Derivasjon : Kjerneregelen

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$F(x) = f(g(x))$$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

Alternativ notasjon :

$$\frac{dF}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

Eksempel :

Deriver $F(x) = e^{\cos^2 x}$ — $g(x)$

$$F(x) = f(g(x)) = e^{g(x)}$$

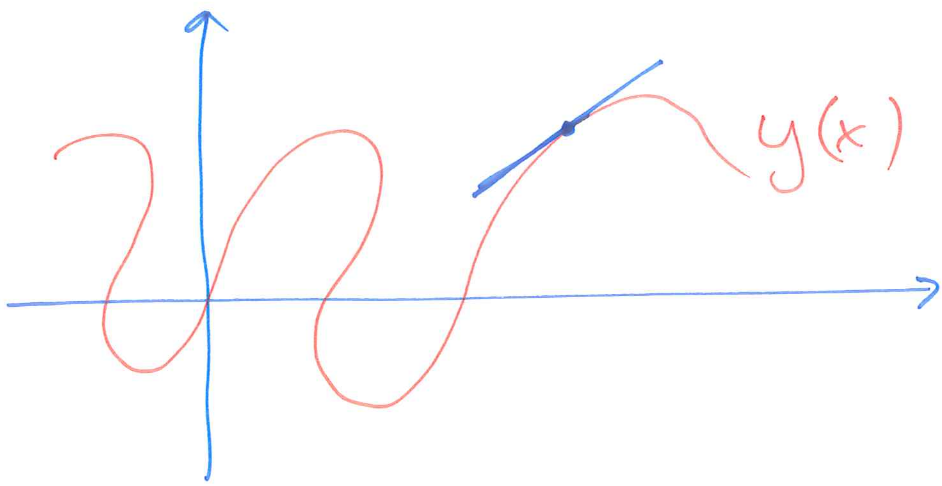
$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$= e^{g(x)} \cdot g'(x)$$

$$= \underline{e^{\cos^2 x} \cdot 2\cos x \cdot (-\sin x)}$$

Implisitt derivasjon

En funksjon $y(x)$ kan være implisitt gitt (eks: $x^2 + y^2 = 1$)



Eksempel:

a) Vis at $(1,1)$ ligger på grafen til

$$x^2y^2 + x^3(y-2) = 0$$

Løsning:

$x = y = 1$ og setter inn:

$$VS = 1^2 1^2 + 1^3(1-2) = 1 + (-1) = 0$$

$$HS = 0$$

Stemmer!

b) Bestem stigningstallet til tangenten til grafen i (1,1).

Bruker implisitt derivasjon:

$$\frac{d}{dx}(Vs) = \frac{d}{dx}(x^2y^2) + \frac{d}{dx}\left(x^3(y-2)\right)$$

$$= 2x \cdot y^2 + x^2 \cdot 2y \cdot \frac{dy}{dx}$$

$$+ 3x^2(y-2) + x^3\frac{dy}{dx}$$

$$\frac{d}{dx}(Hs) = 0$$

$$\frac{d}{dx}(Vs) = \frac{d}{dx}(Hs):$$

$$2x \cdot y^2 + x^2 \cdot 2y\frac{dy}{dx} + 3x^2(y-2) + x^3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2x^2y + x^3) + 2xy^2 + 3x^2(y-2) = 0$$

Setter inn $x = y = 1$:

$$\frac{dy}{dx}\underbrace{(2\cdot 1^2 \cdot 1 + 1^3)}_{3} + \underbrace{2\cdot 1 \cdot 1^2 + 3\cdot 1^2(1-2)}_{-1} = 0$$

