2^x\,dx = \frac{2^x}{\ln(2)} + c$$

$$2 = e^{\ln 2} \qquad 2^x = \left(e^{\ln 2}\right)^x = e^{x\cdot\ln 2}$$

$$\underline{(2^x)'} = \left(e^{x\cdot\ln 2}\right)' = \underline{\ln 2 \cdot 2^x}$$

$$\int \sin x \cdot e^{\cos x}\,dx \qquad u = \cos x, \quad u' = -\sin x$$

$$= \int (-1)\,u' \cdot e^{u}\,dx$$

$$= -\int u' e^{u}\,dx = -e^{u(x)} + c$$

$$\int \sin x\, e^{\cos x}\,dx = \underline{-e^{\cos x} + c}$$

$$\int \frac{1}{-2x+3}\,dx \qquad u = -2x+3, \quad u' = -2$$

$$= \int \frac{\left(\frac{-1}{2}\right)(-2)}{-2x+3}\,dx = \frac{-1}{2}\int \frac{-2}{-2x+3}\,dx = \underline{\frac{-1}{2}\ln|-2x+3| + c}$$