$$3\frac{dy}{dx} - 1 = 0 \Rightarrow \underline{\frac{dy}{dx} = \frac{1}{3}}$$

Stigningstallet til tangenten er $\underline{\frac{1}{3}}$

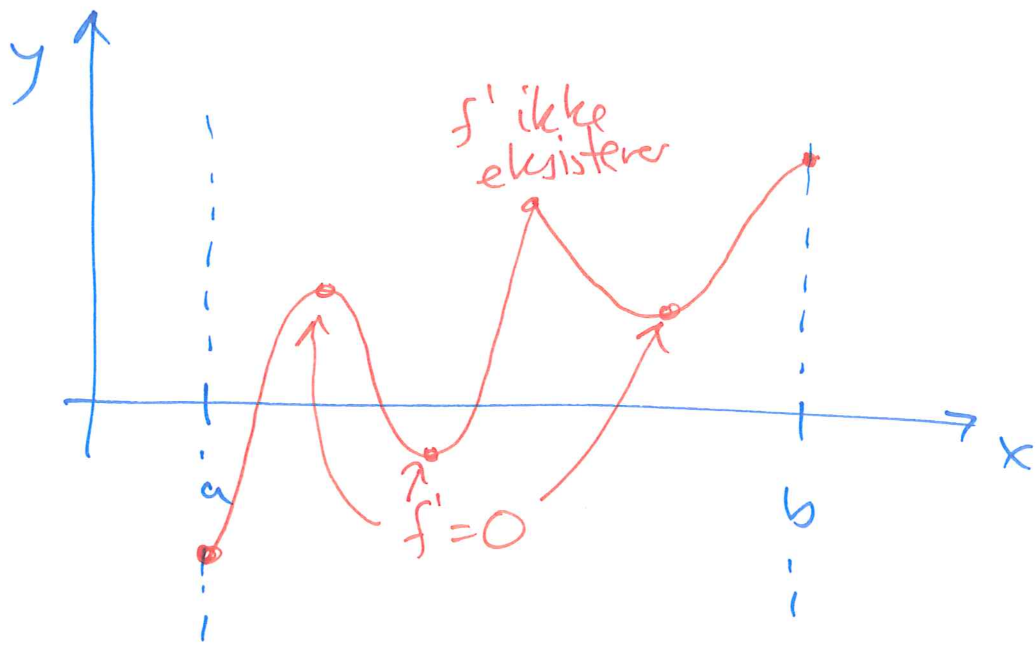
Bestemmelse av maks- og min-punkter

Lokale maks- og min-punkter er enten

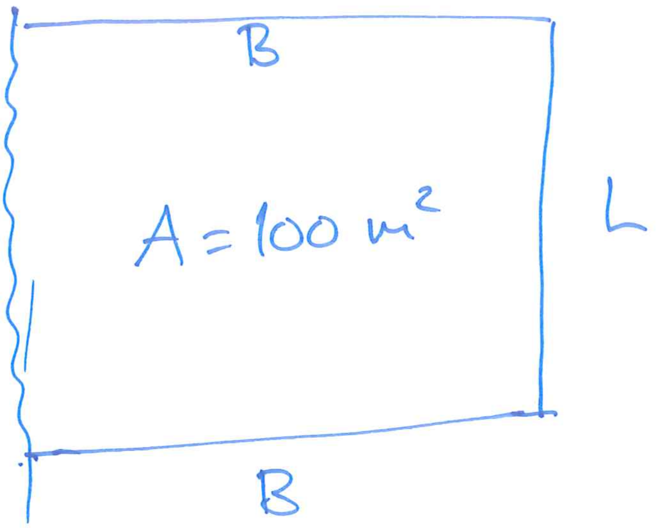
1. punkter hvor $f'(x) = 0$

2. punkter hvor f' ikke eksisterer

3. randpunkter (endepunkter).



Optimalisering, eksempel



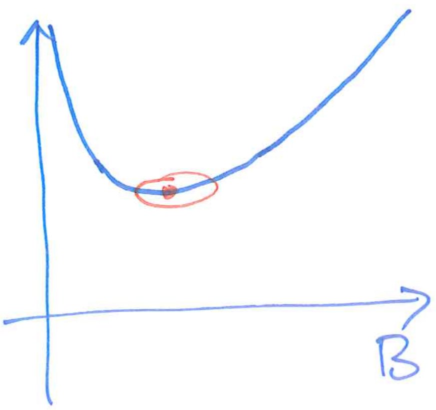
Hvordan blir gjerdet så kort som mulig?

Tot. lengde av gjerdet = $L + 2B$

$A = 100\ m^2$: $\underbrace{L \cdot B = 100}$ $\Rightarrow L = \frac{100}{B}$

$$\Rightarrow \text{Lengden} = f(B) = \underbrace{\frac{100}{B} + 2B}_{\text{skal minimeres}}$$

$100 \cdot B^{-1}$



Setter $f'(B) = 0$:

$$f'(B) = 100 \cdot (-1) B^{-1-1} + 2$$

$$= \frac{-100}{B^2} + 2$$

$f'(B) = 0$: $\quad -\frac{100}{B^2} + 2 = 0$

$$\frac{100}{B^2} = 2$$

$$\frac{100}{2} = \frac{2B^2}{2}$$

$$B^2 = 50$$

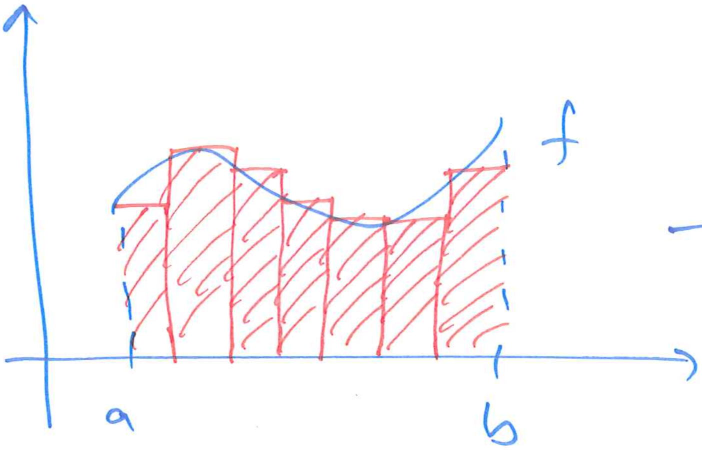
$$B = \sqrt{50} = \underline{\underline{5\sqrt{2}}} \text{ (m)}$$

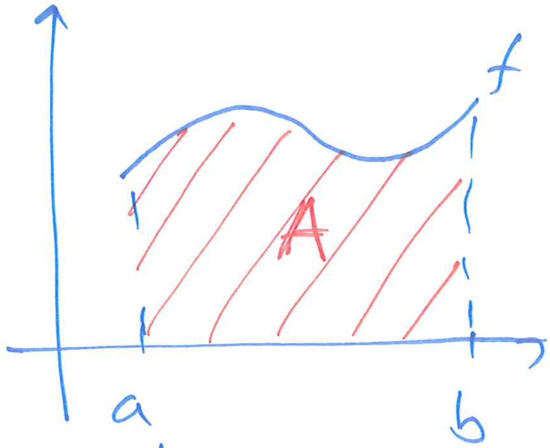
$$\approx \underline{\underline{7{,}1 \text{ m}}}$$

$$\Rightarrow L = \frac{100}{7{,}1} \approx \cancel{28{,}3 \text{ m}}\ \underline{\underline{14{,}2 \text{ m}}}$$

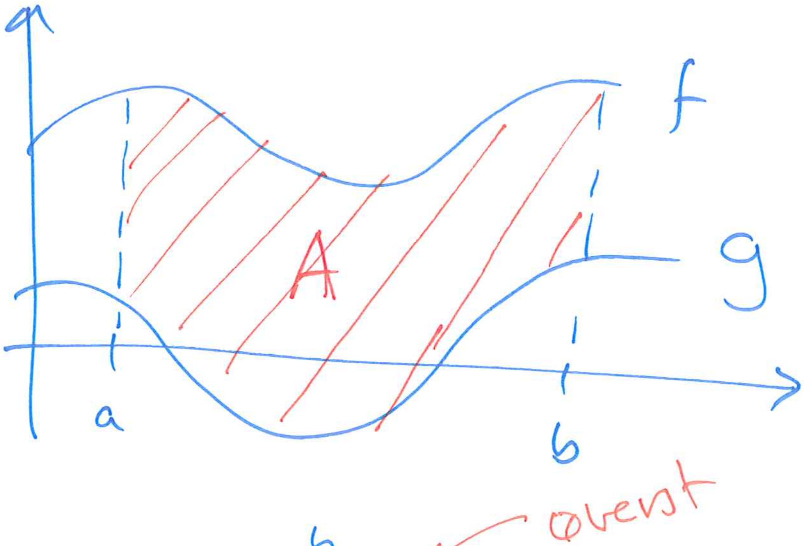
$\Rightarrow$ Lengden av gjerdet $\approx \underline{28{,}3 \text{ m}}$

Det bestemte integralet





$$\underbrace{\sum_k A_{stolpe_k}}_{\text{Riemannsum}} \longrightarrow \underbrace{\int_a^b f(x)\,dx}_{A}$$



øverst

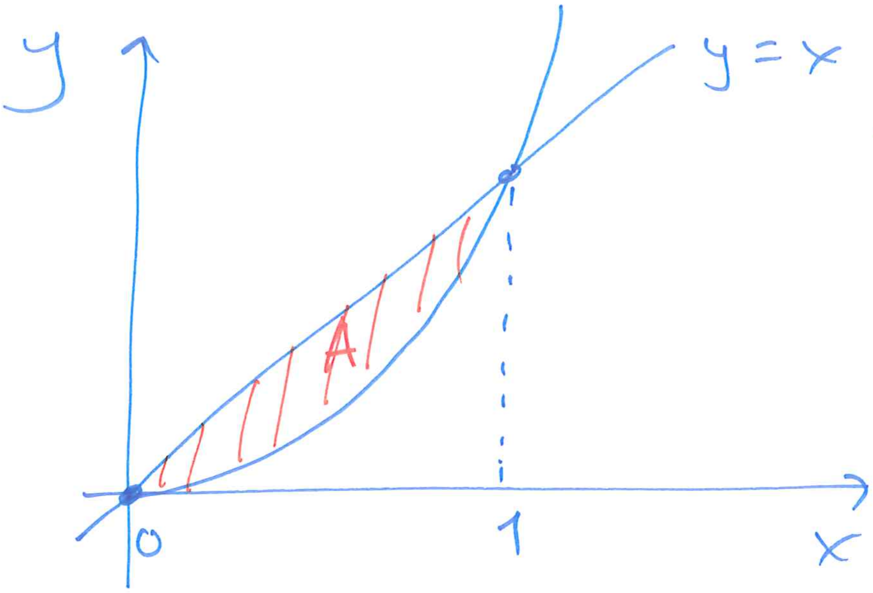
$$A = \int_a^b \big(f(x) - g(x)\big)\,dx$$

nederst

Eksempel:

Bestem arealet mellom grafene til

$y = x$ og $y = x^2$ (for $x \geq 0$).



$x = x^2$

$x = 0$

$x = 1$

$$A = \int_0^1 (y_{øverst} - y_{nederst})\, dx$$

$$= \int_0^1 (x - x^2)\, dx = \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1$$

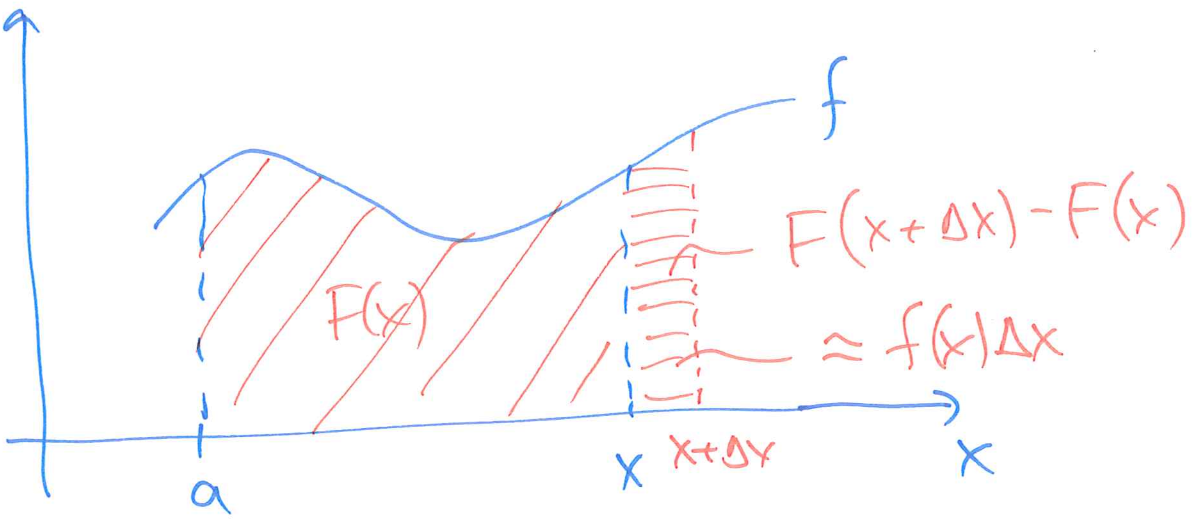
$$= \tfrac{1}{2}\cdot 1^2 - \tfrac{1}{3}1^3 - \left(\tfrac{1}{2}\cdot 0^2 - \tfrac{1}{3}\cdot 0^3\right)$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \underline{\underline{\frac{1}{6}}}$$

Analysens fundamentalteorem

$$F(x) = \int_a^x f(t)\,dt$$

$$\Rightarrow \quad F'(x) = f(x)$$



$$\frac{f(x)\cancel{\Delta x}}{\cancel{\Delta x}} \approx \frac{F(x+\Delta x) - F(x)}{\Delta x} \longrightarrow \underline{F'(x)}$$

$$\int_a^b f(t)\,dt = \underline{F(b) - F(a)}$$

Eksempel:

$$F(x) = \int_0^x \underbrace{\cos(t^2)}_{f(t)}\, dt$$

$$F'(x) = f(x) = \underline{\underline{\cos(x^2)}}$$

$$F(x) = \int_1^{x^2} \underbrace{(y-1)^2}_{f(y)}\, dy$$

$$F(x) = G(h(x)) = \int_1^{h(x) = x^2} (y-1)^2\, dy$$

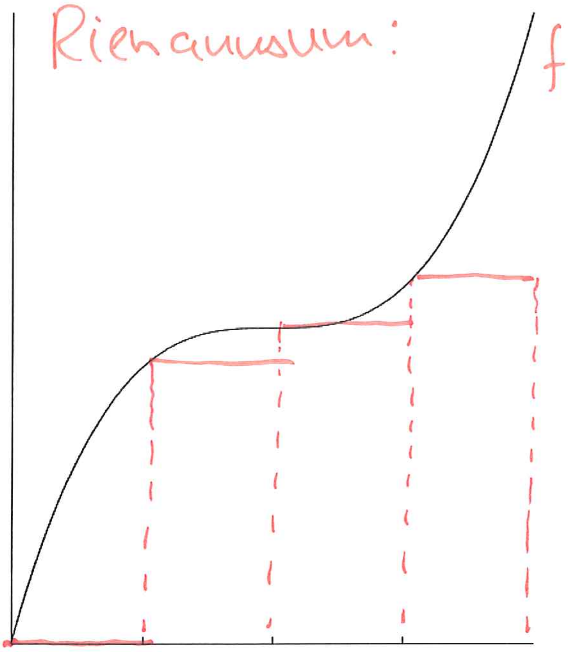
$$F'(x) = G'(h(x)) \cdot h'(x)$$

$$= (h(x)-1)^2 \cdot 2x$$

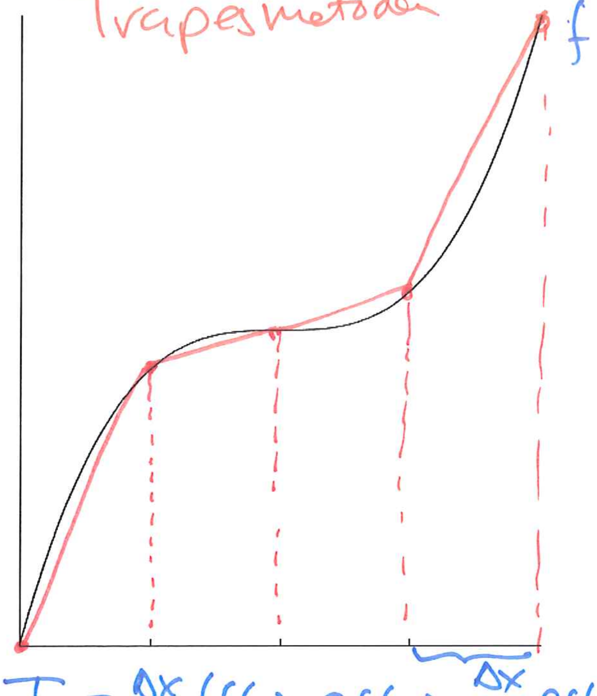
$$= \underline{\underline{(x^2-1)^2 \cdot 2x}}$$

Numerisk integrasjon

Riemannsum:

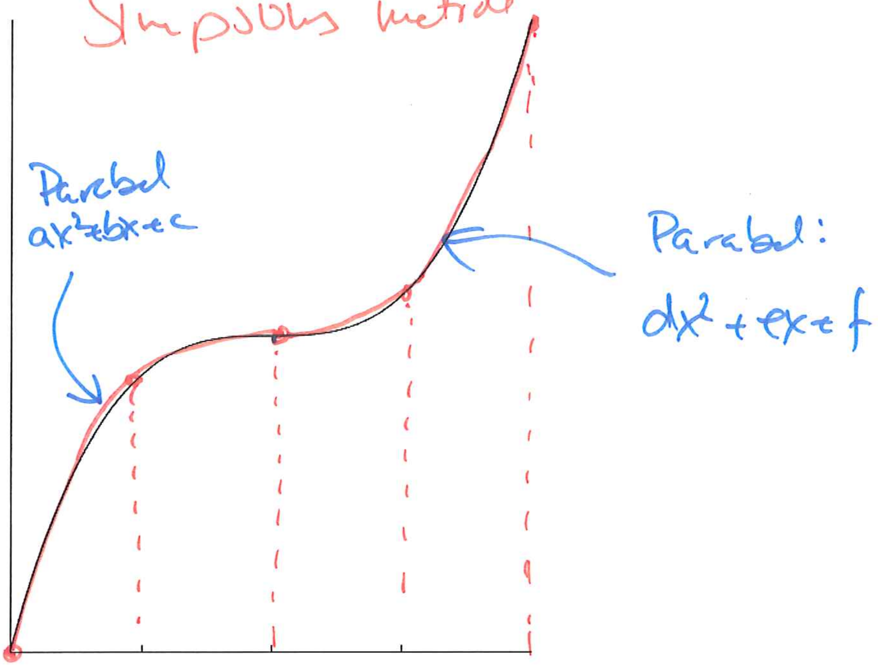


Trapesmetoden



$$T = \frac{\Delta x}{2}\Big(f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)\Big)$$

Simpsons metode



$$S = \frac{\Delta x}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})\Big)$$

Integrasjon ved substitusjon

$$\frac{d}{dx}\left[F(u(x))\right] = F'(u)\cdot u'(x)$$

$$\boxed{\int f(u)\,u'(x)\,dx = \int f(u)\,du}$$

Integrasjon ved substitusjon

Eksempel:

Regn ut $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$ $\qquad e^{2x} = (e^x)^2$

$$\int \frac{e^x}{1+(e^x)^2}\,dx$$

Velg $u = e^x \Rightarrow dx = \dfrac{du}{u'} = \dfrac{du}{e^x}$

$$\int \frac{e^x}{1+(e^x)^2}\,dx = \int \frac{\cancel{e^x}}{1+u^2}\,\frac{du}{\cancel{e^x}}$$

$$= \int \frac{1}{1+u^2}\,du = \arctan u + C$$

$$= \underline{\arctan(e^x) + C}$$

Delvis integrasjon

$$(uv)' = u'v + uv'$$

$$uv = \int u'v\,dx + \int uv'\,dx$$

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

Delvis integrasjon

Eksempel: Regn ut $\int \underbrace{\sin x}_{u} \cdot \underbrace{\cos x}_{v'}\,dx$

$u = \sin x \quad , \quad u' = \cos x$

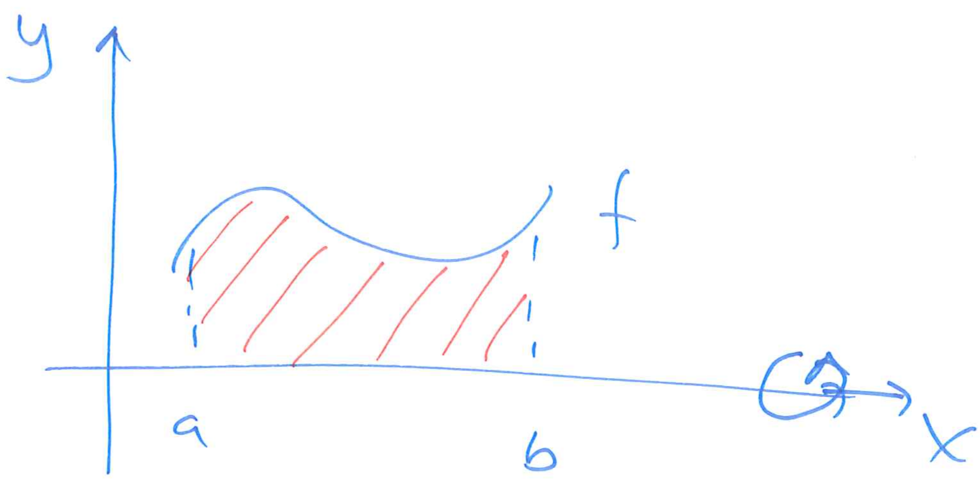
$v' = \cos x \quad , \quad v = \sin x$

$$\int \sin x \cdot \cos x\,dx = \sin x \cdot \sin x - \int \cos x \cdot \sin x\,dx$$

$$2\int \sin x \cos x\,dx = \sin^2 x + C_1$$

$$\int \sin x \cos x\,dx = \underline{\frac{1}{2}\sin^2 x + C_2}$$

Volum av omdreiningslegemer

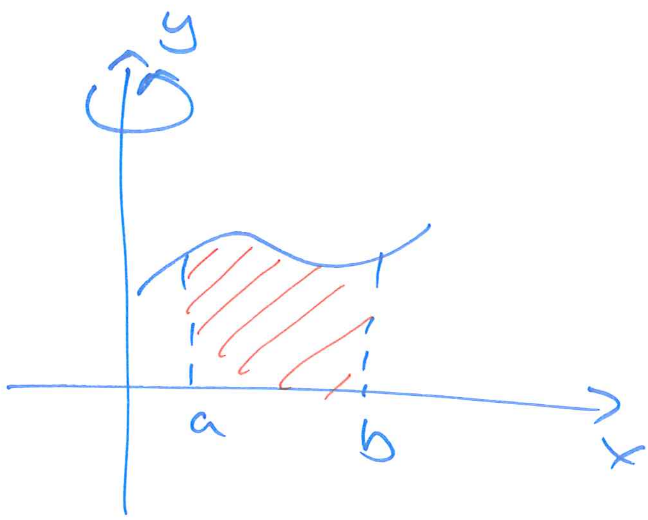


Om x-aksen:

$$V = \pi \int_a^b \left(f(x)\right)^2 dx$$

Om y-aksen:

$$V = 2\pi \int_a^b x\,|f(x)|\,dx$$



Koblede hastigheter

Når to størrelser kan relateres til hverandre og vi har en observasjon av endringsraten til den ene, så kan vi regne ut endringsraten til den andre.

den deriverte

Relasjon: $F = f(g(t))$

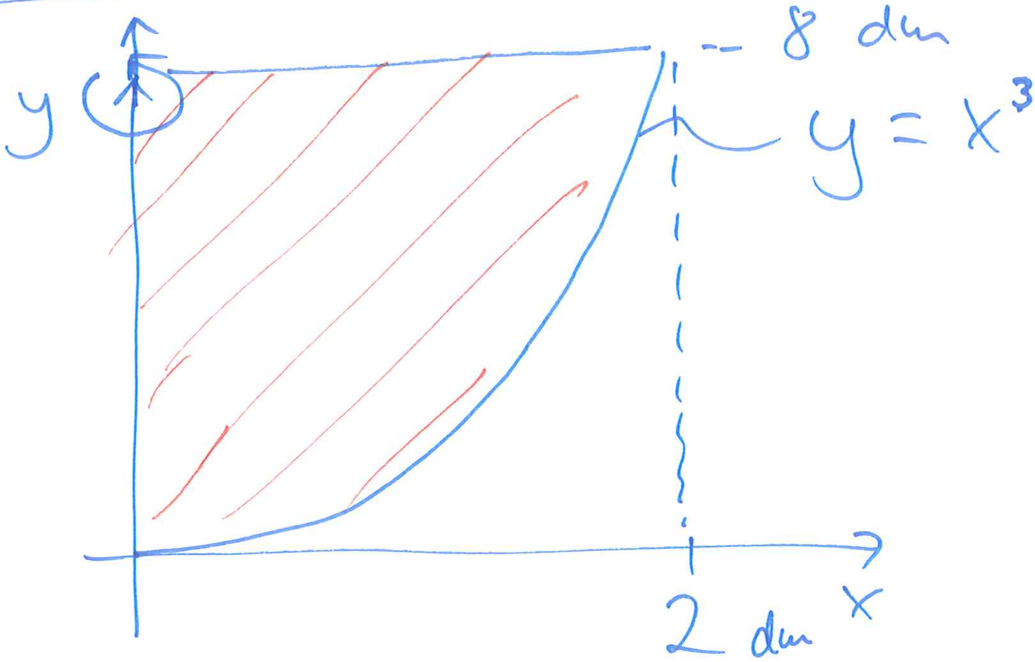
$$\frac{dF}{dt} = \frac{df}{dg} \frac{dg}{dt}$$

Løser mht denne

Kjent

Regnes ut

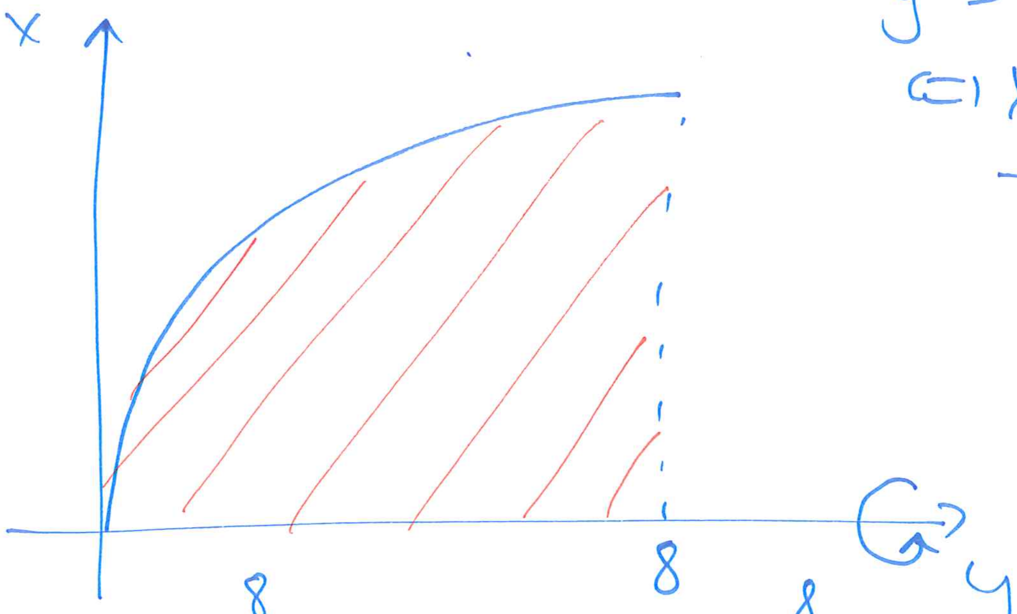
Eksamen desember 2015 oppg 8



a) Vis at volumet er $\frac{96\pi}{5}$ (dm^3).

$y = x^3$

$\Leftrightarrow x = y^{1/3}$



$$V = \pi \int_0^8 \left(y^{1/3}\right)^2 dy = \pi \int_0^8 y^{2/3}\, dy$$

$$= \pi \left[\frac{1}{2/3+1} y^{2/3+1}\right]_0^8 = \pi \frac{1}{5/3} 8^{5/3} = \frac{96\pi}{5}$$

b) $\frac{dV}{dt} = \text{konstant} = \frac{\frac{96\pi}{5}\,dm^3}{2\ min}$

$= \frac{48\pi}{5}\ dm^3/min$

Vet at

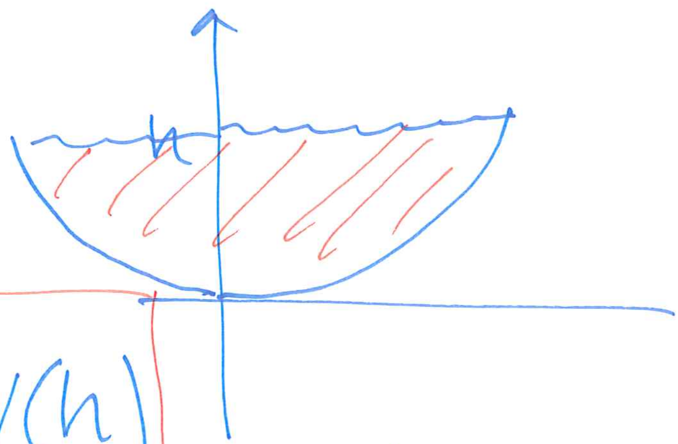
Kjent

$$\boxed{\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}}$$

Skal bestemmes

Regn ut.

$$V = \pi\int_0^h y^{2/3}\,dy$$



$$\boxed{= \frac{3\pi}{5}\,h^{5/3} = V(h)}$$

$$\Rightarrow \frac{dV}{dh} = \frac{3\pi}{\cancel{5}}\,h^{5/3-1}\cdot\frac{\cancel{5}}{\cancel{3}} = \pi h^{2/3}$$

$= 1\ dm$

$$\frac{48\pi}{5} = \cancel{\pi}\cdot h^{2/3}\cdot\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{48/5}{1^{2/3}} \approx 9{,}6\ dm/min$$

Neste uke (9/5, 11/5):

Gjennomgang av eksamen

februar 2014.

Regneøvinger mandag $14^{\underline{30}}$ - $16^{\underline{00}}$

" onsdag $8^{\underline{30}}$ - $12^{\underline{00}}$